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On the Split Closure of the Periodic Timetabling Polytope

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On the Split Closure of the Periodic Timetabling Polytope

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Abstract


The Periodic Event Scheduling Problem (PESP) is the central mathematical tool for periodic timetable optimization in public transport. PESP can be formulated in several ways as a mixed-integer linear program with typically general integer variables. We investigate the split closure of these formulations and show that split inequalities are identical with the recently introduced flip inequalities. While split inequalities are a general mixed-integer programming technique, flip inequalities are defined in purely combinatorial terms, namely cycles and arc sets of the digraph underlying the PESP instance. It is known that flip inequalities can be separated in pseudo-polynomial time. We prove that this is best possible unless $P = NP$, but also observe that the complexity becomes linear-time if the cycle defining the flip inequality is fixed. Moreover, introducing mixed-integer-compatible maps, we compare the split closures of different formulations, and show that reformulation or binarization by subdivision do not lead to stronger split closures. Finally, we estimate computationally how much of the optimality gap of the instances of the benchmark library PESPlib can be closed exclusively by split cuts, and provide better dual bounds for five instances.


Keywords: Periodic Event Scheduling Problem, Periodic Timetabling, Split Closure, Mixed-Integer Programming

Mathematics Subject Classification (MSC2020): 90C11, 90C35, 90B35, 90B20

1 Introduction

The timetable is the core of a public transportation system. It serves as a basis for cost-sensitive tasks such as vehicle and crew scheduling, and is required for accurate planning of passenger routes. A high-quality timetable is thus of utmost importance for a well-planned transportation system. Particularly in the context of urban traffic, a large number of transportation networks are operated with a periodic pattern, creating the demand to optimize periodic timetables. The standard mathematical model for this task is the *Periodic Event Scheduling Problem* (PESP) introduced by Serafini and Ukovich (1989). PESP is a combinatorial optimization problem on a digraph with respect to a certain period time, and it is notoriously hard: Deciding whether a feasible periodic timetable exists is NP-complete for any fixed period time $T \geq 3$ (Odijk, 1994). The feasibility problem remains NP-hard on graphs with bounded treewidth

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(N. Lindner & Reisch, 2022). The difficulty of PESP is also reflected in the fact that since its establishment in 2012, none of the instances of the benchmark library PESPLib could be solved to proven optimality up to date (Goerigk, 2022). Nevertheless, many primal heuristics have been developed (Borndörfer, Lindner, & Roth, 2020; Bortoletto, Lindner, & Masing, 2022; Goerigk & Liebchen, 2017; Großmann et al., 2012; N. Lindner & Liebchen, 2022; Nachtigall & Opitz, 2008; Pätzold & Schöbel, 2016), and there are success stories concerning the implementation of mathematically optimized timetables in practice (Kroon et al., 2009; Liebchen, 2008).

PESP can be formulated as a mixed-integer linear program (MIP) in a multitude of ways (Liebchen, 2006). Several studies of the *periodic timetabling polytope* have been conducted, leading to the discovery of families of cutting planes, such as, e.g., *cycle inequalities* (Odijk, 1994), *change-cycle inequalities* (Nachtigall, 1996), and more recently, *flip inequalities* (N. Lindner & Liebchen, 2020). The separation of cycle and change-cycle inequalities is known to be NP-hard (Borndörfer, Hoppmann, Karbstein, & Lindner, 2020), and flip inequalities are a superset of both cycle and change-cycle inequalities (N. Lindner & Liebchen, 2020). A common theme that cycle, change-cycle and flip inequalities share as well with other families of cutting planes (T. Lindner, 2000; Nachtigall, 1996) is that they are all described in purely combinatorial terms. For example, flip inequalities are determined by a cycle and a set of arcs of the underlying digraph of the PESP instance.

In this paper, we pursue a somewhat opposite strategy: Rather than starting with a combinatorial analysis, we investigate *split inequalities*, a general-purpose tool for treating MIPs introduced by Cook, Kannan, and Schrijver (1990) as an analogon to the Chvátal closure for pure integer programs. The *split closure* given by these inequalities has several nice properties: It is a polyhedron (Conforti, Cornuéjols, & Zambelli, 2010; Cook et al., 1990), coincides with the closure given by mixed-integer rounding and Gomory mixed-integer cuts (Cornuéjols & Li, 2001; Nemhauser & Wolsey, 1990), and leads to finite cutting plane algorithms for binary MIPs (Balas, Ceria, & Cornuéjols, 1993).

While the second Chvátal closure for a pure IP formulation has already been investigated by Liebchen and Swarat (2008), we apply split closure techniques to proper mixed-integer formulations of PESP. Our first result is the following correspondence (Theorem 3.1): Every non-trivial split inequality is a non-trivial flip inequality, and vice versa. The split closure of the periodic timetabling polytope is therefore identical with the closure given by the flip inequalities. Moreover, the split inequalities coming from split disjunctions, where one of the two sides of the split is empty, coincide with Odijk’s cycle inequalities (Theorem 3.3).

In general, the separation of split inequalities is NP-hard (Caprara & Letchford, 2003). In the periodic timetabling situation, we show in Theorem 4.4 that it is weakly NP-hard to separate maximally violated split/flip inequalities. This is best possible unless $P = NP$, as N. Lindner and Liebchen (2020) have already outlined a pseudo-polynomial-time algorithm. The separation problem can however be solved by a parametric IP in the spirit of Balas and Saxena (2008) and Bonami (2012), which in the special case of PESP boils down to a sequence of $\lfloor T/2 \rfloor - 1$ standard IPs (Theorem 4.5). In the event that the cycle defining a flip inequality is fixed, the separation becomes linear-time (Theorem 4.7).

So far, the results on the split closure of the periodic timetabling polytope apply for the cycle-based MIP formulation of PESP (Liebchen & Peeters, 2009; Nachtigall, 1996). Another popular formulation is the incidence-based formulation that is straightforward from the original problem definition by Serafini and Ukovich (1989). In order to compare the split closures of two different MIP formulations, we introduce *mixed-integer-compatible maps*, i.e., affine maps that map mixed-integer points to mixed-integer points. These maps have the general property that they map split closures into split closures (Theorem 5.3). For PESP, the polytope defined by the cycle-based formulation turns out to be a mixed-integer-compatible projection of the

polytope defined by the incidence-based formulation. However, we show that the restriction of this projection to split closures is surjective, so that there is no gain in information concerning split cuts when switching to a different formulation (Theorem 5.11). More results that can be proven using mixed-integer-compatible maps are the following: The split closure commutes with Cartesian products (Theorem 5.5). This enables us to show that the split closure of PESP instances on cactus graphs is exact (Theorem 5.6).

The behavior of split or lift-and-project closures with respect to binarizations, i.e., MIP reformulations with only binary integer variables, have received some attention lately (Aprile, Conforti, & Di Summa, 2021; Dash, Günlük, & Hildebrand, 2018). In the context of PESP, the incidence-based MIP formulation can be binarized in a combinatorial manner by subdivision of arcs. Although split closures of binary MIPs are known to be much better behaved (Balas et al., 1993), we prove by another application of mixed-integer-compatible maps that this binarization procedure does also not lead to stronger split closures (Theorem 5.15).

Finally, we evaluate split closures in practice. To this end, we consider the 22 PESPLib instances and some derived subinstances. We devise an algorithmic procedure to optimize over the split closure making use of our theoretical insights. Our separation algorithm consists of a heuristic and an exact part. The outcome is that although the split closure closes a significant part of the primal-dual gap, it is almost never exact. However, our separation method produces incumbent dual bounds for 5 of the PESPLib instances.

The paper is structured as follows: We summarize the relevant definitions and notions for PESP in Section 2. The correspondence between split and flip inequalities follows in Section 3. The subsequent Section 4 is devoted to separation of split/flip inequalities. Mixed-integer compatible maps and the results on comparing split closures of different formulations are presented in Section 5. Our computational results can be found in Section 6. We conclude the paper in Section 7.

2 Periodic Event Scheduling

The Periodic Event Scheduling Problem has originally been introduced by Serafini and Ukovich (1989), and has gained much attention ever since. In this chapter, we establish the basics, formally state the problem, introduce two equivalent model formulations and introduce our main object of interest, the periodic timetabling polytope.

2.1 Problem Definition

An instance of the *Periodic Event Scheduling Problem* (PESP) is given by a 5-tuple (G, T, ℓ, u, w) , where

- $G = (V, A)$ is a directed graph,
- $T \in \mathbb{N}$, $T \geq 2$, is a *period time*,
- $\ell \in \mathbb{Z}^A$ is a vector of *lower bounds*,
- $u \in \mathbb{Z}^A$ is a vector of *upper bounds*,
- $w \in \mathbb{R}^A$ is a vector of *weights*.

A *periodic tension* is a vector $x \in \mathbb{R}^A$ with $\ell \leq x \leq u$ such that

$$\exists \pi \in [0, T)^V : \quad \forall a = (i, j) \in A : \quad x_a \equiv \pi_j - \pi_i \pmod{T}. \quad (1)$$

In this case, the vector π is called a *periodic timetable*. In the context of periodic timetabling in public transport, the vertices of G typically correspond to arrival or departure *events* of vehicles at some station. The arcs of G are *activities*; they model relations between the events such as, e.g., driving between two stations, dwelling at a station, or passenger transfers (Liebchen & Möhring, 2007). A periodic timetable π thus assigns timings in $[0, T)$ to each event, repeating periodically with period T . The periodic tension x collects the activity durations, which are supposed to lie within the feasible interval $[\ell, u]$. A typical source for the weight of an arc is the estimated number of passengers using the corresponding activity. A reasonable quality indicator of a periodic timetable is hence $w^\top x$, the total travel time of all passengers.

Definition 2.1 (Serafini and Ukovich 1989). Given (G, T, ℓ, u, w) as above, the *Periodic Event Scheduling Problem* is to find a periodic tension x such that $w^\top x$ is minimum, or to decide that none exists.

Example 2.2. Figure 1 shows a small PESP instance together with an optimal periodic tension and a compatible periodic timetable.

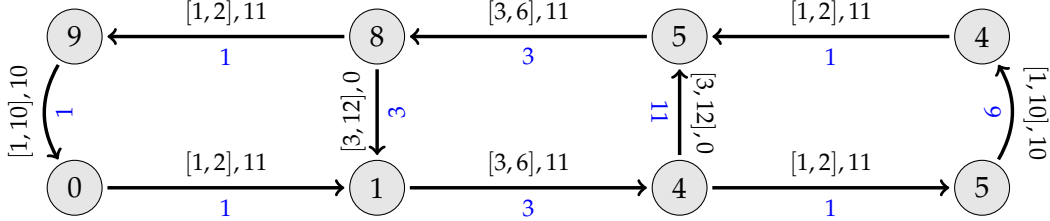


Figure 1: A PESP instance on a digraph $G = (V, A)$ with $T = 10$. The upper label of an arc $a \in A$ is $[\ell_a, u_a], w_a$. The blue lower arc labels indicate a periodic tension x compatible with the periodic timetable π as given by the vertex labels.

Remark 2.3. As described, e.g., by Liebchen (2006), any PESP instance can be preprocessed in such a way that G contains no loops and is weakly connected, $0 \leq \ell < T$ and $\ell \leq u < \ell + T$.

2.2 Mixed-Integer Programming Formulations

PESP can be formulated as a mixed-integer linear program in several ways (Liebchen, 2006). The *incidence-based* model is a straightforward interpretation of the problem definition, introducing auxiliary integer *periodic offsets* to resolve the modulo constraints (1):

$$\begin{aligned}
& \text{Minimize} && w^\top x \\
& \text{s.t.} && x_a = \pi_j - \pi_i + T p_a && a = (i, j) \in A \\
& && \ell_a \leq x_a \leq u_a, && a \in A \\
& && 0 \leq \pi_i \leq T - 1, && i \in V, \\
& && p_a \text{ integer}, && a \in A.
\end{aligned} \tag{2}$$

When all periodic offsets p_a are fixed, (2) becomes a linear program with a totally unimodular constraint matrix. It is hence no restriction to assume that x and π are integral, so that the bound $\pi < T$ in (1) can safely be replaced with $\pi \leq T - 1$. For the purpose of this paper, we will however not treat (2) as a pure integer program, as was done by Liebchen and Swarat (2008). We will instead investigate proper mixed-integer formulations, where the periodic tension variables x and the periodic timetable variables π are considered as continuous variables.

An alternative MIP formulation for PESP is the *cycle-based* formulation, which has been reported to be computationally beneficial (see., e.g., Borndörfer, Lindner, and Roth 2020; Liebchen 2008; Liebchen and Peeters 2009; Peeters 2003; Schiewe and Schöbel 2020):

$$\begin{aligned}
& \text{Minimize} && w^\top x \\
& \text{s.t.} && \Gamma x = Tz, \\
& && \ell \leq x \leq u, \\
& && z \text{ integer.}
\end{aligned} \tag{3}$$

In (3), x represents a periodic tension, and z is an integral *cycle offset*. A periodic timetable π can be recovered from x by a graph traversal.

To explain the further ingredients of the formulation (3), we will require more definitions about cycles, cycle spaces and cycle bases, see Kavitha et al. (2009) for an overview. The *cycle space* \mathcal{C} of G is the abelian group

$$\mathcal{C} := \left\{ \gamma \in \mathbb{Z}^A \mid \forall i \in V : \sum_{a \in \delta^+(i)} \gamma_a = \sum_{a \in \delta^-(i)} \gamma_a \right\}.$$

In terms of linear algebra, \mathcal{C} is the kernel over the integers of the incidence matrix of G ; in the language of network flows, \mathcal{C} is the space of all integer-valued (and arbitrarily signed) circulations in G . The rank of \mathcal{C} is the *cyclomatic number* μ of G . We assume that G is weakly connected (Remark 2.3), so that $\mu = |A| - |V| + 1$.

A vector $\gamma \in \mathcal{C} \cap \{-1, 0, 1\}^A$ will be called an *oriented cycle*. When ignoring arc directions, the support $\{a \in A \mid \gamma_a \neq 0\}$ makes up a possibly non-simple cycle in G . We call arcs a with $\gamma_a > 0$ *forward* and those with $\gamma_a < 0$ *backward*. Any $\gamma \in \mathcal{C}$ can be decomposed into its positive resp. negative part γ_+ resp. γ_- , i.e., $\gamma_+ := \max(\gamma, 0)$ and $\gamma_- := \max(-\gamma, 0)$. The *length* of an oriented cycle γ is $|\gamma| := |\{a \in A \mid \gamma_a \neq 0\}|$.

A set B of μ oriented cycles is called an *integral cycle basis* of G if B is a basis for \mathcal{C} as an abelian group, i.e., if every element of the cycle space \mathcal{C} can be written as a unique integral linear combination of the oriented cycles in B . A particular class of integral cycle bases are the (*strictly*) *fundamental cycle bases*: Let \mathcal{T} be some spanning tree of G . Then the fundamental cycle induced by the co-tree arc a of \mathcal{T} is the unique cycle γ obtained by adding a to \mathcal{T} with the convention that $\gamma_a = 1$. A fundamental cycle basis is then given by the collection of μ fundamental cycles of \mathcal{T} . Arranging the oriented cycles of an integral cycle basis B as rows of a matrix, we obtain a *cycle matrix* $\Gamma \in \{-1, 0, 1\}^{B \times A}$.

Example 2.4. In the example from Figure 1, we have $\mu = 3$. An integral cycle basis B is outlined in Figure 2.

The following theorem shows that the MIP (3) is indeed a valid formulation of PESP.

Theorem 2.5 (Cycle periodicity property, Liebchen & Peeters, 2009). *For a vector $x \in \mathbb{R}^A$, the following are equivalent:*

- (a) x satisfies condition (1),
- (b) $\gamma^\top x \equiv 0 \pmod T$ for all $\gamma \in \mathcal{C}$,
- (c) $\Gamma x \equiv 0 \pmod T$ for the cycle matrix Γ of an integral cycle basis of G .

In the sequel, we will focus on the cycle-based formulation (3), which is justified by the following remark.

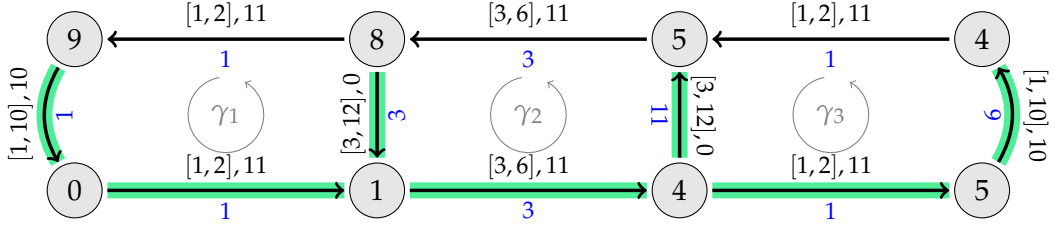


Figure 2: In the instance from Figure 1, the oriented cycles $\gamma_1, \gamma_2, \gamma_3$ constitute an integral cycle basis, as they are the fundamental cycles of the highlighted spanning tree. The cycle γ_2 uses only forward arcs, while γ_1 and γ_3 have both forward and backward arcs. The tension $\gamma_3^\top x$ along γ_3 is $1 + 1 + 9 - 1 = 10 \equiv 0 \pmod{10}$, and $\gamma_1^\top x$ and $\gamma_2^\top x$ are integer multiples of $T = 10$ as well.

Remark 2.6. The incidence-based formulation (2) is a particular incarnation of the cycle-based formulation (3) in the following sense: Let $I = (G, T, \ell, u, w)$ be a PESP instance. We can augment I to an instance I' such that the incidence-based MIP formulation (2) for I coincides with the cycle-based MIP formulation (3) for I' for a certain integral cycle basis B with cycle matrix Γ . To this end, we add a new vertex s and connect it to every original vertex $i \in V$. Set $\ell_{si} := 0, u_{si} := T - 1, w_{si} := 0$. The subgraph \mathcal{T} on the arcs $\{(s, i) \mid i \in V\}$ is a spanning tree of the augmented graph. Each fundamental cycle has the vertex sequence (s, i, j, s) for some arc $a = (i, j) \in A$; we assume that the arcs (s, i) and (i, j) are forward, and that the arc (s, j) is backward. The constraint in (3) for the cycle (s, i, j, s) is then given by $x_{si} + x_a - x_{sj} = Tz_a$. Relabeling x_{si} as π_i for $i \in V$ and z_a as p_a for $a \in A$, the formulation (3) for the augmented instance and the cycle matrix Γ given by the fundamental cycle basis with respect to \mathcal{T} indeed turns out to be the same as the formulation (2) for the original instance I . In particular, the PESP instances I and I' can be considered equivalent.

Example 2.7. Figure 3 shows the augmented instance I' obtained from the instance I from Figure 1 according to Remark 2.6.

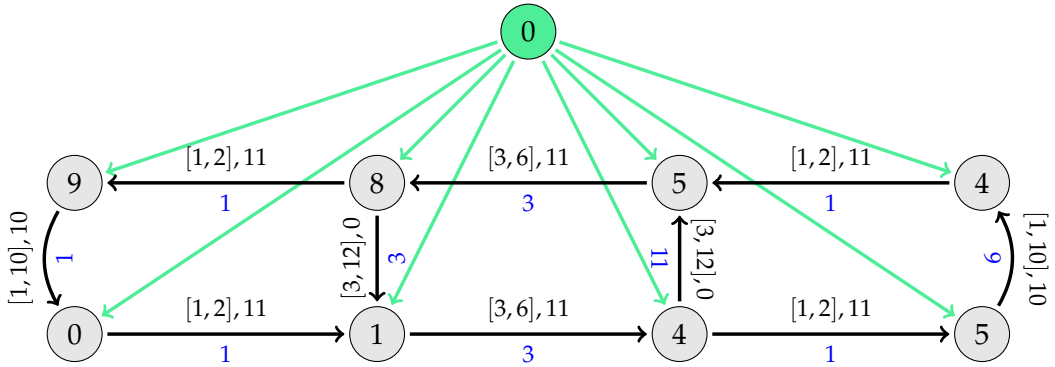


Figure 3: Augmentation of the instance in Figure 1 according to Remark 2.6. The new vertex s and the new arcs (s, i) are highlighted in green. The highlighted arcs form a spanning tree of the augmented instance. The periodic tension x_{si} of a highlighted arc (s, i) can be read off the timetable value π_i given as vertex label at the gray vertex i .

2.3 The Periodic Timetabling Polytope

Before analyzing the split closure, we need to understand the geometric object behind the feasible region of a PESP instance, and also of its natural LP relaxation.

Definition 2.8. For a PESP instance (G, T, ℓ, u, w) and a cycle matrix Γ of an integral cycle basis B , define

$$\begin{aligned}\mathcal{P} &:= \{(x, z) \in \mathbb{R}^A \times \mathbb{R}^B \mid \Gamma x = Tz, \ell \leq x \leq u\}, \\ \mathcal{P}_1 &:= \text{conv}\{(x, z) \in \mathbb{R}^A \times \mathbb{Z}^B \mid \Gamma x = Tz, \ell \leq x \leq u\}.\end{aligned}$$

We will call \mathcal{P} the *fractional periodic timetabling polytope* and \mathcal{P}_1 the *integer periodic timetabling polytope*.

\mathcal{P}_1 is the convex hull of the feasible solutions to (3), and the fractional periodic timetabling polytope \mathcal{P} is the polyhedron associated to the natural linear programming relaxation of (3). Observe that this relaxation is very weak: \mathcal{P} is combinatorially equivalent to the hyperrectangle $\prod_{a \in A} [\ell_a, u_a]$, and an optimal vertex of the LP relaxation of (3) is given by $(\ell, \Gamma \ell / T)$.

Remark 2.9. The choice of a cycle basis Γ is not essential for the definition of \mathcal{P} and \mathcal{P}_1 : If Γ' is the cycle matrix of another integral cycle basis, then there is a unimodular matrix U such that $\Gamma' = U\Gamma$, and $(x, z) \mapsto (x, Uz)$ is a \mathbb{Z} -linear isomorphism.

Several classes of valid inequalities for \mathcal{P}_1 are known (N. Lindner & Liebchen, 2020; T. Lindner, 2000; Nachtigall, 1996, 1998; Odijk, 1994). We will focus on those that are defined in terms of elements of the cycle space \mathcal{C} . The cycle periodicity property (Theorem 2.5) immediately shows:

Theorem 2.10 (Odijk, 1994). *Let $\gamma \in \mathcal{C}$. Then the following cycle inequality holds for all $(x, z) \in \mathcal{P}_1$:*

$$\left\lfloor \frac{\gamma_+^\top \ell - \gamma_-^\top u}{T} \right\rfloor \leq \frac{\gamma^\top x}{T} \leq \left\lceil \frac{\gamma_+^\top u - \gamma_-^\top \ell}{T} \right\rceil. \quad (4)$$

Since the rows of the cycle matrix Γ are oriented cycles, Theorem 2.10 implies bounds on the z -variables in Definition 2.8 as well, so that \mathcal{P} and \mathcal{P}_1 are indeed polytopes.

Let $[\cdot]_T$ denote the modulo T operator with values in $[0, T)$. Another well-known class of inequalities is the following:

Theorem 2.11 (Nachtigall, 1996). *Let $\gamma \in \mathcal{C}$ and $\alpha_\gamma := [-\gamma^\top \ell]_T$. Then the following change-cycle inequality holds for all $(x, z) \in \mathcal{P}_1$:*

$$(T - \alpha_\gamma) \gamma_+^\top (x - \ell) + \alpha_\gamma \gamma_-^\top (x - \ell) \geq \alpha_\gamma (T - \alpha_\gamma). \quad (5)$$

A class generalizing both cycle and change-cycle inequalities are the *flip inequalities* introduced by N. Lindner and Liebchen (2020). Let I be a PESP instance and let $F \subseteq A$ be an arbitrary subset of arcs. We construct a new PESP instance I_F from I by “flipping” the arcs in F : We replace each arc $a = (i, j) \in F$ by an arc $\bar{a} = (j, i)$, and set $\ell_{\bar{a}} := -u_a$, $u_{\bar{a}} := -\ell_a$, and $w_{\bar{a}} := -w_a$. From any periodic tension x for I , we obtain a periodic tension x_F for I_F by defining $x_{F,a} := x_a$ for $a \in A \setminus F$, and $x_{F,\bar{a}} := -x_a$ for $a \in F$. In particular, I is feasible if and only if I_F is feasible, and in case of feasibility, both I and I_F have the same optimal objective value. Moreover, for any $\gamma \in \mathcal{C}$, we obtain an element γ_F in the cycle space of I_F by setting $\gamma_{F,a} := \gamma_a$ for $a \in A \setminus F$, and $\gamma_{F,\bar{a}} := -\gamma_a$ for $a \in F$. We can hence consider the change-cycle inequality for γ_F on I_F and transform it back to I :

Theorem 2.12 (N. Lindner & Liebchen, 2020). Let $\gamma \in \mathcal{C}$ and $F \subseteq A$. Set

$$\alpha_{\gamma,F} := \left[- \sum_{a \in A \setminus F} \gamma_a \ell_a - \sum_{a \in F} \gamma_a u_a \right]_T.$$

Then the following flip inequality holds for all $(x, z) \in \mathcal{P}_I$:

$$\begin{aligned} & (T - \alpha_{\gamma,F}) \sum_{\substack{a \in A \setminus F: \\ \gamma_a > 0}} \gamma_a (x_a - \ell_a) + \alpha_{\gamma,F} \sum_{\substack{a \in A \setminus F: \\ \gamma_a < 0}} (-\gamma_a) (x_a - \ell_a) \\ & + \alpha_{\gamma,F} \sum_{\substack{a \in F: \\ \gamma_a > 0}} \gamma_a (u_a - x_a) + (T - \alpha_{\gamma,F}) \sum_{\substack{a \in F: \\ \gamma_a < 0}} (-\gamma_a) (u_a - x_a) \geq \alpha_{\gamma,F} (T - \alpha_{\gamma,F}). \end{aligned} \quad (6)$$

Remark 2.13. The flip inequalities (6) for $F = \emptyset$ give exactly the change-cycle inequalities (5). Moreover, by flipping all backward resp. all forward arcs of some $\gamma \in \mathcal{C}$, we obtain Odijk's cycle inequalities (4). Since the left-hand side of (6) is always non-negative for $(x, z) \in \mathcal{P}$, flip inequalities with $\alpha_{\gamma,F} = 0$ are trivial. Due to symmetry reasons, the flip inequalities for (γ, F) and $(-\gamma, F)$ coincide, and $\alpha_{\gamma,F} = T - \alpha_{-\gamma,F}$ when $\alpha_{\gamma,F} \geq 1$.

Definition 2.14. We define the *flip polytope* as

$$\mathcal{P}_{\text{flip}} := \{(x, z) \in \mathcal{P} \mid (x, z) \text{ satisfies the flip inequality for all } \gamma \in \mathcal{C} \text{ and } F \subseteq A\}.$$

Apart from the trivial relation $\mathcal{P}_I \subseteq \mathcal{P}_{\text{flip}} \subseteq \mathcal{P}$, the flip polytope has some interesting properties (N. Lindner & Liebchen, 2020): Every vertex of \mathcal{P}_I is a vertex of $\mathcal{P}_{\text{flip}}$, but in general not a vertex of \mathcal{P} . Moreover, if G is a cactus graph, i.e., every arc is contained in at most one simple cycle, then $\mathcal{P}_{\text{flip}} = \mathcal{P}_I$. However, there are PESP instances with $\mu = 2$ and $\mathcal{P}_{\text{flip}} \neq \mathcal{P}_I$.

3 The Split Closure of the Periodic Timetabling Polyhedron

The relation between the periodic timetabling polytope and the flip polytope seems close and deserves more attention. In fact, in this section, we will establish that the flip polytope can be identified with the split closure.

3.1 Preliminaries

We will now recall the definition of split inequalities, split disjunctions, and the split closure, following the treatment by Conforti, Cornuéjols, and Zambelli (2014). To two matrices $A_C \in \mathbb{Q}^{m \times n}$, $A_I \in \mathbb{Q}^{m \times p}$ and a vector $b \in \mathbb{Q}^p$, we associate the mixed-integer set

$$S := \{(x, z) \in \mathbb{R}^n \times \mathbb{Z}^p \mid A_C x + A_I z \leq b\},$$

and the two polyhedra

$$P := \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^p \mid A_C x + A_I z \leq b\}, \quad P_I := \text{conv}(S).$$

A *split* is a pair $(\beta, \beta_0) \in \mathbb{Z}^p \times \mathbb{Z}$. The disjunction

$$\beta^\top z \leq \beta_0 \quad \vee \quad \beta^\top z \geq \beta_0 + 1$$

is satisfied for all $(x, z) \in \text{conv}(S)$ and is called a *split disjunction*. In particular, the polyhedron

$$P^{(\beta, \beta_0)} := \text{conv}(\{(x, z) \in P \mid \beta^\top z \leq \beta_0\} \cup \{(x, z) \in P \mid \beta^\top z \geq \beta_0 + 1\})$$

contains $\text{conv}(S)$. The *split closure* is now defined as

$$P_{\text{split}} := \bigcap_{(\beta, \beta_0) \in \mathbb{Z}^p \times \mathbb{Z}} P^{(\beta, \beta_0)} = \bigcap_{\beta \in \mathbb{Z}^p} \text{conv}(\{(x, z) \in P \mid \beta^\top z \in \mathbb{Z}\}). \quad (7)$$

The split closure P_{split} is a polyhedron (Cook et al., 1990) with the property that $\text{conv}(S) \subseteq P_{\text{split}} \subseteq P$. It is identical to the closure given by Gomory's mixed-integer (GMI) cuts or mixed-integer rounding (MIR) cuts (Nemhauser & Wolsey, 1990). On the downside, optimization and hence separation are in general NP-hard (Caprara & Letchford, 2003).

The split closure can be described as well in terms of defining inequalities: Define

$$\Lambda := \left\{ \lambda \in \mathbb{R}^m \mid \begin{array}{l} \lambda^\top A_C = 0, \lambda^\top A_I \in \mathbb{Z}^p, \lambda^\top b \notin \mathbb{Z}, \text{ and the rows of } (A_C \ A_I) \\ \text{corresponding to non-zero entries of } \lambda \text{ are linearly independent} \end{array} \right\}.$$

Each multiplier vector $\lambda \in \Lambda$ defines the *split inequality*

$$\frac{\lambda_+^\top (b - A_C x - A_I z)}{[\lambda^\top b]_1} + \frac{\lambda_-^\top (b - A_C x - A_I z)}{1 - [\lambda^\top b]_1} \geq 1. \quad (8)$$

Here, we decompose λ into its positive part λ_+ and negative part λ_- , and denote by $[\cdot]_1$ the fractional part in $[0, 1)$. The split inequality (8) for $\lambda \in \Lambda$ is valid for $P^{(\lambda^\top A_I, [\lambda^\top b]_1)}$. Conversely, any facet of $P^{(\beta, \beta_0)}$ is defined by an inequality of the form (8) for some $\lambda \in \Lambda$ with $\lambda^\top A_I = \beta$ and $[\lambda^\top b] = \beta_0$. Consequently,

$$P_{\text{split}} = \{(x, z) \in P \mid (x, z) \text{ satisfies the split inequality (8) for all } \lambda \in \Lambda\}.$$

3.2 Flipping is Splitting for Periodic Timetabling

We investigate now the split closure for the cycle-based MIP formulation (3) for the Periodic Event Scheduling Problem. Thus let $(G = (V, A), T, \ell, u, w)$ be a PESP instance, and let B be an integral cycle basis of G with cycle matrix Γ . Rewriting in the form $A_C x + A_I z \leq b$, the fractional periodic timetabling polytope \mathcal{P} is defined by

$$\underbrace{\begin{pmatrix} \Gamma & -TI \\ -\Gamma & TI \\ I & 0 \\ -I & 0 \end{pmatrix}}_{A_C} \underbrace{\begin{pmatrix} x \\ z \end{pmatrix}}_{A_I} \leq \begin{pmatrix} 0 \\ 0 \\ u \\ -\ell \end{pmatrix}, \quad (9)$$

where I denotes the identity matrix. We will write a multiplier vector $\lambda \in \Lambda$ as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^B \times \mathbb{R}^B \times \mathbb{R}^A \times \mathbb{R}^A$$

corresponding to the four row blocks in (9).

Theorem 3.1. *Every flip inequality with $\alpha_{\gamma, F} \neq 0$ is a split inequality for the cycle-based MIP formulation of PESP (3) and vice versa. In particular, $\mathcal{P}_{\text{split}} = \mathcal{P}_{\text{flip}}$.*

Proof. We analyze the set Λ for the MIP (3). For $\lambda \in \Lambda$, we have

$$\lambda^\top A_C = (\lambda_1 - \lambda_2)^\top \Gamma + (\lambda_3 - \lambda_4)^\top \quad \text{and} \quad \lambda^\top A_I = -T(\lambda_1 - \lambda_2)^\top.$$

As $\lambda^\top A_I$ is integer, we find that $\gamma := T(\lambda_1 - \lambda_2)^\top \Gamma$ is an integer linear combination of the rows of the cycle matrix Γ , so that $\gamma \in \mathcal{C}$. From $\lambda^\top A_C = 0$ we infer that $\lambda_3 - \lambda_4 = -\gamma/T$. By

the linear independence condition, for an arc $a \in A$, not both of $\lambda_{3,a}$ and $\lambda_{4,a}$ can be non-zero. Hence, when we set $F := \{a \in A \mid \lambda_{3,a} \neq 0\}$, we have

$$\lambda_{3,a} = \begin{cases} -\gamma_a/T & \text{if } a \in F, \\ 0 & \text{if } a \in A \setminus F, \end{cases} \quad \text{and} \quad \lambda_{4,a} = \begin{cases} \gamma_a/T & \text{if } a \in A \setminus F, \\ 0 & \text{if } a \in F. \end{cases} \quad (10)$$

With that, the fractional part $[\lambda^\top b]_1$ evaluates to

$$[\lambda^\top b]_1 = [\lambda_3^\top u - \lambda_4^\top \ell]_1 = \left[-\sum_{a \in F} \frac{\gamma_a u_a}{T} - \sum_{a \in A \setminus F} \frac{\gamma_a \ell_a}{T} \right]_1.$$

Observe that for any $y \in \mathbb{R}$, we have

$$T \left[\frac{y}{T} \right]_1 = T \left(\frac{y}{T} - \left\lfloor \frac{y}{T} \right\rfloor \right) = y - T \left\lfloor \frac{y}{T} \right\rfloor = [y]_T,$$

so that

$$T[\lambda^\top b]_1 = T[\lambda_3^\top u - \lambda_4^\top \ell]_1 = \alpha_{\gamma,F}, \quad (11)$$

where $\alpha_{\gamma,F}$ is as in Theorem 2.12, and we have $\alpha_{\gamma,F} \neq 0$ because $\lambda^\top b \notin \mathbb{Z}$.

We now consider the expressions $\lambda_\pm^\top (b - A_C x - A_I z)$. Since for $(x, z) \in \mathcal{P}$,

$$b - A_C x - A_I z = (-\Gamma x + Tz, \Gamma x - Tz, u - x, x - \ell)^\top = (0, 0, u - x, x - \ell)^\top,$$

we have

$$\begin{aligned} \lambda_+^\top (b - A_C x - A_I z) &= \frac{1}{T} \sum_{\substack{a \in F \\ \gamma_a < 0}} \gamma_a (u_a - x_a) + \frac{1}{T} \sum_{\substack{a \in A \setminus F \\ \gamma_a > 0}} \gamma_a (x_a - \ell_a), \\ \lambda_-^\top (b - A_C x - A_I z) &= \frac{1}{T} \sum_{\substack{a \in F \\ \gamma_a > 0}} \gamma_a (u_a - x_a) + \frac{1}{T} \sum_{\substack{a \in A \setminus F \\ \gamma_a < 0}} \gamma_a (x_a - \ell_a). \end{aligned} \quad (12)$$

It is now evident from (11) and (12) that multiplying the split inequality (8) for λ with $\alpha_{\gamma,F}(T - \alpha_{\gamma,F})$ yields the flip inequality (6) for (γ, F) .

To prove the converse, starting from $\gamma \in \mathcal{C}$ and $F \subseteq A$ with $\alpha_{\gamma,F} \neq 0$, we define λ_3 and λ_4 as in (10), so that $\lambda_3 - \lambda_4 = -\gamma/T$. Moreover, as (11) holds, we have that $\lambda^\top b$ is not an integer. Since $\gamma \in \mathcal{C}$, there is an integral vector $\eta \in \mathbb{Z}^B$ with $\eta^\top \Gamma = \gamma$. Then $\lambda := (\eta/T, 0, \lambda_3, \lambda_4) \in \Lambda$, and the split inequality for λ is equivalent to the flip inequality for (γ, F) . \square

N. Lindner and Liebchen (2020) proved that $\mathcal{P}_{\text{flip}} = \mathcal{P}_I$ if the cyclomatic number μ of G is at most one by analyzing the combinatorial structure of $\mathcal{P}_{\text{flip}}$. In terms of the split closure, this result becomes almost trivial:

Corollary 3.2. *Suppose that $\mu \leq 1$. Then $\mathcal{P}_{\text{split}} = \mathcal{P}_I$.*

Proof. This is clear for $\mu = |B| = 0$, as

$$\mathcal{P} = \mathcal{P}_{\text{split}} = \mathcal{P}_I = \{(x, z) \in \mathbb{R}^A \mid \ell \leq x \leq u\}.$$

For $\mu = |B| = 1$, there is only a single integer variable z , and by virtue of (7),

$$\mathcal{P}_{\text{split}} = \bigcap_{\beta \in \mathbb{Z}} \text{conv} \{(x, z) \in \mathcal{P} \mid \beta z \in \mathbb{Z}\} = \text{conv} \{(x, z) \in \mathcal{P} \mid z \in \mathbb{Z}\} = \mathcal{P}_I. \quad \square$$

3.3 Chvátal Closure

For any mixed-integer set S defined by (A_C, A_I, b) with associated polyhedron P , one can define the *Chvátal closure* as a “one-side split closure” by

$$P_{\text{Ch}} := \bigcap \left\{ P^{(\beta, \beta_0)} \mid (\beta, \beta_0) \in \mathbb{Z}^p \times \mathbb{Z} \text{ s.t. } P \cap \{\beta^\top z \leq \beta_0\} = \emptyset \text{ or } P \cap \{\beta^\top z \geq \beta_0 + 1\} = \emptyset \right\},$$

see, e.g., Conforti et al. (2010, 2014). It is clear that $P_{\text{split}} \subseteq P_{\text{Ch}} \subseteq P$. For Periodic Event Scheduling, we find:

Theorem 3.3. *The Chvátal closure of the MIP (3) is given by*

$$P_{\text{Ch}} = \{(x, z) \in \mathcal{P} \mid (x, z) \text{ satisfies the cycle inequality (4) for all } \gamma \in \mathcal{C}\}.$$

Proof. We need to determine those $(\beta, \beta_0) \in \mathbb{Z}^B \times \mathbb{Z}$ for which one of $\mathcal{P} \cap \{\beta^\top z \leq \beta_0\}$ or $\mathcal{P} \cap \{\beta^\top z \geq \beta_0 + 1\}$ is empty. Since $\Gamma x = Tz$ holds for all $(x, z) \in \mathcal{P}$, we have $\beta^\top z = \frac{\gamma^\top x}{T}$ for $\gamma := \beta^\top \Gamma \in \mathcal{C}$ for an arbitrary choice of $\beta \in \mathbb{Z}^B$. Let

$$k_1 := \left\lceil \min \left\{ \frac{\gamma^\top x}{T} \mid (x, z) \in \mathcal{P} \right\} \right\rceil = \left\lceil \frac{\gamma_+^\top \ell - \gamma_-^\top u}{T} \right\rceil,$$

$$k_2 := \left\lfloor \max \left\{ \frac{\gamma^\top x}{T} \mid (x, z) \in \mathcal{P} \right\} \right\rfloor = \left\lfloor \frac{\gamma_+^\top u - \gamma_-^\top \ell}{T} \right\rfloor.$$

Then $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \leq \beta_0\} = \emptyset$ for all $\beta_0 \leq k_1 - 1$ and $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \geq \beta_0 + 1\} = \emptyset$ for $\beta_0 \geq k_2$. If $k_1 \geq k_2 + 1$, then $P_{\text{Ch}} = \emptyset$, and no $(x, z) \in \mathcal{P}$ satisfies the cycle inequality (4) for γ . Otherwise, both polyhedra $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \geq k_1\}$ and $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \leq k_2\}$ are non-empty, and they are defined by \mathcal{P} and Odijk’s cycle inequalities (4).

Moreover, since $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \geq k_1\} \subseteq \mathcal{P} \cap \{\frac{\gamma^\top x}{T} \geq \beta_0\}$ for any $\beta_0 \leq k_1$ and $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \leq k_2\} \subseteq \mathcal{P} \cap \{\frac{\gamma^\top x}{T} \leq \beta_0\}$ for $\beta_0 \geq k_2$, we can conclude that

$$\bigcap_{\beta_0 \leq k_1 - 1} P^{(\beta, \beta_0)} = P^{(\beta, k_1 - 1)} \quad \text{and} \quad \bigcap_{\beta_0 \geq k_2} P^{(\beta, \beta_0)} = P^{(\beta, k_2)}.$$

We conclude that for each $\beta \in \mathbb{Z}^B$ and k_1 and k_2 as above,

$$\begin{aligned} & \bigcap \left\{ P^{(\beta, \beta_0)} \mid \beta_0 \in \mathbb{Z} \text{ s.t. } P \cap \{\beta^\top z \leq \beta_0\} = \emptyset \text{ or } P \cap \{\beta^\top z \geq \beta_0 + 1\} = \emptyset \right\} \\ &= P^{(\beta, k_1 - 1)} \cap P^{(\beta, k_2)}, \end{aligned}$$

from which the claim follows. □

4 Separation of Split Cuts

From a practical point of view, the split closure can be a valuable tool to provide dual bounds for mixed integer programs. Of course, this requires efficient separation methods. As we have established that the split closure is of a specific form in the case of periodic timetabling, we can make use of the combinatorial structure behind flip inequalities to separate cuts.

4.1 Simple Cycles

We show at first that for separating split/flip inequalities, it suffices to consider *simple* oriented cycles, i.e., oriented cycles $\gamma \in \mathcal{C} \cap \{-1, 0, 1\}^A$ that yield a simple cycle on the underlying undirected graph of G .

Lemma 4.1 (Orientation-preserving cycle decomposition). *Let $\gamma \in \mathcal{C}$. Then there are simple oriented cycles $\delta_1, \dots, \delta_r \in \mathcal{C}$ such that $\gamma = \sum_{k=1}^r \delta_k$, $\gamma_+ = \sum_{k=1}^r \delta_{k,+}$, and $\gamma_- = \sum_{k=1}^r \delta_{k,-}$.*

Proof. Let $A^+ := \{a \in A \mid \gamma_a > 0\}$ and $A^- := \{a \in A \mid \gamma_a < 0\}$ be the set of forward and backward arcs of γ , respectively. Construct a digraph G_γ , whose set of arcs A_γ is given by

$$A_\gamma := A^+ \cup \{(j, i) \mid (i, j) \in A^-\}.$$

Define $g_{ij} := \gamma_{ij}$ if $(i, j) \in A^+$ and $g_{ji} := -\gamma_{ij}$ if $(i, j) \in A^-$. Then $g \geq 0$ is a circulation in G_γ , so that it decomposes into simple directed cycles d_1, \dots, d_r . Finally set $\delta_{k,ij} := d_{k,ij}$ if $(i, j) \in A^+$ and $\delta_{k,ij} := -d_{k,ji}$ if $(i, j) \in A^-$. \square

Theorem 4.2. *Let $F \subseteq A$ and $(x, z) \in \mathcal{P}$. If (x, z) satisfies all flip inequalities w.r.t. F and all simple oriented cycles γ , then it satisfies all flip inequalities w.r.t. F and all $\gamma \in \mathcal{C}$. In particular,*

$$\mathcal{P}_{\text{split}} = \mathcal{P}_{\text{flip}} = \left\{ (x, z) \in \mathcal{P} \mid \begin{array}{l} (x, z) \text{ satisfies the flip inequality} \\ \text{for all simple oriented cycles } \gamma \in \mathcal{C} \text{ and all } F \subseteq A \end{array} \right\}.$$

Proof. The inclusion (\subseteq) is clear, it remains to show (\supseteq). Suppose that (x, z) satisfies the flip inequalities w.r.t. F and all simple oriented cycles. Moving to the flipped instance I_F as in Section 2.3, we can assume that $F = \emptyset$, so that it suffices to consider the change-cycle inequality (5) for an arbitrary $\gamma \in \mathcal{C}$. Let $\gamma = \delta_1 + \dots + \delta_r$ be an orientation-preserving decomposition as in Lemma 4.1. We proceed by induction on r .

If $r \leq 1$, then $\gamma = 0$ or γ is simple, and there is nothing to show.

Now assume $r \geq 2$. There is nothing to show if $\alpha_\gamma = 0$, as the left-hand side of the change-cycle inequality is always non-negative. We hence assume $\alpha_\gamma > 0$.

By induction hypothesis, (x, z) satisfies the change-cycle inequality for the cycles $\delta := \delta_1$ and $\varepsilon := \delta_2 + \dots + \delta_r$. If $\alpha_\delta = 0$ resp. $\alpha_\varepsilon = 0$, then we have $\alpha_\gamma = \alpha_\varepsilon$ resp. $\alpha_\gamma = \alpha_\delta$, as $\alpha_\gamma = [\alpha_\delta + \alpha_\varepsilon]_T$. The validity of the change-cycle inequality for γ then follows immediately because the right-hand side equals the one for ε resp. δ , while the left-hand side can only become larger.

We are hence left with the case $\alpha_\gamma, \alpha_\delta, \alpha_\varepsilon > 0$. In this case we can rewrite the change-cycle inequality (5) so that it is of the form

$$\frac{\gamma_+^\top(x - \ell)}{\alpha_\gamma} + \frac{\gamma_-^\top(x - \ell)}{T - \alpha_\gamma} \geq 1,$$

which will be the key ingredient in our argumentation. Define

$$\kappa_\delta := \min \left\{ \frac{\alpha_\delta}{\alpha_\gamma}, \frac{T - \alpha_\delta}{T - \alpha_\gamma} \right\} \quad \text{and} \quad \kappa_\varepsilon := \min \left\{ \frac{\alpha_\varepsilon}{\alpha_\gamma}, \frac{T - \alpha_\varepsilon}{T - \alpha_\gamma} \right\}.$$

With $y := x - \ell$, using that (x, z) satisfies the change-cycle inequality w.r.t. both δ and ε ,

$$\begin{aligned} \frac{\gamma_+^\top y}{\alpha_\gamma} + \frac{\gamma_-^\top y}{T - \alpha_\gamma} &= \frac{\delta_+^\top y}{\alpha_\gamma} + \frac{\delta_-^\top y}{T - \alpha_\gamma} + \frac{\varepsilon_+^\top y}{\alpha_\gamma} + \frac{\varepsilon_-^\top y}{T - \alpha_\gamma} \\ &= \frac{\alpha_\delta}{\alpha_\gamma} \frac{\delta_+^\top y}{\alpha_\delta} + \frac{T - \alpha_\delta}{T - \alpha_\gamma} \frac{\delta_-^\top y}{T - \alpha_\delta} + \frac{\alpha_\varepsilon}{\alpha_\gamma} \frac{\varepsilon_+^\top y}{\alpha_\varepsilon} + \frac{T - \alpha_\varepsilon}{T - \alpha_\gamma} \frac{\varepsilon_-^\top y}{T - \alpha_\varepsilon} \\ &\geq \kappa_\delta \left(\frac{\delta_+^\top y}{\alpha_\delta} + \frac{\delta_-^\top y}{T - \alpha_\delta} \right) + \kappa_\varepsilon \left(\frac{\varepsilon_+^\top y}{\alpha_\varepsilon} + \frac{\varepsilon_-^\top y}{T - \alpha_\varepsilon} \right) \\ &\geq \kappa_\delta + \kappa_\varepsilon, \end{aligned}$$

Claim. $\kappa_\delta + \kappa_\varepsilon = 1$.

If the claim holds, then the change-cycle inequality w.r.t. γ holds for (x, z) , and we are done. Recall that $\alpha_\gamma = [\alpha_\delta + \alpha_\varepsilon]_T$, so that

$$\alpha_\gamma = \begin{cases} \alpha_\delta + \alpha_\varepsilon & \text{if } \alpha_\delta + \alpha_\varepsilon < T, \\ \alpha_\delta + \alpha_\varepsilon - T & \text{otherwise.} \end{cases}$$

If $\alpha_\delta + \alpha_\varepsilon < T$, then $\alpha_\gamma = \alpha_\delta + \alpha_\varepsilon$, hence $\alpha_\delta \leq \alpha_\gamma$ and $T - \alpha_\gamma = T - \alpha_\delta - \alpha_\varepsilon \leq T - \alpha_\delta$, so that $\kappa_\delta = \alpha_\delta / \alpha_\gamma$. Analogously, $\kappa_\varepsilon = \alpha_\varepsilon / \alpha_\gamma$, so that $\kappa_\delta + \kappa_\varepsilon = 1$.

In the other case, we have $\alpha_\gamma = \alpha_\delta + \alpha_\varepsilon - T$. From this, we infer $\alpha_\delta = \alpha_\gamma + T - \alpha_\varepsilon \geq \alpha_\gamma$ and $T - \alpha_\gamma = 2T - \alpha_\delta - \alpha_\varepsilon \geq (T - \alpha_\delta) + (T - \alpha_\varepsilon) \geq T - \alpha_\delta$, so that $\kappa_\delta = (T - \alpha_\delta) / (T - \alpha_\gamma)$. Analogously, $\kappa_\varepsilon = (T - \alpha_\varepsilon) / (T - \alpha_\gamma)$, so that again $\kappa_\delta + \kappa_\varepsilon = 1$. \square

Using Theorem 3.3 and Remark 2.13, Theorem 4.2 implies the analogous result for the Chvátal split closure and the cycle inequalities:

Corollary 4.3. *Let $(x, z) \in \mathcal{P}$. If (x, z) satisfies all cycle inequalities w.r.t. all simple oriented cycles γ , then it satisfies all cycle inequalities for all $\gamma \in \mathcal{C}$. In particular,*

$$\mathcal{P}_{\text{Ch}} = \left\{ (x, z) \in \mathcal{P} \mid \begin{array}{l} (x, z) \text{ satisfies the cycle inequality} \\ \text{for all simple oriented cycles } \gamma \in \mathcal{C} \end{array} \right\}.$$

4.2 Separation Hardness

N. Lindner and Liebchen (2020) outline a pseudo-polynomial time algorithm based on the dynamic program by Borndörfer, Hoppmann, et al. (2020) that finds a maximally violated flip inequality (if there is any), i.e., a simple cycle γ and a set $F \subseteq A$ such that the difference of the right-hand and left-hand sides of (6) is maximum. We prove here that pseudo-polynomial time is best possible unless $P = NP$:

Theorem 4.4. *Given $(x, z) \in \mathcal{P}$ and $M \geq 0$, it is weakly NP-hard to decide whether there exist a simple cycle γ and a subset $F \subseteq A$ such that (x, z) violates the flip inequality for (γ, F) by at least M .*

Proof. We reduce the weakly NP-hard Ternary Partition Problem (Borndörfer, Hoppmann, et al., 2020): Given $m \in \mathbb{N}$ and $c \in \mathbb{N}^m$, is there $a \in \{-1, 0, 1\}^m$ such that $\sum_{i=1}^m a_i c_i = \pm \frac{1}{2} \sum_{i=1}^m c_i$? For a Ternary Partition instance (m, c) , we define a PESP instance (G, T, ℓ, u, w) as follows: The digraph $G = (V, A)$ is given by a complete directed graph on the vertex set

$$V := \{1^+, 1^-, 2^+, 2^-, \dots, m^+, m^-\},$$

where we delete the arcs $(1^-, 1^+), (2^-, 2^+), \dots, (m^-, m^+)$. We set $T := \sum_{i=1}^m c_i$ and

$$\ell_{i^+i^-} := c_i, \quad u_{i^+i^-} := T, \quad w_{i^+i^-} := 1 \quad \text{for all } i \in \{1, \dots, m\}.$$

For all other arcs a , we set $\ell_a := u_a := w_a := 0$. As for any PESP instance, the optimal solution to the LP relaxation of (3) is given by $x^* = \ell$.

Suppose now that $x^* = \ell$ violates some flip inequality (6) for some simple oriented cycle γ and some $F \subseteq A$ by at least $M := \frac{T^2}{4}$. Since $x^* = \ell$, only arcs in F contribute non-trivially to the left-hand side of (6), moreover, these arcs are all of the form (i^+, i^-) . We hence obtain

$$\alpha_{\gamma,F} \sum_{(i^+, i^-) \in F, \gamma_{i^+i^-} = 1} (T - c_i) + (T - \alpha_{\gamma,F}) \sum_{(i^+, i^-) \in F, \gamma_{i^+i^-} = -1} (T - c_i) \leq \alpha_{\gamma,F}(T - \alpha_{\gamma,F}) - M, \quad (13)$$

where $\alpha_{\gamma,F} = [-\sum_{(i^+, i^-) \notin F} \gamma_{i^+i^-} c_i]_T$. As the left-hand side of (6) is non-negative, we have that $\alpha_{\gamma,F}(T - \alpha_{\gamma,F}) \geq M = T^2/4$, which implies $\alpha_{\gamma,F} = T/2$. Set $a_i := -\gamma_{i^+, i^-}$ for all $i \in \{1, \dots, m\}$. Then $[\sum_{i=1}^m a_i c_i]_T = \alpha_{\gamma,F} = T/2$, and as $-T \leq \sum_{i=1}^m a_i c_i \leq T$, we find that $\sum_{i=1}^m a_i c_i = \pm T/2$. In particular, a violated flip inequality leads to a positive answer to the Ternary Partition instance.

Conversely, suppose that there is $a \in \{-1, 0, 1\}^m$ such that $\sum_{i=1}^m a_i c_i = \pm T/2$. Construct a simple oriented cycle γ with $\gamma_{i^+i^-} := -a_i$ for all $i \in \{1, \dots, m\}$. Then $\alpha_{\gamma, \emptyset} = T/2$, and the flip inequality for γ and $F = \emptyset$ (i.e., the change-cycle inequality for γ) is violated by at least $T^2/4 = M$, because the left-hand side of (13) vanishes. \square

In practice, the dynamic program indicated in (N. Lindner & Liebchen, 2020) consumes too much memory. It is therefore advantageous to switch to a cut-generating MIP. Balas and Saxena (2008) describe a parametric MIP with a single parameter $\theta \in [0, 1]$ for this purpose. We are however in a better situation: Translating to periodic timetabling via Theorem 3.1, the parameter θ essentially corresponds to $\alpha_{\gamma,F}$, which is always an integer between 0 and $T - 1$. This means that the parametric MIP can be replaced by a finite sequence of standard IPs for each such integer $\alpha_{\gamma,F}$. The formulation of the IP (14) is straightforward from the definition (6) of flip inequalities:

Theorem 4.5. *Let $(x, z) \in \mathcal{P} \setminus \mathcal{P}_{\text{flip}}$ and $\alpha \in \{1, \dots, T\}$. Then a maximally violated flip inequality w.r.t. (x, z) with $\alpha_{\gamma,F} = \alpha$ among all oriented cycles γ and all $F \subseteq A$ is found by the following integer program:*

$$\begin{aligned} \text{Minimize} \quad & (T - \alpha) \sum_{a \in A} (x_a - \ell_a) y_a^+ + \alpha \sum_{a \in A} (x_a - \ell_a) y_a^- \\ & + \alpha \sum_{a \in A} (x_a - \ell_a) f_a^+ + (T - \alpha) \sum_{a \in A} (u_a - x_a) f_a^- \\ \text{s.t.} \quad & \sum_{a \in A} \ell_a (y_a^- - y_a^+) + \sum_{a \in A} u_a (f_a^- - f_a^+) + kT = \alpha \\ & \sum_{a \in \delta^+(v)} \gamma_a - \sum_{a \in \delta^-(v)} \gamma_a = 0, \quad v \in V, \\ & f^+ - f^- + y^+ - y^- = \gamma, \\ & 0 \leq f^+ + f^- + y^+ + y^- \leq 1, \\ & f^+, f^-, y^+, y^- \in \{0, 1\}^A, \\ & \gamma \in \{-1, 0, 1\}^A, \\ & k \in \mathbb{Z}. \end{aligned} \quad (14)$$

Any feasible solution of (14) with objective value less than $\alpha(T - \alpha)$ will produce a violated flip inequality. Recall from Remark 2.13 that flip inequalities with $\alpha_{\gamma,F} = 0$ are trivial and cannot be violated, and that due to symmetry, it is not necessary to consider the IP (14) for $\alpha \geq T/2$.

4.3 Separation for a Fixed Cycle

We discuss now how to find a maximally violated flip inequality in linear time when the cycle γ is already fixed. To this end, we take the perspective of split cuts. Consider again a mixed-integer set defined by (A_C, A_I, b) and the associated polyhedron $P = \{A_C x + A_I z \leq b\}$. When a split (β, β_0) is fixed, then the separation problem on $P^{(\beta, \beta_0)}$ can be solved as follows (Balas et al., 1993; Bonami, 2012; Conforti et al., 2014): Given $(x, z) \in P$, check whether $\beta^\top z \leq \beta_0$ or $\beta^\top z \geq \beta_0 + 1$. If yes, then $(x, z) \in P^{(\beta, \beta_0)}$. Otherwise, solve the linear program

$$\begin{aligned} \text{Minimize} \quad & (s - t)^\top b + \frac{1}{\beta^\top z - \beta_0} \cdot t^\top (b - A_C x - A_I z) \\ \text{s.t.} \quad & (s - t)^\top A_C = 0 \\ & (s - t)^\top A_I = \beta^\top \\ & s, t \geq 0. \end{aligned} \tag{15}$$

If the value of (15) is at least $\beta_0 + 1$, then $(x, z) \in P^{(\beta, \beta_0)}$, otherwise it is not. In the latter case, if we take a basic optimal solution (s^*, t^*) , then (x, z) is separated by the split inequality w.r.t. $s^* - t^*$. This cut-generating LP (15) finds a maximally violated split inequality in the following sense:

Lemma 4.6. *Suppose that $(x, z) \in P \setminus P^{(\beta, \beta_0)}$. Let (s^*, t^*) be an optimal basic solution of (15), $\lambda^* := s^* - t^*$. Then*

$$[\lambda^{*\top} b]_1 (1 - [\lambda^{*\top} b]_1) - (1 - [\lambda^{*\top} b]_1) \lambda_+^{*\top} (b - A_C x - A_I z) - [\lambda^{*\top} b]_1 \lambda_-^{*\top} (b - A_C x - A_I z) \tag{16}$$

is maximum among all $\lambda = s - t$ such that (s, t) is feasible for (15) and $\lambda^\top b \in [\beta_0, \beta_0 + 1)$.

Proof. Since (s^*, t^*) is basic, we have $\lambda_+^* = s$ and $\lambda_-^* = t$. As $(x, z) \in P \setminus P^{(\beta, \beta_0)}$, $\beta^\top z - \beta_0 > 0$. Then λ^* maximizes

$$-(\beta^\top z - \beta_0) \lambda^\top b - \lambda_-^\top (b - A_C x - A_I z).$$

Adding a constant term, λ^* also maximizes

$$\begin{aligned} & (\beta^\top z - \beta_0)(\beta_0 + 1) - (\beta^\top z - \beta_0) \lambda^\top b - \lambda_-^\top (b - A_C x - A_I z) \\ & = (\beta^\top z - \beta_0)(\beta_0 + 1 - \lambda^\top b) - \lambda_-^\top (b - A_C x - A_I z). \end{aligned}$$

Observing that $\lambda^\top (b - A_C x - A_I z) = \lambda^\top b - \beta^\top z$, this is the same as

$$\begin{aligned} & = (\lambda^\top b - \lambda^\top (b - A_C x - A_I z) - \beta_0)(\beta_0 + 1 - \lambda^\top b) - \lambda_-^\top (b - A_C x - A_I z) \\ & = (\lambda^\top b - \beta_0)(\beta_0 + 1 - \lambda^\top b) - (\beta_0 + 1 - \lambda^\top b) \lambda^\top (b - A_C x - A_I z) - \lambda_-^\top (b - A_C x - A_I z) \\ & = (\lambda^\top b - \beta_0)(\beta_0 + 1 - \lambda^\top b) - (\beta_0 + 1 - \lambda^\top b) \lambda_+^\top (b - A_C x - A_I z) \\ & \quad - (\lambda^\top b - \beta_0) \lambda_-^\top (b - A_C x - A_I z). \end{aligned}$$

Since $\lambda^\top b \in [\beta_0, \beta_0 + 1)$, $[\lambda^\top b]_1 = \lambda^\top b - \beta_0$ and $1 - [\lambda^\top b]_1 = \beta_0 + 1 - \lambda^\top b$, and we arrive at (16). \square

Note that the condition $\lambda^\top b \in [\beta_0, \beta_0 + 1)$ in Lemma 4.6 is no restriction, since it suffices to consider λ for which $[\lambda^\top b] = \beta_0$ (cf. Section 3). We obtain the following in the context of periodic timetabling:

Theorem 4.7. Let \mathcal{P} be a fractional periodic timetabling polytope. Let $(x, z) \in \mathcal{P}$, $\gamma \in \mathcal{C}$ with $\gamma^\top x \notin T\mathbb{Z}$, and set $g := T / \lfloor -\gamma^\top x \rfloor_T$. Then the flip inequality w.r.t. γ and

$$F := \{a \in A \mid \gamma_a > 0 \text{ and } u_a - \ell_a \geq g(u_a - x_a)\} \cup \{a \in A \mid \gamma_a < 0 \text{ and } u_a - \ell_a \leq g(x_a - \ell_a)\}$$

is maximally violated by (x, z) among the flip inequalities w.r.t. γ . In particular, a maximally violated flip inequality w.r.t. γ can be found in $O(|\gamma|)$ time.

Proof. We first write down the cut-generating LP (15) for the PESP situation (9):

$$\begin{aligned} \text{Minimize} \quad & s_3^\top u - t_3^\top u - s_4^\top \ell + t_4^\top \ell + \frac{1}{\beta^\top z - \beta_0} \cdot (t_3^\top (u - x) + t_4^\top (x - \ell)) \\ \text{s.t.} \quad & (s_1 - t_1 - s_2 + t_2)^\top \Gamma + s_3^\top - t_3^\top - s_4^\top + t_4^\top = 0, \\ & -s_1 + t_1 + s_2 - t_2 = \frac{\beta}{T}, \\ & s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4 \geq 0. \end{aligned}$$

Recall from Theorem 3.1 that a flip inequality w.r.t. γ corresponds to a split inequality derived from $P^{(\beta, \beta_0)}$ with $\beta^\top \Gamma = -\gamma$. Since $(x, z) \in \mathcal{P}$, we have $\beta_0 = \lfloor \beta^\top z \rfloor = \lfloor -\gamma^\top x / T \rfloor$. Eliminating the variables s_1, s_2, t_1, t_2 , and setting

$$g := \frac{1}{\beta^\top z - \beta_0} = \frac{1}{\lfloor \beta^\top z \rfloor_1} = \frac{1}{\lfloor \beta^\top \Gamma x / T \rfloor_1} = \frac{1}{\lfloor -\gamma^\top x / T \rfloor_1} = \frac{T}{\lfloor -\gamma^\top x \rfloor_T},$$

this becomes

$$\begin{aligned} \text{Minimize} \quad & s_3^\top u - t_3^\top u - s_4^\top \ell + t_4^\top \ell + g \cdot (t_3^\top (u - x) + t_4^\top (x - \ell)) \\ \text{s.t.} \quad & s_3 - t_3 - s_4 + t_4 = -\frac{\gamma}{T}, \\ & s_3, s_4, t_3, t_4 \geq 0. \end{aligned}$$

This linear program is trivial to solve: In each basic solution, for each arc $a \in A$ at most one of $s_{3,a}, s_{4,a}, t_{3,a}, t_{4,a}$ will be non-zero, and $s_{3,a} = s_{4,a} = t_{3,a} = t_{4,a} = 0$ for all $a \in A$ with $\gamma_a = 0$. We examine the contribution to the objective for each arc a in γ in such a basic solution:

If $\gamma_a > 0$, then either $t_{3,a} > 0$ or $s_{4,a} > 0$. In the first case, the contribution to the objective is $\gamma_a(g(u_a - x_a) - u_a)/T$, otherwise $-\gamma_a \ell_a / T$. Otherwise, if $\gamma_a < 0$, then either $s_{3,a} > 0$ or $t_{4,a} > 0$, the contribution being $-\gamma_a u_a / T$ resp. $-\gamma_a(\ell_a + g(x_a - \ell_a))/T$. In particular, an optimal solution is given by

$$\begin{aligned} t_{3,a} &:= \frac{\gamma_a}{T} && \text{for all } a \text{ s.t. } \gamma_a > 0 \text{ and } -\ell_a \geq g(u_a - x_a) - u_a, \\ s_{4,a} &:= \frac{\gamma_a}{T} && \text{for all } a \text{ s.t. } \gamma_a > 0 \text{ and } -\ell_a < g(u_a - x_a) - u_a, \\ s_{3,a} &:= -\frac{\gamma_a}{T} && \text{for all } a \text{ s.t. } \gamma_a < 0 \text{ and } u_a \leq g(x_a - \ell_a) + \ell_a, \\ t_{4,a} &:= -\frac{\gamma_a}{T} && \text{for all } a \text{ s.t. } \gamma_a < 0 \text{ and } u_a > g(x_a - \ell_a) + \ell_a, \end{aligned}$$

and $s_{3,a} := s_{4,a} := t_{3,a} := t_{4,a} := 0$ otherwise. The cut derived from this solution is the split inequality for $\lambda = s - t$, which by Theorem 3.1 corresponds to the flip inequality for γ and

$$\begin{aligned} F &= \{a \in A \mid \lambda_{3,a} \neq 0\} \\ &= \{a \in A \mid \gamma_a > 0 \text{ and } u_a - \ell_a \geq g(u_a - x_a)\} \cup \{a \in A \mid \gamma_a < 0 \text{ and } u_a - \ell_a \leq g(x_a - \ell_a)\}. \end{aligned}$$

Observe that by (11), $T[\lambda^\top b]_1 = \alpha_{\gamma, F}$. Using (12) and multiplying (16) with T^2 therefore yields the violation of the flip inequality w.r.t. (γ, F) . By Lemma 4.6, we conclude that the violation is indeed maximal. \square

5 Comparing Split Closures

Recall that the Periodic Event Scheduling Problem can be formulated in two ways as a MIP, where the incidence-based formulation (2) is essentially a special case of the cycle-based formulation (3) by virtue of Remark 2.6. The methods of Section 3 therefore apply to both formulations, and the question arises whether one of the two split closures is stronger. We will show that both closures are in fact of the same strength in Section 5.3.

Typically, the integer variables in both formulations are general. However, under certain circumstances, the periodic offset variables p_a in (2) can be assumed to be binary (Liebchen, 2006). We will discuss Section 5.4 how to achieve binary variables by a subdivision procedure. We will show that this binarization approach does not lead to a stronger split closure.

We show in Section 5.2 that split closures commute with Cartesian products, which means in the PESP situation that the split closures can be considered on blocks of G individually.

However, to be able to compare split closures of different polyhedra, we need to develop a few technicalities first in Section 5.1.

5.1 Mixed-Integer-Compatible Maps

We begin with two mixed-integer sets

$$S_i := \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{Z}^{p_i} \mid A_C^i x + A_I^i z \leq b^i\}, \quad i \in \{1, 2\},$$

and the associated polyhedra

$$P_i := \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \mid A_C^i x + A_I^i z \leq b^i\}, \quad (P_i)_I := \text{conv}(S_i), \quad i \in \{1, 2\}.$$

Definition 5.1. A map $\varphi : \mathbb{R}^{n_1} \times \mathbb{R}^{p_1} \rightarrow \mathbb{R}^{n_2} \times \mathbb{R}^{p_2}$ is *mixed-integer-compatible* if φ is affine and $\varphi(\mathbb{R}^{n_1} \times \mathbb{Z}^{p_1}) \subseteq \mathbb{R}^{n_2} \times \mathbb{Z}^{p_2}$.

In particular, if $\varphi(P_1) \subseteq P_2$ for a mixed-integer-compatible map φ , then $\varphi(S_1) \subseteq S_2$ and $\varphi((P_1)_I) \subseteq (P_2)_I$.

Lemma 5.2. Let $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{p_1} \rightarrow \mathbb{R}^{n_2} \times \mathbb{R}^{p_2}$ be a linear map and let $\psi^* : \mathbb{R}^{n_2} \times \mathbb{R}^{p_2} \rightarrow \mathbb{R}^{n_1} \times \mathbb{R}^{p_1}$ be the corresponding dual linear map, identifying dual vector spaces choosing standard bases. Then the following are equivalent:

- (1) ψ is mixed-integer-compatible.
- (2) $\psi(\mathbb{R}^{n_1} \times \{0\}) \subseteq \mathbb{R}^{n_2} \times \{0\}$ and $\psi^*(\{0\} \times \mathbb{Z}^{p_2}) \subseteq \{0\} \times \mathbb{Z}^{p_1}$.

Proof. (1) \Rightarrow (2): For the first statement consider for $i \in [n_1]$ the i -th standard basis vector $e_i \in \mathbb{R}^{n_1}$. Then $\psi(e_i, 0) = (x, z)$ for some $x \in \mathbb{R}^{n_2}$ and $z \in \mathbb{R}^{p_2}$. But as ψ is linear and mixed-integer-compatible, $\psi(\lambda e_i, 0) = (\lambda x, \lambda z)$ with $\lambda z \in \mathbb{Z}^{p_2}$ for all $\lambda \in \mathbb{R}$, so that $z = 0$.

For the second statement, consider for $j \in [p_2]$ the j -th standard basis vector e_j . Then for $i \in [n_1]$, the i -th coordinate of $\psi^*(0, e_j)$ is given by $(0, e_j)^\top \psi(e_i, 0) = 0$ by the first statement. For $i \in [p_1]$, the $(n_1 + i)$ -th coordinate of $\psi^*(0, e_j)$ is given by $(0, e_j)^\top \psi(0, e_i)$, which is integral as ψ is mixed-integer-compatible.

(2) \Rightarrow (1): Let $(x, z) \in \mathbb{R}^{n_1} \times \mathbb{Z}^{p_1}$. Then $\psi(x, z) = \psi(x, 0) + \psi(0, z)$, so using linearity and the first statement in (2), it suffices to consider $\psi(0, e_i)$ for $i \in [p_1]$. But now for $j \in [p_2]$, the $(n_1 + i)$ -th coordinate of $\psi^*(0, e_j)$ is integral by the second statement in (2), and since it is given by $(0, e_j)^\top \psi(0, e_i)$, we conclude that the $(n_2 + j)$ -th coordinate of $\psi(0, e_i)$ is integer. Consequently, ψ must be mixed-integer-compatible. \square

The following is a generalization of Theorem 1 in (Dash et al., 2018).

Theorem 5.3. *Let φ be a mixed-integer-compatible map with $\varphi(P_1) \subseteq P_2$. Then $\varphi((P_1)_{\text{split}}) \subseteq (P_2)_{\text{split}}$.*

Proof. Consider $(x_1, z_1) \in (P_1)_{\text{split}}$ and $\beta_2 \in \mathbb{Z}^{p_2}$. We need to show that $\varphi(x_1, z_1)$ is a convex combination of points $(x_2^i, z_2^i) \in P_2$ with $\beta_2^\top z_2^i$ integral. Since φ is mixed-integer-compatible, the last p_2 entries of $\varphi(0, 0)$ are integral, and so the linear map $\psi := \varphi - \varphi(0, 0)$ is mixed-integer-compatible as well. By Lemma 5.2, $\psi^*(0, \beta_2) = (0, \beta_1)$ for some $\beta_1 \in \mathbb{Z}^{p_1}$. Since $(x_1, z_1) \in (P_1)_{\text{split}}$, it is a convex combination of $(x_1^i, z_1^i) \in P_1$ with $\beta_1^\top z_1^i \in \mathbb{Z}$. Write

$$(x_2^i, z_2^i) := \varphi(x_1^i, z_1^i) = \psi(x_1^i, z_1^i) + \varphi(0, 0) \in P_2.$$

Then

$$\begin{aligned} \beta_2^\top z_2^i &= (0, \beta_2)^\top (x_2^i, z_2^i) = (0, \beta_2)^\top \psi(x_1^i, z_1^i) + (0, \beta_2)^\top \varphi(0, 0) \\ &= \psi^*(0, \beta_2)^\top (x_1^i, z_1^i) + (0, \beta_2)^\top \varphi(0, 0) \\ &= (0, \beta_1)^\top (x_1^i, z_1^i) + (0, \beta_2)^\top \varphi(0, 0) \\ &= \beta_1^\top z_1^i + (0, \beta_2)^\top \varphi(0, 0) \\ &\in \mathbb{Z}. \end{aligned}$$

As φ is affine and hence preserve convex combinations, $\varphi(x_1, z_1)$ is a convex combination of the $(x_2^i, z_2^i) \in P_2$. \square

Example 5.4. An example for a mixed-integer-compatible map is provided by the change of the cycle basis in the context of periodic timetabling. Let $I = (G, T, \ell, u, w)$ be a PESP instance and let Γ, Γ' be two cycle matrices of integral cycle bases of G . As in Remark 2.9, there is a unimodular matrix U such that $\Gamma' = U\Gamma$. The map $\varphi : (x, z) \mapsto (x, Uz)$ maps the fractional periodic timetabling polytope \mathcal{P}_1 defined by Γ to the fractional periodic timetabling polytope \mathcal{P}_2 defined by Γ' . The map φ is clearly linear and maps mixed-integer points to mixed-integer points, so that φ is mixed-integer-compatible by definition. We conclude that $\varphi((\mathcal{P}_1)_{\text{split}}) \subseteq (\mathcal{P}_2)_{\text{split}}$. Since U is unimodular, φ has a mixed-integer compatible inverse, so that φ provides a “mixed-integer” isomorphism of $(P_1)_{\text{split}}$ with $(P_2)_{\text{split}}$.

5.2 Split Closure of Cartesian Products

As first application of mixed-integer-compatible maps, we prove that split closures are compatible with Cartesian products.

Theorem 5.5. *Consider two mixed-integer sets*

$$S_i = \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{Z}^{p_i} \mid A_C^i x + A_I^i z \leq b^i\}, \quad i \in \{1, 2\},$$

and the associated polyhedra

$$P_i := \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \mid A_C^i x + A_I^i z \leq b^i\}, \quad i \in \{1, 2\}.$$

Then $(P_1 \times P_2)_{\text{split}} = (P_1)_{\text{split}} \times (P_2)_{\text{split}}$.

Proof. We first prove $(P_1)_{\text{split}} \times (P_2)_{\text{split}} \subseteq (P_1 \times P_2)_{\text{split}}$ using the characterization (7):

$$\begin{aligned}
& (P_1)_{\text{split}} \times (P_2)_{\text{split}} \\
&= \left(\bigcap_{\beta_1 \in \mathbb{Z}^{p_1}} \text{conv}(\{(x_1, z_1) \in P_1 \mid \beta_1^\top z_1 \in \mathbb{Z}\}) \right) \times \left(\bigcap_{\beta_2 \in \mathbb{Z}^{p_2}} \text{conv}(\{(x_2, z_2) \in P_2 \mid \beta_2^\top z_2 \in \mathbb{Z}\}) \right) \\
&= \bigcap_{\beta_1 \in \mathbb{Z}^{p_1}} \bigcap_{\beta_2 \in \mathbb{Z}^{p_2}} \left(\text{conv}(\{(x_1, z_1) \in P_1 \mid \beta_1^\top z_1 \in \mathbb{Z}\}) \times \text{conv}(\{(x_2, z_2) \in P_2 \mid \beta_2^\top z_2 \in \mathbb{Z}\}) \right) \\
&= \bigcap_{(\beta_1, \beta_2) \in \mathbb{Z}^{p_1} \times \mathbb{Z}^{p_2}} \text{conv}(\{(x_1, z_1) \in P_1 \mid \beta_1^\top z_1 \in \mathbb{Z}\} \times \{(x_2, z_2) \in P_2 \mid \beta_2^\top z_2 \in \mathbb{Z}\}) \\
&= \bigcap_{(\beta_1, \beta_2) \in \mathbb{Z}^{p_1} \times \mathbb{Z}^{p_2}} \text{conv}(\{(x_1, z_1, x_2, z_2) \in P_1 \times P_2 \mid \beta_1^\top z_1 \in \mathbb{Z}, \beta_2^\top z_2 \in \mathbb{Z}\}) \\
&\subseteq \bigcap_{(\beta_1, \beta_2) \in \mathbb{Z}^{p_1} \times \mathbb{Z}^{p_2}} \text{conv}(\{(x_1, z_1, x_2, z_2) \in P_1 \times P_2 \mid (\beta_1, \beta_2)^\top (z_1, z_2) \in \mathbb{Z}\}) \\
&= (P_1 \times P_2)_{\text{split}}.
\end{aligned}$$

To show the reverse inclusion, we consider the natural projections $\varphi_i : P_1 \times P_2 \rightarrow P_i$ for $i \in \{1, 2\}$. Both φ_i are mixed-integer-compatible, so that by Theorem 5.3, $\varphi_i((P_1 \times P_2)_{\text{split}}) \subseteq (P_i)_{\text{split}}$. In particular, the map (φ_1, φ_2) , which is the identity map, maps $(P_1 \times P_2)_{\text{split}}$ into $(P_1)_{\text{split}} \times (P_2)_{\text{split}}$. \square

We apply now Theorem 5.5 to periodic timetabling. Consider for an arbitrary digraph G its decomposition into blocks. Since each cycle is part of a unique block, the cycle space of G decomposes into the direct sum of the cycle spaces of its blocks. This has the consequence that any cycle matrix Γ of G has a block structure as well, so that the fractional periodic timetabling polytope \mathcal{P} is the Cartesian product of the fractional periodic timetabling polytopes associated to the subinstances of each block.

Theorem 5.6 (cf. N. Lindner & Liebchen, 2020). *If G_1, \dots, G_k are the blocks of G and $\mathcal{P}_1, \dots, \mathcal{P}_k$ are the fractional periodic timetabling polytopes of the subinstances of G_1, \dots, G_k , respectively, then*

$$\mathcal{P}_{\text{split}} = (\mathcal{P}_1)_{\text{split}} \times \dots \times (\mathcal{P}_k)_{\text{split}}.$$

In particular, if G is a cactus graph, then $\mathcal{P}_{\text{split}} = \mathcal{P}_1$.

Proof. By the above discussion, this is a direct consequence of Theorem 5.5. If G is a cactus graph, then each block satisfies $\mu \leq 1$. It remains to apply Corollary 3.2. \square

5.3 Incidence-Based vs. Cycle-Based Formulation

Recall from Remark 2.6 that the incidence-based formulation (2) of a PESP instance is identical to a particular cycle-based formulation (3) of an augmented instance, where the augmentation consists in successively adding arcs a with $\ell_a = 0$ and $u_a = T - 1$. Such arcs with $u_a - \ell_a = T - 1$ are sometimes called *free* (e.g., Goerigk and Liebchen 2017), as they do not impact the feasibility of a PESP instance. The augmentation procedure in Remark 2.6 hence decomposes as a sequence of $|V|$ *simple free augmentations*, which we formally define as follows:

Definition 5.7. Let $I = (G, T, \ell, u, w)$ be a PESP instance. Let $I' = (G', T, \ell', u', w')$ be a PESP instance such that I arises from I' by deleting a *free* arc \bar{a} , i.e., $u'_{\bar{a}} - \ell'_{\bar{a}} = T - 1$. We say that I' is a *simple free augmentation* of I by \bar{a} .

We will first investigate a trivial case of a simple augmentation I' of I by \bar{a} : If \bar{a} is a bridge, then \bar{a} constitutes a block of G' , so that we conclude by Theorem 5.6 that

$$\mathcal{P}'_{\text{split}} = \mathcal{P}_{\text{split}} \times [\ell'_{\bar{a}}, u'_{\bar{a}}]_{\text{split}} = \mathcal{P}_{\text{split}} \times [\ell'_{\bar{a}}, u'_{\bar{a}}], \quad (17)$$

where \mathcal{P} and \mathcal{P}' are the fractional periodic tension polytopes of I and I' , respectively. Since \bar{a} is a bridge, any cycle basis for G is a cycle basis for G' , so that the choice of any integral cycle basis yields a natural projection $\mathcal{P}'_{\text{split}} \rightarrow \mathcal{P}_{\text{split}}, (x, x_{\bar{a}}, z) \mapsto (x, z)$, which is well-defined and surjective by (17). Thus any split inequality for I' is trivially a split inequality for I and vice versa.

We will hence turn our interest to the more interesting case that \bar{a} is not a bridge. We start with an observation about cycle bases:

Lemma 5.8. *Let I' be a simple free augmentation of I by \bar{a} such that \bar{a} is not a bridge of G' . Then there is an integral cycle basis B of G and an oriented cycle $\bar{\gamma}$ such that $B' := B \cup \{\bar{\gamma}\}$ is an integral cycle basis of G' and $\bar{a} \in \bar{\gamma}$.*

Proof. Since \bar{a} is not a bridge, G and G' have the same set of nodes, so that any spanning tree of G is a spanning tree of G' . Hence, if B is any fundamental cycle basis of G , we can augment B by the fundamental cycle $\bar{\gamma}$ induced by \bar{a} in G' . \square

Choose cycle bases B, B' and an oriented cycle $\bar{\gamma}$ as in Lemma 5.8. We assume that \mathcal{P} is defined using the cycle matrix Γ of B , and that \mathcal{P}' is defined using the cycle matrix Γ' of B' , so that Γ' arises from Γ by appending the row $\bar{\gamma}^\top$.

Lemma 5.9. *Let I' be a simple free augmentation of I by \bar{a} such that \bar{a} is not a bridge of G' . The natural projection $\varphi : \mathcal{P}' \rightarrow \mathcal{P}, (x, x_{\bar{a}}, z, z_{\bar{\gamma}}) \mapsto (x, z)$ is mixed-integer-compatible. In particular, $\varphi(\mathcal{P}'_{\text{split}}) \subseteq \mathcal{P}_{\text{split}}$.*

Proof. The map φ is linear and maps mixed-integer points to mixed-integer points. That φ descends to split closures follows from Theorem 5.3. \square

In view of Lemma 5.9, the split closure of the simple free augmentation is hence never worse, but could provide a potentially tighter relaxation by additional “projected split inequalities”. We show now that this is not the case.

Lemma 5.10. *Let $\varphi : \mathcal{P}' \rightarrow \mathcal{P}$ denote the natural projection as in Lemma 5.9. Then $\varphi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$.*

Proof. Since I' is an augmentation of I by \bar{a} , we note at first, using the interpretation of split inequalities as flip inequalities from Theorem 3.1, that the set of defining inequalities of $\mathcal{P}'_{\text{split}}$ can be partitioned into the set of defining inequalities of $\mathcal{P}_{\text{split}}$, which cannot contain the variable $x_{\bar{a}}$, and a remaining set of inequalities, which do all contain $x_{\bar{a}}$. The image of $\varphi(\mathcal{P}'_{\text{split}})$ can be described by Fourier-Motzkin elimination of the variable $x_{\bar{a}}$. It is therefore sufficient to show that all inequalities generated by the Fourier-Motzkin procedure are redundant for $\mathcal{P}_{\text{split}}$. Since the redundancy is clear for those inequalities that do not contain $x_{\bar{a}}$, we will hence consider only the remaining inequalities where $x_{\bar{a}}$ has a non-zero coefficient.

Among the defining inequalities of $\mathcal{P}'_{\text{split}}$, $x_{\bar{a}}$ occurs precisely in the bound inequalities $x_{\bar{a}} \geq \ell'_{\bar{a}}$ and $x_{\bar{a}} \leq u'_{\bar{a}}$, and in the flip inequalities of simple cycles containing \bar{a} . Fourier-Motzkin considers pairs of these inequalities, one of them giving a lower bound, and the other an upper bound on $x_{\bar{a}}$. That is, the following types of pairs have to be considered:

- (1) $x_{\bar{a}} \geq \ell'_{\bar{a}}$ and $x_{\bar{a}} \leq u'_{\bar{a}}$,

- (2) $x_{\bar{a}} \geq \ell'_{\bar{a}}$ and a flip inequality for (γ, F) with $\bar{a} \in \gamma$ and $\bar{a} \in F$,
- (3) $x_{\bar{a}} \leq u'_{\bar{a}}$ and a flip inequality for (γ, F) with $\bar{a} \in \gamma$ and $\bar{a} \notin F$,
- (4) two flip inequalities for (γ, F_γ) and (δ, F_δ) with $\bar{a} \in \gamma, \bar{a} \notin F_\gamma, \bar{a} \in \delta, \bar{a} \in F_\delta$.

In all those flip inequalities, we can assume that the cycles are simple and that the parameter α is at least 1. Moreover, using the symmetry in Remark 2.13, we can without loss of generality fix the direction of \bar{a} as forward or backward, replacing γ by $-\gamma$ if necessary. Let us proceed with Fourier-Motzkin:

- (1) Elimination yields $\ell'_{\bar{a}} \leq u'_{\bar{a}}$, which is trivially true.
- (2) Assume that \bar{a} is forward in γ . Then we can write the flip inequality (6) for (γ, F) with $\alpha := \alpha_{\gamma, F} \geq 1$ as

$$\alpha(u'_{\bar{a}} - x_{\bar{a}}) + f(x) \geq \alpha(T - \alpha),$$

where $f(x) \geq 0$ for all $(x, z) \in \mathcal{P}$. Fourier-Motzkin elimination with $x_{\bar{a}} \geq \ell'_{\bar{a}}$ yields

$$\alpha u'_{\bar{a}} + f(x) \geq \alpha(T - \alpha) + \alpha \ell'_{\bar{a}},$$

or equivalently, recalling that $u'_{\bar{a}} - \ell'_{\bar{a}} = T - 1$,

$$f(x) \geq \alpha(T - \alpha - u'_{\bar{a}} + \ell'_{\bar{a}}) = \alpha(1 - \alpha),$$

but this is redundant for $\mathcal{P}_{\text{split}}$, since $(x, z) \in \mathcal{P}$ and $\alpha \geq 1$ imply $f(x) \geq 0 \geq \alpha(1 - \alpha)$.

- (3) is analogous to (2).
- (4) This is the most tedious part. We assume without loss of generality that \bar{a} is backward in γ and forward in δ . We will show that the Fourier-Motzkin inequality is valid for all points $(x, z) \in \mathcal{P}$ with $(\gamma + \delta)^\top x \in T\mathbb{Z}$. Since $\gamma + \delta$ is an element of the cycle space \mathcal{C} of G , the Fourier-Motzkin inequality is hence valid for the convex hull of those points and in particular for $\mathcal{P}_{\text{split}}$ by virtue of (7).

We first write down the flip inequalities, omitting F_γ and F_δ in the subscripts of α :

$$\begin{aligned} \alpha_\gamma(x_{\bar{a}} - \ell'_{\bar{a}}) + f(x) &\geq \alpha_\gamma(T - \alpha_\gamma), \\ \alpha_\delta(u'_{\bar{a}} - x_{\bar{a}}) + g(x) &\geq \alpha_\delta(T - \alpha_\delta), \end{aligned}$$

where $f(x), g(x) \geq 0$ for all $(x, z) \in \mathcal{P}$. Elimination produces

$$\alpha_\delta f(x) + \alpha_\gamma g(x) \geq \alpha_\gamma \alpha_\delta (2T - \alpha_\gamma - \alpha_\delta - u'_{\bar{a}} + \ell'_{\bar{a}}) = \alpha_\gamma \alpha_\delta (T + 1 - \alpha_\gamma - \alpha_\delta). \quad (18)$$

The inequality (18) is trivially redundant if $\alpha_\gamma + \alpha_\delta \geq T + 1$. We hence assume from now on $\alpha_\gamma + \alpha_\delta \leq T$. Let $(x, z) \in \mathcal{P}$ with $(\gamma + \delta)^\top x \in T\mathbb{Z}$. Then

$$\sum_{a \in A \setminus F_\gamma} \gamma_a x_a + \sum_{a \in F_\gamma} \gamma_a x_a + \sum_{a \in A \setminus F_\delta} \delta_a x_a + \sum_{a \in F_\delta} \delta_a x_a \equiv 0 \pmod{T}.$$

Since $\gamma_{\bar{a}} + \delta_{\bar{a}} = 0, \bar{a} \notin F_\gamma, \bar{a} \in F_\delta$, this implies

$$\sum_{a \in A \setminus (F_\gamma \cup \{\bar{a}\})} \gamma_a x_a + \sum_{a \in F_\gamma} \gamma_a x_a + \sum_{a \in A \setminus F_\delta} \delta_a x_a + \sum_{a \in F_\delta \setminus \{\bar{a}\}} \delta_a x_a \equiv 0 \pmod{T},$$

so that, using the definition of $\alpha_\gamma, \alpha_\delta$ (cf. Theorem 2.12),

$$\begin{aligned}
& \sum_{a \in A \setminus (F_\gamma \cup \{\bar{a}\})} \gamma_a(x_a - \ell_a) - \sum_{a \in F_\gamma} \gamma_a(u_a - x_a) + \sum_{a \in A \setminus F_\delta} \delta_a(x_a - \ell_a) - \sum_{a \in F_\delta \setminus \{\bar{a}\}} \delta_a(u_a - x_a) \\
& \equiv - \sum_{a \in A \setminus (F_\gamma \cup \{\bar{a}\})} \gamma_a \ell_a - \sum_{a \in F_\gamma} \gamma_a u_a - \sum_{a \in A \setminus F_\delta} \delta_a \ell_a - \sum_{a \in F_\delta \setminus \{\bar{a}\}} \delta_a u_a \pmod{T} \\
& \equiv \alpha_\gamma + \alpha_\delta + u'_{\bar{a}} - \ell'_{\bar{a}} \pmod{T} \\
& \equiv \alpha_\gamma + \alpha_\delta - 1 \pmod{T}
\end{aligned}$$

As we can assume $\alpha_\gamma, \alpha_\delta \geq 1$, we have that $\alpha := \alpha_\gamma + \alpha_\delta - 1 \geq 0$. This implies that

$$D := \sum_{a \in A \setminus (F_\gamma \cup \{\bar{a}\})} \gamma_a(x_a - \ell_a) - \sum_{a \in F_\gamma} \gamma_a(u_a - x_a) + \sum_{a \in A \setminus F_\delta} \delta_a(x_a - \ell_a) - \sum_{a \in F_\delta \setminus \{\bar{a}\}} \delta_a(u_a - x_a)$$

is either $\leq \alpha - T$ (a) or $\geq \alpha$ (b). Before showing that (18) is redundant in both cases, we write down the left-hand side of (18) explicitly:

$$\begin{aligned}
& \alpha_\delta f(x) + \alpha_\gamma g(x) \\
& = \alpha_\delta(T - \alpha_\gamma) \sum_{a \in A \setminus (F_\gamma \cup \{\bar{a}\}), \gamma_a=1} (x_a - \ell_a) + \alpha_\delta \alpha_\gamma \sum_{a \in A \setminus (F_\gamma \cup \{\bar{a}\}), \gamma_a=-1} (x_a - \ell_a) \\
& + \alpha_\delta \alpha_\gamma \sum_{a \in F_\gamma, \gamma_a=1} (u_a - x_a) + \alpha_\delta(T - \alpha_\gamma) \sum_{a \in F_\gamma, \gamma_a=-1} (u_a - x_a) \\
& + \alpha_\gamma(T - \alpha_\delta) \sum_{a \in A \setminus F_\delta, \delta_a=1} (x_a - \ell_a) + \alpha_\gamma \alpha_\delta \sum_{a \in A \setminus F_\delta, \delta_a=-1} (x_a - \ell_a) \\
& + \alpha_\gamma \alpha_\delta \sum_{a \in F_\delta \setminus \{\bar{a}\}, \delta_a=1} (u_a - x_a) + \alpha_\gamma(T - \alpha_\delta) \sum_{a \in F_\delta \setminus \{\bar{a}\}, \delta_a=-1} (u_a - x_a).
\end{aligned} \tag{19}$$

(a) Expanding $\alpha_\delta(T - \alpha_\gamma)$ and $\alpha_\gamma(T - \alpha_\delta)$ in (19), and then bounding all summands with T as a factor by 0 from below, we obtain

$$\alpha_\delta f(x) + \alpha_\gamma g(x) \geq -\alpha_\gamma \alpha_\delta D \geq -\alpha_\gamma \alpha_\delta (\alpha - T) = \alpha_\gamma \alpha_\delta (T + 1 - \alpha_\gamma - \alpha_\delta).$$

Hence (18) holds in the case that $D \leq \alpha - T$.

(b) Let $\nu := \min(\alpha_\gamma, \alpha_\delta)$. Expanding $\alpha_\delta(T - \alpha_\gamma)$ and $\alpha_\gamma(T - \alpha_\delta)$ in (19), bounding $T\alpha_\gamma, T\alpha_\delta$ from below by $T\nu$, we find

$$\alpha_\delta f(x) + \alpha_\gamma g(x) \geq (T\nu - \alpha_\gamma \alpha_\delta) D.$$

Since ν is one of $\alpha_\gamma, \alpha_\delta$ and $\alpha_\gamma, \alpha_\delta \leq T - 1$, we have $T\nu - \alpha_\gamma \alpha_\delta \geq 0$. This implies with $D \geq \alpha$ that

$$\begin{aligned}
& \alpha_\delta f(x) + \alpha_\gamma g(x) \\
& \geq (T\nu - \alpha_\gamma \alpha_\delta) \alpha \\
& = (T\nu - \alpha_\gamma \alpha_\delta) (\alpha_\gamma + \alpha_\delta - 1) \\
& = T\nu(\alpha_\gamma + \alpha_\delta - 1) + \alpha_\gamma \alpha_\delta (1 - \alpha_\gamma - \alpha_\delta) \\
& = T\nu(\alpha_\gamma + \alpha_\delta - 1) - T\alpha_\gamma \alpha_\delta + \alpha_\gamma \alpha_\delta (T + 1 - \alpha_\gamma - \alpha_\delta)
\end{aligned}$$

It remains to show that $T\nu(\alpha_\gamma + \alpha_\delta - 1) - T\alpha_\gamma \alpha_\delta \geq 0$. This is true since ν is one of $\alpha_\gamma, \alpha_\delta \geq 1$.

We conclude that the image $\varphi(\mathcal{P}'_{\text{split}})$ is fully described by the flip inequalities of cycles not containing \bar{a} and the variable bounds for all arcs except \bar{a} . Hence $\varphi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$. \square

As a corollary to Lemma 5.10, we obtain the following result.

Theorem 5.11. *Let I and I' be PESP instances with fractional periodic timetabling polyhedra \mathcal{P} and \mathcal{P}' , respectively. Suppose that I' arises from I by a sequence of simple free augmentations. If $\varphi : \mathcal{P}' \rightarrow \mathcal{P}$ denotes the natural projection, then $\varphi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$.*

In particular, recalling Remark 2.6, the incidence-based formulation (2) is not stronger than the cycle-based formulation (3) in terms of split closures. Consequently, it is of no use to develop methods which augment an instance by a free arc, obtain a flip/split inequality and project down again, as this will not lead to information which cannot already be obtained from the split closure of the original instance.

5.4 Binarization by Subdivision

A reformulation of a MIP general variables into one with binary variables can exhibit stronger split closures (Dash et al., 2018) or lift-and-project closures (Aprile et al., 2021). For the application of periodic timetabling, there is a combinatorial binarization method: Let $I = (G, T, \ell, u, w)$ be a PESP instance, $G = (V, A)$. We assume that $0 \leq \ell < T$ and $\ell \leq u < \ell + T$ by preprocessing (see Remark 2.3), so that the integer periodic offset variables p_a in the incidence-based formulation (2) of PESP can only take values in $\{0, 1, 2\}$. Moreover, if $u_a \leq T$ for some $a \in A$, then $p_a \in \{0, 1\}$ for any integer feasible solution (x, π, p) (Liebchen, 2006).

Definition 5.12. Let $I' = (G', T, \ell', u', w')$ be a PESP instance that arises from I by subdividing an arc $\bar{a} \in A$ with $\ell_{\bar{a}} < u_{\bar{a}}$ into two new arcs a_1, a_2 such that:

$$\begin{aligned} 0 &\leq \ell'_{a_1} \leq u'_{a_1}, \\ 0 &\leq \ell'_{a_2} \leq u'_{a_2}, \\ \ell'_{a_1} + \ell'_{a_2} &= \ell_{\bar{a}}, \\ u'_{a_1} + u'_{a_2} &= u_{\bar{a}}, \\ w'_{a_1} = w'_{a_2} &= w_{\bar{a}}. \end{aligned}$$

We call I' a *simple subdivision* of I at \bar{a} .

Observe that if the bounds on the arc \bar{a} are such that $u_{\bar{a}} > T$, one can always construct a simple subdivision I' of I at \bar{a} such that $u'_{a_i} - \ell'_{a_i} > 0$ and $u'_{a_i} \leq T$ for $i \in \{1, 2\}$, due to the assumption that $\ell < T$ and $u < \ell + T$. As a result, for the instance I' arising from subdividing each arc \bar{a} with $u_{\bar{a}} > T$ as above, the incidence-based MIP formulation (2) will then have exclusively binary variables.

Example 5.13. Figure 4 shows the instance obtained from the instance I from Figure 1 by subdividing every arc with $u_a > T$.

Let I' be a simple subdivision of a PESP instance I at \bar{a} , introducing new arcs a_1 and a_2 . The cycle spaces of G and G' are isomorphic: If γ is an element of the cycle space of G , then γ' with

$$\gamma'_a := \begin{cases} \gamma_{\bar{a}} & \text{if } a \in \{a_1, a_2\}, \\ \gamma_a & \text{if } a \notin \{a_1, a_2\}, \end{cases}$$

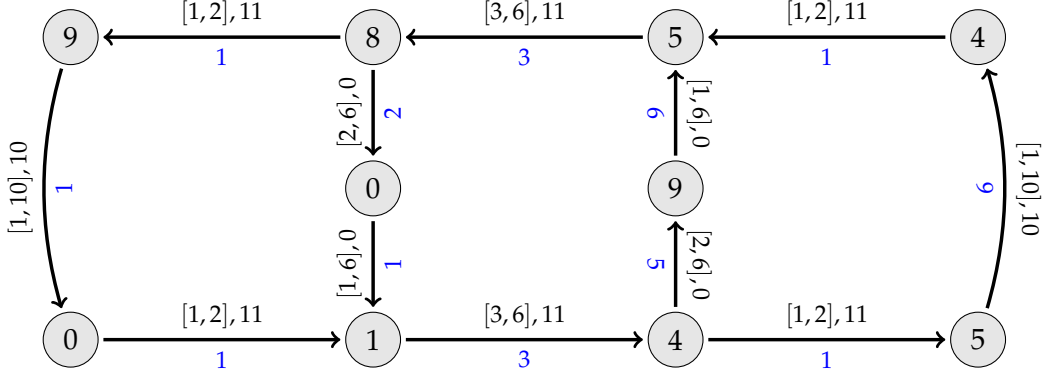


Figure 4: Subdivision of the instance in Figure 1 obtained from two simple subdivisions such that $u_a \leq T$ for all arcs a .

defines an element of the cycle space of G' , and the whole cycle space of G' arises this way. We can therefore associate to an integral cycle basis $B = \{\gamma_1, \dots, \gamma_\mu\}$ of G the integral cycle basis $B' = \{\gamma'_1, \dots, \gamma'_\mu\}$. Then any cycle offset z in (3) w.r.t. B defines a cycle offset z' w.r.t. B' by $z'_{\gamma'_\mu} := z_{\gamma_\mu}$, so that cycle offsets are essentially the same. We will use Γ and Γ' to define \mathcal{P} and \mathcal{P}' , the fractional periodic tension polytopes of I' and I , respectively.

Lemma 5.14. *Consider a simple subdivision I' of I at an arc \bar{a} with notation as above.*

- (1) *The map $\rho : \mathcal{P}' \rightarrow \mathcal{P}, (x, x_{a_1}, x_{a_2}, z) \mapsto (x, x_{a_1} + x_{a_2}, z)$ is well-defined and mixed-integer-compatible.*
- (2) *The map $s : \mathcal{P} \rightarrow \mathcal{P}', (x, x_{\bar{a}}, z) \mapsto \left(x, \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\bar{a}} - \ell_{\bar{a}}}(x_{\bar{a}} - \ell_{\bar{a}}), \ell'_{a_2} + \frac{u'_{a_2} - \ell'_{a_2}}{u_{\bar{a}} - \ell_{\bar{a}}}(x_{\bar{a}} - \ell_{\bar{a}}), z \right)$ is well-defined and mixed-integer-compatible.*
- (3) *$\rho \circ s : \mathcal{P} \rightarrow \mathcal{P}$ is the identity map.*
- (4) *$\rho(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$.*

Proof. (1) The map is well-defined: The hypothesis $\ell'_{a_1} + \ell'_{a_2} = \ell_{\bar{a}}$ and $u'_{a_1} + u'_{a_2} = u_{\bar{a}}$ implies that $\ell_{\bar{a}} \leq x_{a_1} + x_{a_2} \leq u_{\bar{a}}$ holds for all $(x, x_{a_1}, x_{a_2}, z) \in \mathcal{P}'$. As ρ is linear and does not affect the integrality of z , it is mixed-integer-compatible.

(2) The map is well defined: Due to the assumption of subdividing arcs with $u_{\bar{a}} > T$ only, we have $u_{\bar{a}} - \ell_{\bar{a}} > 0$ and $u'_{a_1} - \ell'_{a_1} \geq 0$. Since $x_{\bar{a}} - \ell_{\bar{a}} \geq 0$ for all $(x, x_{\bar{a}}, z) \in \mathcal{P}$, we conclude

$$\ell'_{a_1} = \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\bar{a}} - \ell_{\bar{a}}}(x_{\bar{a}} - \ell_{\bar{a}}) \leq \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\bar{a}} - \ell_{\bar{a}}}(x_{\bar{a}} - \ell_{\bar{a}}) \leq \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\bar{a}} - \ell_{\bar{a}}}(u_{\bar{a}} - \ell_{\bar{a}}) = u'_{a_1}.$$

The argument for the x_{a_2} entry is analogous. We note that s is affine and maps point with integral z to points with integral z , so that s is mixed-integer-compatible.

(3) This follows since

$$\ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\bar{a}} - \ell_{\bar{a}}}(x_{\bar{a}} - \ell_{\bar{a}}) + \ell'_{a_2} + \frac{u'_{a_2} - \ell'_{a_2}}{u_{\bar{a}} - \ell_{\bar{a}}}(x_{\bar{a}} - \ell_{\bar{a}}) = \ell_{\bar{a}} + \frac{u_{\bar{a}} - \ell_{\bar{a}}}{u_{\bar{a}} - \ell_{\bar{a}}}(x_{\bar{a}} - \ell_{\bar{a}}) = x_{\bar{a}}.$$

- (4) Since ρ and s are mixed-integer compatible, $\rho(\mathcal{P}'_{\text{split}}) \subseteq \mathcal{P}_{\text{split}}$ and $s(\mathcal{P}_{\text{split}}) \subseteq \mathcal{P}'_{\text{split}}$. The composition $\rho|_{\mathcal{P}'_{\text{split}}} \circ s|_{\mathcal{P}_{\text{split}}}$ of the restrictions to split closures is hence well-defined, and by (3), it is the identity map on $\mathcal{P}_{\text{split}}$. We conclude that $\rho|_{\mathcal{P}'_{\text{split}}}$ is surjective. \square

A repeated application of Lemma 5.14 together with Theorem 5.11 yields:

Theorem 5.15. *Let I and I' be PESP instances with fractional periodic timetabling polyhedra \mathcal{P} and \mathcal{P}' , respectively. Suppose that I' arises from I by a sequence of simple subdivisions and simple free augmentations. If $\psi : \mathcal{P}' \rightarrow \mathcal{P}$ denotes the composition of the summation maps ρ in Lemma 5.14 (1) for the subdivisions and the projection maps φ in Lemma 5.9 for the free augmentations, then $\psi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$.*

In particular, when we binarize the MIP (3) by first performing simple subdivisions and then move to the formulation (2), we gain no further insight about split inequalities.

6 Computational Experiments

We want to assess how useful the split closure is for obtaining dual bounds for PESP in practice. To that end we introduce a procedure, which exploits Theorem 4.7 in a heuristic way, and proceeds to find cuts systematically once the heuristic fails, such that we optimize over the entire split closure by means of Theorem 4.5. We will also examine the performance of the heuristic in comparison to the systematic exploration.

6.1 Separation Procedure

Our goal is to optimize over the entire split closure. We do so with our custom separator which proceeds as illustrated by the flowchart in Figure 5: At first, it tries to heuristically generate violated flip inequalities (highlighted in blue in the chart): We compute a minimum spanning tree with respect to the periodic slack $x - \ell$ of the current LP solution $(x, z) \in \mathcal{P}$, and determine a most violated flip inequality for the fundamental cycles of that tree by Theorem 4.7. When no more heuristic cuts are found, the parametric IP (14) as in Theorem 4.5 is solved. During the solution process of the IP, a callback retrieves intermediate incumbent solutions and generates the corresponding cuts. The procedure terminates when no more violated cuts can be found, or the time limit is hit. Since the amount of cuts found by the heuristic is rather larger in the beginning, we apply the filtering mechanisms of SCIP to detect effective cuts. However, cuts found by the parametric IP will always be enforced, so that the whole procedure is correct up to numerical tolerances: If the procedure terminates because no more violated cuts can be detected, then the optimal solution over the split closure has been found.

6.2 Methodology

To conduct our computational experiments, we use the benchmark library PESPLib (Goerigk, 2022), whose instances are derived from real-world scenarios. Although significant progress has been made in the past, no instance could be solved to proven optimality up to date.

Since the PESPLib instances are computationally very hard, we consider not only the full instances, but also two subinstances per instance whose cyclomatic number μ has been restricted to 25, and 100, respectively. Note that μ is the number of integer variables in (3). The restriction procedure for an instance (G, T, ℓ, u, w) works by iteratively removing arcs, deleting in each step one arc a with highest span $u_a - \ell_a$ and breaking ties by preferring lowest weight

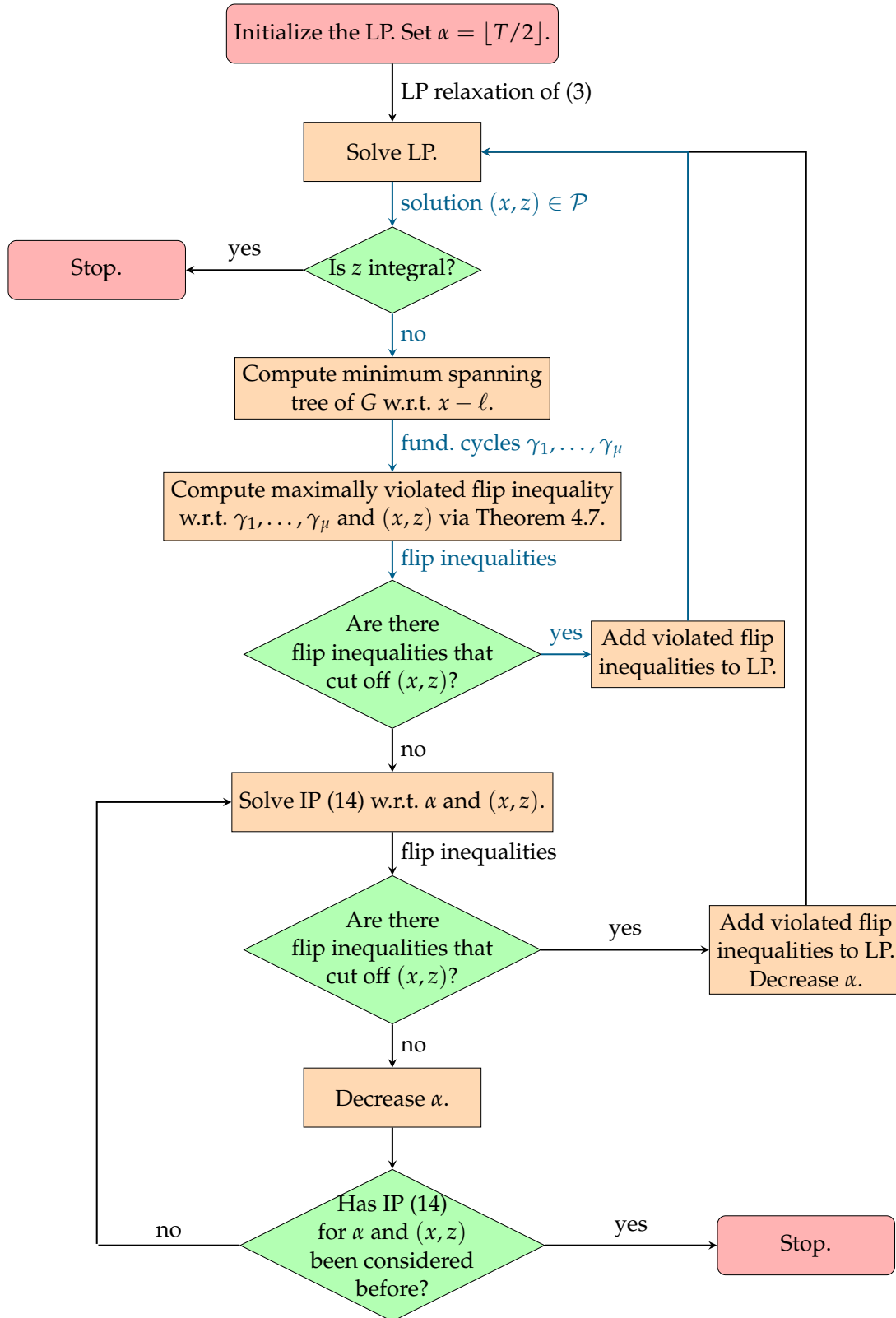


Figure 5: Flowchart of the split cut generation procedure. “Decrease α ” means to set $\alpha := \alpha - 1$ if $\alpha \geq 2$ and $\alpha := \lfloor T/2 \rfloor$ otherwise. We start with $\alpha = \lfloor T/2 \rfloor$, as this is likely to produce cuts with large violation.

w_a (cf. Goerigk and Liebchen 2017). In contrast to the full PESplib instances, these restricted variants can be solved to optimality within a reasonable amount of time.

We first preprocess each instance, so that in particular the assumptions as in Remark 2.3 hold. For each instance $I = (G, T, \ell, u, w)$, we consider the cycle-based formulation (3) using an integral cycle basis B that minimizes $\sum_{\gamma \in B} \sum_{a \in \gamma} (u_a - \ell_a)$. This choice of cycle basis is motivated by its good performance for computing dual bounds (Borndörfer, Lindner, & Roth, 2020; Masing, Lindner, & Ebert, 2023). By Theorem 5.11, the cycle-based formulation is not weaker than the incidence-based formulation, and by Remark 2.6, it is more compact. We then invoke the branch-cut-and-price framework SCIP (Achterberg, 2009). The advantage of using SCIP is that it is highly customizable and we can disable everything that does not come from split cuts: We disable the built-in presolving, branching, heuristics, propagators separators and merely call a custom separation callback during the cutting loop at the root node.

In all experiments, we use SCIP 8.0.3 (Bestuzheva et al., 2021) with Gurobi 9.5.2 (Gurobi Optimization, LLC, 2023) as LP solver. We also use Gurobi to solve the parametric IP from Theorem 4.5. Gurobi is allowed to use 6 threads on an Intel Xeon E3-1270 v6 CPU running at 3.8 GHz with 32 GB RAM. The time limit has been set to 4 hours wall time for each instance.

6.3 Results

6.3.1 Restriction to $\mu = 25$

Table 1 shows the results for the restrictions of the PESplib instances to the cyclomatic number $\mu = 25$. For all but one instance, the cut generation procedure of Section 6.2 terminates within 22 minutes, only R1L1v hits the time limit due to a hard parametric IP (14). Optimizing over the split closure is exact for R4L1 and R4L4v, but R4L4v is trivial in the sense that $x = \ell$ is an optimal solution. The average relative optimality gap with respect to the optimal objective value in terms of weighted slack $w^\top(x - \ell)$ and the best bound obtained by split cuts, taken over all 22 instances, is 6.61 %.

6.3.2 Restriction to $\mu = 100$

The results for the restriction to $\mu = 100$ are summarized in Table 2. Again, we can determine the optimal solution of (3) for all these restricted instances. The cut generation procedure of Section 6.2 terminates within the time limit for 20 out of 22 instances. R4L4v is again almost trivial to solve, because two cuts suffice to produce an integral solution. The second smallest gap is at R1L1v, although the time limit is hit. The average optimality gap is 13.37 %, which is about twice as much as in the case $\mu = 25$.

6.3.3 Full instances

Finally, the results for the full PESplib instances are given in Table 3 in comparison to the best known primal bounds and in Table 4 in comparison to the best known dual bounds. All instances hit the time limit. Compared to the restricted instances, relatively few cuts are generated by the IP (14), which is both due to the large supply of heuristically generated cuts, and the difficulty of the IP. The time limit is not sufficient to unfold the power of the IP, on the other hand, increasing the time limit to 8 or 24 hours empirically produced only marginal improvements. This effect is also illustrated in Figure 6: The plot shows an exemplary progression of the dual bound and the number of applied cuts for the instance R2L1 with a logarithmic time axis. The heuristic separation procedure finds no more cuts for the first time after roughly 45 minutes (about $10^{3.43}$ seconds), and then the parametric IP takes over, causing a sudden and

Instance	μ	Opt. Val. (\mathcal{P}_1)	Dual Bd. ($\mathcal{P}_{\text{split}}$)	Gap [%]	Cuts	IP Cuts	Time [s]
BL1	25	479 501	455 492	5.01	114	33	98
BL2	25	582 203	529 247	9.10	128	32	116
BL3	25	614 544	513 344	16.47	122	28	197
BL4	25	581 688	507 168	12.81	176	65	106
R1L1	25	1 469 763	1 314 105	10.59	284	123	747
R1L2	25	1 271 066	1 235 774	2.78	226	96	857
R1L3	25	1 704 349	1 693 441	0.64	238	114	1 281
R1L4	25	1 543 182	1 429 795	7.35	294	118	936
R2L1	25	2 598 725	2 171 855	16.43	212	83	255
R2L2	25	2 726 109	2 471 181	9.35	238	75	335
R2L3	25	1 698 794	1 661 074	2.22	116	12	91
R2L4	25	2 417 447	2 325 110	3.82	244	64	119
R3L1	25	1 110 721	1 055 499	4.97	170	83	513
R3L2	25	1 283 884	1 148 551	10.54	152	67	201
R3L3	25	1 617 501	1 478 034	8.62	196	58	389
R3L4	25	1 063 438	987 067	7.18	143	56	399
R4L1	25	1 053 623	1 053 623	0.00	102	14	205
R4L2	25	1 394 526	1 313 700	5.80	136	39	231
R4L3	25	1 718 591	1 648 388	4.08	148	47	213
R4L4	25	498 913	488 043	2.18	171	60	701
R1L1v	25	1 741 592	1 645 779	5.50	128	58	14 400
R4L4v	25	3 660 000	3 660 000	0.00	0	0	0

Table 1: Results for the PESPlib instances restricted to $\mu = 25$. The table lists the optimal objective value of the MIP (3) in terms of weighted slack $w^\top(x - \ell)$, the best dual bound obtained by split cuts, the primal-dual gap, the total number of applied split cuts, the number of cuts provided by the parametric IP (14), and the running time in seconds.

Instance	μ	Opt. Val. (\mathcal{P}_1)	Dual Bd. ($\mathcal{P}_{\text{split}}$)	Gap [%]	Cuts	IP Cuts	Time [s]
BL1	100	1 341 151	1 216 355	9.31	1 092	357	954
BL2	100	1 733 429	1 451 049	16.29	910	307	1 287
BL3	100	1 747 063	1 461 798	16.33	922	304	1 864
BL4	100	1 605 968	1 427 228	11.13	975	349	1 399
R1L1	100	5 481 154	4 582 018	16.40	1 300	493	6 903
R1L2	100	4 873 559	3 952 695	18.90	1 138	348	7 453
R1L3	100	6 256 521	5 151 095	17.67	998	324	4 742
R1L4	100	5 008 640	4 202 959	16.09	1 407	415	6 184
R2L1	100	8 284 107	6 881 776	16.93	1 021	294	2 453
R2L2	100	7 099 578	6 244 993	12.04	1 366	406	4 648
R2L3	100	6 722 776	5 982 798	11.01	1 102	342	6 038
R2L4	100	5 516 243	4 996 368	9.42	1 217	317	3 242
R3L1	100	4 366 123	3 770 709	13.64	927	355	7 180
R3L2	100	4 666 798	3 796 483	18.65	764	253	8 554
R3L3	100	4 719 345	3 890 774	17.56	921	301	7 492
R3L4	100	2 950 612	2 730 898	7.45	885	348	11 242
R4L1	100	4 428 800	3 715 032	16.12	717	179	2 947
R4L2	100	4 101 438	3 492 759	14.84	789	236	14 400
R4L3	100	4 302 565	3 740 673	13.06	875	226	8 785
R4L4	100	1 994 572	1 676 547	15.94	607	184	7 448
R1L1v	100	10 253 906	9 715 723	5.25	191	0	14 400
R4L4v	100	14 880 000	14 880 000	0.00	2	0	1

Table 2: Results for the PESplib instances restricted to $\mu = 100$. The table lists the optimal objective value of the MIP (3) in terms of weighted slack $w^\top(x - \ell)$, the best dual bound obtained by split cuts, the primal-dual gap, the total number of applied split cuts, the number of cuts provided by the parametric IP (14), and the running time in seconds.

persisting drop in performance. We can observe that once the parametric IP came into effect, the heuristic stage provides only few further cuts. This could be due to the initial high quality results provided by the heuristic, such that the improvement through a cut from the parametric IP results in only a marginal change in the new solution. The subsequent spanning tree in the following heuristic stage could then be similar to the previous one, such that from this point on, only little to no improvement is found in the heuristic stage; and the costly parametric IP is the main contributor.

With respect to all instances, the average optimality gap is 40.83 %. As expected, the quality of the results obtained by our method is dependent on the problem size. In particular for the 16 RiLj instances there is a strong correlation between the size of μ and the optimality gap. This is also evidenced by the Pearson correlation coefficient, which is approximately 95 %.

On the dual side, the split closure provides at least 91.10 % of the currently best known dual bound. This underlines the good performance of our method – most of the incumbent dual bounds have been obtained by longer computation times, and in contrast to our study, neither branching nor other types of cutting planes apart from split cuts have been forbidden. Despite being at a disadvantage in this regard, our method provides better dual bounds for five out of the 22 instances, with improvements up to 25 %. Other bounds have been obtained with the help of heuristically separated flip inequalities as well, by, e.g., Borndörfer, Lindner, and Roth (2020); N. Lindner and Liebchen (2020); N. Lindner, Liebchen, and Masing (2021); Masing et al. (2023), such that our procedure can be seen as an advancement of previous methods in the sense that our heuristic unlocks more potential due to exploiting Theorem 4.7.

6.4 Insights

From our experiments we have gained two main insights: On one hand, we have seen that our procedure is indeed useful in computing qualitative dual bounds, as we were able to improve five instances of the benchmarking library PESPLib significantly. But also for the other instances, some of which have been treated excessively in the past, a high percentage of the bound could be reached in comparably little time by our procedure.

On the other hand, our tests help us to assess the quality of the split closure for computing the lower bounds independent of the procedure chosen: The instances where the optimal solution could be obtained and our procedure terminates give an indication of how well suited the split closure is for dual bounds in the context of PESP. Here, we were able to observe that the split closure provided fairly low optimality gaps on average, and even certified optimality in three cases, a non-negligible gap remains: E.g., in the worst case, namely for R1L2 with $\mu = 100$, there is a gap of 18.9 % between the optimal dual bound of the split closure and the optimal solution. We reach the conclusion that the split closure is essential for raising the dual bound. However, in order to close the primal-gap entirely, further methods, e.g., higher rank split cuts, will have to be applied.

Considering that usually $\mathcal{P}_I \subsetneq \mathcal{P}_{\text{split}}$, such that any bound obtained from the split closure will not be sufficient to prove optimality, one could ask the question, whether it is worth it to explore it to its full extent, or whether the fast, heuristic section of our procedure would be sufficient. For an indication, we analyzed the instances where our procedure terminated before the time limit was reached: We found that the best bound before the parametric IP came into effect reached at least 89.1 %, and on average even 95.6 % of the final dual bound. We conclude that indeed the heuristic approach of separating flip inequalities is quite effective, as it is able to cover the majority of the dual bound that can be obtained from the split closure quickly. In our case, the addition of the parametric IP in the procedure was essential for the assessment of the split closure and might be helpful to find new cuts, so that the heuristic can produce

Instance	μ	Primal Bd. (\mathcal{P}_1)	Dual Bd. ($\mathcal{P}_{\text{split}}$)	Gap [%]	Cuts	IP Cuts	Time [s]
BL1	5 298	6 333 641	4 252 778	32.85	42 927	15	14 400
BL2	4 880	6 799 331	4 299 517	36.77	37 498	84	14 400
BL3	6 265	6 675 098	4 290 946	35.72	58 628	20	14 400
BL4	9 684	6 562 147	3 923 974	40.20	88 640	265	14 400
R1L1	2 722	29 894 745	19 041 890	36.30	21 965	60	14 400
R1L2	2 876	30 507 180	19 059 669	37.52	23 767	45	14 400
R1L3	2 848	29 319 593	18 193 974	37.95	23 468	61	14 400
R1L4	3 769	26 516 727	16 441 121	38.00	30 460	18	14 400
R2L1	3 206	42 422 038	24 806 675	41.52	27 739	163	14 400
R2L2	3 360	40 642 186	24 464 467	39.81	28 842	159	14 400
R2L3	3 239	38 558 371	22 645 939	41.27	28 816	95	14 400
R2L4	5 514	32 483 894	19 102 410	41.19	47 958	0	14 400
R3L1	4 630	43 271 824	25 343 534	41.43	38 725	17	14 400
R3L2	4 800	45 220 083	25 963 773	42.58	41 951	19	14 400
R3L3	5 446	40 483 617	22 273 090	44.98	46 099	6	14 400
R3L4	7 478	33 335 852	17 027 192	48.92	46 773	0	14 400
R4L1	5 331	49 426 919	27 938 824	43.47	42 505	6	14 400
R4L2	5 688	48 764 793	27 585 028	43.43	45 946	7	14 400
R4L3	6 871	45 493 081	23 849 465	47.58	46 277	0	14 400
R4L4	9 371	36 703 391	16 488 684	55.08	42 579	0	14 400
R1L1v	2 832	42 591 141	28 544 123	32.98	20 326	22	14 400
R4L4v	9 637	61 968 380	38 307 814	38.18	45 916	0	14 400

Table 3: Results for the full PESPlib instances. The table lists the best known primal bound for the MIP (3) in terms of weighted slack $w^\top(x - \ell)$ according to (Goerigk, 2022), the best dual bound obtained by split cuts, the primal-dual gap, the total number of applied split cuts, the number of cuts provided by the parametric IP (14), and the running time in seconds.

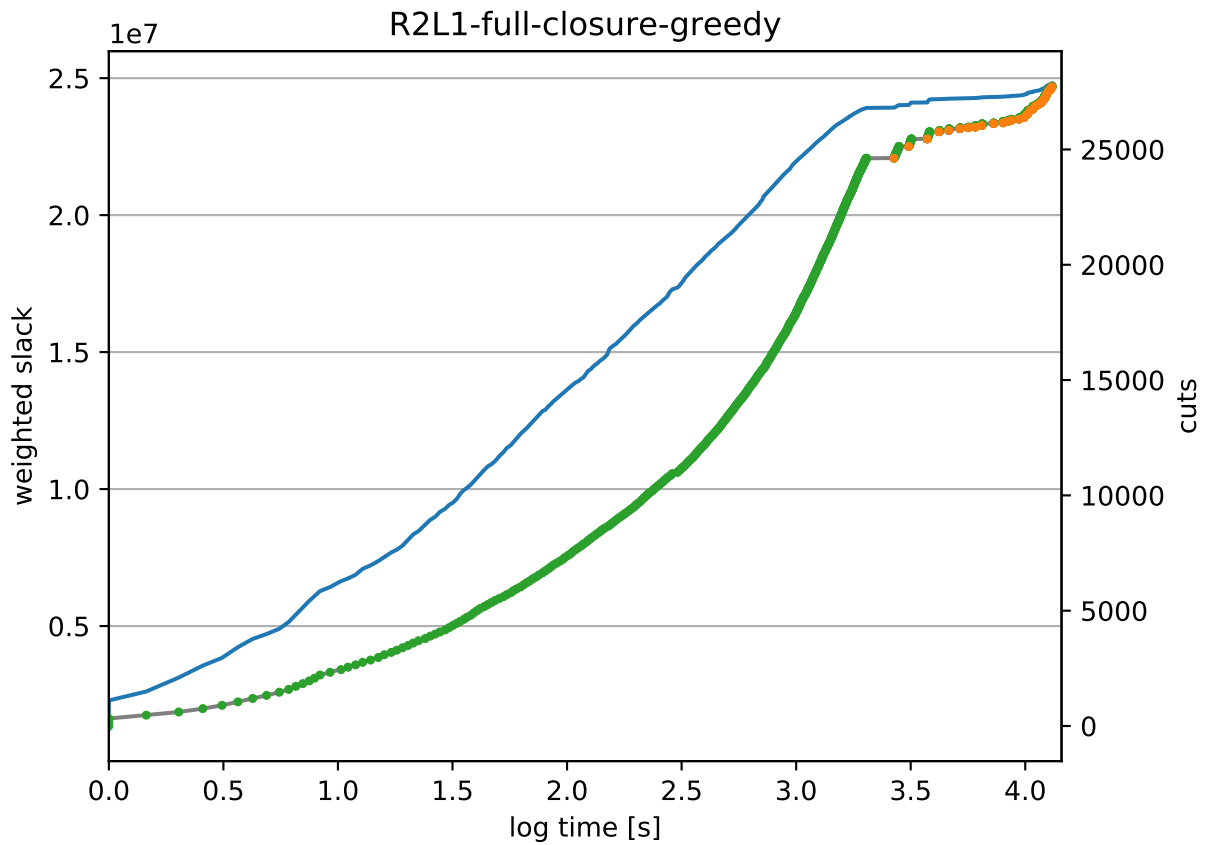


Figure 6: Evolution of the dual bound in terms of weighted slack $w^\top(x - \ell)$ (blue, left axis) and the number of applied split cuts (grey, right axis) for the instance R2L1. Green markers correspond to cuts obtained from the heuristic, orange to cuts from the parametric IP. The time axis is logarithmic.

Instance	μ	Dual Bd. (\mathcal{P}_1)	Dual Bd. ($\mathcal{P}_{\text{split}}$)	Gap [%]	Dual Bd. Source
BL1	5 298	3 668 148	4 252 778	−15.94	Borndörfer, Lindner, and Roth (2020)
BL2	4 880	3 943 811	4 299 517	−9.02	Borndörfer, Lindner, and Roth (2020)
BL3	6 265	3 571 976	4 290 946	−20.13	Borndörfer, Lindner, and Roth (2020)
BL4	9 684	3 131 491	3 923 974	−25.31	Borndörfer, Lindner, and Roth (2020)
R1L1	2 722	20 901 883	19 041 890	8.90	N. Lindner et al. (2021)
R1L2	2 876	19 886 799	19 059 669	4.16	Masing et al. (2023)
R1L3	2 848	19 323 821	18 193 974	5.85	Masing et al. (2023)
R1L4	3 769	17 283 850	16 441 121	4.88	Masing et al. (2023)
R2L1	3 206	25 929 643	24 806 675	4.33	Masing et al. (2023)
R2L2	3 360	25 642 692	24 464 467	4.59	Masing et al. (2023)
R2L3	3 239	23 941 492	22 645 939	5.41	Masing et al. (2023)
R2L4	5 514	19 793 447	19 102 410	3.49	Masing et al. (2023)
R3L1	4 630	26 825 864	25 343 534	5.53	Masing et al. (2023)
R3L2	4 800	27 178 406	25 963 773	4.47	Masing et al. (2023)
R3L3	5 446	23 007 043	22 273 090	3.19	Masing et al. (2023)
R3L4	7 478	17 432 725	17 027 192	2.33	Masing et al. (2023)
R4L1	5 331	29 174 444	27 938 824	4.24	Masing et al. (2023)
R4L2	5 688	28 664 399	27 585 028	3.77	Masing et al. (2023)
R4L3	6 871	24 293 621	23 849 465	1.83	Masing et al. (2023)
R4L4	9 371	17 961 400	16 488 684	8.20	N. Lindner and Liebchen (2020)
R1L1v	2 832	29 620 775	28 544 123	3.63	Goerigk (2022)
R4L4v	9 637	32 296 041	38 307 814	−18.61	Goerigk (2022)

Table 4: Comparison of dual bounds for the full PESplib instances. The table lists the best known dual bound for the MIP (3) in terms of weighted slack $w^\top(x - \ell)$ according to the source in the last column, the best dual bound obtained by split cuts, and the primal-dual gap.

effective cuts again. However for practical purposes, particularly when other methods aimed at improving the dual bounds are used in parallel, the time-consuming parametric IP might be too costly. The heuristic part could be sufficient, particularly in light of the realization that – also in practice – the split closure is not enough to close the dual gap entirely.

7 Conclusion

We have shown that in the context of periodic timetabling, the split closure can be expressed in combinatorial terms, namely via flip inequalities with respect to simple cycles. Consequently, this means that a dual bound obtained from flip inequalities is as good as from split cuts. However, flip inequalities are – in a way – easier to grasp: We show that for a fixed cycle, a separating flip inequality can be found in linear time. This can be used to obtain a heuristic, which turned out to be powerful in practice. In combination with a systematic exploration of violated flip inequalities, we were able to improve the dual bounds of five instances of the benchmark library PESPlib – proving both the effectiveness of our approach, but also of the benefit of the split closure in the context of PESP. One of our main contributions is also in the insight that the split closures of various equivalent PESP formulations are all equivalent as well, meaning that neither the specific MIP formulation, nor any amount of subdivision or augmentation will lead to a stronger split closure.

Our computational experiments also indicate that even with a full exploration of the flip polytope, a certain gap will remain. To close the primal-dual gap entirely, further research into stronger cuts is needed, which will have to be different from first-order split cuts.

References

- Achterberg, T. (2009). SCIP: solving constraint integer programs. *Mathematical Programming Computation*, 1(1), 1–41. doi: 10.1007/s12532-008-0001-1
- Aprile, M., Conforti, M., & Di Summa, M. (2021). *Binary extended formulations and sequential convexification*. arXiv. (arXiv:2106.00354 [cs, math])
- Balas, E., Ceria, S., & Cornuéjols, G. (1993). A lift-and-project cutting plane algorithm for mixed 0–1 programs. *Mathematical Programming*, 58(1), 295–324. doi: 10.1007/BF01581273
- Balas, E., & Saxena, A. (2008). Optimizing over the split closure. *Mathematical Programming*, 113(2), 219–240. doi: 10.1007/s10107-006-0049-5
- Bestuzheva, K., Besançon, M., Chen, W.-K., Chmiela, A., Donkiewicz, T., Doornmalen, J. v., ... Witzig, J. (2021). *The SCIP Optimization Suite 8.0* (ZIB-Report No. 21-41). Zuse Institute Berlin. Retrieved from <http://nbn-resolving.de/urn:nbn:de:0297-zib-85309>
- Bonami, P. (2012). On optimizing over lift-and-project closures. *Mathematical Programming Computation*, 4(2), 151–179. doi: 10.1007/s12532-012-0037-0
- Borndörfer, R., Hoppmann, H., Karbstein, M., & Lindner, N. (2020). Separation of cycle inequalities in periodic timetabling. *Discrete Optimization*, 35, 100552. doi: 10.1016/j.disopt.2019.100552
- Borndörfer, R., Lindner, N., & Roth, S. (2020). A concurrent approach to the periodic event scheduling problem. *Journal of Rail Transport Planning & Management*, 15, 100175. doi: 10.1016/j.jrtpm.2019.100175
- Bortoletto, E., Lindner, N., & Masing, B. (2022). Tropical Neighbourhood Search: A New Heuristic for Periodic Timetabling. In M. D’Emidio & N. Lindner (Eds.), *22nd Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS)*

- 2022) (Vol. 106, pp. 3:1–3:19). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik. (ISSN: 2190-6807) doi: 10.4230/OASICS.ATMOS.2022.3
- Caprara, A., & Letchford, A. N. (2003). On the separation of split cuts and related inequalities. *Mathematical Programming*, 94(2), 279–294. doi: 10.1007/s10107-002-0320-3
- Conforti, M., Cornuéjols, G., & Zambelli, G. (2010). Polyhedral Approaches to Mixed Integer Linear Programming. In M. Jünger et al. (Eds.), *50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art* (pp. 343–385). Berlin, Heidelberg: Springer. doi: 10.1007/978-3-540-68279-0_11
- Conforti, M., Cornuéjols, G., & Zambelli, G. (2014). *Integer Programming* (Vol. 271). Cham: Springer International Publishing. doi: 10.1007/978-3-319-11008-0
- Cook, W., Kannan, R., & Schrijver, A. (1990). Chvátal closures for mixed integer programming problems. *Mathematical Programming*, 47(1), 155–174. doi: 10.1007/BF01580858
- Cornuéjols, G., & Li, Y. (2001). Elementary closures for integer programs. *Operations Research Letters*, 28(1), 1–8. doi: 10.1016/S0167-6377(00)00067-5
- Dash, S., Günlük, O., & Hildebrand, R. (2018). Binary extended formulations of polyhedral mixed-integer sets. *Mathematical Programming*, 170(1), 207–236. doi: 10.1007/s10107-018-1294-0
- Goerigk, M. (2022). *PESPlib – A benchmark library for periodic event scheduling*. (<http://num.math.uni-goettingen.de/~m.goerigk/pesplib/>, last update on 11/07/2022)
- Goerigk, M., & Liebchen, C. (2017). An Improved Algorithm for the Periodic Timetabling Problem. In G. D’Angelo & T. Dollevoet (Eds.), *17th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2017)* (Vol. 59, pp. 12:1–12:14). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. (ISSN: 2190-6807) doi: 10.4230/OASICS.ATMOS.2017.12
- Großmann, P., Hölldobler, S., Manthey, N., Nachtigall, K., Opitz, J., & Steinke, P. (2012). Solving Periodic Event Scheduling Problems with SAT. In H. Jiang, W. Ding, M. Ali, & X. Wu (Eds.), *Advanced Research in Applied Artificial Intelligence* (pp. 166–175). Berlin, Heidelberg: Springer. doi: 10.1007/978-3-642-31087-4_18
- Gurobi Optimization, LLC. (2023). *Gurobi Optimizer Reference Manual*. Retrieved from <https://www.gurobi.com>
- Kavitha, T., Liebchen, C., Mehlhorn, K., Michail, D., Rizzi, R., Ueckerdt, T., & Zweig, K. A. (2009). Cycle bases in graphs characterization, algorithms, complexity, and applications. *Computer Science Review*, 3(4), 199–243. doi: 10.1016/j.cosrev.2009.08.001
- Kroon, L., Huisman, D., Abbink, E., Fioole, P.-J., Fischetti, M., Maróti, G., . . . Ybema, R. (2009). The New Dutch Timetable: The OR Revolution. *Interfaces*, 39(1), 6–17. doi: 10.1287/inte.1080.0409
- Liebchen, C. (2006). *Periodic timetable optimization in public transport* (Unpublished doctoral dissertation). Technische Universität Berlin.
- Liebchen, C. (2008). The First Optimized Railway Timetable in Practice. *Transportation Science*, 42(4), 420–435. (Publisher: INFORMS) doi: 10.1287/trsc.1080.0240
- Liebchen, C., & Möhring, R. H. (2007). The Modeling Power of the Periodic Event Scheduling Problem: Railway Timetables — and Beyond. In F. Geraets, L. Kroon, A. Schoebel, D. Wagner, & C. D. Zaroliagis (Eds.), *Algorithmic Methods for Railway Optimization* (pp. 3–40). Berlin, Heidelberg: Springer. doi: 10.1007/978-3-540-74247-0_1
- Liebchen, C., & Peeters, L. (2009). Integral cycle bases for cyclic timetabling. *Discrete Optimization*, 6(1), 98–109. doi: 10.1016/j.disopt.2008.09.003
- Liebchen, C., & Swarat, E. (2008). The Second Chvatal Closure Can Yield Better Railway Timetables. In M. Fischetti & P. Widmayer (Eds.), *8th Workshop on Algorithmic Approaches for Transportation Modeling, Optimization, and Systems (ATMOS’08)* (Vol. 9). Dagstuhl,

- Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. (ISSN: 2190-6807) doi: 10.4230/OASICS.ATMOS.2008.1580
- Lindner, N., & Liebchen, C. (2020). Determining All Integer Vertices of the PESP Polytope by Flipping Arcs. In D. Huisman & C. D. Zaroliagis (Eds.), *20th Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2020)* (Vol. 85, pp. 5:1–5:18). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum für Informatik. (ISSN: 2190-6807) doi: 10.4230/OASICS.ATMOS.2020.5
- Lindner, N., & Liebchen, C. (2022). Timetable merging for the Periodic Event Scheduling Problem. *EURO Journal on Transportation and Logistics*, 11, 100081. doi: 10.1016/j.ejtl.2022.100081
- Lindner, N., Liebchen, C., & Masing, B. (2021). Forward Cycle Bases and Periodic Timetabling. In M. Müller-Hannemann & F. Perea (Eds.), *21st Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2021)* (Vol. 96, pp. 2:1–2:14). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik. (ISSN: 2190-6807) doi: 10.4230/OASICS.ATMOS.2021.2
- Lindner, N., & Reisch, J. (2022). An analysis of the parameterized complexity of periodic timetabling. *Journal of Scheduling*, 25(2), 157–176. doi: 10.1007/s10951-021-00719-1
- Lindner, T. (2000). *Train Scheduling in Public Rail Transport* (Unpublished doctoral dissertation). Technische Universität Braunschweig.
- Masing, B., Lindner, N., & Ebert, P. (2023). *Forward and Line-Based Cycle Bases for Periodic Timetabling* (ZIB-Report No. 23-05). Zuse Institute Berlin. Retrieved from <https://nbn-resolving.org/urn:nbn:de:0297-zib-89731>
- Nachtigall, K. (1996). *Cutting Planes for a Polyhedron Associated with a Periodic Network*. (Tech. Rep. No. IB 112-96/17). Deutsches Zentrum für Luft- und Raumfahrt e. V.
- Nachtigall, K. (1998). *Periodic Network Optimization and Fixed Interval Timetables* (Habilitation Thesis). Universität Hildesheim.
- Nachtigall, K., & Opitz, J. (2008). Solving Periodic Timetable Optimisation Problems by Modulo Simplex Calculations. In M. Fischetti & P. Widmayer (Eds.), *8th Workshop on Algorithmic Approaches for Transportation Modeling, Optimization, and Systems (ATMOS'08)* (Vol. 9). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. (ISSN: 2190-6807) doi: 10.4230/OASICS.ATMOS.2008.1588
- Nemhauser, G. L., & Wolsey, L. A. (1990). A recursive procedure to generate all cuts for 0–1 mixed integer programs. *Mathematical Programming*, 46(1), 379–390. doi: 10.1007/BF01585752
- Odiijk, M. A. (1994). *Construction of periodic timetables, Part 1: A cutting plane algorithm* (Tech. Rep. No. 94-61). TU Delft.
- Peeters, L. (2003). *Cyclic Railway Timetable Optimization* (Unpublished doctoral dissertation). Erasmus Universiteit Rotterdam.
- Pätzold, J., & Schöbel, A. (2016). A Matching Approach for Periodic Timetabling. In M. Goerigk & R. Werneck (Eds.), *16th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2016)* (Vol. 54, pp. 1:1–1:15). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. (ISSN: 2190-6807) doi: 10.4230/OASICS.ATMOS.2016.1
- Schiewe, P., & Schöbel, A. (2020). Periodic Timetabling with Integrated Routing: Toward Applicable Approaches. *Transportation Science*, 54(6), 1714–1731. (Publisher: INFORMS) doi: 10.1287/trsc.2019.0965
- Serafini, P., & Ukovich, W. (1989). A Mathematical Model for Periodic Scheduling Problems. *SIAM Journal on Discrete Mathematics*, 2(4), 550–581. (Publisher: Society for Industrial and Applied Mathematics) doi: 10.1137/0402049