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# On the implementation and strengthening of intersection cuts for QCQPs

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# On the implementation and strengthening of intersection cuts for QCQPs

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## Abstract

The generation of strong linear inequalities for QCQPs has been recently tackled by a number of authors using the intersection cut paradigm—a highly studied tool in integer programming whose flexibility has triggered these renewed efforts in non-linear settings. In this work, we consider intersection cuts using the recently proposed construction of *maximal quadratic-free sets*. Using these sets, we derive closed-form formulas to compute intersection cuts which allow for quick cut-computations by simply plugging-in parameters associated to an arbitrary quadratic inequality being violated by a vertex of an LP relaxation. Additionally, we implement a cut-strengthening procedure that dates back to Glover and evaluate these techniques with extensive computational experiments.

## 1 Introduction

Nowadays, the reach of state-of-the-art optimization solvers is vast, and certain classes of non-convex optimization problems that years ago seemed impenetrable can now be solved in moderate running times. The current state-of-the-art can be roughly described as a combination of outer-approximation techniques that can provide strong dual bounds, heuristics that can find strong feasible solutions and spatial branch-and-bound—a generalization of the classical integer programming method that allows branching on continuous variables. However, we still encounter computational challenges preventing us to solve many non-convex optimization instances to provable optimality.

In this work, we focus on obtaining dual bounds by constructing *linear* outer-approximation to a rich family of non-convex optimization problems: quadrat-

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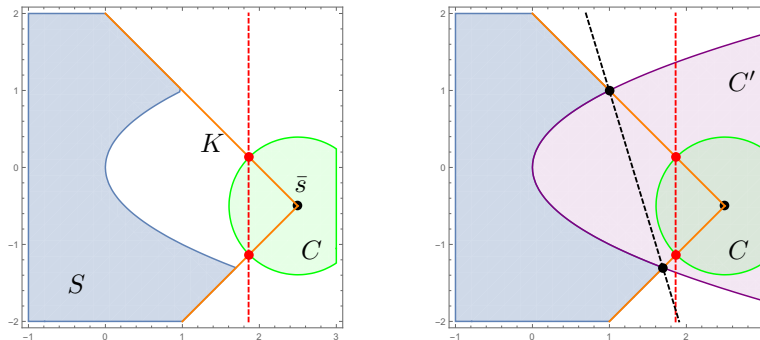


Figure 1: On the left, an intersection cut (red) separating  $\bar{s}$  from  $S$  (blue). The cut is computed using the intersection points of an  $S$ -free set  $C$  (green) and the rays of a simplicial cone  $K \supseteq S$  (boundary in orange) with apex  $\bar{s} \notin S$ . On the right, the effect of using another  $S$ -free set  $C' \supseteq C$ .

ically constrained quadratic programs (QCQPs). These are optimization problems whose objective and constraints are defined solely in terms of quadratic functions. One can assume, without loss of generality, that the objective function is linear, thus we refer as QCQP a problem of the form

$$\min \quad \bar{c}^\top s \quad (1a)$$

$$\text{s.t.} \quad s \in S \subseteq \mathbb{R}^p, \quad (1b)$$

where

$$S = \{s \in \mathbb{R}^p : s^\top Q_i s + b_i^\top s + c_i \leq 0, i = 1, \dots, m\}.$$

In order to find linear outer-approximations to (1) we follow the *intersection cut* paradigm [35, 4, 21] which requires the following ingredients. We assume we have  $\bar{s} \notin S$ , a basic optimal solution of a *linear programming* (LP) relaxation of (1)<sup>1</sup>. Additionally, we require a simplicial conic relaxation  $K \supseteq S$  with apex  $\bar{s}$ , and an  $S$ -free set  $C$ —a convex set satisfying  $\text{int}(C) \cap S = \emptyset$ —such that  $\bar{s} \in \text{int}(C)$ . With these ingredients, we can find a cutting plane: a valid inequality guaranteed to separate  $\bar{s}$  from  $S$ . In Figure 1(left) we show a simple intersection cut in the case when all  $p$  rays of  $K$  intersect the boundary of the  $S$ -free set  $C$ . In such case, the intersection cut is simply defined by the hyperplane containing all such intersection points (hence its name).

When the intersection cuts are computed using the intersection points between the  $S$ -free set and the extreme rays of  $K$ , the larger the  $S$ -free set the better. In other words, if two  $S$ -free sets  $C, C'$  are such that  $C \subsetneq C'$ , the intersection cut derived from  $C'$  is stronger than the one derived from  $C$  [12]. In Figure 1(right) we show this phenomenon. This makes inclusion-wise *maximality* of an  $S$ -free set a desirable goal. We provide the appropriate details of this procedure in Section 2.1, and we also refer the reader to [13] for additional background.

Note that if  $\bar{s} \notin S$ , there exists  $i \in \{1, \dots, m\}$  such that

$$\bar{s} \notin S_i := \{s \in \mathbb{R}^p : s^\top Q_i s + b_i^\top s + c_i \leq 0\},$$

<sup>1</sup>If  $\bar{s} \in S$  the problem would be solved.

and constructing an  $S_i$ -free set containing  $\bar{s}$  suffices to ensure separation. We refer to these sets as *quadratic-free sets*. Recently, Muñoz and Serrano [28] provided a method for constructing *maximal* quadratic-free sets for any arbitrary quadratic inequality. The focus of [28] lies on the structure, construction and maximality proofs for quadratic-free sets, leaving aside the actual calculation of the intersection cut and its computational impact.

## 1.1 Contribution

The first contribution of this work is an implementation and extensive testing of intersection cuts based on the maximal quadratic-free sets proposed by Muñoz and Serrano. In [28], they showed how to construct maximal  $S^h$ -free and  $S^g$ -free sets for the quadratic-representable sets

$$S^h := \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \|y\|\} \quad (2)$$

$$S^g := \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \|y\|, a^\top x + d^\top y = -1\}, \quad (3)$$

where  $\max\{\|a\|, \|d\|\} = 1$ . They also argued that one can always transform

$$S := \{s \in \mathbb{R}^p : s^\top Qs + b^\top s + c \leq 0\}, \quad (4)$$

a generic quadratic set, into  $S^h$  or  $S^g$  and use the maximal  $S^h$ -free and  $S^g$ -free to construct maximal  $S$ -free sets. The maximal  $S$ -free sets of Muñoz and Serrano, however, are described only with respect to  $S^h$  and  $S^g$ .

Here, we provide transformations from  $S$  into  $S^h$  and  $S^g$  explicitly, and show descriptions of the resulting  $S$ -free sets. Moreover, we derive closed-form expressions to compute a valid inequality violated by  $\bar{s} \notin S$ .

Additionally, we implement and test a well known *cut strengthening* procedure designed to improve the intersection cut coefficient of an extreme ray of  $K$  that never intersects the boundary of the  $S$ -free set. We show that to compute the strengthened cuts one needs to solve single-variable convex optimization problems, which we provide explicitly.

In our computational experiments, not only we consider the quadratic constraints that appear in a QCQP instance, but we also implemented a family of cuts obtained from *implied quadratic constraints* in an extended space. These were used in the construction of maximal *outer-product-free* sets by Bienstock et al. [8, 7], which here we reinterpret as maximal quadratic-free sets. All our ideas are tested using the general-purpose solver SCIP [30].

## 1.2 Literature review

The basic idea behind intersection cuts can be traced back to Tuy [35]. These cuts, also known as *Tuy cuts* or *concavity cuts*, were introduced for the problem of minimizing a concave function over a polytope. Later on, intersection cuts were introduced in integer programming by Balas [4] and have been largely

studied since. Intersection cuts deduced from arbitrary convex  $S$ -free set is due to Glover [21], although the term  $S$ -free was coined by Dey and Wolsey [16].

Intersection cuts has been an active research stream in the mixed-integer linear programming literature for decades. See e.g. [12, 14, 6] for in-depth analyses of the relation of intersection cuts using maximal  $\mathbb{Z}^n$ -free sets and the generation of facets of  $\text{conv}(S)$ , when  $S$  is a mixed-integer set. We also refer the reader to [3, 5, 2, 10, 15, 22] and references therein. Intersection cuts have also been extended to the mixed-integer conic case: see e.g. [1, 23, 26, 27]. A different, but related, method was recently proposed by Towle and Luedtke [34].

Lately, there has been a number of methods proposed for the use of the intersection cut framework in non-linear non-convex settings. Fischetti et al. [17] applied intersection cuts to bilevel optimization. Bienstock et al. [8, 7] studied outer-product-free sets, which can be used for generating intersection cuts for polynomial optimization when using an extended formulation. Serrano [33] showed how to construct a concave underestimator of any factorable function and from them one can build intersection cuts for factorable mixed integer non-linear programs. Fischetti and Monaci [18] constructed bilinear-free sets through a bound disjunction and, in each term of the disjunction, underestimating the bilinear term with McCormick inequalities [24]. The complement of this disjunction is the bilinear-free set. Finally, Muñoz and Serrano [28] constructed multiple families of maximal quadratic-free sets that can be used to compute intersection cuts for a QCQP.

Alternative cutting-plane-generation approaches to intersection cuts can be seen in [23, 11], and in [29], where the authors show that the convex hull of a single quadratic constraint over a polytope is second-order cone representable. We refer to the survey [9] and the references therein for other efforts of extending cutting planes to the non-linear setting.

### 1.3 Notation

We mostly follow standard notation.  $\|\cdot\|$  is the euclidean norm in  $\mathbb{R}^n$ .  $B_r(x)$  and  $D_r(x)$  denote the euclidean ball centered at  $x$  of radius  $r$  and its boundary, respectively, i.e.,  $B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$  and  $D_r(x) = \{y \in \mathbb{R}^n : \|y - x\| = r\}$ .  $\text{conv}(\cdot)$ ,  $\text{int}(\cdot)$ , and  $\text{rec}(\cdot)$  denote the convex hull, interior and recession cone of a set, respectively.

### 1.4 Outline

The rest of the paper is organized as follows. In Section 2 we review the concepts of intersection cuts, the negative edge extension strengthening and maximal quadratic-free sets. In Section 3 we show transformations mapping  $S$  to a set  $S^h$  or  $S^g$ , and use these to explicitly construct maximal  $S$ -free sets. We also discuss how to use implied quadratic constraints to obtain cutting planes. In Section 4 we show the computations needed for obtaining the intersection cut

coefficients and their strengthening using the maximal  $S$ -free sets. Finally, in Section 5 we show our computational results.

## 2 Preliminaries

In this section we review the main tools we work with in this paper, namely the notions of *intersection cuts*, *negative edge extension* and *quadratic-free-sets*.

### 2.1 Intersection cuts

Suppose we have a vector  $\bar{s} \notin S$ , a simplicial cone  $K \supseteq S$  with apex  $\bar{s}$  and an  $S$ -free set  $C$  containing  $\bar{s}$  in its interior. As briefly described in the introduction, what the intersection cut paradigm provides is a way of separating  $\bar{s}$  from  $S$  by computing an inequality separating  $\bar{s}$  from  $\text{conv}(K \setminus \text{int}(C))$ .

This tool is typically used from a *linear programming* perspective. In this case, the vector  $\bar{s}$  is a basic solution of a polyhedron  $P \supseteq S$ . Since separation from  $\text{conv}(P \setminus \text{int}(C))$  is NP-hard [19], a simplicial conic relaxation  $K$  is used instead of  $P$  to alleviate the computational burden. This simplicial cone is normally obtained from a basis defining  $\bar{s}$ , and thus there is a correspondence between rays and nonbasic variables.

#### 2.1.1 Basic intersection cut

Since  $K$  is a simplicial cone with apex  $\bar{s}$ , it can be described as

$$K = \{s \in \mathbb{R}^p : A(s - \bar{s}) \leq 0\},$$

where  $A$  is an *invertible* matrix. Using this notation, the extreme rays of  $K$  are simply given by the columns of  $-A^{-1}$ . Let us call these rays  $r^j, j \in [p]$ . Define  $\alpha_j^* \in (0, \infty]$  as

$$\alpha_j^* = \sup_{\alpha \geq 0} \{\alpha : \bar{s} + \alpha r^j \in C\}. \quad (5)$$

These are simply the *step-lengths* required to reach the intersection points of the rays  $r^j$  and  $C$ . The *intersection cut* is defined as the hyperplane that contains all points of the form  $\bar{s} + \alpha_j^* r^j$  when  $\alpha_j^* < \infty$ , and that is parallel to every ray  $r^j$  when  $\alpha_j^* = \infty$ .

The following closed-form expression for the inequality  $\pi^T z \leq \pi_0$  describing the intersection cut can be obtained (see [4]):

$$\sum_{i=1}^p \frac{1}{\alpha_i^*} A_i (s - \bar{s}) \leq -1. \quad (6)$$

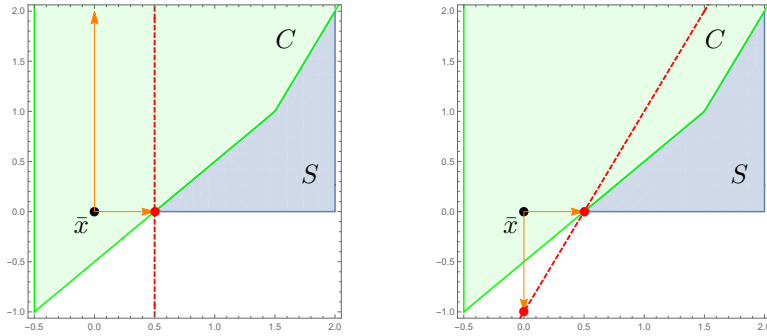


Figure 2: On the left, an intersection cut (red) separating  $\bar{x}$  from  $S$  (blue) in a case a ray of  $K$  (orange arrows) is contained in  $C$  (green). On the right, the strengthened cut when taking an appropriate negative step.

where  $1/\infty := 0$  and  $A_i$  is the  $i$ -th row of  $A$ . This formula shows that all computational burden of computing an intersection lies on computing the corresponding step-lengths  $\alpha_i^*$  defined in (5).

For the sake of completeness, we describe two important properties of intersection cuts that can be considered folklore results in optimization nowadays. Proofs can be found e.g. in [4].

**Lemma 1.** *Let  $\pi^\top z \leq \pi_0$  represent the inequality in (6). This inequality is valid for  $S$ :*

$$\{z : \pi^\top z > \pi_0\} \cap K \supseteq \text{int}(C).$$

*Additionally, if all step-lengths are finite, i.e.  $\alpha_i^* < \infty \forall i \in [p]$ , then*

$$\{z : \pi^\top z \leq \pi_0\} \cap K = \text{conv}(K \setminus \text{int}(C)).$$

The second statement of the lemma shows that, when all rays of  $K$  intersect the boundary of  $C$ , the intersection cut is, in a sense, the best we can hope for. This is not always the case, however, especially when the  $S$ -free sets are unbounded as they are in our setting.

### 2.1.2 Strengthening intersection cuts

In the case a ray of  $K$  does not intersect the boundary of the  $S$ -free set  $C$ , i.e. when  $r^j \in \text{rec}(C)$  for some  $j$ , the resulting intersection cut may not define a facet of  $\text{conv}(K \setminus \text{int}(C))$  and strengthening it may be possible. In Figure 2(left) we illustrate this situation.

Glover [20] proposed using *negative* step-lengths for rays  $r^j \in \text{rec}(C)$  as a way of strengthening the intersection cut. This would replace the terms  $1/\alpha_j^* = 0$  in (6) with a negative value, thus improving the cut coefficients. This idea is known as *negative edge extension*, and it has been carefully studied since, showing that choosing the right negative step-lengths can provide facets of  $\text{conv}(K \setminus \text{int}(C))$



[31, 32, 8]. In Figure 2(right) we show the resulting strengthened cut after applying the negative edge extension.

In [31], the authors show how to use the negative edge extension in order to obtain facets of  $\text{conv}(K \setminus \text{int}(C))$  when  $C$  is a polyhedron. Later on, in [32] it was shown that for a non-polyhedral  $S$ -free set  $C$ , there is always a polyhedron  $C'$  such that  $\text{conv}(K \setminus \text{int}(C')) = \text{conv}(K \setminus \text{int}(C))$ , thus providing a way of obtaining facets of  $\text{conv}(K \setminus \text{int}(C))$ . Finally, in [8] it was shown how to obtain a facet of  $\text{conv}(K \setminus \text{int}(C))$  directly, without going through an intermediate polyhedron. This last derivation is the one we use here.

Let  $F = \{i : \alpha_i^* < \infty\}$ . The negative edge extension technique aims at finding  $\rho_i < 0$  such that

$$\sum_{i \in F} \frac{1}{\alpha_i^*} A_i(s - \bar{s}) + \sum_{i \notin F} \frac{1}{\rho_i} A_i(s - \bar{s}) \leq -1. \quad (7)$$

is a valid cut. We know that  $\rho_i = -\infty$  provides a valid cut, since we recover (6) in such case. Moreover, it can be proved that, as long as  $\alpha_i^* r^i - \rho_i r^j \in \text{rec}(C) \forall i \in F, j \notin F$ , the cut is valid. This can be seen in Figure 2(right): a “more negative” step on the vertical ray maintains validity of the cut, while a “less negative” step yields a non-valid cut. We define the strengthened cut as (7) with

$$\rho_j = \max_{\rho < 0} \{\rho : \alpha_i^* r^i - \rho r^j \in \text{rec}(C) \quad \forall i \in F\}. \quad (8)$$

The resulting cut is a facet of  $\text{conv}(K \setminus \text{int}(C))$ . For details we refer the reader to [31, 32, 8].

For the purposes of this article, the main takeaway message of Section 2.1 is that, for computing an intersection cut and a potential strengthening, one needs to solve problems (5) and (8) efficiently. To achieve this, counting with adequate descriptions of the  $S$ -free set  $C$  and of  $\text{rec}(C)$  is key.

## 2.2 Maximal quadratic-free sets

Here, we review the maximal quadratic-free sets proposed by Muñoz and Serrano [28]. As mentioned before, these sets are constructed explicitly for the sets

$$S^h := \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \|y\|\} \quad (9a)$$

$$S^g := \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \|y\|, a^\top x + d^\top y = -1\}, \quad (9b)$$

with  $\max\{\|a\|, \|d\|\} = 1$ . For a generic quadratic set

$$S = \{s \in \mathbb{R}^p : s^\top Q s + b^\top s + c \leq 0\}$$

the authors argue that a maximal  $S$ -free set can be obtained from  $S^h$  and  $S^g$  by noting that one can always map  $S$  to a set of either of the following forms

$$\{(x, y, z) \in \mathbb{R}^{n+m+l} : \|x\| \leq \|y\|\}, \quad (10a)$$

$$\{(x, y, z) \in \mathbb{R}^{n+m+l} : \|x\| \leq \|y\|, a^\top x + d^\top y + h^\top z = -1\}, \quad (10b)$$

depending on whether the quadratic function defining  $S$  is homogeneous or not. Such a transformation guarantees that constructing a maximal (10a)- or (10b)-free sets yields a maximal  $S$ -free set. Since one of these sets is contained in a hyperplane, we need the following definition in order to avoid trivial (10b)-free sets such as  $\{(x, y) \in \mathbb{R}^{n+m} : a^\top x + d^\top y + h^\top z \leq -1\}$ .

**Definition 1.** Given  $S, C, H \subseteq \mathbb{R}^n$  where  $S$  is closed,  $C$  is closed and convex and  $H$  is an affine hyperplane, we say that  $C$  is  $S$ -free with respect to  $H$  if  $C \cap H$  is  $S \cap H$ -free w.r.t the induced topology in  $H$ . We say  $C$  is maximal  $S$ -free with respect to  $H$ , if for any  $C' \supseteq C$  that is  $S$ -free with respect to  $H$  it holds that  $C' \cap H \subseteq C \cap H$ .

To complete the transformation to  $S^h$  and  $S^g$  and discard the  $z$  variables, Muñoz and Serrano argue the following.

**Proposition 1.** *Let  $C$  be a maximal  $S^h$ -free. Then,  $C \times \mathbb{R}^l$  is maximal (10a)-free and maximal (10b)-free (with respect to the corresponding hyperplane).*

As for the vector  $\bar{s} \notin S$ , applying the same reductions from  $S$  to either  $S^h$  or  $S^g$ , we can transform a vector  $\bar{s}$  to a vector  $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+m}$  such that  $\|\bar{x}\| > \|\bar{y}\|$ . We can further assume  $a^\top \bar{x} + d^\top \bar{y} = -1$  when considering  $S^g$ .

The following result shows a family of maximal  $S^h$ -free sets

**Theorem 1.** [28] *Consider the set  $S^h$  defined in (9a). For any  $\lambda \in D_1(0)$ , the set*

$$C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} : \|y\| \leq \lambda^\top x\}, \quad (11)$$

*is maximal  $S^h$ -free. Moreover, if  $\lambda = \frac{\bar{x}}{\|\bar{x}\|}$  then  $C_\lambda$  contains  $(\bar{x}, \bar{y})$  in its interior.*

To list the maximal  $S^g$ -free sets we need the following definition

**Definition 2.** Let  $a \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^m$  and  $\lambda \in D_1(0)$ . We define the function  $\phi_{\lambda, a, d} : \mathbb{R}^m \rightarrow \mathbb{R}$  as

$$\phi_{\lambda, a, d}(y) = \begin{cases} \|y\|, & \text{if } \lambda^\top a \|y\| + d^\top y \leq 0 \\ \sqrt{(\|y\|^2 - (d^\top y)^2)(1 - (\lambda^\top a)^2)} - d^\top y \lambda^\top a, & \text{otherwise.} \end{cases} \quad (12)$$

Additionally, the following lemma will be of use later.

**Lemma 2.** *Let  $\phi_{\lambda, a, d}$  defined as (12), with  $\max\{\|a\|, \|d\|\} = 1$ . Then*

$$\phi_{\lambda, a, d}(y) \leq \|y\| \quad \forall y \in \mathbb{R}^m.$$

*Moreover, if  $\lambda^\top a \|y\| + d^\top y > 0$ , this inequality is strict.*

*Proof.* The non-strict bound can be deduced from the construction of the function  $\phi_{\lambda, a, d}$  in [28]. We provide a direct proof here.

Clearly, it suffices to show that

$$\sqrt{(\|y\|^2 - (d^\top y)^2)(1 - (\lambda^\top a)^2) - d^\top y \lambda^\top a} \leq \|y\|$$

Since  $\max\{\|a\|, \|d\|\} = 1$ , by Cauchy-Schwarz

$$|d^\top y \lambda^\top a| \leq \|y\| \Rightarrow \|y\| + d^\top y \lambda^\top a \geq 0$$

and thus it suffices to show that

$$(\|y\|^2 - (d^\top y)^2)(1 - (\lambda^\top a)^2) \leq (\|y\| + d^\top y \lambda^\top a)^2.$$

Rearranging terms this becomes

$$0 \leq 2\|y\|d^\top y \lambda^\top a + (d^\top y)^2 + \|y\|^2(\lambda^\top a)^2.$$

Since the right-hand side is  $(\lambda^\top a \|y\| + d^\top y)^2$ , we conclude the desired statement. For the strict inequality, we can follow the same procedure.  $\square$

The maximal  $S^g$ -free sets we work with are as follows.

**Theorem 2.** [28] Consider a non-convex set  $S^g$  defined as in (9b), with  $\max\{\|a\|, \|d\|\} = 1$ , and  $(\bar{x}, \bar{y})$  satisfying  $\|\bar{x}\| > \|\bar{y}\|$  and  $a^\top \bar{x} + d^\top \bar{y} = -1$ . Consider  $\phi_{\lambda,a,d}$  defined in (12). Let  $H = \{(x, y) \in \mathbb{R}^{n+m} : a^\top x + d^\top y = -1\}$ , and  $\lambda = \frac{\bar{x}}{\|\bar{x}\|}$ .

If  $\|a\| \leq \|d\| = 1$ , the set

$$C_{\phi_{\lambda,a,d}} := \{(x, y) \in \mathbb{R}^n : \phi_{\lambda,a,d}(y) \leq \lambda^\top x\} \quad (13)$$

is maximal  $S^g$ -free with respect to  $H$  and contains  $(\bar{x}, \bar{y})$  in its interior.

If  $\|d\| < \|a\| = 1$ , the set

$$C_{\phi_{\lambda,a,d}}^g := \left\{ (x, y) : \begin{array}{ll} \|y\| \leq \lambda^\top x & \text{if } \lambda^\top a \|y\| + d^\top y \leq 0 \\ \phi_{\lambda,a,d}\left(y - \frac{d}{1 - \|d\|^2}\right) \leq \lambda^\top \left(x + \frac{a}{1 - \|d\|^2}\right) & \text{otherwise} \end{array} \right\}.$$

is maximal  $S^g$ -free with respect to  $H$  and contains  $(\bar{x}, \bar{y})$  in its interior.

**Remark 1.** In [28] the authors show that the set  $C_{\phi_{\lambda,a,d}}$  is also maximal  $S_{\leq 0}$ -free for

$$S_{\leq 0} = \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \|y\|, a^\top x + d^\top y \leq 0\}. \quad (14)$$

which allows the computation of intersection cuts for a set defined using a *homogeneous quadratic* intersected with a *linear homogeneous inequality*. While most of our constructions below only use  $S^h$  and  $S^g$ , the set  $S_{\leq 0}$  will be of use when we implement cutting planes using implied quadratic inequalities in an extended space. We provide more details in Section 3.1  $\square$

This completes the description of the base sets we use. These represent the building blocks in our construction below.

### 3 Explicit computation of maximal quadratic-free sets

In this section, we show how to use the maximal  $S^h$ - and  $S^g$ -free sets in order to construct a maximal quadratic-free for an arbitrary quadratic. The overall strategy is to first diagonalize and then homogenize the quadratic inequality in order to map  $S$  onto the desired sets. In what follows, we explicitly compute such transformations and show the resulting maximal  $S$ -free sets. Let

$$S = \{s \in \mathbb{R}^p : s^\top Qs + b^\top s + c \leq 0\},$$

with  $Q$  a symmetric matrix, and let  $Q = V\Theta V^\top$  be its eigenvalue decomposition. Then,

$$S = \{s \in \mathbb{R}^p : s^\top V\Theta V^\top s + b^\top s + c \leq 0\}.$$

Defining  $\psi(s) := V^\top s$  and  $\bar{b} = V^\top b$ , we have

$$S = \{s \in \mathbb{R}^p : \psi(s)^\top \Theta \psi(s) + \bar{b}^\top \psi(s) + c \leq 0\}.$$

Let  $\theta_i$   $i = 1, \dots, p$  be the eigenvalues of  $Q$ , and define  $I_+ = \{i : \theta_i > 0\}$ ,  $I_- = \{i : \theta_i < 0\}$ , and  $I_0 = \{i : \theta_i = 0\}$ . Then,

$$\begin{aligned} S &= \{s \in \mathbb{R}^p : \sum_{i \in I_+ \cup I_-} (\theta_i \psi_i(s)^2 + \bar{b}_i \psi_i(s)) + \sum_{i \in I_0} \bar{b}_i \psi_i(s) + c \leq 0\} \\ &= \{s \in \mathbb{R}^p : \sum_{i \in I_+} \left( \sqrt{\theta_i} \left( \psi_i(s) + \frac{\bar{b}_i}{2\theta_i} \right) \right)^2 - \sum_{i \in I_-} \left( \sqrt{-\theta_i} \left( \psi_i(s) + \frac{\bar{b}_i}{2\theta_i} \right) \right)^2 \\ &\quad + \sum_{i \in I_0} \bar{b}_i \psi_i(s) + \left( c - \frac{1}{4} \sum_{i \in I_+ \cup I_-} \frac{\bar{b}_i^2}{\theta_i} \right) \leq 0\} \end{aligned}$$

From this last expression, we obtain the following equivalent description of  $S$

$$S = \{s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 + (\bar{b}_{I_0})^\top z(s) + \kappa \leq 0\},$$

where  $\bar{b}_{I_0}$  is the sub-vector of  $\bar{b}$  with entries in  $I_0$  and

$$\begin{aligned} x_i(s) &= \sqrt{\theta_i} \left( \psi_i(s) + \frac{\bar{b}_i}{2\theta_i} \right) = \sqrt{\theta_i} v_i^\top \left( s + \frac{b}{2\theta_i} \right), & \forall i \in I_+, \\ y_i(s) &= \sqrt{-\theta_i} \left( \psi_i(s) + \frac{\bar{b}_i}{2\theta_i} \right) = \sqrt{-\theta_i} v_i^\top \left( s + \frac{b}{2\theta_i} \right), & \forall i \in I_-, \\ z_i(s) &= \psi_i(s) = v_i^\top s & \forall i \in I_0, \\ \kappa &= c - \frac{1}{4} \sum_{i \in I_+ \cup I_-} \frac{\bar{b}_i^2}{\theta_i} = c - \frac{1}{4} \sum_{i \in I_+ \cup I_-} \frac{(v_i^\top b)^2}{\theta_i}, \end{aligned}$$

where  $v_i$  is the  $i$ -th eigenvector, that is, the  $i$ -th column of  $V$ .

**Remark 2.** If the quadratic expression  $s^\top Qs + b^\top s + c$  has some purely linear variables, i.e., it can be written as  $s_q^\top Q_q s_q + b_q^\top s_q + b_l^\top s_l + c$ , then we only need

to compute the factorization of  $Q_q$  and we can just place  $s_l$  as the definition of  $z(s)$ . This is correct since in such a case,

$$Q = \begin{pmatrix} Q_q & 0 \\ 0 & 0 \end{pmatrix}$$

and its eigenvalue decomposition is

$$Q = \begin{pmatrix} V_q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Theta_q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_q & 0 \\ 0 & I \end{pmatrix}^\top.$$

□

**Remark 3.** We would like to point out that using an eigenvalue decomposition is not crucial. Other factorizations of the  $Q$  matrix can have the same effect, and can potentially lead to other maximal quadratic-free sets. We chose the eigenvalue decomposition since it can be computed efficiently, and it is available within SCIP without extra computations. □

Let us assume now that we have a point  $\bar{s} \notin S$ . In the following, we construct a maximal  $S$ -free containing  $\bar{s}$  by distinguishing four different scenarios of  $S$ .

**Case 1:**  $\bar{b}_{I_0} = 0$  and  $\kappa = 0$ . In this case,  $S$  simplifies to

$$S = \{s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 \leq 0\}.$$

Note that since  $\bar{s} \notin S$ ,

$$\|x(\bar{s})\|^2 > \|y(\bar{s})\|^2 \geq 0.$$

Given that the map  $s \rightarrow (x(s), y(s), z(s))$  is affine and invertible, Theorem 1 along with Proposition 1 imply that

$$\mathcal{C} = \left\{ s \in \mathbb{R}^p : \frac{x(\bar{s})^\top}{\|x(\bar{s})\|} x(s) \geq \|y(s)\| \right\}$$

is a maximal  $S$ -free set and contains  $\bar{s}$  in its interior.

**Case 2:**  $\bar{b}_{I_0} = 0$  and  $\kappa > 0$ . In this case we homogenize the quadratic expression using a new variable  $\zeta$ .

$$\begin{aligned} S &= \{s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 + \kappa \leq 0\}. \\ &= \left\{ s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 + \zeta^2 \leq 0, \frac{\zeta}{\sqrt{\kappa}} = 1 \right\}. \end{aligned}$$

Let

$$\hat{x}(s) = \frac{x(s)}{\sqrt{\kappa}}, \quad \hat{y}(s) = \frac{y(s)}{\sqrt{\kappa}}, \quad \text{and} \quad \hat{\zeta} = \frac{\zeta}{\sqrt{\kappa}}.$$

Then, the following reformulation of  $S$  has the form  $S^g$ :

$$S = \{s \in \mathbb{R}^p : \|(\hat{x}(s), \hat{\zeta})\|^2 - \|\hat{y}(s)\|^2 \leq 0, a^\top(\hat{x}(s), \hat{\zeta}) + d^\top \hat{y}(s) = -1\},$$

where  $a = -e_{p_++1}$ ,  $d = 0$ , and  $p_+ = |I_+|$ . Note that in this case  $\|a\| = 1 > \|d\| = 0$ . Since the map  $s \rightarrow (\hat{x}(z), \hat{y}(z), z(s), 1)$  is affine and one-to-one, Theorem 2 and Proposition 1 imply that

$$\mathcal{C} = \left\{ s : \begin{array}{ll} \|\hat{y}(s)\| \leq \lambda^\top(\hat{x}(s), 1) & \text{if } \lambda^\top a \|\hat{y}(s)\| + d^\top \hat{y}(s) \leq 0 \\ \phi_{\lambda,a,d} \left( \hat{y}(s) - \frac{d}{1 - \|d\|^2} \right) \leq \lambda^\top \left( \hat{x}(s) + \frac{a}{1 - \|d\|^2} \right) & \text{otherwise} \end{array} \right\},$$

where  $\lambda = \frac{(\hat{x}(\bar{s}), 1)}{\|(\hat{x}(\bar{s}), 1)\|} = \frac{(x(\bar{s}), \sqrt{\kappa})}{\|(x(\bar{s}), \sqrt{\kappa})\|}$ , is a maximal  $S$ -free set and contains  $\bar{s}$  in its interior.

We can greatly simplify this expression and obtain a computationally more appealing description of  $\mathcal{C}$ . First, note that  $\lambda^\top a = -\lambda_{p_++1} = -\frac{\sqrt{\kappa}}{\|(x(\bar{s}), \sqrt{\kappa})\|}$ . Thus,  $\lambda^\top a < 0$ . Since  $d = 0$ , we conclude  $\lambda^\top a \|\hat{y}(s)\| + d^\top \hat{y}(s) \leq 0$  for every  $s$ . This implies that

$$\mathcal{C} = \{s \in \mathbb{R}^p : \phi_{\lambda,a,d}(\hat{y}(s)) \leq \lambda^\top(\hat{x}(s), 1)\}.$$

Furthermore, as

$$\phi_{\lambda,a,d}(y) = \begin{cases} \|y\|, & \text{if } \lambda^\top a \|y\| + d^\top y \leq 0 \\ \sqrt{(\|y\|^2 - (d^\top y)^2)(1 - (\lambda^\top a)^2)} - d^\top y \lambda^\top a, & \text{otherwise.} \end{cases}$$

a similar analysis allows us to conclude that  $\phi_{\lambda,a,d}(y) = \|y\|$ . Hence,

$$\mathcal{C} = \{s \in \mathbb{R}^p : \|\hat{y}(s)\| \leq \lambda^\top(\hat{x}(s), 1)\} = \{s \in \mathbb{R}^p : \|y(s)\| \leq \lambda^\top(x(s), \sqrt{\kappa})\}$$

is a maximal  $S$ -free set.

**Case 3:**  $\bar{b}_{I_0} = 0$  and  $\kappa < 0$ . This case requires a similar homogenization to the last case, although the sign of  $\kappa$  yields a different expression of the maximal  $S$ -free set. As before, we rewrite  $S$  using an additional variable  $\zeta$

$$\begin{aligned} S &= \{s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 + \kappa \leq 0\}. \\ &= \{s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 - \zeta^2 \leq 0, \frac{\zeta}{\sqrt{-\kappa}} = 1\}. \end{aligned}$$

Then,

$$S = \{s \in \mathbb{R}^p : \|\hat{x}(s)\|^2 - \|(\hat{y}(s), \hat{\zeta})\|^2 \leq 0, a^\top \hat{x}(s) + d^\top(\hat{y}(s), \hat{\zeta}) = -1\},$$

where  $a = 0$ ,  $d = -e_{p_-+1}$ ,  $p_- = |I_-|$  and

$$\hat{x}(s) = \frac{x(s)}{\sqrt{-\kappa}}, \quad \hat{y}(s) = \frac{y(s)}{\sqrt{-\kappa}}, \quad \text{and } \hat{\zeta} = \frac{\zeta}{\sqrt{-\kappa}}.$$

In this case  $\|d\| = 1 > \|a\| = 0$ . Since the map  $s \rightarrow (\hat{x}(s), \hat{y}(s), z(s), 1)$  is affine and one-to-one, Theorem 2 and Proposition 1 imply that a maximal  $S$ -free set is given by

$$\mathcal{C} = \{s \in \mathbb{R}^p : \phi_{\lambda,a,d}((\hat{y}(s), 1)) \leq \lambda^\top \hat{x}(s)\},$$

where  $\lambda = \frac{\hat{x}(\bar{s})}{\|\hat{x}(\bar{s})\|} = \frac{x(\bar{s})}{\|x(\bar{s})\|}$ . As before, using the definition of  $\phi_{\lambda,a,d}$  we can greatly simplify this expression. First, note that  $d^\top(\hat{y}(s), 1) = -1 < 0$ . Since  $a = 0$ , we conclude  $\lambda^\top a \|\hat{y}(s), 1\| + d^\top(\hat{y}(s), 1) \leq 0$  for every  $s$ . This implies that

$$\mathcal{C} = \{s \in \mathbb{R}^p : \|(\hat{y}(s), 1)\| \leq \lambda^\top \hat{x}(s)\} = \{s \in \mathbb{R}^p : \|(y(s), \sqrt{-\kappa})\| \leq \lambda^\top x(s)\}.$$

**Case 4:**  $\bar{b}_{I_0} \neq 0$ . In this case, we rely on the following fact:

**Lemma 3.** *Let*

$$\begin{aligned} S_1 &= \{(x, y, z) \in \mathbb{R}^{n+m+l} : \|x\|^2 - \|y\|^2 + \beta^\top z + \gamma \leq 0\}, \\ S_2 &= \{(x, y, w) \in \mathbb{R}^{n+m+1} : \|x\|^2 - \|y\|^2 + w + \gamma \leq 0\}, \end{aligned}$$

with  $\beta \neq 0$  and consider a maximal  $S_2$ -free set  $C_2$ . Then

$$C_1 := \{(x, y, z) \in \mathbb{R}^{n+m+l} : (x, y, \beta^\top z) \in C_2\}$$

is maximal  $S_1$ -free.

*Proof.* The set  $C_1$  is clearly  $S_1$ -free. Maximality follows from the projection argument in [28, Theorem 3.6], as the set  $C_1$  has a lineality space given by  $\{0\} \times \{0\} \times \langle \beta \rangle^\perp$   $\square$

This last lemma, and the fact that  $\bar{b}_{I_0} \neq 0$ , motivate the following definition

$$w(s) := (\bar{b}_{I_0})^\top z(s)$$

We rewrite  $S$  using  $w(s)$  and homogenize

$$\begin{aligned} S &= \{s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 + w(s) + \kappa \leq 0\}. \\ &= \{s \in \mathbb{R}^p : \|x(s)\|^2 - \|y(s)\|^2 + w(s)\zeta + \kappa\zeta^2 \leq 0, \zeta = 1\}. \end{aligned}$$

Notice that

$$\begin{aligned} w(s)\zeta + \kappa\zeta^2 &= \frac{1}{4\sqrt{1+\kappa^2}}(w(s) + (\kappa + \sqrt{1+\kappa^2})\zeta)^2 \\ &\quad - \frac{1}{4\sqrt{1+\kappa^2}}(w(s) + (\kappa - \sqrt{1+\kappa^2})\zeta)^2. \end{aligned}$$

Let

$$\begin{aligned} \hat{x}(s) &= \frac{1}{\sqrt[4]{1+\kappa^2}} \left( x(s), \frac{1}{2\sqrt[4]{1+\kappa^2}}(w(s) + (\kappa + \sqrt{1+\kappa^2})\zeta) \right) \\ \hat{y}(s) &= \frac{1}{\sqrt[4]{1+\kappa^2}} \left( y(s), \frac{1}{2\sqrt[4]{1+\kappa^2}}(w(s) + (\kappa - \sqrt{1+\kappa^2})\zeta) \right) \end{aligned}$$

Noting that

$$\zeta = e_{p+1}^\top \hat{x} - e_{p-1}^\top \hat{y}$$

we obtain the following representation of  $S$

$$S = \{s \in \mathbb{R}^p : \|\hat{x}(s)\|^2 - \|\hat{y}(s)\|^2 \leq 0, a^\top \hat{x}(s) + d^\top \hat{y}(s) = -1\},$$

where  $a = -e_{p_++1}$ ,  $d = e_{p_-+1}$ ,  $p_+ = |I_+|$ , and  $p_- = |I_-|$ . At this point we would like to summarize how we use the construction on [28], since we have accumulated several transformations. The sequence of variable changes can be represented as

$$s \xrightarrow{\text{Aff. inv.}} (x, y, z) \xrightarrow{\text{Lemma 3}} (x, y, w) \xrightarrow{\text{Homog.}} (x, y, w, \zeta) \xrightarrow{\text{Linear inv.}} (\hat{x}, \hat{y})$$

In all affine one-to-one transformations, maximality clearly follows through. We remark that mapping a set to a homogenized version of it is an affine and one-to-one transformation. The only transformation that is not one-to-one is the one given by Lemma 3, but in this case simply replacing the  $w$  variable with the corresponding expression involving  $z$  suffices to maintain maximality.

In summary, Theorem 2 implies that a maximal  $S$ -free set is given by

$$\mathcal{C} = \{s \in \mathbb{R}^p : \phi_{\lambda,a,d}(\hat{y}(s)) \leq \lambda^\top \hat{x}(s)\},$$

where  $\lambda = \frac{\hat{x}(\bar{s})}{\|\hat{x}(\bar{s})\|}$ . From the definition of  $a$  and  $d$ , we have

$$\phi_{\lambda,a,d}(y) = \begin{cases} \|y\|, & \text{if } -\lambda_{p_++1}\|y\| + y_{p_-+1} \leq 0 \\ \frac{\|y\|}{\sqrt{(1 - \lambda_{p_++1}^2)(\|y\|^2 - y_{p_-+1}^2)} + \lambda_{p_++1}y_{p_-+1}}, & \text{otherwise.} \end{cases}$$

Note that  $\sqrt{\|\hat{y}(s)\|^2 - \hat{y}_{p_-+1}(z)^2} = \frac{\|y(s)\|}{\sqrt{1+\kappa^2}}$  and  $\sqrt{1 - \lambda_{p_++1}^2} = \frac{1}{\sqrt{1+\kappa^2}} \frac{\|x(\bar{s})\|}{\|\hat{x}(\bar{s})\|}$ , thus

$$\mathcal{C} = \left\{ s : \begin{cases} \|\hat{y}(s)\| \leq \lambda^\top \hat{x}(s), & \text{if } -\lambda_{p_++1}\|\hat{y}(s)\| + \hat{y}_{p_-+1}(s) \leq 0 \\ \frac{\|x(\bar{s})\|}{\sqrt{1+\kappa^2}}\|y(s)\| + \hat{x}_{p_++1}(\bar{s})\hat{y}_{p_-+1}(s) \leq \hat{x}(\bar{s})^\top \hat{x}(s), & \text{otherwise} \end{cases} \right\}.$$

### 3.1 Implied quadratics in an extended space

As we mentioned in the introduction, we additionally incorporated cutting planes obtained with our approach using *implied quadratic constraints* of an extended formulation. Most LP relaxations for QCQPs involve linearizing bilinear terms  $x_i x_j$  using a new variable  $X_{i,j} = x_i x_j$ , therefore these variables must satisfy the following quadratic equality:

$$X_{i_1,j_1} X_{i_2,j_2} = X_{i_1,j_2} X_{i_2,j_1}. \quad (15)$$

We interpret (15) as two inequalities which fall into our Case 1 above. Whenever  $i_1 \neq j_1$ ,  $i_1 \neq j_2$ ,  $i_2 \neq j_1$  and  $i_2 \neq j_2$ , the maximal quadratic-free set we construct, which is based on  $C_\lambda$  (11), is exactly one of the maximal outer-product-free sets constructed by Bienstock et al. [8, 7].

On the other hand if, for instance,  $i_1 = j_1$ , Bienstock et al. show that the (15)-free we would construct with  $C_\lambda$  is not maximal unless  $\lambda$  is chosen appropriately.



From the perspective of maximal quadratic-free sets, we can use the extra valid inequality  $X_{i_1, j_1} \geq 0$  to enlarge the set. Consider

$$\begin{aligned} S_M &:= \{s \in \mathbb{R}^4 : s_1 s_2 - s_3 s_4 \leq 0, s_1 \geq 0\} \\ &= \{s \in \mathbb{R}^4 : \|x(s)\| - \|y(s)\| \leq 0, a^\top x(s) + d^\top y(s) \leq 0\} \end{aligned}$$

where  $x(s) = (s_1 + s_2, s_3 - s_4)$ ,  $y(s) = (s_1 - s_2, s_3 + s_4)$ ,  $a = (-1, 0)$  and  $d = (-1, 0)$ . The last description is of the form  $S_{\leq 0}$ , and we thus construct a maximal  $S_M$ -free set from our discussion in Remark 1. This construction is analogous to Case 4 above, and we can thus use the same upcoming computations for Case 4 in order to handle these special sets.

This completes the description of our maximal  $S$ -free sets. In what follows, we provide the closed-form expressions for computing the cutting planes obtained from each one of these sets.

## 4 Intersection cuts coefficients and their strengthening

### 4.1 Cut coefficient computations

All our maximal  $S$ -free sets are described as  $\mathcal{C} = \{s : g(s) \leq 0\}$ . For a ray  $r$  of the simplicial conic relaxation obtained from a basis associated to  $\bar{s}$ , the cut coefficient of the non-basic variable associated to  $r$  is found through the smallest  $t > 0$  such that  $g(\bar{s} + tr) = 0$  (recall that  $g(\bar{s}) < 0$ ). As described in Section 2.1, if  $t^*$  is such a root (the *step-length*), the cut coefficient is  $1/t^*$ . If no such  $t$  exists, then  $t^* = \infty$  and the cut coefficient is 0. For convenience, we uniformly use the term “smallest positive root”, defining it as  $\infty$  if none exists and consider  $1/\infty := 0$ . Here we make explicit what needs to be computed in order to find  $t^*$ .

**Case 1:  $\bar{b}_{I_0} = 0$  and  $\kappa = 0$ .** In this case the maximal  $S$ -free set is given by

$$\mathcal{C} = \{s \in \mathbb{R}^p : \frac{x(\bar{s})^\top}{\|x(\bar{s})\|} x(s) \geq \|y(s)\|\}.$$

Then, we need to find the smallest positive solution to

$$\|y(\bar{s} + tr)\| - \frac{x(\bar{s})^\top}{\|x(\bar{s})\|} x(\bar{s} + tr) = 0. \quad (16)$$

Note that this yields an equation of the form

$$\sqrt{A_r t^2 + B_r t + C_r} - (D_r t + E_r) = 0.$$

Let  $\tau(t) = \sqrt{A_r t^2 + B_r t + C_r} - (D_r t + E_r)$ . By construction,  $\tau$  is convex. Since  $\bar{s} \in \text{int } \mathcal{C}$ , we have that  $\tau(0) < 0$ , that is,

$$\sqrt{C_r} - E_r < 0. \quad (17)$$

This implies, in particular, that there is at most one positive root. Additionally, it is not hard to see that

$$\lim_{t \rightarrow \infty} \tau(t) = \begin{cases} \infty, & \text{if } \sqrt{A_r} > D_r \\ -\infty, & \text{if } \sqrt{A_r} < D_r \\ \frac{B_r}{2\sqrt{A_r}} - E_r, & \text{if } \sqrt{A_r} = D_r. \end{cases}$$

We claim that the limit in the case  $\sqrt{A_r} = D_r$  must be negative. To see this, note that since the radicand  $A_r t^2 + B_r t + C_r$  is always nonnegative, its minimum value is nonnegative as well, that is,  $C_r \geq \frac{B_r^2}{4A_r}$ . This implies that  $\sqrt{C_r} \geq \frac{B_r}{2\sqrt{A_r}}$ . Therefore,

$$\frac{B_r}{2\sqrt{A_r}} - E_r < \frac{B_r}{2\sqrt{A_r}} - \sqrt{C_r} \leq 0,$$

where the first inequality follows from (17).

From this, we conclude that  $\tau$  has a positive root if and only if  $\sqrt{A_r} - D_r > 0$ .

Assume that  $\sqrt{A_r} - D_r > 0$  and let  $\bar{t}$  be the positive root. Since  $\tau(t) < 0$  for all  $t \in [0, \bar{t})$ ,  $\sqrt{A_r t^2 + B_r t + C_r} < (D_r t + E_r)$  for all  $t \in [0, \bar{t})$ . Thus,

$$0 \leq 2\sqrt{A_r t^2 + B_r t + C_r} < \sqrt{A_r t^2 + B_r t + C_r} + (D_r t + E_r), \quad \forall t \in [0, \bar{t}).$$

Hence, the first positive root of

$$\tau(t)(\sqrt{A_r t^2 + B_r t + C_r} + (D_r t + E_r)) = A_r t^2 + B_r t + C_r - (D_r t + E_r)^2$$

is  $\bar{t}$ . This means that we can find  $\bar{t}$  by finding the smallest positive root of the last quadratic equation.

We now find the coefficients of the equation we need to solve. Recall that,

$$\begin{aligned} x(s) &= \left( \sqrt{\theta_i} v_i^\top \left( s + \frac{b}{2\theta_i} \right) \right)_{i \in I_+} \\ y(s) &= \left( \sqrt{-\theta_i} v_i^\top \left( s + \frac{b}{2\theta_i} \right) \right)_{i \in I_-}. \end{aligned}$$

Thus,

$$\begin{aligned} x(\bar{s} + tr) &= x(\bar{s}) + t \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} \\ y(\bar{s} + tr) &= y(\bar{s}) + t \left( \sqrt{-\theta_i} v_i^\top r \right)_{i \in I_-} \end{aligned}$$

With this, we compute the components of (16).

$$\begin{aligned} \|y(\bar{s} + tr)\|^2 &= \left\| y(\bar{s}) + t \left( \sqrt{-\theta_i} v_i^\top r \right)_{i \in I_-} \right\|^2 \\ &= \|y(\bar{s})\|^2 + 2ty(\bar{s})^\top \left( \sqrt{-\theta_i} v_i^\top r \right)_{i \in I_-} + t^2 \left\| \left( \sqrt{-\theta_i} v_i^\top r \right)_{i \in I_-} \right\|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} x(\bar{s})^\top x(\bar{s} + tr) &= x(\bar{s})^\top \left( x(\bar{s}) + t \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} \right) \\ &= tx(\bar{s})^\top \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} + \|x(\bar{s})\|^2, \end{aligned}$$

from where we obtain

$$\frac{x(\bar{s})^\top}{\|x(\bar{s})\|} x(\bar{s} + tr) = t \frac{x(\bar{s})^\top}{\|x(\bar{s})\|} \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} + \|x(\bar{s})\|.$$

Altogether, we see that

$$A_r = \left\| \left( \sqrt{-\theta_i} v_i^\top r \right)_{i \in I_-} \right\|^2 = - \sum_{i \in I_-} \theta_i (v_i^\top r)^2 \quad (18a)$$

$$B_r = 2y(\bar{s})^\top \left( \sqrt{-\theta_i} v_i^\top r \right)_{i \in I_-} = -2 \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (18b)$$

$$C_r = \|y(\bar{s})\|^2 = - \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2 \quad (18c)$$

$$D_r = \frac{x(\bar{s})^\top}{\|x(\bar{s})\|} \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} = \frac{1}{E} \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (18d)$$

$$E_r = \|x(\bar{s})\| = \sqrt{\sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2} \quad (18e)$$

**Remark 4.** Computing these coefficients can be done efficiently. For the coefficients of the quadratic equation, we just need to compute and store  $v_i^\top r$ ,  $v_i^\top \bar{s}$ , and  $v_i^\top b$  for all  $i \in I_+ \cup I_-$ . Basically,  $V^\top r$ ,  $V^\top \bar{s}$ , and  $V^\top b$ . After computing these coefficients, we just need to compute the roots of a single-variable quadratic.  $\square$

We summarize our discussion in the following lemma.

**Lemma 4.** *Consider  $S$  and  $\bar{s}$  as defined above and  $r$  an arbitrary ray. The step-length associated to  $r$  for Case 1 is the smallest positive root of a single-variable quadratic equation of the form*

$$A_r t^2 + B_r t + C_r - (D_r t + E_r)^2 = 0, \quad (19)$$

where the coefficients  $A_r, B_r, C_r, D_r, E_r$  for each case are displayed in (18). Equation (19) has at most one positive root, and it has no such root if and only if  $\sqrt{A_r} \leq D_r$ .

**Case 2:**  $\bar{b}_{I_0} = 0$  and  $\kappa > 0$ . In this case, we showed that a maximal  $S$ -free set is given by

$$\mathcal{C} = \left\{ s : \|y(s)\| \leq \frac{(x(\bar{s}), \sqrt{\kappa})^\top}{\|(x(\bar{s}), \sqrt{\kappa})\|} (x(s), \sqrt{\kappa}) \right\}.$$

Then, we need to find the smallest positive  $t$  such that

$$\|y(\bar{s} + tr)\| - \frac{(x(\bar{s}), \sqrt{\kappa})^\top}{\|(x(\bar{s}), \sqrt{\kappa})\|} (x(\bar{s} + tr), \sqrt{\kappa}) = 0.$$

which is again an equation of the form

$$\sqrt{A_r t^2 + B_r t + C_r} - (D_r t + E_r) = 0,$$

as in the previous case. The same reasoning applies, which shows that we can focus on the case  $\sqrt{A_r} > D_r$  and compute the smallest positive root of

$$A_r t^2 + B_r t + C_r - (D_r t + E_r)^2$$

Actually, it is not hard to see that the  $A_r, B_r,$  and  $C_r$  coefficients are the same as in the previous case. For the rest, we compute the following

$$\begin{aligned} (x(\bar{s}), \sqrt{\kappa})^\top (x(\bar{s} + tr), \sqrt{\kappa}) &= x(\bar{s})^\top x(\bar{s} + tr) + \kappa \\ &= tx(\bar{s})^\top \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} + \|x(\bar{s})\|^2 + \kappa \end{aligned}$$

thus,

$$\frac{(x(\bar{s}), \sqrt{\kappa})^\top}{\|(x(\bar{s}), \sqrt{\kappa})\|} (x(\bar{s} + tr), \sqrt{\kappa}) = t \frac{x(\bar{s})^\top \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+}}{\|(x(\bar{s}), \sqrt{\kappa})\|} + \|(x(\bar{s}), \sqrt{\kappa})\|.$$

Collecting all terms we obtain that in this case

$$A_r = - \sum_{i \in I_-} \theta_i (v_i^\top r)^2 \quad (20a)$$

$$B_r = -2 \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (20b)$$

$$C_r = - \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2 \quad (20c)$$

$$D_r = \frac{1}{E} \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (20d)$$

$$E_r = \|(x(\bar{s}), \sqrt{\kappa})\| = \sqrt{\kappa + \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2}. \quad (20e)$$

Similarly to the previous case we summarize our discussion in the following lemma.

**Lemma 5.** *Consider  $S$  and  $\bar{s}$  as defined above and  $r$  an arbitrary ray. The step-length associated to  $r$  for Case 2 is obtained as the smallest positive root of a single-variable quadratic equation of the form (19) where the coefficients  $A_r, B_r, C_r, D_r, E_r$  for each case are displayed in (20). Equation (19) has at most one positive root, and it has no such root if and only if  $\sqrt{A_r} \leq D_r$ .*

**Case 3:**  $\bar{b}_{I_0} = 0$  and  $\kappa < 0$ . In this case,

$$\mathcal{C} = \left\{ s : \|(y(s), \sqrt{-\kappa})\| \leq \frac{x(\bar{s})^\top}{\|x(\bar{s})\|} x(s) \right\}.$$

As before, we need to find the smallest positive solution of

$$\|(y(\bar{s} + tr), \sqrt{-\kappa})\| - \frac{x(\bar{s})^\top}{\|x(\bar{s})\|} x(\bar{s} + tr) = 0.$$

This case is almost identical as the previous case. Indeed, only the expressions defining the  $C$  and  $E$  coefficients change. As

$$\|(y(\bar{s} + tr), \sqrt{-\kappa})\|^2 = \|y(\bar{s} + tr)\|^2 - \kappa,$$

we have that

$$A_r = - \sum_{i \in I_-} \theta_i (v_i^\top r)^2 \quad (21a)$$

$$B_r = -2 \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (21b)$$

$$C_r = -\kappa - \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2 \quad (21c)$$

$$D_r = \frac{1}{E} \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (21d)$$

$$E_r = \sqrt{\sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2} \quad (21e)$$

As before, a summarizing lemma follows:

**Lemma 6.** *Consider  $S$  and  $\bar{s}$  as defined above and  $r$  an arbitrary ray. The step-length associated to  $r$  for Case 3 is the smallest positive root of a single-variable quadratic equation of the form (19) where the coefficients  $A_r, B_r, C_r, D_r, E_r$  for each case are displayed in (21). Equation (19) has at most one positive root, and it has no such root if and only if  $\sqrt{A_r} \leq D_r$ .*

**Case 4:**  $\bar{b}_{I_0} \neq 0$ . In this case, we constructed the following  $S$ -free set

$$\mathcal{C} = \{ s \in \mathbb{R}^p : \phi_{\lambda, a, d}(\hat{y}(s)) \leq \lambda^\top \hat{x}(s) \},$$

where  $\lambda = \frac{\hat{x}(\bar{s})}{\|\hat{x}(\bar{s})\|}$  and

$$\phi_{\lambda, a, d}(y) = \begin{cases} \|y\|, & \text{if } -\lambda_{p+1}\|y\| + y_{p-+1} \leq 0 \\ \sqrt{(1 - \lambda_{p+1}^2)(\|y\|^2 - y_{p-+1}^2)} + \lambda_{p+1}y_{p-+1}, & \text{otherwise.} \end{cases}$$

We also obtained the equivalent description,

$$\mathcal{C} = \left\{ s : \begin{array}{l} \|\hat{y}(s)\| \leq \lambda^\top \hat{x}(s), \text{ if } -\lambda_{p+1} \|\hat{y}(s)\| + \hat{y}_{p+1}(s) \leq 0 \\ \frac{\|x(\bar{s})\|}{\sqrt{1+\kappa^2}} \|y(s)\| + \hat{x}_{p+1}(\bar{s}) \hat{y}_{p+1}(s) \leq \hat{x}(\bar{s})^\top \hat{x}(s), \text{ otherwise} \end{array} \right\}. \quad (22)$$

We begin our computations using the first description. We need to find the smallest positive root of  $\phi_{\lambda,a,d}(\hat{y}(\bar{s} + tr)) - \lambda^\top \hat{x}(\bar{s} + tr)$ . This function is convex and negative at  $t = 0$ , thus, it has at most one such root. Let us consider the following two equations (one associated to each piece in the definition of  $\mathcal{C}$ ):

$$\|\hat{y}(\bar{s} + tr)\| - \lambda^\top \hat{x}(\bar{s} + tr) = 0 \quad (23)$$

$$\frac{\|x(\bar{s})\|}{\sqrt{1+\kappa^2}} \|y(\bar{s} + tr)\| + \hat{x}_{p+1}(\bar{s}) \hat{y}_{p+1}(\bar{s} + tr) - \hat{x}(\bar{s})^\top \hat{x}(\bar{s} + tr) = 0. \quad (24)$$

The following lemma indicates how to orderly use both parts in the definition of  $\mathcal{C}$  to compute the desired cut coefficient based on the solutions to (23) and (24).

**Lemma 7.** *Both (23) and (24) have at most one positive solution. If (23) does not have a positive solution, neither does (24). If (23) has a positive solution  $\bar{t}_1$ , then it is the desired step-length if and only if*

$$-\lambda_{p+1} \|\hat{y}(\bar{s} + \bar{t}_1 r)\| + \hat{y}_{p+1}(\bar{s} + \bar{t}_1 r) \leq 0. \quad (25)$$

*Lastly, if (25) does not hold, the smallest positive root  $\bar{t}_2$  of (24) is the desired step-length.*

*Proof.* First, we note that both (23) and (24) have at most one root since the left-hand side functions are convex and negative at  $t = 0$ . By Lemma 2, we know that  $\phi_{\lambda,a,d}(y) \leq \|y\|$ , thus, if  $\|\hat{y}(\bar{s} + tr)\| - \lambda^\top \hat{x}(\bar{s} + tr)$  does not have a positive root, neither does  $\phi_{\lambda,a,d}(\hat{y}(\bar{s} + tr)) - \lambda^\top \hat{x}(\bar{s} + tr)$ .

Now assume that  $\bar{t}_1$  is a non-negative root of  $\|\hat{y}(\bar{s} + tr)\| - \lambda^\top \hat{x}(\bar{s} + tr)$ . Using the strict inequality in Lemma 2,  $\bar{t}_1$  is also root of  $\phi_{\lambda,a,d}(\hat{y}(\bar{s} + tr)) - \lambda^\top \hat{x}(\bar{s} + tr)$  if and only if (25) holds.

If case (25) does not hold, we must compute the positive root using the second case in the definition of  $\phi_{\lambda,a,d}$ , which yields (24). Assume that  $\bar{t}_2$  is such root. We can immediately conclude that  $\bar{t}_2$  is the root of  $\phi_{\lambda,a,d}(\hat{y}(\bar{s} + tr)) - \lambda^\top \hat{x}(\bar{s} + tr)$  and that  $\bar{t}_2 > \bar{t}_1$ . This follows by Lemma 2, since  $\phi_{\lambda,a,d}(\hat{y}(\bar{s} + \bar{t}_1 r)) < 0$  when (25) does not hold.  $\square$

In conclusion, we just need to find two positive roots, one for each part in the definition of  $\mathcal{C}$ , and take the smallest one that is valid.

Let us look at the equation induced by  $\|\hat{y}(\bar{s} + tr)\| - \lambda^\top \hat{x}(\bar{s} + tr)$ . As before, this is an expression of the form

$$\sqrt{A_r t^2 + B_r t + C_r} - (D_r t + E_r)$$

for which we compute the coefficients next. We have

$$\begin{aligned}\hat{x}_{p+1}(s) &= \frac{1}{2\sqrt{1+\kappa^2}}(w(s) + \kappa + \sqrt{1+\kappa^2}) \\ \hat{y}_{p+1}(s) &= \frac{1}{2\sqrt{1+\kappa^2}}(w(s) + \kappa - \sqrt{1+\kappa^2}).\end{aligned}$$

Thus,

$$\begin{aligned}\hat{x}_{p+1}(\bar{s} + tr) &= t \left( \frac{w(r)}{2\sqrt{1+\kappa^2}} \right) + \hat{x}_{p+1}(\bar{s}) \\ \hat{y}_{p+1}(\bar{s} + tr) &= t \left( \frac{w(r)}{2\sqrt{1+\kappa^2}} \right) + \hat{y}_{p+1}(\bar{s})\end{aligned}$$

Then,

$$\begin{aligned}\|(\hat{y}(\bar{s} + tr))\|^2 &= \frac{1}{\sqrt{1+\kappa^2}} \|y(\bar{s} + tr)\|^2 + \left( t \left( \frac{w(r)}{2\sqrt{1+\kappa^2}} \right) + \hat{y}_{p+1}(\bar{s}) \right)^2 \\ &= \frac{1}{\sqrt{1+\kappa^2}} \|y(\bar{s} + tr)\|^2 + t^2 \left( \frac{w(r)^2}{4(1+\kappa^2)} \right) + 2t \left( \frac{w(r)}{2\sqrt{1+\kappa^2}} \right) \hat{y}_{p+1}(\bar{s}) + \hat{y}_{p+1}(\bar{s})^2\end{aligned}$$

Thus,

$$A_r = \frac{w(r)^2}{4(1+\kappa^2)} - \frac{1}{\sqrt{1+\kappa^2}} \sum_{i \in I_-} \theta_i (v_i^\top r)^2 \quad (26a)$$

$$B_r = 2 \left( \frac{w(r)}{2\sqrt{1+\kappa^2}} \right) \hat{y}_{p+1}(\bar{s}) - \frac{2}{\sqrt{1+\kappa^2}} \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (26b)$$

$$C_r = \hat{y}_{p+1}(\bar{s})^2 - \frac{1}{\sqrt{1+\kappa^2}} \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2 \quad (26c)$$

$$(26d)$$

For the other coefficients we compute:

$$\begin{aligned}\hat{x}(\bar{s})^\top \hat{x}(\bar{s} + tr) &= \frac{1}{\sqrt{1+\kappa^2}} x(\bar{s})^\top x(\bar{s} + tr) + \hat{x}_{p+1}(\bar{s}) \hat{x}_{p+1}(\bar{s} + tr) \\ &= \frac{1}{\sqrt{1+\kappa^2}} x(\bar{s})^\top x(\bar{s} + tr) + \hat{x}_{p+1}(\bar{s}) \left( t \left( \frac{w(r)}{2\sqrt{1+\kappa^2}} \right) + \hat{x}_{p+1}(\bar{s}) \right) \\ &= \frac{1}{\sqrt{1+\kappa^2}} \left( tx(\bar{s})^\top \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} + \|x(\bar{s})\|^2 \right) \\ &\quad + \hat{x}_{p+1}(\bar{s}) \left( t \left( \frac{w(r)}{2\sqrt{1+\kappa^2}} \right) + \hat{x}_{p+1}(\bar{s}) \right) \\ &= \frac{t}{\sqrt{1+\kappa^2}} \left( x(\bar{s})^\top \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} + \hat{x}_{p+1}(\bar{s}) \frac{w(r)}{2} \right) + \|\hat{x}(\bar{s})\|^2\end{aligned}$$

so

$$\begin{aligned} & \lambda^\top \hat{x}(\bar{s} + tr) \\ &= \frac{t}{\sqrt{1 + \kappa^2} \|\hat{x}(\bar{s})\|} \left( x(\bar{s})^\top \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} + \hat{x}_{p_+ + 1}(\bar{s}) \frac{w(r)}{2} \right) + \|\hat{x}(\bar{s})\| \end{aligned}$$

Thus,

$$\begin{aligned} D_r &= \frac{1}{E\sqrt{1 + \kappa^2}} \left( \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) + \hat{x}_{p_+ + 1}(\bar{s}) \frac{w(r)}{2} \right) \\ &= \frac{1}{E\sqrt{1 + \kappa^2}} \left( \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) + \frac{(w(\bar{s}) + \kappa + \sqrt{1 + \kappa^2})w(r)}{4\sqrt{1 + \kappa^2}} \right) \end{aligned} \quad (27a)$$

$$E_r = \frac{1}{\sqrt[4]{1 + \kappa^2}} \sqrt{\frac{(w(\bar{s}) + \kappa + \sqrt{1 + \kappa^2})^2}{4\sqrt{1 + \kappa^2}} + \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2} \quad (27b)$$

With these coefficients we obtain the root  $\bar{t}_1$ . After finding this root, we have to check whether (25) holds, that is

$$-\lambda_{p_+ + 1} \|\hat{y}(\bar{s} + \bar{t}_1 r)\| + \hat{y}_{p_- + 1}(\bar{s} + \bar{t}_1 r) \leq 0.$$

With the same  $A_r, B_r, C_r, D_r$ , and  $E_r$  we have just computed, we have

$$\|\hat{y}(\bar{s} + tr)\| = \sqrt{A_r t^2 + B_r t + C_r}$$

and

$$\begin{aligned} \hat{y}_{p_- + 1}(\bar{s} + tr) &= t \left( \frac{w(r)}{2\sqrt{1 + \kappa^2}} \right) + \hat{y}_{p_- + 1}(\bar{s}) \\ \lambda_{p_+ + 1} &= \frac{1}{E} \frac{1}{2\sqrt{1 + \kappa^2}} (w(\bar{s}) + \kappa + \sqrt{1 + \kappa^2}). \end{aligned}$$

Thus, we have to check that

$$-\frac{1}{E_r} \frac{1}{2\sqrt{1 + \kappa^2}} (w(\bar{s}) + \kappa + \sqrt{1 + \kappa^2}) \sqrt{A_r t^2 + B_r t + C_r} + t \left( \frac{w(r)}{2\sqrt{1 + \kappa^2}} \right) + \hat{y}_{p_- + 1}(\bar{s}) \leq 0.$$

for  $t = \bar{t}_1$ . As  $\hat{y}_{p_- + 1}(\bar{s}) = \frac{1}{2\sqrt{1 + \kappa^2}} (w(\bar{s}) + (\kappa - \sqrt{1 + \kappa^2}))$ , we can simplify the above inequality to

$$-\frac{1}{E_r} (w(\bar{s}) + \kappa + \sqrt{1 + \kappa^2}) \sqrt{A_r t^2 + B_r t + C_r} + t w(r) + (w(\bar{s}) + (\kappa - \sqrt{1 + \kappa^2})) \leq 0.$$

We summarize our discussion up to here in the following lemma

**Lemma 8.** *The smallest positive root  $\bar{t}_1$  of (23) can be found using the quadratic equation (19) with the coefficients displayed in (26) and (27). We have  $\bar{t}_1 = \infty$  if and only if  $\sqrt{A_r} \leq D_r$ . Additionally, (25) holds if and only if*

$$-\frac{1}{E_r} (w(\bar{s}) + \kappa + \sqrt{1 + \kappa^2}) \sqrt{A_r \bar{t}_1^2 + B_r \bar{t}_1 + C_r} + \bar{t}_1 w(r) + (w(\bar{s}) + (\kappa - \sqrt{1 + \kappa^2})) \leq 0.$$



The last computation we need to perform is the root of the second case (24) (which before we referred to as  $\bar{t}_2$ ). For this, we look at the equation

$$\frac{\|x(\bar{s})\|}{\sqrt{1+\kappa^2}} \|y(\bar{s}+tr)\| + \hat{x}_{p_++1}(\bar{s}) \hat{y}_{p_--1}(\bar{s}+tr) - \hat{x}(\bar{s})^\top \hat{x}(\bar{s}+tr) = 0.$$

We rewrite it as

$$\frac{\|x(\bar{s})\|}{\sqrt{1+\kappa^2}} \|y(\bar{s}+tr)\| - (\hat{x}(\bar{s})^\top \hat{x}(\bar{s}+tr) - \hat{x}_{p_++1}(\bar{s}) \hat{y}_{p_--1}(\bar{s}+tr)) = 0.$$

which is also an equation of the form

$$\sqrt{A_r t^2 + B_r t + C_r} - (D_r t + E_r) = 0.$$

From the term involving  $\|y(\bar{s}+tr)\|$  we see that

$$A_r = -\frac{\|x(\bar{s})\|^2}{1+\kappa^2} \sum_{i \in I_-} \theta_i (v_i^\top r)^2 \quad (28a)$$

$$B_r = -2 \frac{\|x(\bar{s})\|^2}{1+\kappa^2} \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (28b)$$

$$C_r = -\frac{\|x(\bar{s})\|^2}{1+\kappa^2} \sum_{i \in I_-} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right)^2 \quad (28c)$$

To find the remaining coefficients, we compute the term

$$\begin{aligned} & \hat{x}(\bar{s})^\top \hat{x}(\bar{s}+tr) - \hat{x}_{p_++1}(\bar{s}) \hat{y}_{p_--1}(\bar{s}+tr) \\ &= \frac{1}{\sqrt{1+\kappa^2}} x(\bar{s})^\top x(\bar{s}+tr) + \hat{x}_{p_++1}(\bar{s}) (\hat{x}_{p_++1}(\bar{s}+tr) - \hat{y}_{p_--1}(\bar{s}+tr)) \\ &= \frac{1}{\sqrt{1+\kappa^2}} x(\bar{s})^\top x(\bar{s}+tr) + \hat{x}_{p_++1}(\bar{s}) \\ &= \frac{1}{\sqrt{1+\kappa^2}} \left( tx(\bar{s})^\top \left( \sqrt{\theta_i} v_i^\top r \right)_{i \in I_+} + \|x(\bar{s})\|^2 \right) + \hat{x}_{p_++1}(\bar{s}) \end{aligned}$$

Thus,

$$D_r = \frac{1}{\sqrt{1+\kappa^2}} \sum_{i \in I_+} \theta_i \left( v_i^\top \left( \bar{s} + \frac{b}{2\theta_i} \right) \right) (v_i^\top r) \quad (29a)$$

$$E_r = \frac{1}{\sqrt{1+\kappa^2}} \left( \|x(\bar{s})\|^2 + \frac{w(\bar{s}) + \kappa + \sqrt{1+\kappa^2}}{2} \right). \quad (29b)$$

**Lemma 9.** *The smallest positive root  $\bar{t}_2$  of (24) can be found using the quadratic equation (19) with the coefficients displayed in (28) and (29). We have  $\bar{t}_2 = \infty$  if and only if  $\sqrt{A_r} \leq D_r$ .*

## 4.2 Strengthening computations

As mentioned in Section 2.1.2, when a ray  $r$  of the simplicial conic relaxation lies in  $\text{rec}(\mathcal{C})$ , the corresponding cut-coefficient is 0 and may be strengthened

using a *negative edge extension*. Let  $r^i$ ,  $i = 1, \dots, p$ , be the extreme rays of the simplicial conic relaxation, and  $\alpha_i^* \in (0, \infty]$  the step-length computed for  $r^i$ . Let  $F = \{i : \alpha_i^* < \infty\}$ . The negative edge extension computes, for each  $j \notin F$ ,

$$\rho_j = \max_{\rho < 0} \{\rho : \alpha_i^* r^i - \rho r^j \in \text{rec}(\mathcal{C}) \quad \forall i \in F\}. \quad (30)$$

and uses the cut-coefficient  $1/\rho_j < 0$  instead of 0. It is not hard to see that  $\rho_j = \min\{\rho_j^i : i \in F\}$ , where

$$\rho_j^i = \max_{\rho < 0} \{\rho : \alpha_i^* r^i - \rho r^j \in \text{rec}(\mathcal{C})\}. \quad (31)$$

Using what we describe below, we could solve (31) directly through a single-variable convex optimization problem. However, we can reformulate the problem so as to not consider all  $\rho < 0$ .

**Lemma 10.** *If  $r^i$  and  $r^j$  are linearly independent<sup>2</sup>,*

$$\rho_j^i = (\bar{\mu} - 1) \frac{\alpha_i^*}{\bar{\mu}},$$

where

$$\bar{\mu} := \max\{\mu \in [0, 1] : \mu r^i + (1 - \mu)r^j \in \text{rec}(\mathcal{C})\}. \quad (32)$$

Otherwise,  $\rho_j^i = -\alpha_i^* \|r^i\| / \|r^j\|$ .

*Proof.* The case when  $r^i$  and  $r^j$  are linearly dependent follows by noting that, since  $r^j \in \text{rec}(\mathcal{C})$  and  $r^i \notin \text{rec}(\mathcal{C})$ , it must be that

$$r^j = -\frac{\|r^j\|}{\|r^i\|} r^i.$$

and  $\rho_j^i$  must satisfy

$$\alpha_i^* r^i - \rho_j^i r^j = 0.$$

For the other case, using a rescaling argument we can see that there must exist  $\beta > 0$  such that

$$\begin{aligned} \bar{\mu} r^i + (1 - \bar{\mu}) r^j &= (\alpha_i^* r^i - \rho_j^i r^j) \beta \\ \Leftrightarrow (\bar{\mu} - \alpha_i^* \beta) r^i + (1 - \bar{\mu} + \rho_j^i \beta) r^j &= 0 \end{aligned}$$

Since  $r^i$  and  $r^j$  are linearly independent, this last equality implies

$$\bar{\mu} - \alpha_i^* \beta = 1 - \bar{\mu} + \rho_j^i \beta = 0.$$

Rearranging and combining these two equations yields

$$\rho_j^i = (\bar{\mu} - 1) \frac{\alpha_i^*}{\bar{\mu}}.$$

□

<sup>2</sup>Since we are considering rays of a simplicial cone of dimension  $p$ , they are all linearly independent. However, in practice, the set  $S$  is usually of dimension  $\ll p$ . In these cases, one can either extend the  $S$ -free set to dimension  $p$ , or restrict the rays to the support of  $S$  for computational purposes. The latter might create linear dependence.

This last lemma shifts the computation of (31) to the computation of (32). While these look similar, in our experiments the latter proved to be computationally better: the domain of the variable to optimize is bounded, which resulted in a faster and numerically more stable strengthening routine.

Since (32) is a single-variable problem over a bounded domain, we use a binary-search approach to solve it, and thus having an efficient membership oracle of  $\text{rec}(\mathcal{C})$  suffices. Using that our  $S$ -free sets have the form  $\mathcal{C} = \{s : g(s) \leq 0\}$ , from our previous discussion we further note that  $r \in \text{rec}(\mathcal{C})$  if and only if  $g(\bar{s} + tr) = 0$  has no positive solution. What follows is based on this fact.

**Cases 1, 2 and 3.** Since  $r \in \text{rec}(\mathcal{C})$  is equivalent to determining if  $g(\bar{s} + tr) = 0$  has no positive solution, we see that  $r \in \text{rec}(\mathcal{C})$  if and only if  $\sqrt{A_r} \leq D_r$  (see Lemmas 4, 5 and 6). Let us denote  $A_i$  the coefficient  $A_{r^i}$ , and similarly for the other coefficients. In order to solve (32) we show

**Lemma 11.**  $\mu r^i + (1 - \mu)r^j \in \text{rec}(\mathcal{C})$  if and only if

$$\sqrt{\mu^2 A_i + (1 - \mu)^2 A_j + 2\mu(1 - \mu) \sum_{k \in I_-} \theta_k (v_k^\top r^i)(v_k^\top r^j) - \mu D_i - (1 - \mu) D_j} \leq 0.$$

*Proof.* We know that  $r = \mu r^i + (1 - \mu)r^j \in \text{rec}(\mathcal{C})$  if and only if  $\sqrt{A_r} \leq D_r$ . From the expression of  $A_r$  and  $D_r$  in cases 1, 2 and 3, we see that

$$\begin{aligned} A_r &= \mu^2 A_i + (1 - \mu)^2 A_j + 2\mu(1 - \mu) \sum_{k \in I_-} \theta_k (v_k^\top r^i)(v_k^\top r^j) \\ D_r &= \mu D_i + (1 - \mu) D_j \end{aligned}$$

which shows the result. □

This results casts (32) as a single-variable single-constraint convex problem.

**Case 4.** In this case we have to proceed differently since  $\mathcal{C}$  is piecewise-defined. Using that each part of the definition is a convex function we can prove:

**Lemma 12.** Let  $\tau(t) = -\lambda_{p_+ + 1} \|\hat{y}(\bar{s} + tr)\| + \hat{y}_{p_- + 1}(\bar{s} + tr)$ . If  $\lim_{t \rightarrow \infty} \tau(t) > 0$ , then  $r \in \text{rec}(\mathcal{C})$  if and only if (23) has a positive root. Otherwise,  $r \in \text{rec}(\mathcal{C})$  if and only if (24) has a positive root.

This lemma frames how we can check for a ray  $r$  to be in  $\text{rec}(\mathcal{C})$ . The two following results show precisely how to verify each condition.

**Lemma 13.** Let  $A_r, B_r, C_r, D_r, E_r$  be defined as in (26) and (27), and  $\tau(t)$  as in the previous lemma. Then

$$\lim_{t \rightarrow \infty} \tau(t) = \begin{cases} \text{sgn}(-\lambda_{p_+ + 1})\infty, & \text{if } \sqrt{\bar{A}_r} > \bar{D}_r \\ \text{sgn}(\lambda_{p_+ + 1})\infty, & \text{if } \sqrt{\bar{A}_r} < \bar{D}_r \\ \text{sgn}(-\lambda_{p_+ + 1})\left(\frac{\bar{B}_r}{2\sqrt{\bar{A}_r}} - \bar{E}_r\right), & \text{if } \sqrt{\bar{A}_r} = \bar{D}_r. \end{cases}$$

where

$$\bar{A}_r := \lambda_{p_+ + 1}^2 A_r, \bar{B}_r := \lambda_{r+1}^2 B_r, \bar{C}_r := \lambda_{p_+ + 1}^2 C_r, \bar{D}_r := -\frac{w(r)}{2\sqrt{1 - \kappa^2}}, \bar{E}_r := \hat{y}_{p_+ + 1}(\bar{s}).$$

*Proof.* Recall that  $\|\hat{y}(\bar{s} + tr)\| = \sqrt{A_r t^2 + B_r t + C_r}$ . Therefore,  $\tau(t) = -\lambda_{p_+ + 1} \|\hat{y}(\bar{s} + tr)\| + \hat{y}_{p_+ + 1}(\bar{s} + tr)$  is of the form

$$\tau(t) = \sqrt{\bar{A}_r t^2 + \bar{B}_r t + \bar{C}_r} - (\bar{D}_r t + \bar{E}_r)$$

with  $\bar{A}_r, \bar{B}_r, \bar{C}_r, \bar{D}_r, \bar{E}_r$  defined in the lemma statement. Depending on the sign of  $\lambda_{r+1}$ , the function  $\tau(t)$  is either convex or concave. Hence, using the same reasoning as in Section 4.1, we obtain the desired limits.  $\square$

Finally, the only missing ingredient is the verification of when (23) or (24) have a positive root for the ray  $r = \mu r^i + (1 - \mu)r^j$ . We show

**Lemma 14.** Let  $r = \mu r^i + (1 - \lambda)r^j$ . Then, (23) has a positive root if and only if

$$\sqrt{\mu^2 A_i + (1 - \mu)^2 A_j + 2\mu(1 - \mu) \left( \frac{\|x(\bar{s})\|^2}{4(1 + \kappa)^2} \sum_{k \in I_-} \theta_k (v_k^\top r^i)(v_k^\top r^j) \right)} - \mu D_i + (1 - \mu) D_j \leq 0.$$

where the coefficients are defined as in (26) and (27). Similarly, (24) has a positive root if and only if

$$\sqrt{\mu^2 A_i + (1 - \mu)^2 A_j - 2\mu(1 - \mu) \left( \frac{\omega(r^i)\omega(r^j)}{4(1 + \kappa)^2} - \frac{1}{\sqrt{1 + \kappa^2}} \sum_{k \in I_-} \theta_k (v_k^\top r^i)(v_k^\top r^j) \right)} - \mu D_i + (1 - \mu) D_j \leq 0.$$

where the coefficients are defined as in (28) and (29).

*Proof.* The results follow from computing the condition  $\sqrt{\bar{A}_r} \leq \bar{D}_r$  in each corresponding case.  $\square$

## 5 Computational Results

In this section we show our computational experiments testing the efficacy of the cutting planes we propose. In all our cutting plane computations we use the formulas we describe in this article within the SCIP optimization suite.

To test our approach, we use what is commonly known as *root node experiments*: we start from an LP relaxation of a QCQP (providing a dual bound

Table 1: Summary of gap closed in root node experiments between intersection cuts and default SCIP. The columns *rel* denote the corresponding relative improvement with respect to default SCIP. The *#solved* row shows the number of instances solved in the root node.

subset	DEFAULT		ICUTS		ICUTS-S		MINOR	
	mean	rel	mean	rel	mean	rel	mean	rel
clean	0.56		0.61	1.08	0.60	1.07	0.59	1.04
affected	0.52		0.59	1.12	0.58	1.11	0.55	1.06
#solved	90		116		114		92	

subset	MINOR-S		ICUTS+MINOR		ICUTS+MINOR-B	
	mean	rel	mean	rel	mean	rel
clean	0.58	1.04	0.61	1.09	0.62	1.09
affected	0.55	1.06	0.60	1.14	0.60	1.15
#solved	91		116		117	

$d_1$ ) and incorporate our cutting planes to SCIP via a *separator*. After SCIP stops, we compute the *gap closed* measure of performance: if  $d_2$  is the dual bound obtained when the algorithm finishes, and  $p$  a reference primal bound, the function  $GC(p, d_1, d_2) = \frac{d_2 - d_1}{p - d_1}$  is the *gap closed* improvement of  $d_2$  with respect to  $d_1$ .

We embedded the computation of the intersection cuts in a development version of SCIP with CPLEX 12.10.0.0 as the underlying LP solver. For testing, we used a Linux cluster of Intel Xeon CPU E5-2660 v3 2.60GHz with 25MB cache and 128GB main memory. The time limit in all experiments was set to one hour. The test set we used consists of the publicly available instances of the MINLP Lib [25]. We selected all non-convex instances with at least one quadratic constraint, leaving us with 705 out of 1625 instances. Since we are interested in the amount of primal-dual gap closed, we discarded instances for which no primal solution was available (including infeasible instances) or no dual solution was found. Furthermore, we filtered-out instances where SCIP failed with at least one setting. This resulted in a test set of 587 instances.

We refer to the following settings: **DEFAULT** refers to SCIP’s default settings, **ICUTS** refers to including our intersections cuts, **MINOR** refers to including the cuts in extended space obtained from (15), **ICUTS-S** and **MINOR-S** refers to their strengthened versions. Finally, **MINOR-B** refers to cuts obtained from (15) including the non-negativity bounds in the extended space, when appropriate, as discussed in Section 3.1. Combinations of these settings are displayed with a ‘+’ sign. The overall best performing setting was **ICUTS+MINOR-B**, thus we mainly report comparisons of this setting with respect to variations of it.

In Table 1 we show summarized results for various settings. First of all, we can observe the improvement of **ICUTS+MINOR-B** with respect to **DEFAULT**. On average, we see an improvement of 8% in the gap closed. This improvement becomes 12% if we restrict to *affected* instances, i.e., instances for which at least one of the non-default settings added cutting planes. Considering the heterogeneity of these instances, these improvements are significant. Additionally, using our cutting planes SCIP was able to solve 27 more instances in the root node.

When comparing ICUTS-S and MINOR-S with their non-strengthened versions ICUTS and MINOR, respectively, we can see a (slightly) negative effect of the strengthening. This was an unexpected phenomenon, which we examined in detail on a number of instances. From these, we observe that the cut coefficients are not improved significantly in many cases. If we also consider that the strengthening increases the density of a cut, we come to the conclusion that, overall, the coefficient's modest improvement does not compensate the extra difficulties associated to a dense LP. Preliminary experiments we conducted with these cutting planes in spatial branch-and-bound support this claim: when using the strengthening routine, around 10% less LP iterations were executed per second compared to default. This means that each iteration when solving LPs gets slower. In contrast, applying the intersection cuts without the strengthening reduced the LP iterations in only 4%.

When comparing ICUTS+MINOR-B and ICUTS, as well as ICUTS+MINOR-B and MINOR we see that both cutting plane families are complementing each other well. Although most of the improvement is due to ICUTS, the contribution of MINOR-B is non-negligible. This good complementary behavior was expected: these two families are using quadratic inequalities that lie in different spaces, hence combining them provide significantly different violated constraints to separate.

In the comparison between ICUTS+MINOR-B and ICUTS+MINOR, we see that also considering the non-negativity bounds when constructing the  $S$ -free sets has a positive impact not only in theory, but also in practice.

In Figure 3 we show scatter plots comparing different settings. These plots support our previous analysis, and also show that the results are stable: if a setting improves the performance on average, the improvement is relatively consistent among the whole test set.

Overall, we believe these results are encouraging and promising for our families of cutting planes. Given the heterogeneity of the instances, and how generic the cutting planes are, the results we are obtaining advocate for our approach as a viable alternative for QCQPs.

## 6 Final remarks

In this work, we have shown an implementation of intersection cuts for QCQPs using the newly developed maximal quadratic-free sets. We show a detailed framework on how to construct cutting planes using any violated quadratic, and the necessary results showing the correctness of our computations. Our results allow for efficient cut computations that any researcher can embed in their optimization routines by simply plugging into our formulas the necessary parameters of a generic quadratic inequality.

Our careful implementation resulted in encouraging results: we were able to close more gap in a significant number of instances, and we also showed that

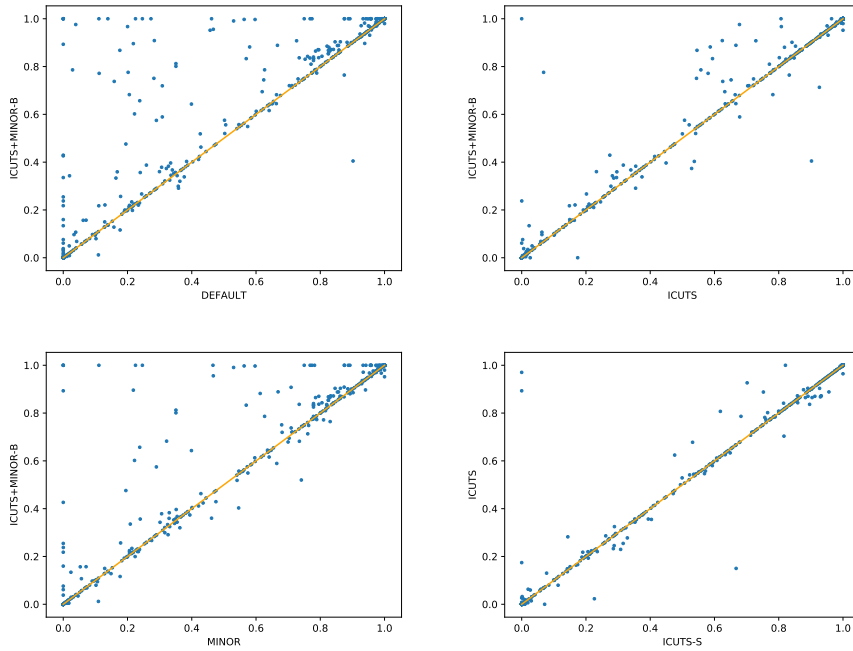


Figure 3: Scatter plots showing comparisons of gap closed in root node experiments between various pairs of settings.

these and the cuts proposed by Bienstock et al. are complementing each other well. While, unfortunately, the strengthening procedure did not yield good results, we believe it still provides valuable insights for the optimization community.

Our current and future work involves a full incorporation of these cutting planes in spatial branch-and-bound. This will require a much more careful handling of the density of the cuts we create, as well as special cut selection rules.

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