





E. NAVA-YAZDANI¹, M. HANIK², F. AMBELLAN³, C. V.
TYCOWICZ⁴

On Gradient Formulas in an Algorithm for the Logarithm of the Sasaki Metric

¹  0000-0003-4895-739X

²  0000-0002-7120-4081

³  0000-0001-9415-0859

⁴  0000-0002-1447-4069

Zuse Institute Berlin
Takustr. 7
14195 Berlin
Germany

Telephone: +49 30 84185-0
Telefax: +49 30 84185-125

E-mail: bibliothek@zib.de
URL: <http://www.zib.de>

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

On Gradient Formulas in an Algorithm for the Logarithm of the Sasaki Metric

E. Nava-Yazdani, M. Hanik F. Ambellan
C. v. Tycowicz

Here we explain the computation of the gradient in the algorithm to compute the logarithm of the Sasaki metric that is used in geomstats. It is an implementation of the method Muralidharan and Fletcher proposed in [1, p. 1030]. The notation in this note follows their article¹; all definitions can also be found there.

Setting $\varepsilon := 1/L$, we use a re-scaled energy

$$E(p, u) := \frac{\varepsilon}{2} \sum_{k=1}^L (\|v_k\|^2 + \|w_k\|^2),$$

which has the same extrema as the one of Muralidharan and Fletcher. Dividing through the number of discrete steps makes computations numerically more stable as the length of the gradient scales only linearly with L (instead of quadratically). Consequently, we obtain

$$\nabla_{p_k^i} E = -\varepsilon \left(\nabla_{v_k^i} v_k^i + R(u_k^i, w_k^i) v_k^i \right), \quad (1)$$

$$\nabla_{u_k^i} E = -\varepsilon \nabla_{v_k^i} w_k^i. \quad (2)$$

Note that in the paper a minus sign is missing for both gradients, which appears when differentiating the squared norms in E .

We approximate the vectors v_k^i as in the paper by the forward finite difference

$$v_k^i \approx \frac{\text{Log}(p_k^i, p_{k+1}^i)}{\varepsilon}. \quad (3)$$

For consistency we also use the forward difference

$$w_k^i \approx \frac{\phi(u_{k+1}^i, p_k^i) - u_k^i}{\varepsilon} \quad (4)$$

instead of the central difference that is proposed in [1] to approximate w_k^i .

For the following note that $\phi(\text{Log}(p, q), q) = -\text{Log}(q, p)$ for all $p, q \in M$ such that $\text{Log}(p, q)$ exists.

As suggested in the paper, we approximate the covariant derivative $\nabla_{v_k^i} v_k^i$

¹In particular, $\text{Log}(p, q) \in T_p M$ for points p, q close enough in a manifold M , in contrast to the convention in geomstats.

by a central difference

$$\begin{aligned}
\nabla_{v_k^i} v_k^i &\approx \frac{\phi(v_{k+1}^i, p_k^i) - \phi(v_{k-1}^i, p_k^i)}{2\varepsilon} \\
&= \frac{\phi(-\text{Log}(p_{k+1}^i, p_k^i), p_k^i) - \phi(\text{Log}(p_{k-1}^i, p_k^i), p_k^i)}{2\varepsilon^2} \\
&= \frac{\text{Log}(p_k^i, p_{k+1}^i) + \text{Log}(p_k^i, p_{k-1}^i)}{2\varepsilon^2}.
\end{aligned} \tag{5}$$

Note that in this formula we approximate the “upper” vector v_{k+1}^i by a *backward* finite difference. The advantage over consistently using forward differences is that we do not need a boundary condition.

The approximation of $\nabla_{v_k^i} w_k^i$ is done via a second-order central difference, that is,

$$\nabla_{v_k^i} w_k^i \approx \frac{\phi(u_{k+1}^i, p_k^i) - 2u_k^i + \phi(u_{k-1}^i, p_k^i)}{\varepsilon^2}. \tag{6}$$

Inserting (3), (4), (5), and (6) into (1) and (2) then yields

$$\begin{aligned}
\nabla_{p_k^i} E &\approx \frac{-\text{Log}(p_k^i, p_{k+1}^i) - \text{Log}(p_k^i, p_{k-1}^i)}{2\varepsilon} \\
&\quad - \frac{R(u_k^i, \phi(u_{k+1}^i, p_k^i) - u_k^i) \text{Log}(p_k^i, p_{k+1}^i)}{\varepsilon}, \\
\nabla_{u_k^i} E &\approx \frac{2u_k^i - \phi(u_{k+1}^i, p_k^i) - \phi(u_{k-1}^i, p_k^i)}{\varepsilon}.
\end{aligned}$$

Literatur

- [1] P. MURALIDHARAN AND P. T. FLETCHER, *Sasaki metrics for analysis of longitudinal data on manifolds*, in Proceedings of the 2012 IEEE Conference on Computer Vision and Pattern Recognition, IEEE, 2012, pp. 1027–1034.