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# A Safe Computational Framework for Integer Programming applied to Chvátal’s Conjecture

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## Abstract

We describe a general and safe computational framework that provides integer programming results with the degree of certainty that is required for machine-assisted proofs of mathematical theorems. At its core, the framework relies on a rational branch-and-bound certificate produced by an exact integer programming solver, SCIP, in order to circumvent floating-point roundoff errors present in most state-of-the-art solvers for mixed-integer programs. The resulting certificates are self-contained and checker software exists that can verify their correctness independently of the integer programming solver used to produce the certificate. This acts as a safeguard against programming errors that may be present in complex solver software. The viability of this approach is tested by applying it to finite cases of Chvátal’s conjecture, a long-standing open question in extremal combinatorics. We take particular care to verify also the correctness of the input for this specific problem, using the Coq formal proof assistant. As a result we are able to provide a first machine-assisted proof that Chvátal’s conjecture holds for all downsets whose union of sets contains seven elements or less.

## 1 Introduction

The work on algorithms and software for mathematical optimization is often motivated by the solution of real-world applications. This sometimes overshadows the value that these methods can have for answering questions in mathematics itself. Some of them can quite naturally be cast in the form of optimization problems. Prominent examples are the extensive use of linear programming to settle Kepler’s conjecture [20] or the

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use of semidefinite programming in discrete geometry [6]. In addition, first attempts at using integer programming have been made to settle open questions in extremal combinatorics [28], graph theory [25], and graph pebbling [22].

The use of integer programming (IP) for constructing rigorous mathematical proofs, however, is faced with two main computational difficulties. First, virtually all state-of-the-art IP solvers rely on fast floating-point arithmetic, hence their results are compromised by roundoff errors. Second, most solvers do not provide sufficient output that would allow to check and verify the correctness of their result. These limitations are unfortunate given that the presence of integer variables allows for expressive models and that solvers for integer programming problems have strongly increased in computational power over the last years [1].

In marked contrast, satisfiability solving (SAT) has been employed with considerable success to answer open questions in discrete mathematics. A recent milestone is the solution of the boolean Pythagorean triples problem [21]. Satisfiability solving and integer programming share the theoretical difficulty that compact certificates are in general not available since both SAT and IP are not known to be in  $\text{co-}\mathcal{NP}$ . However, over the last years the SAT community has established standards and tools for proof logging and solver-independent verification of results [33].

In comparison, exact IP software with verifiable results is still in its infancy. Besides domain-specific work such as for the traveling salesman solver Concorde [3] or partial functionality within software libraries targeted towards polyhedral analysis [5, 7], the only exact, general IP solver we are aware of and whose results are not compromised by floating-point errors is an extension of the solver SCIP [13]. Exact SCIP has recently been further extended by the possibility to print certificates that can be verified independently from the solution process [10]. The goal of this paper is to demonstrate how these tools can be employed to create a *safe* computational framework for investigating a particular mathematical application by integer programming. To this end, we couple

1. an exact rational IP *solver*, SCIP [13], with
2. IP certificates for its branch-and-bound tree *output* that can be verified independently from the solution process by the checker VIPR [10],
3. and verification procedures for the correctness of the *input* implemented in a formal proof assistant, Coq [14].

Notably, this framework features the combined use of an exact IP certificate and a formal proof assistant to ensure the correctness of the certificate's input data. To the best of our knowledge, this has not been explored in the literature before.

We apply this framework to Chvátal's conjecture, which is a well-known open problem in extremal set theory dating back to 1974 and contained in Erdős's list of favorite combinatorial problems [17]. Using our framework, we obtain machine-assisted proofs for low-dimensional cases that were previously unknown.

The rest of this paper is organized as follows. Section 2 outlines the general methodology for using exact rational integer programming together with input/output verification for machine-assisted theorem proving. Section 3 describes the IP formulations that

we use to model Chvátal’s conjecture and presents valid inequalities for the underlying polytopes and “cuts” from the literature that reduce the number of integral solutions to the IP formulations. Section 4 contains a detailed description of our experimental results and Section 5 concludes with an outlook on future work. The implementation and results are freely available to the public [16].

## 2 Verifiable Proofs for Integer Programming Results

In the following, we outline our computational methodology used to solve the integer programs presented in Section 3 such that the results can be trusted and both input and output can be verified independently of the IP solver used. Figure 1 illustrates the four components.

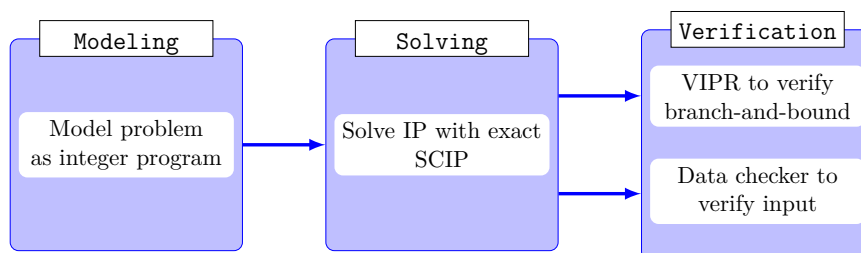


Figure 1: General framework for modeling, solving, and verifying the results of integer programs.

**Modeling.** As the first step, we use the modeling language ZIMPL [24] to formulate the integer program. ZIMPL employs exact rational arithmetic when instantiating the model in order to ensure that no roundoff errors are introduced before passing the model to a solver.

**Solving.** Next, we solve the IP using the exact rational variant of the MIP solver SCIP [13]. Exact SCIP implements a hybrid branch-and-bound algorithm that combines floating-point and exact rational arithmetic in a safe manner. Several methods are used in order to obtain safe dual bounds by correcting relaxation solutions from fast floating-point linear programming (LP) solvers. An exact rational LP solver, QSopt\_ex [4], is used as sparingly as possible. Although exact SCIP still lacks many more sophisticated techniques implemented in state-of-the-art floating-point solvers such as presolving reductions, cutting planes, or symmetry handling, its design helps to yield superior performance compared to a naïve branch-and-bound method solely relying on rational LP solves.

**Output Verification.** Although exact SCIP is designed to provide safe results, the correctness of the algorithm and implementation cannot easily be verified externally.

To address this issue, we use VIPR [10], a recently developed certificate format that consists of the problem definition followed by an encoding of the branch-and-bound proof as a list of valid inequalities. It rests on three simple inference steps that allow for elementary, stepwise verification: aggregation of inequalities, rounding of right-hand sides, and resolution of a binary disjunction. In this sense, a VIPR certificate can, in theory, be checked by hand, although in practice this may be prohibitive for larger certificates. Hence, the VIPR project comes with an automatic, standalone checker, but the simplicity of the format allows for the implementation of alternative checkers.

Exact SCIP can be configured to generate VIPR certificates during the solving process such that its result must not be trusted blindly. Its correctness can be verified completely independently of the solving process.

**Input Verification.** VIPR verification only ensures the correctness of the branch-and-bound certificate with respect to the integer program encoded in the problem section of the certificate file. However, due to implementation errors, the problem section of the certificate file may actually not match the integer program of interest. Therefore we implemented a safe input-checker that internally creates its own representation of the constraint matrix for Problem  $P_{red}(n)$ . It then reads the problem section of the certificate file and checks if the two constraint matrices coincide. This input checker is written using the Coq proof assistant [14], a mathematical proof management system. The matrix creation in this input checker is problem-specific, nevertheless it can easily be adapted to formulations for similar problems.

All in all, we are confident that this framework ensures a high level of trust in the computational proof of Theorem 1. All tools are made publicly available for review [16], including the certificate files for the computational results presented in Section 4.

### 3 A Polyhedral Approach to Chvátal’s Conjecture

Chvátal’s conjecture is a well-known open problem in extremal set theory from 1974, later earning a spot among Erdős’ favorite combinatorial problems [17]. Despite its popularity, research efforts have yielded limited progress, mostly restricted to special cases and related variants of the original conjecture. Before continuing in more detail we need the following definitions.

Let  $[n] := \{1, 2, \dots, n\}$ . A *family*  $\mathcal{F}$  is a set of subsets of  $[n]$ . Let  $U(\mathcal{F})$  denote the union of all sets in  $\mathcal{F}$ . A family  $\mathcal{F}$  is a *downset* if and only if  $A \in \mathcal{F}$  and  $B \subseteq A$  implies  $B \in \mathcal{F}$ . If  $\mathcal{F}$  is a downset then  $F \in \mathcal{F}$  is a *base* if and only if no strict supersets of  $F$  are contained, i.e.,  $F \subseteq D$  and  $D \in \mathcal{F}$  implies  $F = D$ . A family  $\mathcal{F}$  is called *intersecting* if and only if the intersection of any pairwise sets in  $\mathcal{F}$  is nonempty. A family  $\mathcal{F}$  is a *star* if and only if there exists an element in  $U(\mathcal{F})$  contained in all sets of  $\mathcal{F}$ . A family  $\mathcal{F}$  has the *star property* if and only if some maximum-sized intersecting family in  $\mathcal{F}$  is a star. These definitions are visualized in Figure 2. We are ready to state Chvátal’s conjecture as follows:

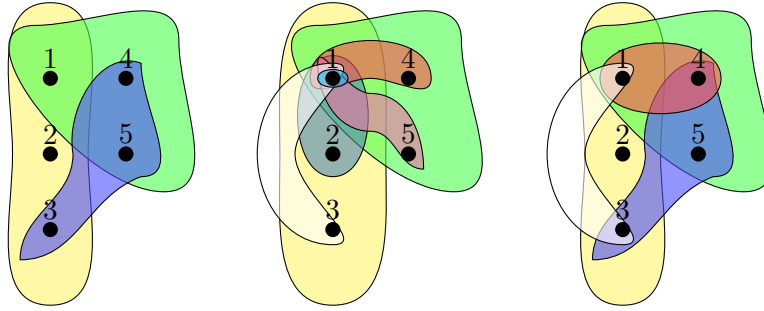


Figure 2: Visualization of set-families as hypergraphs. On the left is the base  $\{\{1, 2, 3\}, \{1, 4, 5\}, \{3, 4, 5\}\}$  of a down-set with 17 elements. In the middle is one star in that downset of maximal size. On the right is an example of an intersecting family that is not a star.

**Conjecture 1** (Chvátal [12]). *Every downset has the star property.*

Schönheim [29] showed that Chvátal’s conjecture holds for all downsets  $\mathcal{D}$  whose bases have a nonempty intersection. Stein [31] proved the conjecture holds for all downsets in which all but one of the bases is a *simple* star, i.e., a star in which the intersection of all of its sets is equal to the intersection of any of its two sets. Miklós [26] showed that Chvátal’s conjecture holds for any downset  $\mathcal{D}$  that contains an intersection family of size  $\lfloor \mathcal{D}/2 \rfloor$ . Sterboul [32] proved that any downsets whose sets have three or less elements always satisfy Conjecture 1. The last result was recently proven again in different ways by Czabarka, Hurlbert, Kamat [15] and Olarte, Santos, Spreer [27]. Furthermore, Chvátal maintains a website dedicated to the conjecture with a substantial list of publications on the topic [11].

Our main result on Chvátal’s conjecture is the following:

**Theorem 1.** *Conjecture 1 holds for all downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq 7$ .*

To the best of our knowledge there is no known computational methodology in the literature that investigates Conjecture 1 even for small ground sets. The previously known best bound on the cardinality of the ground set was  $|U(\mathcal{D})| = 5$ , which follows directly from [32] as we show in Proposition 3.

Already Fishburn [18] highlighted the connection between combinatorial optimization and Chvátal’s conjecture and investigated related problems. Thus modeling the conjecture as an IP and using solvers that safeguard against numerical issues is a natural step to further investigate Conjecture 1. Furthermore, relying on an IP framework to investigate Chvátal’s conjecture for small ground sets has other advantages. First, the rich and well-developed theory of polyhedral combinatorics, that is inherent in an IP approach, may lead to new insights on Conjecture 1. Second, as we see in Section 3.3, known partial results on Chvátal’s conjecture can be encoded as “cuts” in our framework. This improves the performance of the exact rational solver and may allow strengthening of Theorem 1 in the future.

In the following sections we develop integer programming formulations for Chvátal's conjecture over fixed-size ground sets. The formulations are based on decision variables that index members of the power set. Due to the exponential nature of power sets, the size of the formulations is bound to grow quickly with the size of the ground set. However, even for small ground sets little is known and the results in Section 4 improve upon what is known today.

### 3.1 An Infeasibility-Based Formulation

The first IP model is formulated such that Chvátal's conjecture holds for the considered ground set if and only if the IP has no solution. In other words, a feasible solution to the IP formulation for any fixed  $n$  would yield a counterexample to Chvátal's conjecture. Let  $2^{[n]}$  denote the power set of  $[n]$ , then we consider the integer program  $P_{inf}(n)$ ,

$$\max \sum_{S \in 2^{[n]}} x_S \tag{1a}$$

$$x_T \leq x_S \quad \forall T \in 2^{[n]}, \forall S \in 2^{[n]} : S \subset T, \tag{1b}$$

$$y_T + y_S \leq 1 \quad \forall T \in 2^{[n]} \setminus \{\emptyset\}, \forall S \in 2^{[n]} \setminus \{\emptyset\} : T \cap S = \emptyset, \tag{1c}$$

$$y_S \leq x_S \quad \forall S \in 2^{[n]}, \tag{1d}$$

$$\sum_{S \in 2^{[n]} : i \in S} x_S + 1 \leq \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S \quad \forall i \in [n], \tag{1e}$$

$$x_S, y_S \in \{0, 1\} \quad \forall S \in 2^{[n]}.$$

Here,  $x$  encodes the set family  $\mathcal{S}(x) := \{S \subseteq [n] : x_S = 1\}$  and  $y$  encodes the subfamily  $\mathcal{S}(y) := \{S \subseteq [n] : y_S = 1\}$ . The first class of *downset* inequalities (1b) ensures that  $\mathcal{S}(x)$  is a downset. The second class of *intersecting* inequalities (1c) ensures that  $\mathcal{S}(y) \setminus \{\emptyset\}$  is an intersecting family. The third class of *containment* inequalities (1d) ensures that the intersecting family is contained in the chosen downset,  $\mathcal{S}(y) \subseteq \mathcal{S}(x)$ . Finally, the fourth class of *star* inequalities (1e) requires that the intersecting family has greater cardinality than any star in the downset.

**Theorem 2.** *Let  $n$  be a positive integer. All downsets  $\mathcal{F}$  such that  $|U(\mathcal{F})| \leq n$  satisfy Chvátal's conjecture if and only if  $P_{inf}(n)$  is infeasible.*

*Proof.* Fix  $n$ . Suppose that Chvátal's conjecture does not hold, i.e., there exists a downset  $\mathcal{D}$  and an intersecting family  $\mathcal{Y} \subseteq \mathcal{D}$  such that  $|\mathcal{Y}|$  is larger than the size of every star in  $\mathcal{D}$ . W.l.o.g. assume  $\mathcal{D} \subseteq 2^{[n]}$ . Let  $x$  and  $y$  be their incidence vectors, i.e.,  $\mathcal{D} = \mathcal{S}(x)$  and  $\mathcal{Y} = \mathcal{S}(y)$ . By construction,  $x$  and  $y$  satisfy constraints (1b–1d). Furthermore, for each element  $i \in [n]$ ,  $|\mathcal{Y}|$  is larger than the size of all stars that have  $i$  as common element, hence (1e) is equally satisfied. In total,  $x$  and  $y$  constitute a feasible solution to  $P_{inf}(n)$ .

Conversely, suppose all downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq n$  satisfy Chvátal's conjecture. Suppose  $x$  and  $y$  are feasible solutions to  $P_{inf}(n)$ . By (1b),  $\mathcal{S}(x)$  forms a downset and



$|U(\mathcal{S}(x))| \leq n$ . By (1c) and (1d),  $\mathcal{Y} := \mathcal{S}(y) \setminus \{\emptyset\}$  forms an intersecting family contained in  $\mathcal{S}(x)$ . Hence,  $|\mathcal{Y}|$  can be at most the size of the largest star contained in  $\mathcal{S}(x)$ ,

$$|\mathcal{Y}| = \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S \leq \max_{i \in [n]} \sum_{S \in 2^{[n]}: i \in S} x_S. \quad (2)$$

But then constraint (1e) is violated for  $i_0 \in \arg \max_{i \in [n]} \sum_{S \in 2^{[n]}: i \in S} x_S$ .  $\square$

Note that the objective function (1a) of  $P_{inf}(n)$  is, in some sense, arbitrary since we only need to decide whether the integer program has a feasible solution or not. The objective function encodes the cardinality of  $\mathcal{S}(x)$ , hence solving  $P_{inf}(n)$  amounts to searching for a largest counterexample to Conjecture 1. In the following we present a more advanced formulation that uses the optimal value as an essential component.

### 3.2 An Optimality-Based Formulation

As already noted in [27] it is sufficient to only consider downsets generated by an intersecting family. We use this insight in the following, advanced formulation  $P_{opt}(n)$ ,

$$\max \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S - z \quad (3a)$$

$$y_T + y_S \leq 1 \quad \forall T \in 2^{[n]} \setminus \{\emptyset\}, \forall S \in 2^{[n]} \setminus \{\emptyset\} : T \cap S = \emptyset, \quad (3b)$$

$$\sum_{S \in 2^{[n]}: i \in S} x_S \leq z \quad \forall i \in [n], \quad (3c)$$

$$y_T \leq x_S \quad \forall T \in 2^{[n]}, \forall S \in 2^{[n]} : S \subseteq T, \quad (3d)$$

$$x_S, y_S \in \{0, 1\} \quad \forall S \in 2^{[n]},$$

$$z \in \mathbb{Z}_{\geq 0}.$$

The first class of *intersecting* inequalities (3b) is the same as (1c), whereas the second class of *star* inequalities (3c) differs from (1e). It ensures that the largest star is bounded above by the positive integer variable  $z$ . Finally, the third class of *generation* inequalities (3d) ensures, as will be made clearer in the proof of Theorem 3, that an optimal solution of  $P_{opt}(n)$  considers only downsets generated by the intersecting family. We note that the generation inequalities (3d) can also be included in  $P_{inf}(n)$  instead of (1b) and (1d) by the same argument. Before we formally state and prove the correctness of  $P_{opt}(n)$  with regards to Conjecture 1, we need the following observation.

**Observation 1.** *Let  $n$  be a positive integer. An optimal solution of  $P_{opt}(n)$  satisfies at least one star inequality (3c) with equality.*

Observation 1 follows from the objective function of  $P_{opt}(n)$ . Variable  $z$  is restricted only from below by its lower bound zero and the left-hand sides of constraints (3c). Since  $P_{opt}(n)$  is a maximization problem and the objective coefficient of  $z$  is negative, in an optimal solution the variable  $z$  will be as small as possible. This implies that at least one star inequality (3c) is tight.

**Theorem 3.** *Let  $n$  be a positive integer. Downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq n$  satisfy Chvátal's conjecture if and only if the objective function value of an optimal solution of  $P_{opt}(n)$  is zero.*

*Proof.* Fix  $n \in \mathbb{N}$ . First note that  $P_{opt}(n)$  is feasible since any star is also an intersecting family. Thus for any downset  $\mathcal{D} \subseteq 2^{[n]}$ , choosing  $x$  as the indicator vector of  $\mathcal{D}$  and setting the  $y$ -variables such that they represent a maximum-cardinality star in  $\mathcal{D}$  yields a feasible solution. Choosing  $z$  to be the maximum star cardinality, i.e., the smallest value such that constraints (3c) are satisfied, also proves a lower bound of zero on the objective value.

Now suppose all downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq n$  satisfy Chvátal's conjecture and let  $x, y, z$  be an optimal solution for  $P_{opt}(n)$ . Then it suffices to show that  $\sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S \leq z$ . Constraints (3b) ensure that  $\mathcal{Y} := \mathcal{S}(y) \setminus \{\emptyset\}$  forms an intersecting family. Furthermore, the  $x$ -variables do not appear in the objective function and are bounded below only by constraints (3d). Hence, w.l.o.g. we may assume that  $x_S = \max_{T \supseteq S} y_T$ . Then, by constraints (3d),  $\mathcal{D} := \mathcal{S}(x)$  is a downset and  $\mathcal{Y} \subseteq \mathcal{D}$ . Assuming Chvátal's conjecture ensures that  $|\mathcal{Y}| = \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S$  is at most the size of the largest star in  $\mathcal{D}$ , which by constraints (3c) is less than or equal to  $z$ .

Conversely, suppose there exists a counterexample to Chvátal's conjecture, i.e., a downset  $\mathcal{D}$ ,  $|U(\mathcal{D})| \leq n$ , and an intersecting family  $\mathcal{Y} \subseteq \mathcal{D}$  such that  $|\mathcal{Y}|$  is larger than the size of any star in  $\mathcal{D}$ . W.l.o.g. assume  $\mathcal{D} \subseteq 2^{[n]}$  and let  $x$  and  $y$  be the incidence vectors of  $\mathcal{D}$  and  $\mathcal{Y}$ , respectively, i.e.,  $\mathcal{D} = \mathcal{S}(x)$  and  $\mathcal{Y} = \mathcal{S}(y)$ . Let  $z$  be the size of the largest star in  $\mathcal{D}$ , i.e.,  $z = \max \sum_{S \in 2^{[n]}: i \in S} x_S$ . Then by construction,  $x, y, z$  is a feasible solution for  $P_{opt}(n)$ . Because we consider a counterexample, the objective function value is at least one.  $\square$

### 3.3 Valid Inequalities and Model Reductions

As mentioned in Section 1, one of the advantages of an IP approach is that  $P_{inf}(n)$  and  $P_{opt}(n)$  can be studied in greater depth through polyhedral combinatorial techniques. Furthermore known results from the literature can be expressed as valid inequalities and problem reductions for  $P_{inf}(n)$  and  $P_{opt}(n)$ , in the sense that Theorems 2 and 3 still hold with these additional constraints, and the number of feasible solutions is less than or equal to the number of current solutions. This is demonstrated in the following section and may help to increase the size of  $n$  for which the models can be solved.

First, consider the intersecting inequalities of form (1c) and (3b). When  $T = [n] \setminus S$ , these can be interpreted as a special case of the following partition inequalities.

**Proposition 1.** *Let  $n$  be a positive integer and let  $\mathcal{P}$  be a partition of  $[n]$ . Then the inequality*

$$\sum_{S \in \mathcal{P}} y_S \leq 1 \tag{4}$$

*is a valid for  $P_{inf}(n)$  and  $P_{opt}(n)$ .*

*Proof.* Suppose  $\sum_{S \in \mathcal{P}} y_S \geq 2$  for an integer feasible solution  $y$ , then there exist  $S, T \in \mathcal{P}$  with  $y_S = y_T = 1$ , violating (1c) and (3b).  $\square$

As a consequence, intersection inequalities that do not cover the whole ground set can be strengthened.

**Proposition 2.** *Let  $n$  be a positive integer and  $S, T \in 2^{[n]} \setminus \{\emptyset\}$  such that  $T \cap S = \emptyset$ . Suppose  $S \cup T \neq [n]$ . Then the intersecting inequality*

$$y_T + y_S \leq 1 \tag{5}$$

*is dominated by a partition inequality.*

*Proof.*  $S, T$  can be completed to a partition by their complement  $[n] \setminus (S \cup T)$ . Inequality (5) is trivially dominated by the corresponding partition inequality  $y_T + y_S + y_{[n] \setminus (S \cup T)} \leq 1$ .  $\square$

The large number of partitions prohibits the static addition of these inequalities to the formulation. However, modern IP solvers automatically extract the conflicting  $y$ -assignments from constraints (1c) and (3b) and add partition inequalities dynamically. Next, consider the following central result.

**Theorem 4** (Berge [8]). *If  $\mathcal{D}$  is a downset then  $\mathcal{D}$  is a disjoint union of pairs of disjoint sets, together with  $\emptyset$  if  $|\mathcal{D}|$  is odd.*

This yields the following result, that can easily be expressed as a valid inequality for  $P_{inf}(n)$  and  $P_{opt}(n)$ , as in Corollary 2.

**Corollary 1** (Anderson [2] p.105). *Let  $\mathcal{D}$  be a downset and  $\mathcal{Y}$  an intersecting family such that  $\mathcal{Y} \subseteq \mathcal{D}$ . Then  $2|\mathcal{Y}| \leq |\mathcal{D}|$ .*

**Corollary 2.** *Let  $n$  be any positive integer. Suppose the following inequality*

$$\sum_{S \in 2^{[n]} \setminus \{\emptyset\}} 2y_S \leq \sum_{S \in 2^{[n]}} x_S \tag{6}$$

*is added to  $P_{inf}(n)$  and  $P_{opt}(n)$ . Then Theorems 2 and 3 hold for the modified formulations of  $P_{inf}(n)$  and  $P_{opt}(n)$ , respectively.*

The following result is used in [27] to give a simple proof that Chvátal's conjecture holds for all downsets whose sets have three elements or less.

**Theorem 5** (Kleitman, Magnanti [23]). *Any intersecting family that is contained in the union of two stars generates a downset that satisfies Conjecture 1.*

As a consequence,  $y$ -variables for sets with one or two elements can be fixed to zero in  $P_{inf}(n)$  and  $P_{opt}(n)$ .

**Corollary 3.** *Let  $n$  be any positive integer. For  $P_{inf}(n)$  and  $P_{opt}(n)$  fix  $y_S = 0$  for all  $S \in 2^{[n]}$  such that  $1 \leq |S| \leq 2$ . Then Theorems 2 and 3 hold for  $P_{inf}(n)$  and  $P_{opt}(n)$ , respectively, with the given fixings.*

*Proof.* Suppose  $y$  stems from a solution of  $P_{inf}(n)$  or  $P_{opt}(n)$ , then it encodes an intersecting family  $\mathcal{Y} := \mathcal{S}(y) \setminus \{\emptyset\}$ . If  $\{i\} \in \mathcal{Y}$ , then  $\mathcal{Y}$  is a star centered around  $i$ . If  $\{i, j\} \in \mathcal{Y}$ , then  $\mathcal{Y}$  is the union of two stars centered around  $i$  and  $j$ , respectively. By Theorem 5, these do not amount to counterexamples to Chvátal's conjecture and can safely be excluded from the formulations without changing the feasibility status of  $P_{inf}(n)$  and the optimal objective value of  $P_{opt}(n)$ , respectively.  $\square$

Theorem 5 can also be exploited for a short proof, that the conjecture holds for all ground sets of size less or equal than 5.

**Proposition 3.** *Conjecture 1 holds for all downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq 5$ .*

*Proof.* Consider an intersecting family  $\mathcal{Y}$  such that  $|U(\mathcal{Y})| \leq 5$ , w.l.o.g.  $\mathcal{Y} \subseteq 2^{[5]}$ . All 10 sets of size 3 have to be part of the intersecting family. If this is not the case, then  $\mathcal{Y}$  is contained in the union of the stars of the remaining two elements and the conjecture holds by Theorem 5. The maximal star, containing only elements of size 1, 2, 3 has size 11.

Let us now consider sets of size 4. There are  $k$  sets of size 4 in  $\mathcal{Y}$  with  $0 \leq k \leq \binom{5}{4}$ . For any star, there exists at most one set of size 4 that does not contain the common element. Therefore the size of the largest intersecting family is at most  $10 + k$ , whereas the size of the largest star is  $11 + k - 1 = 10 + k$ .

Since the whole ground set can not be part of the intersecting family, this concludes the proof.  $\square$

Variable fixings are certainly the most effective improvements to the problem formulation, since they directly reduce the problem size as opposed to general valid inequalities that increase the number of constraints. If we know that Conjecture 1 holds for all downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq n$  for some fixed  $n$ , then we can use a simple variable fixing scheme for the case when  $|U(\mathcal{D})| = n + 1$ , as follows.

**Proposition 4.** *Let  $n$  be a fixed positive integer. Suppose  $P_{inf}(n)$  is infeasible and the objective function value of an optimal solution of  $P_{opt}(n)$  is zero for all positive integers  $n_0 < n$ . Fix  $x_S = 1$  for all  $S \in 2^{[n]}$  such that  $|S| = 1$ . Then Theorems 2 and 3 hold for  $P_{inf}(n)$  and  $P_{opt}(n)$ , respectively, with the given fixings.*

*Proof.* Consider the  $x$ -vector from a solution of  $P_{inf}(n)$  or  $P_{opt}(n)$  and suppose that  $x_{\{i\}} = 0$  for some element  $i \in [n]$ . As in the proofs of Theorems 2 and 3, we may assume that  $x$  encodes a downset  $\mathcal{D} := \mathcal{S}(x)$  and  $\mathcal{Y} := \mathcal{S}(y) \subseteq \mathcal{D}$  is an intersecting family. By the downset property,  $x_S = 0$  for all  $S \ni i$ . But then  $|U(\mathcal{D})| < n$  and by assumption the solution is not a counterexample to Conjecture 1. Hence, we may fix  $x_S = 1$  for all  $S \in 2^{[n]}$  such that  $|S| = 1$ .  $\square$

We can apply this proposition incrementally by starting from the case  $n = 6$ , with all  $x$ -variables for sets that contain exactly one element fixed to 1. After solving, we increase  $n$  by one and repeat the procedure.

Finally, we discuss how to exploit the fundamental result of [32] as an additional fixing scheme.

**Proposition 5.** *Let  $n \geq 6$  be a fixed integer. In  $P_{inf}(n)$  and  $P_{opt}(n)$  fix  $x_S = 1$  for all  $S \in 2^{[4]}$ . Then Theorems 2 and 3 hold, respectively, with the given fixings.*

*Proof.* According to [32], counterexamples to Chvátal's conjecture must feature a downset that contains at least one set of size four. By permuting the elements in  $[n]$  suitably, we can always ensure that this set is  $\{1,2,3,4\}$ .  $\square$

To summarize, we arrive at the following improved formulation  $P_{red}(n)$ ,

$$\max \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S - z \tag{7a}$$

$$y_T + y_S \leq 1 \quad \forall T \in 2^{[n]} \setminus \{\emptyset\}, \forall S \in 2^{[n]} \setminus \{\emptyset\} : T \cap S = \emptyset, \tag{7b}$$

$$\sum_{S \in 2^{[n]} : i \in S} x_S \leq z \quad \forall i \in [n], \tag{7c}$$

$$y_T \leq x_S \quad \forall T \in 2^{[n]}, \forall S \in 2^{[n]} : S \subseteq T, \tag{7d}$$

$$\sum_{S \in 2^{[n]} \setminus \{\emptyset\}} 2y_S \leq \sum_{S \in 2^{[n]}} x_S, \tag{7e}$$

$$y_S = 0 \quad \forall S \in 2^{[n]} : 1 \leq |S| \leq 2, \tag{7f}$$

$$x_S = 1 \quad \forall S \in 2^{[n]} : |S| = 1, \tag{7g}$$

$$x_S = 1 \quad \forall S \subseteq [4], \tag{7h}$$

$$x_S, y_S \in \{0, 1\} \quad \forall S \in 2^{[n]},$$

$$z \in \mathbb{Z}_{\geq 0}.$$

This formulation serves as the basis for our proof of Theorem 1 that uses the following equivalence incrementally for  $n = 6$  and  $n = 7$ . We summarize our reductions in the following theorem.

**Theorem 6.** *Let  $n$  be a positive integer and suppose Chvátal's conjecture holds for all downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq n - 1$ . Then all downsets  $\mathcal{D}$  such that  $|U(\mathcal{D})| \leq n$  satisfy Chvátal's conjecture if and only if the objective function value of an optimal solution of  $P_{red}(n)$  is zero.*

*Proof.* As follows from Corollary 2, constraint (7e) is a valid inequality for  $P_{opt}(n)$ . Corollary 3 shows that constraints (7f) do not exclude any counterexamples that may have objective function value greater than zero. According to Proposition 4, the same holds for constraints (7g) under the assumption that Chvátal's conjecture is correct for smaller ground sets. According to Proposition 5, constraints (7h) may exclude counterexamples, but only as long as at least one symmetric counterexample remains feasible.  $\square$

## 4 Computational Results

Using the safe computational framework outlined in Section 2, we could solve  $P_{opt}(n)$  and  $P_{red}(n)$  for  $n = 5, 6$ , and 7, producing a machine-assisted proof of Theorem 1. Furthermore, several floating-point MIP solvers could solve  $P_{red}(8)$  to optimality. Although this does not constitute a safe proof, it makes it highly likely that Chvátal’s conjecture holds for  $n = 8$  and a search for counterexamples should focus on larger ground sets.

Beyond the plain question of solvability, in this section we provide details regarding the following questions: *What are the times spent for solving the integer programs and how are they affected by the improvements in the formulations? How large are the resulting certificates and how expensive is their verification? How does the performance of the exact framework compare to the performance of standard floating-point MIP solvers?*

The results for the optimality-based formulations are provided in Table 1. All tests were run on a cluster of computing nodes with Intel Xeon Gold 5122 CPUs with 3.6 GHz and 96 GB of main memory. The exact version of SCIP was built with CPLEX 12.6.3 as floating-point LP solver and QSOpt\_ex 2.5.10 [4] for exact rational LP solves. As floating-point MIP solver, we used SCIP 6.0.0 [19], built with CPLEX 12.8.0 as the underlying LP solver. The time limit was set to 12 hours for all runs and on each computing node only one job was executed at a time.

Table 1: Computational details for solving the Chvátal IPs for the two formulations  $P_{opt}(n)$  and  $P_{red}(n)$ . The sizes reported for the VIPR certificates are for uncompressed text files. The running time for input verification is negligible and always below 5 seconds.

IP	$n$	#vars	#ineqs	SCIP 6.0.0	SCIP exact	VIPR	
				time [s]	time [s]	size [MB]	time [s]
$P_{opt}(n)$	5	63	427	0.2	0.5	0.5	0.07
	6	127	1336	2.3	22.6	73	4.6
	7	255	4125	91.0	4024.2	21000	1258.3
	8	511	12618	–	–	–	–
$P_{red}(n)$	5	31	433	0.1	0.1	0.018	0.005
	6	88	1317	0.1	0.2	0.25	0.2
	7	208	4050	13.5	124.5	163	28.9
	8	455	12424	7278.9	–	–	–

First, we observe the effectiveness of the additional inequalities and fixings applied in  $P_{red}(n)$ . The running times of exact SCIP are significantly reduced, as are the sizes and verification times for the VIPR certificates. Furthermore, only  $P_{red}(8)$  can be solved by floating-point SCIP, while it times out for  $P_{opt}(8)$ . Second, note that the size of the IPs is not large compared to what MIP solvers today can often handle easily in many industrial applications. This underlines the difficulty of the underlying combinatorial question. Third, the sizes and running times of checking the VIPR certificate are significant but

do not constitute a bottleneck for the current framework. Note that the times for input verification in Coq are not reported, because they are negligible and always below 5 seconds.

## 5 Conclusion

The goal of this paper was to bridge a gap that currently divides theory and practice of integer programming. In theory, the integer programming paradigm holds the promise of globally optimal, proven results. In practice, however, there is no established computational framework that rigorously safeguards results that are obtained with integer programming software against numerical and programming errors.

Bridging this gap necessarily involved theoretical and practical aspects. On the one hand, we developed non-trivial integer programming formulations for an exemplary application: Chvátal’s conjecture, a long-standing open question in extremal combinatorics, where even low-dimensional cases are unanswered. One advantage of this approach was the flexibility of the IP formulations to include partial results from the literature by valid inequalities and variable fixings. Solving these formulations, we could show that counterexamples to the conjecture do not seem to exist for the cases  $|U(\mathcal{D})| = 6, 7,$  and  $8,$  which were previously open.

On the other hand, we had to develop and combine mathematical software in order to equip these results with the level of numerical rigor and independent verifiability that is required by computational proofs of mathematical theorems. Such verifiable computer proofs could be produced for the cases  $|U(\mathcal{D})| = 6$  and  $7.$  A proof for the case  $|U(\mathcal{D})| = 8$  is tangible if the performance of the exact rational solver is enhanced in the future. Hence, the results also motivate sustained work on closing the current performance gap between exact and inexact MIP solvers.

We hope that the generality of the computational framework presented makes it useful for the investigation of other open questions in extremal combinatorics and beyond. Most directly, our IP models could be appropriately modified to investigate variations on Chvátal’s conjecture such as proposed by Snevily [30] or a generalization of Chvátal’s conjecture proposed by Borg [9].

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