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## **Bounds for the final ranks during a round robin tournament**

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# Bounds for the final ranks during a round robin tournament

## Abstract

This article answers two kinds of questions regarding the Bundesliga which is Germany's primary football (soccer) competition having the highest average stadium attendance worldwide. First "At any point of the season, what final rank will a certain team definitely reach?" and second "At any point of the season, what final rank can a certain team at most reach?". Although we focus especially on the Bundesliga, the models that we use to answer the two questions can easily be adopted to league systems that are similar to that of the Bundesliga.

## 1 Introduction

The Bundesliga consists of 18 teams and applies a system of promotion and relegation with the 2. Bundesliga. The Bundesliga is a double round-robin tournament where each club plays two matches against every other club. One at home and one away. Nowadays a victory is worth three points and a loss none. A draw yields one point each. The team at the top of the league table at the end of the season is the German Champion. Currently, the top six to seven clubs in the table qualify for profitable international competitions organized by the Union of European Football Associations (UEFA). The two teams at the bottom of the table are relegated into the 2. Bundesliga, while the top two teams in the 2. Bundesliga are promoted. The 16th-placed team (third-last), and the third-placed team in the 2. Bundesliga play a two-leg play-off match. The winner of this match plays the next season in the Bundesliga, and the loser in the 2. Bundesliga. If teams are level on points, tie-breakers are applied (Wikipedia contributors (2019a)).

The Elimination Problem, i.e., to decide whether a given team can still win a football league at some point in the season, was shown to be NP-complete independently by Bernholt, Gülich, Hofmeister, and Schmitt (1999) and Kern and Paulusma (2001). The results were further generalized to various other score allocation rules in Kern and Paulusma (2004). In Raack, Raymond, Schlechte, and Werner (2014) a general ranking integer programming model that calculates the number of points needed to finish  $i$ th in any sport, including football, motor sports and ice hockey is developed. The authors also give a good overview of publications dealing with sport problems. Kendall, Knust, Ribeiro, and Urrutia (2010) deals with the Brazilian national football championship and the calculation of the minimum number of points any team has to win to be sure it will be qualified for the playoffs. Also a method to compute the number of points each team has

to win to have any chance to be qualified is presented. Again, a good overview of previous works is given.

Table 1: Bundesliga standings after matchday 20 season 2017 / 2018. P = position, GP = games played, W = won, D = drawn, L = lost, F = goals for, A = goals against, GD = goal difference, Pts = points

P	Team	GP	W	D	L	F	A	GD	Pts
1	FC Bayern München	20	16	2	2	49	16	33	50
2	Bayer 04 Leverkusen	20	9	7	4	41	27	14	24
3	FC Schalke 04	20	9	7	4	32	25	7	34
4	Eintracht Frankfurt	20	9	6	5	26	20	6	33
5	RB Leipzig	20	9	5	6	32	29	3	32
6	Borussia Dortmund	20	8	7	5	42	27	15	31
7	Borussia Mönchengladbach	20	9	4	7	30	32	-2	31
8	FC Augsburg	20	7	7	6	29	26	3	28
9	TSG Hoffenheim	20	7	6	7	31	32	-1	27
10	Hannover 96	20	7	6	7	28	30	-2	27
11	Hertha BSC	20	6	8	6	27	27	0	26
12	SC Freiburg	20	5	9	6	22	35	-13	24
13	VfL Wolfsburg	20	4	11	5	23	24	-1	23
14	VfB Stuttgart	20	6	2	12	16	26	-10	20
15	1. FSV Mainz 05	20	5	5	10	24	35	-11	20
16	Werder Bremen	20	3	8	9	16	25	-9	17
17	Hamburger SV	20	4	4	12	16	29	-13	16
18	1. FC Köln	20	3	4	13	15	34	-19	13

## 2 Motivation

*Der Postillon* is a German website, featuring satirical articles reporting on international, national, and local news in newspaper and TV format. In October 2015, it had more than 14 million visitors. *Der Postillon* also appears as an English version titled *The Postillon* (Wikipedia contributors (2019b)). On 29 January 2018 *Der Postillon* announced, that the Munich based German sports club *FC Bayern München*—one of the most successful European clubs internationally—celebrated non-relegation at an early stage. Euphoric FC Bayern fans were reportedly singing in the town center of Munich until the early morning (Nießen (2018)). A very similar version of this article was published on 3 February 2014 (Nießen (2014)). At the time of publication, matchday 20 (in 2018) and matchday 19 (in 2014) were over. In February 2018 we became aware of these articles in *Der Postillon*. On the one hand we were astonished that avoidance of relegation was managed at such an early stage of the league. On the other hand we were not able to verify the statement with pencil and paper. Hence, we decided to set up mixed-integer optimization models (Schrijver (1986)) to obtain a mathematical proof for the statements.

### 3 Optimization Model

In this paper, we consider a set  $T$  of  $n$  teams playing a double round-robin tournament where each team receives three points for a victory, one point for a draw, and no points for a loss. Let  $S \subseteq T \times T$  be the set of matches that have not been played yet. Here,  $(i, j) \in S$  represents the home game of team  $i$  against team  $j$ . Further, let  $S_m := \{(m, j) \in S\} \cup \{(i, m) \in S\}$  be the subset of remaining games of team  $m \in T$ . Finally, let  $p_m \in \{0, \dots, 3 \cdot (2 \cdot (n - 1) - |S_m|)\}$  be the number of points that team  $m \in T$  has already collected during the tournament. In the following, we model all possible courses of the tournament together with an implicated ranking of some fixed team  $t \in T$ .

For each  $(i, j) \in S$  we introduce three binary variables  $x_{ij}$ ,  $y_{ij}$ , and  $z_{ij}$ . Equation (1) states, that a game  $(i, j) \in S$  has to either end with a home victory of team  $i$  ( $x_{ij} = 1$ ), a draw ( $y_{ij} = 1$ ), or an away victory of team  $j$  ( $z_{ij} = 1$ ), i.e.,

$$x_{ij} + y_{ij} + z_{ij} = 1, \forall (i, j) \in S. \quad (1)$$

Additionally, for each team  $m \in T$  participating in the tournament we introduce a continuous variable  $p_m^* \in [p_m, p_m + 3 \cdot |S_m|]$  denoting its final number of points at the end of the tournament. Equation (2) is used to ensure the correct final score of team  $m \in T$  depending on the results of its remaining games:

$$p_m^* = p_m + \sum_{(m, j) \in S_m} (3 \cdot x_{mj} + y_{mj}) + \sum_{(i, m) \in S_m} (y_{im} + 3 \cdot z_{im}), \forall m \in T. \quad (2)$$

Finally, we model the ranking of a team  $t \in T$ . A team  $t$  has a higher rank in the final table than another team  $m \in T$  if it has collected more points. In case that both teams are even on points, we assume that we can arbitrarily choose their order. Hence, for each team  $m \in T \setminus \{t\}$  we add a binary variable  $c_m$  indicating whether or not  $t$  has a higher rank than  $m$ . Each of the following two classes of constraints ensure a correct ranking.

$$p_t^* - p_m^* \leq c_m \cdot (p_t - p_m + 3 \cdot |S_t|), \forall m \in T \setminus \{t\}, \quad (3a)$$

$$p_t^* - p_m^* \geq (1 - c_m) \cdot (p_t - p_m - 3 \cdot |S_m|), \forall m \in T \setminus \{t\}. \quad (3b)$$

In case that  $p_t^* - p_m^* > 0$  we have that  $c_m = 1$  due to (3a) and if  $p_t^* - p_m^* < 0$  we have  $c_m = 0$  due to (3b). In case that  $p_t^* = p_m^*$  both values are feasible, i.e., we can arbitrarily choose the order of teams  $t$  and  $m$ .

Using the variables and constraints introduced above, we are able to formulate integer programs that answer the following two questions:

1. Which position in the final table will team  $t \in T$  definitely reach?
2. Which position in the final table can team  $t \in T$  at most reach?

The model giving the answer to the first question is (4a) while the model giving the answer to the second question is (4b).

$$\max \left\{ n - \sum_{m \in T \setminus \{t\}} c_m : (1), (2), (3a) \right\} \quad (4a)$$

$$\min \left\{ n - \sum_{m \in T \setminus \{t\}} c_m : (1), (2), (3b) \right\}. \quad (4b)$$

Due to the absence of constraints (3b),  $c_m = 1$  is always feasible for model (4a) even though the final number of points of team  $t$  may be smaller than or equal to that of team  $m$ . But since we maximize  $-\sum_{m \in T \setminus \{t\}} c_m$ , i.e., minimize the number of teams that end up with strictly less points than  $t$ ,  $c_m = 1$  iff  $p_t^* > p_m^*$  holds for each optimal solution. Hence, the objective of (4a) yields the worst case final position of team  $t$ . An analogous argument shows that (4b) yields the best case final position.

Although we focus especially on the Bundesliga, the models that we use to answer the two questions can easily be adopted to league systems that are similar to that of the Bundesliga.

## 4 Computational Results

To translate the mathematical models of the problems into mathematical programs expressed in the .lp file format which can be read and solved by a MIP solver we used ZIMPL (Koch (2004)). As a MIP solver we used SCIP 5.0.1 (see Achterberg (2009), Gleixner, Eifler, Gally, Gamrath, Gemander, Gottwald, Hendel, Hojny, Koch, Miltenberger, Müller, Pfetsch, Puchert, Rehfeldt, Schlösser, Serrano, Shinano, Viernickel, Vigerske, Weninger, Witt, and Witzig (2017)) with standard settings. All models were solved on a standard Linux PC with an Intel® Core™ i5-3570 CPU @ 3.40 GHz and 16 GB RAM. We solved 14688 models, originating from standings during the seasons 2003 / 2004 – 2017 / 2018. We omitted models for matchdays that have obvious solutions, such as matchdays one to eight. The average solution time of the models was 0.024 s, while 11.84 s was the longest solution time that we observed and it took 909 solving nodes to prove optimality for the respective instance. Surprisingly this instance is just problem (4a) for *FC Bayern* on matchday 21 in 2014. Only 0.89 % of the problems were not solved in the root node.

### 4.1 Problems emerging from the motivational chapter

Now we are ready to get back to our motivating problem from section 2. Is it true that *FC Bayern* managed relegation already on matchday 20 in 2018? To answer this questions, we set up model (4a) such that  $t = \text{"Bayern\_Muenchen"}$ ,  $(p_m)_{m=1}^{18}$  as listed in the Pts-column of table 1 and  $S = \{\{\text{"Eintracht\_Frankfurt"}$ ,  $\text{"Bor\_Moenchengladbach"}$ \},  $\{\text{"Bayern\_Muenchen"}$ ,  $\text{"TSG\_Hoffenheim"}$ \},  $\dots$ ,  $\{\text{"VfL\_Wolfsburg"}$ ,  $\text{"FC\_Koeln"}$ \},  $\{\text{"FC\_Schalke\_04"}$ ,  $\text{"Hannover\_96"}$ \}\} according to the remaining schedule, see Olympia-Verlag GmbH (2019b). It turned out, that after matchday 20 was over, it was still possible to construct a course of the league such that *FC Bayern* would be second to last with 50 points, while 1. *FC Köln* becomes German Champion with 53 Points. To double-check our model, we put its results into an online standings calculator (Olympia-Verlag GmbH (2019a)), see figure 1. Analogously we found out, that in 2014 *FC Bayern* managed relegation not until matchday 20.

TABELLE

Platz	Team	Spiele	g.	u.	v.	Tore	Diff.	Pkt.	Platz	Team	Spiele	g.	u.	v.	Tore	Diff.	Pkt.
1	1. FC Köln	34	16	5	13	28:34	-6	53	10	RB Leipzig	34	15	6	13	38:36	2	51
2	Borussia Dortmund	34	15	7	12	49:34	15	52	11	TSG Hoffenheim	34	15	6	13	39:38	1	51
3	Bayer 04 Leverkusen	34	15	7	12	47:35	12	52	12	Hannover 96	34	15	6	13	36:36	0	51
4	FC Schalke 04	34	15	7	12	38:33	5	52	13	VfB Stuttgart	34	16	3	15	26:29	-3	51
5	Bor. Mönchengladbach	34	16	4	14	37:39	-2	52	14	1. FSV Mainz 05	34	15	6	13	34:38	-4	51
6	FC Augsburg	34	14	9	11	36:31	5	51	15	SC Freiburg	34	14	9	11	31:40	-9	51
7	Eintracht Frankfurt	34	15	6	13	32:28	4	51	16	Bayern München	34	16	2	16	49:30	19	50
8	VfL Wolfsburg	34	13	12	9	32:28	4	51	17	Werder Bremen	34	4	8	22	17:38	-21	20
9	Hertha BSC	34	14	9	11	35:32	3	51	18	Hamburger SV	34	5	4	25	17:42	-25	19

Figure 1: Screenshot: Verification of computed final standing with an online standings calculator (Olympia-Verlag GmbH (2019a)). Platz = position, Spiele = games played, g. = won, u. = drawn, v. = lost, Tore = goals for/against, Diff. = goal difference, Pkt. = points

## 4.2 Further illustrative problems

Additionally, we solved the models (4a) and (4b) for each team  $t$  and for every matchday of the Bundesliga season 2017/2018. Figure 2 contains four charts that visualize the results obtained by the models (4a) and (4b) during the season for the exemplarily chosen teams *FC Bayern München*, *FC Schalke 04*, *Bayer 04 Leverkusen* and *Hamburger SV*. In the first round of the season, we configured the optimization models in a way such that the final position is understood as the final position of the first round. In the chart for *FC Bayern München* the triangle symbolizing the fact that non-relegation was certain is encircled. 14 points from seven games marks the worst start to the season in seven years. For the first time in the club's history, *FC Bayern München* lost a two-goal lead twice in a row. However, already on matchday 29 *FC Bayern München* became German Champion. As the only team in the Bundesliga *Bayer 04 Leverkusen* scored at least one goal in all 17 first-round matches. The team has worked its way up in the first-round matches and showed continuity in the second round. *Hamburger SV*, finally second to last and relegated to second Bundesliga was for almost 55 years the only team in the Bundesliga that was never relegated. But on the last matchday, due to a win of *VfL Wolfsburg* against *1. FC Köln*, relegation became inevitable.

**Remark** All the models examined in this article and many more can be downloaded from [https://cloud.zib.de/s/bundesliga\\_data/download](https://cloud.zib.de/s/bundesliga_data/download).

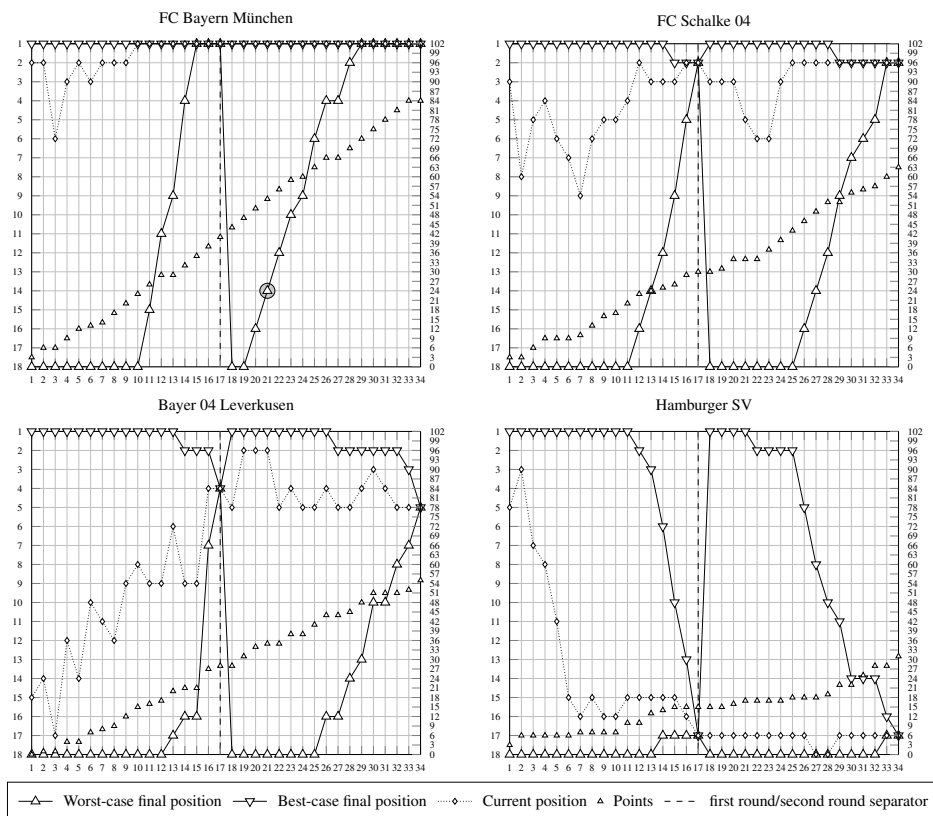


Figure 2: Results obtained by the optimization models on every matchday of the Bundesliga season 2017 / 2018 for four exemplarily chosen teams. The left y-axes are used to depict the current rank, the best case and the worst case final rank according to the respective matchday (x-axes). The right y-axes contain the current points.

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