




RALF BORNDÖRFER<sup>1</sup>, STEPHAN SCHWARTZ<sup>2</sup>, WILLIAM  
SURAU<sup>3</sup>

## **Finding Minimum Balanced Separators - an Exact Approach**

---

<sup>1</sup>  0000-0001-7223-9174

<sup>2</sup>  0000-0003-2901-5065

<sup>3</sup>  0000-0003-1104-3383

Zuse Institute Berlin  
Takustr. 7  
14195 Berlin  
Germany

Telephone: +49 30 84185-0  
Telefax: +49 30 84185-125

E-mail: [bibliothek@zib.de](mailto:bibliothek@zib.de)  
URL: <http://www.zib.de>

ZIB-Report (Print) ISSN 1438-0064  
ZIB-Report (Internet) ISSN 2192-7782

# Finding Minimum Balanced Separators - an Exact Approach

Ralf Borndörfer, Stephan Schwartz, William Surau

*Zuse Institute Berlin, Takustr. 7, 14195 Berlin*

## Abstract

Balanced separators are node sets that split the graph into size bounded components. They find applications in different theoretical and practical problems. In this paper we discuss how to find a minimum set of balanced separators in node weighted graphs. Our contribution is a new and exact algorithm that solves MINIMUM BALANCED SEPARATORS by a sequence of HITTING SET problems. The only other exact method appears to be a mixed-integer program (MIP) for the edge weighted case. We adapt this model to node weighted graphs and compare it to our approach on a set of instances, resembling transit networks. It shows that our algorithm is far superior on almost all test instances.

## 1 Balanced Separators

The definition of balanced separators is not consistent in the literature. Here we use the following:

**Definition 1.** Let  $G = (V, E)$  be a graph with node weights  $w \in \mathbb{R}_{\geq 0}^V$ ,  $X \subseteq V$ , and  $\alpha \in [0, 1]$ . We call  $X$  an  $\alpha$ -**balanced separator** if any connected component  $C$  in  $G - X$  satisfies  $w(C) \leq \alpha \cdot w(G)$ .

Other definitions only consider unit node weights or require that  $X$  separates  $G$  into two disjoint vertex sets  $A, B \subseteq V$  such that no edge between a vertex in  $A$  and  $B$  exists,  $V = A \cup B \cup X$ , and  $|A|, |B| \leq \alpha \cdot |V|$ . In the latter definition we call  $X$  an  $\alpha$ -balanced biseparator. Note that for  $\alpha \geq \frac{2}{3}$  both definitions intersect. We consider the problem of finding balanced separators of minimum cardinality.

**Definition 2.** Let  $G = (V, E)$  be graph with node weights  $w \in \mathbb{R}_{\geq 0}^V$  and  $\alpha \in [0, 1]$ . The **Minimum Balanced Separators** problem is to find a set  $X \subseteq V$  such that  $X$  is an  $\alpha$ -balanced separator and  $|X|$  is minimum.

Observe that MINIMUM BALANCED SEPARATORS is  $\mathcal{NP}$ -hard. By setting  $w \equiv 1$  and  $\alpha = \frac{1}{|V|}$  we basically search for a minimum vertex cover, which is one of the classical  $\mathcal{NP}$ -hard problems.

Solving MINIMUM BALANCED SEPARATORS is usually done by approximation algorithms [3] or even pseudo approximation algorithms with relaxed upper bounds [9]. In [5] the authors found kernels for unweighted graphs. The problem is also studied for different graph classes like planar graphs [10] or graphs with maximum degree 3 [4]. In [13] they propose an exact approach to find minimum balanced biseparators.

Balanced separators have a variety of applications. They are used in approximation algorithms to find tree decompositions [7]. In many graph algorithms based on the divide and conquer paradigm [11] finding balanced separators of small size is important. Separators are also used to indicate bottlenecks in communication systems [1].

Our work is motivated by covering or partitioning a graph with connected subgraphs subject to lower and upper bounds. These problems occur in many districting [12], waste collection [6] or toll control enforcement [2] problems and can be solved by compact MIPs as presented in [8, 14]. Enforcing connectivity in MIPs is crucial and can be done by declaring vertices as roots from which connected subgraphs are formed. A naive approach is to allow every vertex to operate as a root. Since every considered root increases the number of variables and constraints, we aim to minimize the set of roots. Here we can exploit the lower weight bound  $L$  of each connected subgraph. If we set  $\alpha = \frac{L-\varepsilon}{w(G)}$  (for  $\varepsilon$  sufficiently small) then each connected subgraph  $S$  with  $w(S) \geq L$  shares at least one vertex with any  $\alpha$ -balanced separator. Therefore, finding a minimum set of roots reduces to MINIMUM BALANCED SEPARATORS.

## 2 Compact Flow Formulation

Elijazyfer [8] proposes an exact method for finding  $\alpha$ -balanced separators in edge weighted graphs. For the sake of completeness we present the adapted version to the node weighted case. The MIP is based on a flow formulation, which requires directed graphs. To turn  $G$  into a directed graph we define  $A$  as the bidirected arc set of the edges in  $E$ . The idea is to search for an arc partition such that each component induces a connected subgraph and its cumulative weight does not exceed  $\alpha \cdot w(G)$ . The set of vertices occurring in more than one component constitutes an  $\alpha$ -balanced separator. Therefore, the objective is to minimize the cardinality of this vertex set. We introduce variables  $x \in \{0, 1\}^V$  for the indication of the  $\alpha$ -balanced separator, variables  $y \in \{0, 1\}^{V \times A}$  for the arc partition, variables  $q \in \mathbb{N}_0^{V \times A}$  for flows ensuring connectivity of the components, and variables  $s \in \{0, 1\}^{V \times V}$  and  $c \in \{0, 1\}^{V \times V}$  to determine the weight of a component.

Constraints (1b) partition the arcs and (1c) force both directions of an arc to be in the same component. In (1d) it is ensured that both endpoints of an arc are in the same component as the arc. Lines (1e) and (1f) specify a flow to ensure connectivity, for more details see [2]. If a vertex is in more than one subgraph it has to be a separator, which is forced by (1g). The upper bound of each component is set in (1h). The variable  $c_v^r$  indicates if vertex  $v$  in the component rooted at  $r$  is excluded in the

cumulative weight. This is only possible if  $s_v^r = 1$  (1i) and  $v$  is a separator (1j).

$$\min_{x,s,c,y,q} \sum_{v \in V} x_v \quad (1a)$$

$$\text{s.t.} \quad \sum_{r \in V} y_a^r = 1 \quad \forall a \in A, \quad (1b)$$

$$y_{(u,v)}^r = y_{(v,u)}^r \quad \forall r \in V \quad \forall (u,v) \in A, \quad (1c)$$

$$2 \cdot y_{(u,v)}^r \leq s_u^r + s_v^r \quad \forall r \in V \quad \forall (u,v) \in A, \quad (1d)$$

$$q_a^r \leq y_a^r \cdot (|V| - 1) \quad \forall r \in V \quad \forall a \in A, \quad (1e)$$

$$\sum_{v \in \delta^+(v)} q_a^r - \sum_{v \in \delta^-(v)} q_a^r \geq s_v^r \quad \forall r \in V \quad \forall v \in V \setminus \{r\}, \quad (1f)$$

$$1 + x_v \cdot (|V| - 1) \geq \sum_{r \in V} s_v^r \quad \forall v \in V, \quad (1g)$$

$$\sum_{v \in V} (s_v^r - c_v^r) \cdot w_v \leq \alpha \cdot w(G) \quad \forall r \in V. \quad (1h)$$

$$c_v^r \leq s_v^r \quad \forall r \in V \quad \forall v \in V, \quad (1i)$$

$$c_v^r \leq x_v \quad \forall r \in V \quad \forall v \in V. \quad (1j)$$

We can see that the model contains a large number of variables and constraints and uses many big  $M$  constraints. In particular, for the connectivity it considers a flow emerging from every vertex, while our main motivation is to avoid this necessity.

### 3 Exact Algorithm

We propose a different exact approach that circumvents these issues. While our formulation still has a large number of constraints we present a separation routine that dynamically generates necessary constraints. To this end, we define  $\mathcal{S}$  as the set of all connected subgraphs of  $G$  with a cumulative weight greater than  $\alpha \cdot w(G)$ . The master problem has the following form:

$$\min_x \sum_{v \in V} x_v \quad (2a)$$

$$\text{s.t.} \quad \sum_{v \in V(S)} x_v \geq 1 \quad \forall S \in \mathcal{S}, \quad (2b)$$

$$x_v \in \{0, 1\} \quad \forall v \in V. \quad (2c)$$

The variable  $x_v$  indicates if  $v$  is part of the balanced separator. Our objective is to minimize the cardinality of the balanced separator (2a). Constraints (2b) ensure that each subgraph in  $\mathcal{S}$  contains at least one separating vertex.

Model (2) solves MINIMUM BALANCED SEPARATORS. Let  $X \subseteq V$  be a solution of (2), then every connected component  $C$  in  $G - X$  fulfills  $w(C) \leq \alpha \cdot w(G)$ , otherwise  $X$  would not be feasible since  $C \in \mathcal{S}$ . Let  $X^*$  be a solution of MINIMUM BALANCED SEPARATORS. Any subgraph  $S \in \mathcal{S}$  has to have a common vertex with  $X^*$ , otherwise a connected component  $C \supseteq S$  in  $G - X^*$  exists with  $w(C) > \alpha \cdot w(G)$ . Therefore,

model (2) is correct. Observe that this model is the well known HITTING SET, which is defined as:

**Definition 3.** Let  $\mathcal{U}$  be a set and  $\mathcal{S} \subseteq \mathcal{P}(\mathcal{U})$ . The **Hitting Set** problem is to find a set  $X \subseteq \mathcal{U}$  such that for any  $S \in \mathcal{S}$  holds  $X \cap S \neq \emptyset$  and  $|X|$  is minimum.

The algorithm starts by solving HITTING SET( $V, \emptyset$ ) and adds iteratively violated constraints, corresponding to the connected components exceeding the upper bound.

---

**Algorithm 1** MINIMUM BALANCED SEPARATORS

---

**Input:**  $G = (V, E), w \in \mathbb{R}_{\geq 0}^V, \alpha \in [0, 1]$

**Output:**  $X \subseteq V$  s.t.  $X$  is an  $\alpha$ -balanced separator and  $|X|$  is minimum.

```

1:  $S, X \leftarrow \emptyset$ 
2: while in  $G - X$  exists a connected component  $C$  with  $w(C) > \alpha \cdot w(G)$  do
3:   for all connected component  $C$  in  $G - X$  do
4:     if  $w(C) > \alpha \cdot w(G)$  then
5:        $S \leftarrow S \cup \{V(C)\}$ 
6:    $X \leftarrow \text{HITTING SET}(V, S)$ 
7: return  $X$ 

```

---

**Proposition 4.** Algorithm 1 is correct.

*Proof.* Let  $X^*$  be a minimum  $\alpha$ -balanced separator and  $X$  be a solution from Algorithm 1. Since  $G - X$  contains no connected component  $C$  with  $w(C) > \alpha \cdot w(G)$ ,  $X$  is an  $\alpha$ -balanced separator. If  $|X^*| < |X|$ , then  $X$  is not an optimal solution of HITTING SET, because at any iteration  $X^*$  is a feasible solution.

In each iteration one or more subgraphs from  $\mathcal{S}$  are added to  $S$ , otherwise we stop. Since at least one vertex of every subgraph in  $S$  is in  $X$  no subgraph gets added twice. Therefore, and since  $|\mathcal{S}|$  is finite, we conclude that Algorithm 1 terminates.  $\square$

We have improved Algorithm 1 by only adding minimal connected subgraphs that are violating the upper bound. Let  $\mathcal{S}' \subseteq \mathcal{S}$  be the set of all connected subgraphs, such that no proper connected subgraph is in  $\mathcal{S}$  as well. Let  $X \subseteq V$  be the current solution and  $C$  be a connected component in  $G - X$  with  $w(C) > \alpha \cdot w(G)$ . We search for an arbitrary covering of  $C$  with subgraphs of  $\mathcal{S}'$ . This can be done by a simple breadth first search (BFS) on  $C$ . Starting at any vertex we construct a subgraph following the BFS. As soon as the current subgraph violates the upper bound we start to construct a new subgraph from an uncovered vertex. We achieve another improvement by bounding the objective value of HITTING SET in each iteration. Let us say we are at the  $i$ -th iteration of the while loop. Let  $X' \subseteq V$  be the solution of HITTING SET in the previous iteration. We know that the current solution  $X \subseteq V$  fulfills  $|X'| \leq |X|$ . Therefore, we can add the constraint  $\sum_{v \in V} x_v \geq |X'|$ .

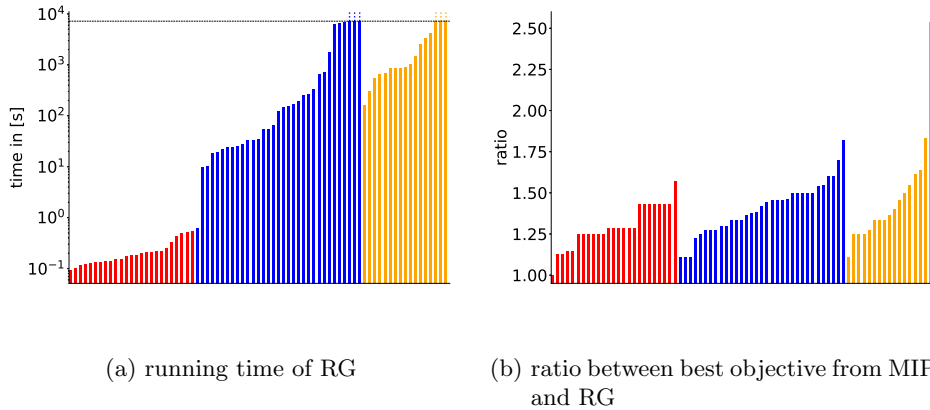


Figure 1: Computational results of RG and MIP on `tree_lg` (red), `voronoi_medium_lg` (blue), and `voronoi_large_lg` (orange). The order of the instances is sorted independently.

## 4 Computational Study

We ran the experiments on machines equipped with Intel Xeon E3-1234 CPUs with 3.7GHz and 32GB RAM. The code is written in Python 3.6 and to solve MIPs Gurobi 9.1 is used. The time limit is set to 2 hours.

We consider test instances that resemble transit networks of different complexity. All test instances are described and can be found in a GitHub repository<sup>1</sup>, and our instance-wise computational results in an online supplement<sup>2</sup>. We only use the node weighted instances labeled with ”\_lg” (for line graph). Inspired by a real world problem from [2], we set parameter  $\alpha$  such that  $\alpha \cdot w(G) = 100 - \varepsilon$ , where  $\varepsilon = 10^{-4}$ . This leads to values of  $\alpha$  ranging from 0.12 to 0.21. The instances are split into three groups, `tree_lg`, `voronoi_medium_lg`, and `voronoi_large_lg`. By MIP we denote model (1) and RG refers to Algorithm 1.

In Figure 1a we see the running time of RG. The time limit is marked by a horizontal dashed line. If an instance exceeds the time limit, the corresponding bar ends with a dotted line. Our algorithm finds an optimal solution within the time limit for all but 6 instances. Note that the scale is logarithmic and the running times lie in a broad range, even within the same group. The MIP, on the other hand, exceeds the time limit on all 75 instances. Our algorithm is not only much faster but also provides better solutions. In Figure 1b we see that the incumbent separator set in MIP after 2 hours is significantly larger than the optimal solution from RG. Again, the instances are ordered by ratio and grouped. While MIP only finds an optimal solution for one instance it could not find a corresponding dual bound better than 0. In fact, the dual

<sup>1</sup>[https://github.com/stephanschwartz/vertex\\_covering\\_with\\_capacitated\\_trees](https://github.com/stephanschwartz/vertex_covering_with_capacitated_trees)

<sup>2</sup>[https://github.com/williamsurau/finding\\_minimum\\_balanced\\_separators-an\\_exact\\_approach](https://github.com/williamsurau/finding_minimum_balanced_separators-an_exact_approach)

bound for MIP is still 0 on all instances after 7200 seconds. Moreover, we ran MIP on several selected instances with a time limit of 24 hours, and it was not able to close the gap for any of those. Once again, this illustrates the superiority of the presented approach.

## References

- [1] Sandeep N Bhatt and Frank Thomson Leighton. “A framework for solving VLSI graph layout problems”. In: *Journal of Computer and System Sciences* 28.2 (1984), pp. 300–343.
- [2] Ralf Borndörfer, Stephan Schwartz, and William Surau. *Vertex Covering with Capacitated Trees*. eng. Tech. rep. 21-14. Takustr. 7, 14195 Berlin: ZIB, 2021.
- [3] Sebastian Brandt and Roger Wattenhofer. “Approximating small balanced vertex separators in almost linear time”. In: *Workshop on Algorithms and Data Structures*. Springer. 2017, pp. 229–240.
- [4] Thang Nguyen Bui and Curt Jones. “Finding good approximate vertex and edge partitions is NP-hard”. In: *Information Processing Letters* 42.3 (1992), pp. 153–159.
- [5] Katrin Casel, Tobias Friedrich, Davis Issac, Aikaterini Niklanovits, and Ziena Zeif. “Balanced Crown Decomposition for Connectivity Constraints”. In: *arXiv preprint arXiv:2011.04528* (2020).
- [6] François Clautiaux, Jeremy Guillot, and Pierre Pesneau. “Exact approaches for solving a covering problem with capacitated subtrees”. In: *Computers & Operations Research* 105 (2019), pp. 85–101.
- [7] Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized algorithms*. Vol. 5. 4. Springer, 2015.
- [8] Ziena Elijazyfer. “Längenbeschränkte Teilgraphenbildung zur Maut-Kontrollstreckenoptimierung”. Masterthesis. Freie Universität Berlin, 2018.
- [9] Uriel Feige, MohammadTaghi Hajiaghayi, and James R Lee. “Improved approximation algorithms for minimum weight vertex separators”. In: *SIAM Journal on Computing* 38.2 (2008), pp. 629–657.
- [10] Martin Holzer, Grigorios Prasinou, Frank Schulz, Dorothea Wagner, and Christos Zaroliagis. “Engineering planar separator algorithms”. In: *European Symposium on Algorithms*. Springer. 2005, pp. 628–639.
- [11] Arnold L Rosenberg and Lenwood S Heath. *Graph separators, with applications*. Springer Science & Business Media, 2001.
- [12] María Angélica Salazar-Aguilar, Roger Z Ríos-Mercado, and Mauricio Cabrera Ríos. “New models for commercial territory design”. In: *Networks and Spatial Economics* 11.3 (2011), pp. 487–507.



- [13] Cid C de Souza and Victor F Cavalcante. “Exact algorithms for the vertex separator problem in graphs”. In: *Networks* 57.3 (2011), pp. 212–230.
- [14] William Surau. “Das homogene langenbeschrankte zusammenhangende Teilgraphenuberdeckungsproblem”. Bachelorthesis. Freie Universitat Berlin, 2020.