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Solving Railway Track Allocation Problems

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Solving Railway Track Allocation Problems*

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Abstract

The *optimal track allocation problem* (OPTRA), also known as the train routing problem or the train timetabling problem, is to find, in a given railway network, a conflict-free set of train routes of maximum value. We propose a novel integer programming formulation for this problem that is based on additional ‘configuration’ variables. Its LP-relaxation can be solved in polynomial time. These results are the theoretical basis for a column generation algorithm to solve large-scale track allocation problems. Computational results for the Hanover-Kassel-Fulda area of the German long distance railway network involving up to 570 trains are reported.

1 Introduction

Routing a maximum number of trains in a conflict-free way through a track network is one of the basic scheduling problems for a railway company. The problem has received growing attention in the operations research literature recently, see, e.g., Brännlund et al. [1998], Caprara et al. [2001], Caprara et al. [2002], Borndörfer et al. [2006], Cacchiani [2007] and Cacchiani et al. [2008]. All of these articles model the track allocation problem in terms of a multi-commodity flow of trains in an appropriate time expanded graph, ruling out conflicts by additional packing constraints.

The main problem with this approach is that the resulting integer programs become notoriously difficult already for small problem sizes. This is due to an enormous number of (weak) packing constraints in the model. The purpose of this article is to propose a new formulation of the ‘extended’ type, that handles conflicts not in terms of constraints, but in terms of additional variables. Our formulation has a constant number of rows, is amenable to standard column generation techniques, and therefore suited for large-scale computation.

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2 The Optimal Track Allocation Problem

The optimal track allocation problem can be formally described in terms of a digraph $D = (V, A)$. Its nodes represent arrivals and departures of trains at a set S of stations at discrete times $T \subseteq \mathbb{Z}$, its arcs model runs of trains between stations. Denote by $s(v) \in S$ the station associated with departure or arrival $v \in V$, and by $t(v) \in T$ the time of this event; we assume $t(u) < t(v)$ for each arc $uv \in A$ such that D is acyclic. Denote by $J = \{s(u)s(v) : uv \in A\}$ the set of all railway tracks. We are further given a set I of requests to route trains through D . More precisely, train $i \in I$ can be routed on a path through some suitably defined subdigraph $D_i = (V_i, A_i) \subseteq D$ from a starting point $s_i \in V_i$ to a terminal point $t_i \in V_i$; let P_i be the set of all routes for train $i \in I$, and $P = \bigcup_{i \in I} P_i$ the set of all train routes (taking the disjoint union). We say that an arc $uv \in A$ *blocks* the underlying track $s(u)s(v)$ during the time interval $[t(u), t(v) - 1]$, that two arcs $a, b \in A$ are *in conflict* if their respective blocking intervals overlap, and that two routes $p, q \in P$ are in conflict if any of their arcs are in conflict. A *track allocation* or timetable is a set of conflict-free routes, at most one for each train. Given arc weights $w_a, a \in A$, the weight of route $p \in P$ is $w_p = \sum_{a \in p} w_a$, and the weight of a track allocation $X \subseteq P$ is $w(X) = \sum_{p \in X} w_p$. The *optimal track allocation problem* (OPTRA) is to find a track allocation of maximum weight.

We refer the reader to the articles Caprara et al. [2001], Caprara et al. [2002], and Borndörfer et al. [2006] for discussions how this basic model can be set up to deal with various technical and operational requirements such as preferences for departure, arrival, and travel times, train driving dynamics, single and double tracks, zero-level crossings, station capacities, headways, dwell and turnover times, routing corridors, correspondences, complementarities, and synergies between trains etc.

OPTRA is \mathcal{NP} -hard Caprara et al. [2002]. It can be seen as a multi-commodity flow problem with additional packing constraints, which can be modeled in terms of inequalities Caprara et al. [2001], Caprara et al. [2002], Borndörfer et al. [2006]. We propose here an alternative formulation that is based on arc ‘configurations’, i.e., sets of arcs on the same underlying track that are mutually not in conflict. Formally, let $A_{st} = \{uv \in A : s(u)s(v) = st\}$ be the set of all arcs associated with some track $st \in J$; a *configuration* for this track st is a set of arcs $q \subseteq A_{st}$ that are mutually conflict-free. Let Q_j denote the set of all configuration associated with track $j \in J$, and $Q = \bigcup_{j \in J} Q_j$ the set of all configurations.

Introducing 0/1-variables $x_p, p \in P$, and $y_q, q \in Q$, OPTRA can be stated as the following integer program.

$$\begin{aligned}
(\text{PCP}) \quad & \text{(i)} \quad \max \sum_{p \in P} w_p x_p \\
& \text{(ii)} \quad \sum_{p \in P_i} x_p \leq 1, \quad \forall i \in I \\
& \text{(iii)} \quad \sum_{q \in Q_j} y_q \leq 1, \quad \forall j \in J \\
& \text{(iv)} \quad \sum_{a \in p \in P} x_p - \sum_{a \in q \in Q} y_q \leq 0, \quad \forall a \in A \\
& \text{(v)} \quad x_p, y_q \geq 0, \quad \forall p \in P, q \in Q \\
& \text{(vi)} \quad x_p, y_q \in \mathbb{Z}, \quad \forall p \in P, q \in Q.
\end{aligned}$$

The objective PCP (i) maximizes the weight of the track allocation. Constraints (ii) state that a train can run on at most one route, constraints (iii) allow at most one configuration for each track. Inequalities (iv) link train routes and track configurations to guarantee a conflict-free allocation, (v) and (vi) are the non-negativity and integrality constraints. Note that the upper bounds $x_p \leq 1$, $p \in P$, and $y_q \leq 1$, $q \in Q$, are redundant.

3 Column Generation

Consider the LP-relaxation PLP of PCP, i.e., $\text{PLP} = \text{PCP (i)-(v)}$; it can be solved by column generation. In fact, it will turn out that the pricing problems for both the route and the configuration variables can be solved in polynomial time by computing longest paths in appropriate acyclic graphs. To see this, consider the dual DLP of PLP.

$$\begin{aligned}
(\text{DLP}) \quad & \text{(i)} \quad \min \sum_{j \in J} \pi_j + \sum_{i \in I} \gamma_i \\
& \text{(ii)} \quad \gamma_i + \sum_{a \in p} \lambda_a \geq w_p \quad \forall p \in P_i, i \in I \\
& \text{(iii)} \quad \pi_j - \sum_{a \in q} \lambda_a \geq 0 \quad \forall q \in Q_j, j \in J \\
& \text{(iv)} \quad \gamma_i, \pi_j, \lambda_a \geq 0 \quad \forall i \in I, j \in J, a \in A.
\end{aligned}$$

Here, γ_i , $i \in I$, π_j , $j \in J$, and λ_a , $a \in A$, are the dual variables associated with constraints PLP (i), (ii), and (iii), respectively. The pricing problem for a route $p \in P_i$ for train $i \in I$ is

$$\exists p \in P_i : \gamma_i + \sum_{a \in p} \lambda_a < w_p \iff \sum_{a \in p} (w_a - \lambda_a) > \gamma_i.$$

This is the same as finding a longest $s_i t_i$ -path in D_i w.r.t. arc weights $w_a - \lambda_a$; as D_i is acyclic, this problem can be solved in polynomial time.

The pricing problem for a configuration $q \in Q_j$ for track $j \in J$ is

$$\exists q \in Q_j : \pi_j - \sum_{a \in q} \lambda_a < 0 \iff \sum_{a \in q} \lambda_a > \pi_j.$$

Let $j = st$ and consider the construction illustrated in Figure 1. Denote by $A_{st} = \{uv \in A : s(u)s(v) = st\}$ the set of arcs that run on track st and by $L_{st} := \{u : uv \in A_{st}\}$ and $R_{st} := \{v : uv \in A_{st}\}$ the associated set of departure and arrival nodes; note that all arcs in A_{st} go from L_{st} to R_{st} . Let $\bar{A}_{st} := \{vu : t(v) \leq t(u), v \in R_{st}, u \in L_{st}\}$ be a set of ‘return’ arcs that go in the opposite direction. It is easy to see that $D_{st} = (L_{st} \cup R_{st}, A_{st} \cup \bar{A}_{st})$ is bipartite and acyclic, and that $L_{st}R_{st}$ -paths $a_1, \bar{a}_1, \dots, \bar{a}_{k-1}, a_k$ in D_{st} and configurations a_1, \dots, a_k in Q_{st} are in 1-1 correspondence. Using arc weights λ_a , $a \in A_{st}$, and 0, $a \in \bar{A}_{st}$, pricing configurations in Q_{st} is equivalent to finding a longest $L_{st}R_{st}$ -path in D_{st} . As D_{st} is acyclic, this is polynomial. It follows

Theorem 3.1 *PLP can be solved in polynomial time.*

In practice, tailing-off prevents the straightforward solution of PLP to optimality. However, the path lengths $\max_{p \in P_i} \sum_{a \in p} (w_a - \lambda_a)$ and $\max_{q \in Q_j} \sum_{a \in q} \lambda_a$ yield the following bound $\beta = \beta(\gamma, \pi, \lambda)$.

Lemma 3.2 *Let $\gamma, \pi, \lambda \geq 0$ be dual variables¹ for PLP and $v(\text{PLP})$ the optimum of PLP. Let $\eta_i := \max_{p \in P_i} \sum_{a \in p} (w_a - \lambda_a) - \gamma_i$, $i \in I$, and $\theta_j := \max_{q \in Q_j} \sum_{a \in q} \lambda_a - \pi_j$, $j \in J$. Then:*

$$v(\text{PLP}) \leq \sum_{i \in I} \max\{\gamma_i + \eta_i, 0\} + \sum_{j \in J} \max\{\pi_j + \theta_j, 0\} =: \beta(\gamma, \pi, \lambda).$$

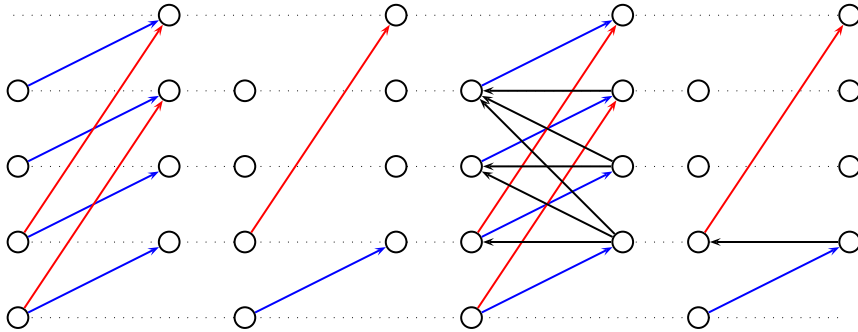


Figure 1: Arc configurations on a track. From left to right: train routing digraph, conflict-free configuration, configuration routing digraph, and LR -path.

¹Note that these will be infeasible during column generation.

4 Computational Results

We have implemented a column generation algorithm for the PCP along the lines of the preceding sections. We have used this code to solve three large-scale railway track allocation problems for the Hannover-Kassel-Fulda area of the German long-distance railway network involving 146, 250, and 570 trains, see Table 1. The instances are based on a common macroscopic infrastructure model with 37 stations and 120 tracks, 6 different train types (ICE, IC, RE, RB, S, ICG), and 4320 headway times, see Figure 2 for an illustration and Borndörfer et al. [2006] for a more detailed discussion.

Figure 3 illustrates the solution of the LP-relaxation PLP for the two large scenarios 2 and 3. It can be seen that the upper bound $\beta(\gamma, \pi, \lambda)$ and the optimal value $v(RPLP)$ of the restricted master-LP converge, i.e., we can indeed solve these LPs close to optimality. This provides a good starting point to compute high-quality integer solutions using standard rounding heuristics, see columns IP and gap in Table 1. All computations were made single-threaded on a Dell Precision 650 PC with 2GB of main memory and a dual Intel Xeon 3.8 GHz CPU running SUSE Linux 10.1. The reduced master-LPs were solved with CPLEX 10.0 using the barrier or dual simplex method, depending on the column generation progress.

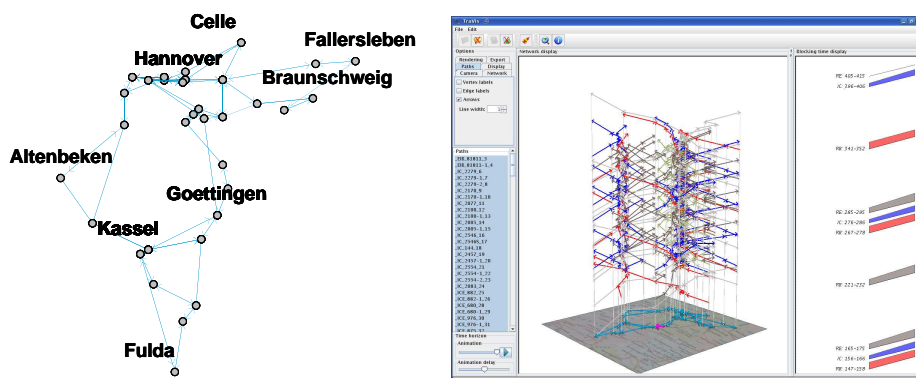


Figure 2: Infrastructure network (left), visualization of an allocation (right).

²*cols* is the max. number of columns in main memory during column generation.

no	$ I $	rows	cols ²	iter	β	$v(RPLP)$	IP	gap	time
								in %	in sec.
1	146	6034	120366	162	93418	93381	93371	0.05	4439
2	250	12461	213218	168	148101	147375	147375	0.75	39406
3	570	11112	250550	148	245278	239772	234538	4.58	59910

Tab. 1: Solving large-scale railway track allocation problems.

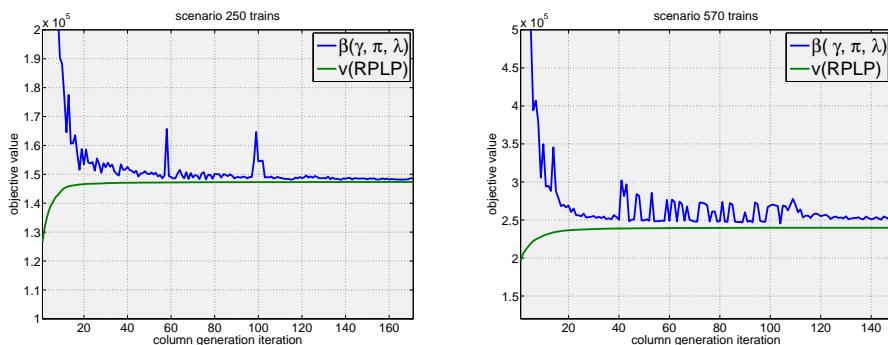


Figure 3: Solving the LP-relaxations of scenario 2 (left) and 3 (right) by column generation.

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