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On the Complexity of the Maximum Minimum Cost Flow Problem

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Abstract

Consider a flow network, i.e., a directed graph where each arc has a nonnegative capacity and an associated length, together with nonempty supply-intervals for the sources and nonempty demand-intervals for the sinks. The Maximum Minimum Cost Flow Problem (MMCF) is to find fixed supply and demand values within these intervals such that the optimal objective value of the induced Minimum Cost Flow Problem (MCF) is maximized. In this paper, we show that MMCF is APX-hard and that the problem remains NP-hard in the uncapacitated case.

Introduction

In the following, we consider a flow network, i.e., a directed graph $G = (V, A)$ with node set V and arc set $A \subseteq V \times V$ where each arc $a \in A$ has an associated nonnegative capacity $c_a \in \mathbb{R}_{\geq 0}$ and length $\ell_a \in \mathbb{R}$. By $V^+ \subseteq V$ and $V^- \subseteq V$ we denote the sources and the sinks of the flow network, respectively. W.l.o.g. we assume that $V^+ \cap V^- = \emptyset$ and the remaining nodes $V^0 := V \setminus (V^+ \cup V^-)$ are called inner nodes.

Furthermore, for each source $u \in V^+$ we are given a nonempty supply interval $[\underline{b}_u, \overline{b}_u] \subseteq \mathbb{R}_{\geq 0}$, where $\underline{b}_u \in \mathbb{R}_{\geq 0}$ is a lower and $\overline{b}_u \in \mathbb{R}_{\geq 0}$ is an upper bound on u's supply. Analogously, for each sink $w \in V^-$ a nonempty demand interval $[\underline{b}_w, \overline{b}_w] \subseteq \mathbb{R}_{\leq 0}$ with lower and upper bound $\underline{b}_w, \overline{b}_w \in \mathbb{R}_{\leq 0}$ is given. In other words, we consider a HOSE-model type demand polytope, which was first introduced by Duffield et al. in the context of virtual private networks, see [5]. Finally, a vector $b \in \mathbb{R}^{|V^+ \cup V^-|}$ is called supply and demand vector if $b_v \in [\underline{b}_v, \overline{b}_v]$ for each $v \in V^+ \cup V^-$ and $\sum_{v \in V^+ \cup V^-} b_v = 0$, i.e, if supplies and demands respect the corresponding bounds and are balanced.

The Maximum Minimum Cost Flow Problem (MMCF) is to find a supply and demand vector such that the objective value of the induced Minimum Cost Flow Problem (MCF) is maximized, see Ahuja et al. [1] for a definition and details on MCF. In the uncapacitated case, where all arc capacities are assumed to be infinite, i.e., $c_a = \infty$ for all $a \in A$, we call the problem Uncapacitated Maximum Minimum Cost Flow Problem (UMMCF).

Similar to Ahuja et al. [1], we assume that G contains an uncapacitated directed path from each source towards each sink, in order to ensure feasibility for the induced MCF instances. We impose this condition, if necessary, by adding direct arcs with large cost and infinite capacity. No such arc appears in an optimal MCF solution unless there exists a supply and demand vector inducing an infeasible MCF instance for the original flow network.

A Bilevel Optimization Model for MMCF

MMCF can be modelled as a linear bilevel optimization program. For details on bilevel optimization we refer to the book by Dempe et al. [4].

$$\max_{b} \quad \sum_{a \in A} \ell_a f_a \tag{1}$$

$$\text{s.t.} \quad b_u \in [\underline{b}_u, \overline{b}_u] \qquad \forall u \in V^+ \tag{2}$$

$$b_w \in [\underline{b}_w, \overline{b}_w] \qquad \forall w \in V^- \tag{3}$$

$$\min_{f} \quad \sum_{a \in A} \ell_a f_a \tag{4}$$

$$\text{s.t.} \quad \sum_{a \in \delta^+(u)} f_a - \sum_{a \in \delta^-(u)} f_a = b_u \qquad \forall u \in V^+ \tag{5}$$

$$\sum_{a \in \delta^+(w)} f_a - \sum_{a \in \delta^-(w)} f_a = b_w \qquad \forall w \in V^- \tag{6}$$

$$\sum_{a \in \delta^+(v)} f_a - \sum_{a \in \delta^-(v)} f_a = 0 \qquad \forall v \in V^0 \tag{7}$$

$$f_a \in [0, c_a] \qquad \forall a \in A. \tag{8}$$

For each source $u \in V^+$ the variable b_u represents its supply and for each sink $w \in V^-$ the variable b_w represents its demand. Their values are chosen by the leader with respect to the corresponding bounds, see (2) and (3). Additionally, the leader has to balance supply and demand, because otherwise the MCF problem, which the follower solves subsequently, does not admit any feasible solution. If supply and demand cannot be balanced, the MMCF instance is itself infeasible. Thus, the b-variables form a supply and demand vector for the corresponding MMCF instance.

In this model, we use the well-known arc flow formulation to model MCF, see (4) - (8). The nonnegative f_a variables describe the amounts of flow on the corresponding arcs $a \in A$, which are bounded by their capacities c_a (8). Constraints (5) and (6) ensure that the supplies and demands of the sources and sinks are satisfied while constraints (7) ensure that flow conservation holds at all inner nodes. The follower routes the flow such that the cost $\sum_{a \in A} \ell_a f_a$ is minimized, while it is the leaders goal to choose a vector maximizing it, see (4) and (1), respectively.

In the remainder of this paper, we denote a solution for MMCF as a tuple (b, f), where b is a supply and demand vector and $f \in \mathbb{R}^{|A|}_{\geq 0}$ is a flow-vector, i.e., a vector containing the amounts of flow on the arcs. A solution is called feasible if f is an optimal solution for the MCF instance induced by b, i.e, if (b, f) is a feasible solution for the bilevel program above. We denote the cost of a solution by $c(b, f) := \sum_{a \in A} \ell_a f_a$ and a solution (b, f) is called optimal if it is feasible and $c(b, f) \geq c(b', f')$ for all feasible solutions (b', f').

Complexity of MMCF

In this section we prove the following theorem:

Theorem 1. MMCF *is APX-hard.*

We reduce from the Maximum Independent Set with Bounded Degree Problem. Its definition, as it is stated here, is adapted from Ausiello et al., see GT23 in [2]:

Definition 1. *Let $H = (W, E)$ be an undirected graph such that the degree of each node is bounded by some constant $B \geq 3$, i.e., we have $\Delta v \leq B$ for each $v \in W$. The goal of the* Maximum Independent Set with Bounded Degree Problem (MIS) *is to find a maximum subset $W' \subseteq W$ w.r.t. the cardinality such that no two vertices in W' are joined by an edge $e \in E$.*

Theorem 2. MIS *is APX-complete.*

Proof. A proof can be found in the papers of Berman and Fujito [3] or Papadimitriou and Yannakakis [7]. □

Given an undirected graph $H = (W, E)$ as instance of MIS, we create a corresponding MMCF instance $I_H = (V, A)$ as follows. Note that we assume w.l.o.g. that H contains no isolated vertices, since those are trivially contained in every maximum independent set.

First of all, for each vertex $v \in W$ we add a source node $v^+ \in V^+$ as well as two inner nodes $v^0, v^1 \in V^0$. Furthermore, for each edge $e \in E$ we add a sink node $e^- \in V^-$. For each source $v^+ \in V^+$ we define $\underline{b}_{v^+} := 0$ as the lower supply bound while the upper supply bound is set to $\overline{b}_{v^+} := \Delta v$, i.e., the degree of the corresponding vertex $v \in W$. Additionally, for each sink $e^- \in V^-$ we define $\underline{b}_{e^-} := -1$ and $\overline{b}_{e^-} := 0$.

Next, we add four different types of arcs to I_H, i.e., $A := A_1 \cup A_2 \cup A_3 \cup A_4$. First, for each vertex $v \in W$ we add an arc from $v^+ \in V^+$ to $v^0 \in V^0$, i.e., $A_1 := \{v^+v^0 \mid v \in W\}$. Second, for $v \in W$ we add an arc from $v^0 \in V^0$ to each sink $e^- \in V^-$ whose corresponding edge $e \in E$ is incident to v, i.e., $A_2 := \{v^0e^- \mid v \in W, e \in \delta(v)\}$. Third, for $v \in W$ we add an arc from $v^+ \in V^+$ to $v^1 \in V^0$, i.e., $A_3 := \{v^+v^1 \mid v \in W\}$. And fourth, for each $v \in W$ we add an arc from $v^1 \in V^+$ to each sink $e^- \in V^-$ whose corresponding edge $e \in E$ is incident to v, i.e., $A_4 := \{v^1e^- \mid v \in W, e \in \delta(v)\}$. Additionally, we define $\ell_a := 0$ for each $a \in A_1 \cup A_2 \cup A_4$ and $\ell_a := 1$ for each $a \in A_3$. Finally, we set $c_a := \Delta v - 1$ for all $a = v^+v^0 \in A_1$, and $c_a := 1$ for all $a \in A_2 \cup A_3 \cup A_4$. This concludes the construction of I_H.

The MMCF instance I_H corresponding to MIS instance H in Figure 1 can be found in Figure 2. Note that I_H is of linear size w.r.t. H, since the number of nodes is equal to $|V| = 3|W| + |E|$ and the number of arcs is equal to $|A| = 2|W| + 4|E|$.

Before we continue with the proof of Theorem 1, we introduce some definitions concerning solutions for instance I_H.

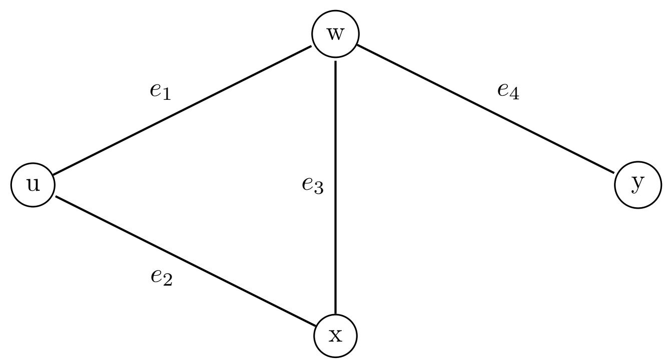

Figure 1: Example graph $H = (W, E)$.

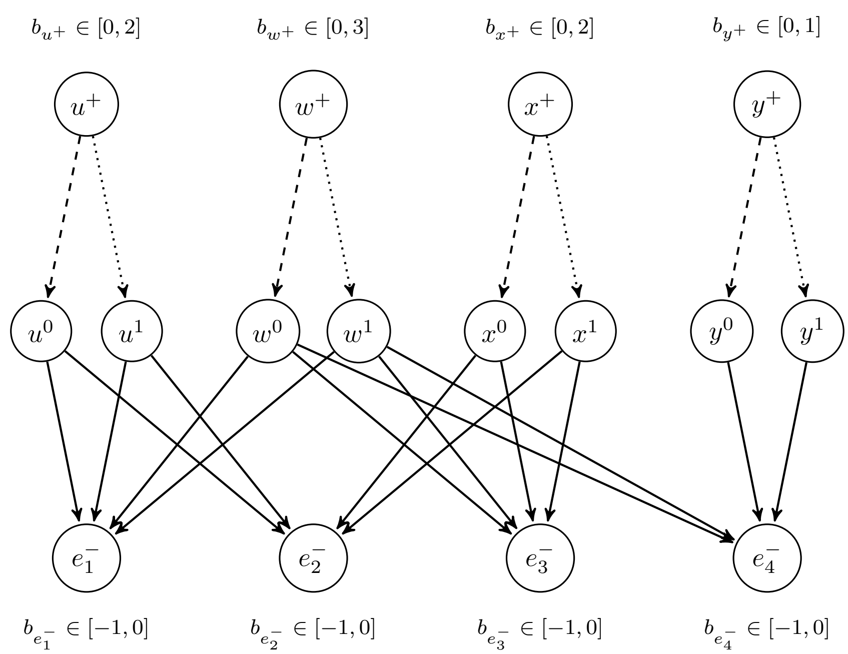

Figure 2: Instance I_H of MMCF constructed for MIS instance H from Figure 1. For a dashed arc $a = (v^+v^0) \in A_1$ we have $c_a = \Delta v - 1$ and $\ell_a = 0$, while for a dotted arc $a \in A_3$ we have $c_a = 1$ and $\ell_a = 1$. Finally, for a solid arc $a \in A_2 \cup A_4$ we have $c_a = 1$ and $\ell_a = 0$.

Consider a feasible solution (b, f) for instance I_H. For each source $v^+ \in V^+$ the flow towards a sink $e^- \in V^-$, whose corresponding edge $e \in E$ is incident to the corresponding node $v \in W$, can be uniquely partitioned into flows on two paths: The short path $v^+ \to v^0 \to e^-$ having total length 0 and the long path $v^+ \to v^1 \to e^-$ having length 1. The amount of flow on each of these paths can directly be read from $f_{v^0e^-}$ and $f_{v^1e^-}$, respectively.

Additionally, we call a solution (b, f) assigning if it is feasible and for each sink $e^- \in V^-$ corresponding to edge $e = \{x, y\} \in E$ with $b_{e^-} < 0$ we have that either $f_{x^0e^-} + f_{x^1e^-} > 0$ or $f_{y^0e^-} + f_{y^1e^-} > 0$ but not both.

Lemma 1. *Let (b, f) be a feasible solution for I_H. There exists an assigning solution $(\tilde{b}, \tilde{f})$, which can be determined in $\mathcal{O}(|E|)$, such that $c(\tilde{b}, \tilde{f}) \geq c(b, f)$.*

Proof. Let (b, f) be a feasible solution and assume there exists a sink $e^- \in V^-$ corresponding to $e = \{x, y\} \in E$ such that we have $f_x^e := f_{x^0e^-} + f_{x^1e^-} > 0$ and $f_y^e := f_{y^0e^-} + f_{y^1e^-} > 0$. W.l.o.g. we assume that $\bar{b}_{x^+} - b_{x^+} \leq \bar{b}_{y^+} - b_{y^+}$. The idea now is to shift the supply routed from y^+ towards e^- over to x^+. Therefore, let $\omega := c_{x^+x^0} - f_{x^+x^0}$ denote the remaining capacity on the short path from x^+ towards e^-. We define

$$\tilde{b}_v := \begin{cases} b_{x^+} + f_y^e & \text{for } v = x^+ \in V^+ \\ b_{y^+} - f_y^e & \text{for } v = y^+ \in V^+ \\ b_v & \text{otherwise} \end{cases}$$

and

$$\tilde{f}_a := \begin{cases} f_{x^+x^0} + \min\{\omega, f_y^e\} & \text{if } a = x^+x^0 \\ f_{x^0e^-} + \min\{\omega, f_y^e\} & \text{if } a = x^0e^- \\ f_{x^+x^1} + \max\{f_y^e - \omega, 0\} & \text{if } a = x^+x^1 \\ f_{x^1e^-} + \max\{f_y^e - \omega, 0\} & \text{if } a = x^1e^- \\ f_{y^+y^0} - f_{y^0e^-} & \text{if } a = y^+y^0 \\ 0 & \text{if } a = y^0e^- \\ f_{y^+y^1} - f_{y^1e^-} & \text{if } a = y^+y^1 \\ 0 & \text{if } a = y^1e^- \\ f_a & \text{otherwise.} \end{cases}$$

Since x^+ can only supply those sinks, whose corresponding edges are incident to $x \in W$, we have

$$\begin{aligned} b_{x^+} &= f_{x^+x^0} + f_{x^+x^1} = \sum_{\tilde{e} \in \delta(x)} f_{x^0\tilde{e}^-} + \sum_{\tilde{e} \in \delta(x)} f_{x^1\tilde{e}^-} = \sum_{\tilde{e} \in \delta(x)} (f_{x^0\tilde{e}^-} + f_{x^1\tilde{e}^-}) \\ &\leq (\sum_{\tilde{e} \in \delta(x)} |b_{\tilde{e}^-}|) - (f_{y^0e^-} + f_{y^1e^-}) \leq \Delta x - f_y^e = \bar{b}_x - f_y^e, \end{aligned}$$

showing that $\tilde{b}_{x^+} = b_{x^+} + f_y^e \leq \bar{b}_x$ and therefore $\tilde{b}$ is a supply and demand vector.

Further, while the flows on the short and long path from y^+ towards e^- are set to 0, we route up to ω units from x^+ towards e^- on the short path and the remaining supply on the long path.

If there was flow on the short path from y^+ towards e^- in f and there is still flow on some long paths starting at y^+ in $\tilde{f}$, we need to shift flow from long onto short paths in order to make $\tilde{f}$ an optimal solution for the induced MCF. This is ensured by the Algorithm 1.

Algorithm 1 Shift path flows to ensure that $\tilde{f}$ is optimal

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for all $e \in \delta(y)$ **do**
  $r \leftarrow \min\{c_{y^+y^0} - \tilde{f}_{y^+y^0}, \tilde{f}_{y^1e^-}\}$
  $\tilde{f}_{y^+y^0} \leftarrow \tilde{f}_{y^+y^0} + r$
  $\tilde{f}_{y^0e^-} \leftarrow \tilde{f}_{y^0e^-} + r$
  $\tilde{f}_{y^+y^1} \leftarrow \tilde{f}_{y^+y^1} - r$
  $\tilde{f}_{y^1e^-} \leftarrow \tilde{f}_{y^1e^-} - r$
end for

Finally, it remains to show that $c(\tilde{b}, \tilde{f}) \geq c(b, f)$. Recall that we assumed that $\bar{b}_{x^+} - b_{x^+} \leq \bar{b}_{y^+} - b_{y^+}$. There are two cases that we consider: First, if $\bar{b}_{y^+} - b_{y^+} \leq 1$ then $\bar{b}_{x^+} - b_{x^+} \leq 1$ and all the shifted supply is routed along the long path from x^+ towards e^-. Second, if we have $\bar{b}_{y^+} - b_{y^+} \geq 1$, all the supply from y^+ towards e^- was routed along the short path in f. In both cases we do not decrease the objective value and therefore $c(\tilde{b}, \tilde{f}) \geq c(b, f)$.

Iteratively applying the construction made above to all sinks e^- results in an assigining solution $(\tilde{b}, \tilde{f})$ such that $c(\tilde{b}, \tilde{f}) \geq c(b, f)$. Further, the procedure can be adjusted such that it runs in linear time w.r.t. the number of edges, i.e., in $\mathcal{O}(|E|)$. This is because for each sink $e^- \in V^-$ the construction of $(\tilde{b}, \tilde{f})$ can be done in constant time and instead of applying Algorithm 1 after each sink e^- separately, it suffices to run it once for each node $y \in V$ after all the supply shifts. □

Next, let us call a solution (b, f) *bound-tight* if it is feasible and $b_{v^+} = 0$ or $b_{v^+} = \Delta v$ holds for all $v^+ \in V^+$.

Lemma 2. *Let (b, f) be a feasible solution for I_H. There exists a bound-tight solution $(\tilde{b}, \tilde{f})$, which can be determined in $\mathcal{O}(|E|)$, such that $c(\tilde{b}, \tilde{f}) \geq c(b, f)$.*

Proof. Let (b, f) be a feasible solution. Further, using Lemma 1, we assume w.l.o.g. that (b, f) is assigning. We define

$$\tilde{b}_v := \begin{cases} \bar{b}_{x^+} & \text{for } v = x^+ \in V^+ \text{ with } b_{x^+} > \Delta x - 1 \\ -1 & \text{for } v = e^- \in V^- \text{ with } e = \{x, y\} \text{ if } b_{x^+} > \Delta x - 1 \text{ or } b_{y^+} > \Delta y - 1 \\ 0 & \text{for } v = x^+ \in V^+ \text{ with } b_{x^+} \leq \Delta x - 1 \\ 0 & \text{for } v = e^- \in V^- \text{ with } e = \{x, y\} \text{ if } b_{x^+} \leq \Delta x - 1 \text{ and } b_{y^+} \leq \Delta y - 1 \end{cases}$$

and

$$\tilde{f}_a := \begin{cases} \Delta x - 1 & \text{if } a = x^+x^0 \text{ and } b_{x^+} > \Delta x - 1 \\ 1 & \text{if } a = x^+x^1 \text{ and } b_{x^+} > \Delta x - 1 \\ f_a & \text{if } a = x^0e^- \text{ and } b_{x^+} > \Delta x - 1 \\ f_a + (1 - b_{e^-}) & \text{if } a = x^1e^- \text{ and } b_{x^+} > \Delta x - 1 \\ 0 & \text{otherwise.} \end{cases}$$

First, the supply of each source $x^+ \in V^+$ with $b_{x^+} > \Delta x - 1$ is increased up to its upper bound, i.e., $\tilde{b}_{x^+} := \overline{b}_{x^+} = \Delta x$, and the demand of each sink e^- with $e \in \delta(x)$ up to -1, i.e., $\tilde{b}_{e^-} := -1$. The additional supply of $1 - b_{e^-}$ of each sink is routed along the corresponding long path.

Second, the supply of all sources x^+ with $b_{x^+} \leq \Delta x - 1$, the demands of their assigned sinks, as well as the flows on the corresponding short paths are set to 0. Since $b_{x^+} \leq \Delta x - 1$, there is no flow on any of the corresponding long paths.

In both cases, the objective value does not decrease and we determined a bound-tight solution $(\tilde{b}, \tilde{f})$ with $c(\tilde{b}, \tilde{f}) \geq c(b, f)$ in $\mathcal{O}(|E|)$. □

Lemma 3. *Let (b, f) be a bound-tight solution for the MMCF instance I_H. Then $W' := \{v \in W \mid b_{v^+} = \overline{b}_{v^+}\}$ is an independent set in H. Furthermore, we have that $c(b, f) = |W'|$.*

Proof. Assume that W' is not independent. Then there exist $x, y \in W'$ such that $b_{x^+} = \overline{b}_{x^+}$, $b_{y^+} = \overline{b}_{y^+}$, and $\{x, y\} = e \in E$. Since only sinks whose corresponding edges are contained in $\delta(x) \cup \delta(y)$ can be supplied by x^+ and y^+ and because $e \in \delta(x) \cap \delta(y)$, we have

$$\overline{b}_{x^+} + \overline{b}_{y^+} = \sum_{e \in \delta(x) \cup \delta(y)} |b_{e^-}| \leq |\delta(x) \cup \delta(y)| \leq \Delta x + \Delta y - 1 < \overline{b}_{x^+} + \overline{b}_{y^+}$$

which is a contradiction. Thus, W' is an independent set and

$$\begin{aligned} c(b, f) &= \sum_{a \in A_1} \ell_a f_a + \sum_{a \in A_2} \ell_a f_a + \sum_{a \in A_3} \ell_a f_a + \sum_{a \in A_4} \ell_a f_a \\ &= \sum_{v \in W} \left(0 \cdot f_{v^+v^0} + 0 \cdot \sum_{e \in \delta(v)} f_{v^0e^-} + 1 \cdot f_{v^+v^1} + 0 \cdot \sum_{e \in \delta(v)} f_{v^1e^-} \right) \\ &= \sum_{v \in W} (0 \cdot f_{v^+v^0} + 0 \cdot f_{v^+v^0} + 1 \cdot f_{v^+v^1} + 0 \cdot f_{v^+v^1}) \\ &= \sum_{v \in W'} (0 \cdot (\Delta v - 1) + 0 \cdot (\Delta v - 1) + 1 \cdot 1 + 0 \cdot 1) \\ &= \sum_{v \in W'} 1 \\ &= |W'|. \end{aligned}$$

□

Lemma 4. *Let W' be an independent set in H. Then there exists a bound-tight solution (b, f) for I_H with $|W'| = c(b, f)$.*

Proof. Consider (b, f) defined as

$$b_v := \begin{cases} \Delta x & \text{if } v = x^+ \in V^+ \text{ and } x \in W' \\ -1 & \text{if } v = e^- \in V^- \text{ and } e \in \delta(W') \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_a := \begin{cases} \Delta x - 1 & \text{if } a = x^+x^0 \text{ and } x \in W' \\ \frac{\Delta x - 1}{\Delta x} & \text{if } a = x^0e^- \text{ and } x \in W' \\ 1 & \text{if } a = x^+x^1 \text{ and } x \in W' \\ \frac{1}{\Delta x} & \text{if } a = x^1e^- \text{ and } x \in W' \\ 0 & \text{otherwise.} \end{cases}$$

By construction, (b, f) is feasible, bound-tight and we have $c(b, f) = |W'|$. □

Lemma 5. *There exists an independent set W' of H with size $|W'| = k$ if and only if there exists a bound-tight solution (b, f) for I_H with $c(b, f) = k$.*

Proof. Given a feasible and bound-tight solution (b, f) for I_H with $c(b, f) = k$, the induced independent set W' from Lemma 3 has size $|W'| = k$.

Conversely, if there exists an independent set W' of H with size k, by Lemma 4 there exists a bound-tight solution (b, f) for I_H with $c(b, f) = k$. □

Corollary 1. *The size of a maximum independent set W' of H is $|W'| = k$ if and only if an optimal solution (b, f) for I_H has objective value $c(b, f) = k$.*

Using the previous results, we can now prove Theorem 1.

Theorem 1. MMCF *is APX-hard.*

Proof. Suppose there exists a PTAS for MMCF yielding a $(1 - \varepsilon)$-factor approximate solution. Let $I_H = (V, A)$ be the corresponding MMCF instance for MIS instance $H = (W, E)$ and let k denote the optimal objective value of I_H, which is equal to the size of a maximum independent set in H by Corollary 1. Such a PTAS would give us a feasible solution (b, f) for I_H with solution value $c(b, f) \geq (1 - \varepsilon)k$. Using Lemma 1 and Lemma 2, we can determine a bound-tight feasible solution $(\tilde{b}, \tilde{f})$ in $\mathcal{O}(|E|)$ with $c(\tilde{b}, \tilde{f}) \geq c(b, f)$. Next, by Lemma 3 we can extract an independent set in $\mathcal{O}(|W|)$, which has size at least $(1 - \varepsilon)k$. Hence, a $(1 - \varepsilon)$-factor PTAS for MMCF together with the algorithms from Lemma 1, Lemma 2, and Lemma 3 would yield a $(1 - \varepsilon)$-factor PTAS for MIS. Thus, no PTAS for MMCF can exist unless P=NP. □

Complexity of UMMCF

In this section we consider the special case of MMCF, where all arcs have infinite (or sufficiently high) capacity, i.e., we assume $c_a = \infty$ for all $a \in A$. We call this variant the Uncapacitated Maximum Minimum Cost Flow Problem (UMMCF). The goal of this section is to show that UMMCF is NP-hard

Theorem 3. UMMCF *is NP-hard.*

Before we prove Theorem 3, we need the following definition and a related result. We call a supply and demand vector b *bound-close* if all values are equal to one of their bounds except for at most one, i.e., $b_v \in (\underline{b}_v, \overline{b}_v)$ for at most one node $v \in V^+ \cup V^-$. For UMMCF we can show that there exists an optimal solution with a bound-close supply and demand vector.

Lemma 6. *For UMMCF there exists an optimal solution with a bound-close supply and demand vector.*

Proof. Let (b, f) be an optimal solution for UMMCF. If b it is not bound-close, there exist two nodes $x, y \in V^+ \cup V^-$ whose demand or supply values are both not equal to one of their bounds, i.e., we have $b_x \in (\underline{b}_x, \overline{b}_x)$ and $b_y \in (\underline{b}_y, \overline{b}_y)$.

The idea of the proof is to derive two new optimal solutions by increasing and decreasing the inflow and outflow in the case that $x \in V^+$ and $y \in V^-$ (or vice versa), or by shifting inflow or outflow in the case that $x, y \in V^+$ or $x, y \in V^-$, respectively. Both solutions are constructed in such a way that either b_x or b_y is set to one of its bounds.

Let $\Omega_1 := \min\{b_x - \underline{b}_x, \overline{b}_y - b_y\} > 0$ and $\Omega_2 := \min\{\overline{b}_x - b_x, b_y - \underline{b}_y\} > 0$. Consider the supply and demand vectors $\tilde{b}$ and $\hat{b}$ defined as

$$\tilde{b}_v := \begin{cases} b_x - \Omega_1 & \text{if } v = x \\ b_y + \Omega_1 & \text{if } v = y \\ b_v & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{b}_v := \begin{cases} b_x + \Omega_2 & \text{if } v = x \\ b_y - \Omega_2 & \text{if } v = y \quad RR \\ b_v & \text{otherwise.} \end{cases}$$

By construction, $\tilde{b}$ and $\hat{b}$ are supply and demand vectors. Additionally, for $\tilde{b}$ it holds that $\tilde{b}_x = \underline{b}_x$ or $\tilde{b}_y = \overline{b}_y$ and analogously for $\hat{b}$ we have that $\hat{b}_x = \overline{b}_x$ or $\hat{b}_y = \underline{b}_y$. An example is visualized in Figure 3.

Next, let $\tilde{f}$ and $\hat{f}$ be optimal solutions for the uncapacitated MCF problem induced by $\tilde{b}$ and $\hat{b}$, respectively. By construction $(\tilde{b}, \tilde{f})$ and $(\hat{b}, \hat{f})$ are feasible solutions and $c(b, f) \geq c(\tilde{b}, \tilde{f})$ and $c(b, f) \geq c(\hat{b}, \hat{f})$ hold because (b, f) is optimal. This observation we denote by $(*)$ in the following.

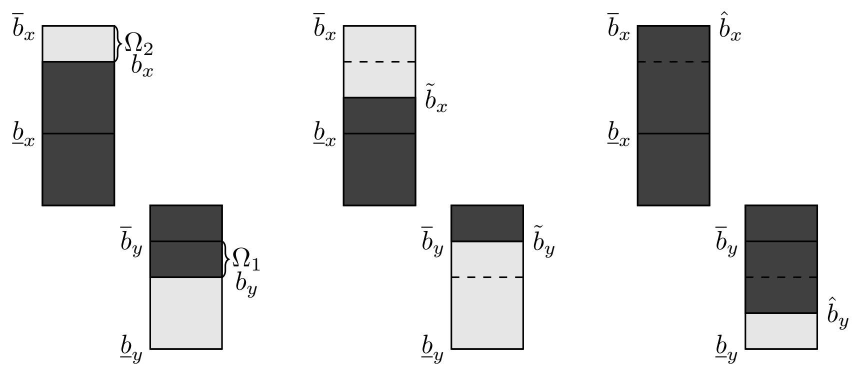

Figure 3: Supply and demand vector b with $x \in V^+$, $y \in V^-$, $b_x \in (\underline{b}_x, \overline{b}_x)$, and $b_y \in (\underline{b}_y, \overline{b}_y)$ (left). For $\tilde{b}$ (middle) the in- and outflows have been decreased, while in $\hat{b}$ (right) both have been increased. In both of the latter supply and demand vectors one of the values equals one of its bounds.

Next, for $\lambda := \frac{\Omega_2}{\Omega_1+\Omega_2} \in (0,1)$ we have that $\lambda\tilde{b} + (1-\lambda)\hat{b} = b$. This is obviously correct for all nodes $v \in (V^+ \cup V^-) \setminus \{x, y\}$. For x it holds that

$$\begin{aligned}
\lambda\,\tilde{b}_x + (1-\lambda)\,\hat{b}_x \;&=\; \frac{\Omega_2}{\Omega_1+\Omega_2}\,(b_x - \Omega_1) + \left(1 - \frac{\Omega_2}{\Omega_1+\Omega_2}\right)(b_x + \Omega_2) \\
&= \frac{\Omega_2}{\Omega_1+\Omega_2}\,(b_x - \Omega_1) + \frac{\Omega_1}{\Omega_1+\Omega_2}\,(b_x + \Omega_2) \\
&= \frac{b_x\Omega_2 - \Omega_1\Omega_2 + b_x\Omega_1 + \Omega_1\Omega_2}{\Omega_1+\Omega_2} \;=\; \frac{b_v(\Omega_1+\Omega_2)}{\Omega_1+\Omega_2} \;=\; b_x,
\end{aligned}$$

and for y an analogous argument can be made. Because we have $\lambda \in (0,1)$, it holds that $\lambda\tilde{f} + (1-\lambda)\hat{f}$ is a feasible flow for the MCF instance induced by b, because

$$(b, \lambda\tilde{f} + (1-\lambda)\hat{f}) \;=\; (\lambda\tilde{b} + (1-\lambda)\hat{b}, \lambda\tilde{f} + (1-\lambda)\hat{f}) \;=\; \lambda(\tilde{b}, \tilde{f}) + (1-\lambda)(\hat{b}, \hat{f}),$$

but it may not be optimal. It follows that $c(b, f) = c(\tilde{b}, \tilde{f}) = c(\hat{b}, \hat{f})$, because

$$\begin{aligned}
c(b, f) &\le c(b, \lambda\tilde{f} + (1-\lambda)\hat{f}) \\
&= \sum_{a \in A} \ell_a(\lambda\tilde{f}_a + (1-\lambda)\hat{f}_a) \\
&= \lambda \sum_{a \in A} \ell_a \tilde{f}_a + (1-\lambda) \sum_{a \in A} \ell_a \hat{f}_a \\
&= \lambda\, c(\tilde{b}, \tilde{f}) + (1-\lambda)\, c(\hat{b}, \hat{f}) \\
&\overset{(*)}{\le} \lambda\, c(b, f) + (1-\lambda)\, c(b, f) \;=\; c(b, f).
\end{aligned}$$

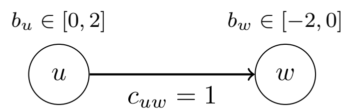

Figure 4: Counterexample showing that Lemma 6 does not hold for MMCF in general. The unique optimal solution here is $b_u = 1$, $b_w = -1$, and $f_{uw} = 1$.

Therefore, $(\tilde{b}, \tilde{f})$ and $(\hat{b}, \hat{f})$ are optimal solutions with the supply or demand value of either x or y being equal to one of their bounds. By iteratively applying this algorithmic procedure to node pairs $x, y \in V^+ \cup V^-$, whose demand or supply values are not at their bounds, we derive an optimal solution with a bound-close supply and demand vector after at most $|V| - 1$ iterations. $\square$

Lemma 6 does not hold for MMCF in general, as the simple counterexample in Figure 4 demonstrates. Next, we are going to prove that UMMCF is NP-hard. To do this, we reduce from PARTITION. The following definition is adapted from SP12 in [6]:

Definition 2. *Given a finite set $Z := \{z_1, \ldots, z_n\}$ and a size $s(z) \in \mathbb{Z}^+$ for each $z \in Z$. Is there a feasible partition of Z, i.e., a set $Z' \subseteq Z$ such that $\sum_{z \in Z'} s(z) = \sum_{z \in Z \setminus Z'} s(z)$.*

Given an instance Z of PARTITION, we construct an instance $I_Z = (V, A)$ of UMMCF as follows:

For each $z_i \in Z$ we add a source $u_i \in V^+$ and a sink $w_i \in V^-$. For each source $u_i \in V^+$ we define $\underline{b}_{u_i} := 0$ and $\overline{b}_{u_i} := s(z_i)$, while for each sink $w_i \in V^-$ we set $\underline{b}_{w_i} = -s(z_i)$ and $\overline{b}_{w_i} := 0$. Finally, we add a single inner node $v \in V^0$.

The arc set A consists of three different types, i.e., $A := A_1 \cup A_2 \cup A_3$. First, for each source $u_i \in V^+$ an arc towards the corresponding sink $w_i \in V^-$ is added, i.e., $A_1 := \{u_i w_i \mid i \in \{1, \ldots, n\}\}$, and we define $\ell_{u_i w_i} := 0$ for each $a \in A_1$. Further, an arc between each source $u_i \in V^+$ and the inner node v is added, i.e., $A_2 := \{u_i v \mid i \in \{1, \ldots, n\}\}$ and we define $\ell_{u_i v} := 1$ for each $a \in A_2$. Finally, an arc between the inner node v and each sink $w_i \in V^-$ is added, i.e., $A_3 := \{v w_i \mid i \in \{1, \ldots, n\}\}$ and we define $\ell_{v w_i} := 1$ for each $a \in A_3$. This concludes the construction of $I_Z = (V, A)$ with $|V| = 2|Z| + 1$ and $|A| = 3|Z|$. Figure 5 shows the UMMCF instance I_Z for the example PARTITION instance Z in its caption.

Next, a supply and demand vector b for I_Z with the property that either $b_{u_i} = 0$ or $b_{w_i} = 0$ for all $i \in \{1, \ldots, n\}$ is called *complementary.*

Lemma 7. *For each feasible solution (b, f) of I_Z there exists a feasible solution $(\tilde{b}, \tilde{f})$ with $c(b, f) = c(\tilde{b}, \tilde{f})$ and $\tilde{b}$ being complementary.*

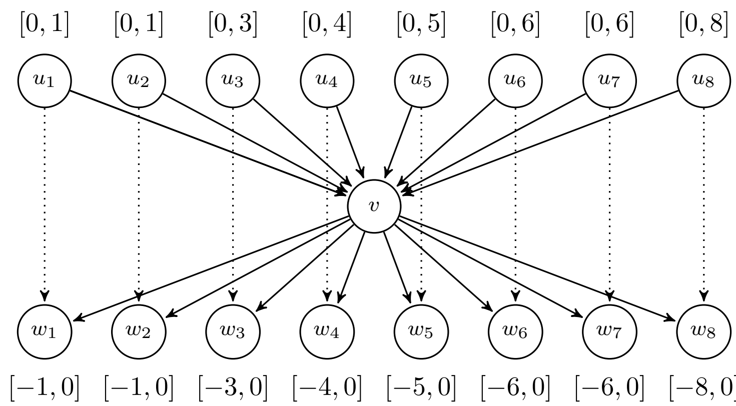

Figure 5: Instance I_Z corresponding to PARTITION instance $Z := \{z_1, \dots z_8\}$ with $s(z_1) = 1$, $s(z_2) = 1$, $s(z_3) = 3$, $s(z_4) = 4$, $s(z_5) = 5$, $s(z_6) = 6$, $s(z_7) = 6$, $s(z_8) = 8$. The dotted arcs represent arc set A_1 and have length $\ell_a = 0$, while the solid arcs represent the sets A_2 and A_3 with length $\ell_a = 1$. The intervals at the nodes describe the range of feasible supplies and demands.

Proof. For each $i \in \{1, \dots, n\}$ let $x_i := \min\{b_{v_i}, |b_{w_i}|\} \geq 0$. Since f is an optimal solution for the MCF instance induced by b, we have $f_{u_i w_i} = x_i$. Thus,

$$\tilde{b}_v := \begin{cases} b_{u_0} & \text{if } v = u_0 \in V^+ \\ b_{u_i} - x_i & \text{if } v = u_i \in V^+ \\ b_{w_i} + x_i & \text{if } v = w_i \in V^- \end{cases} \quad \tilde{f}_a := \begin{cases} f_{u_i w_i} - x_i & \text{if } a = u_i w_i \in A_1 \\ f_a & \text{otherwise} \end{cases}$$

is a feasible solution and $\tilde{b}$ is complementary. Additionally, since $\ell_a = 0$ for all $a \in A_1$, we have that $c(b, f) = c(\tilde{b}, \tilde{f})$. □

Corollary 2. *There exists an optimal solution (b, f) for I_Z with complementary supply and demand vector b.*

Lemma 8. *There exists a feasible partition Z' of Z if and only if there exists a feasible solution (b, f) for I_Z such that $c(b, f) = \sum_{z \in Z} s(z)$.*

Proof. Let $Z' \subseteq Z$ be a feasible partition, i.e., $\sum_{z \in Z'} s(z) = \sum_{z \in Z} \frac{s(z)}{2} = \sum_{z \in Z \setminus Z'} s(z)$. Consider solution (b, f) given as:

$$b_v := \begin{cases} s(z_i) & \text{if } v = u_i,\ z_i \in Z' \\ -s(z_i) & \text{if } v = w_i,\ z_i \in Z \setminus Z' \\ 0 & \text{otherwise} \end{cases} \text{ and } f_a := \begin{cases} z_i & \text{if } a = u_i v,\ z_i \in Z' \\ z_i & \text{if } a = v w_i,\ z_i \in Z \setminus Z' \\ 0 & \text{otherwise.} \end{cases}$$

By construction, b is a supply and demand vector and complementary. Additionally, f is an optimal solution for the induced uncapacitated MCF. Finally, we have

$$\begin{aligned}
c(b,f) &= \sum_{u_i v \in A_2} f_{u_i v} + \sum_{v w_i \in A_3} f_{v w_i} = \sum_{z_i \in Z'} f_{u_i v} + \sum_{z_i \in Z \setminus Z'} f_{v w_i} \\
&= \sum_{z_i \in Z'} b_{u_i} + \sum_{z_i \in Z \setminus Z'} |b_{w_i}| = \sum_{z_i \in Z'} s(z_i) + \sum_{z_i \in Z \setminus Z'} s(z_i) \\
&= \sum_{z_i \in Z} s(z_i).
\end{aligned}$$

Conversely, by Lemma 7 there exists a feasible and complementary solution (b,f) for I_Z such that $c(b,f) = \sum_{z \in Z} s(z)$. Due to the complementarity of b, we have that $f_a = 0$ for all $a \in A_1$. Now, let $Z' := \{z_i \in Z \mid b_{u_i} > 0\} \subseteq Z$. It follows that

$$\begin{aligned}
\sum_{z \in Z} s(z) = c(b,f) &= \sum_{u_i v \in A_2} f_{u_i v} + \sum_{v w_i \in A_3} f_{v w_i} \\
&= \sum_{u_i v \in A_2} b_{u_i} + \sum_{v w_i \in A_3} |b_{w_i}| \\
&= \sum_{z_i \in Z'} b_{u_i} + \sum_{z_i \in Z \setminus Z'} |b_{w_i}| \\
&\leq \sum_{z_i \in Z'} s(z_i) + \sum_{z_i \in Z \setminus Z'} s(z_i) = \sum_{z \in Z} s(z)
\end{aligned}$$

Hence, both $\sum_{z_i \in Z'} b_{u_i} = \sum_{z_i \in Z'} s(z_i)$ and $\sum_{z_i \in Z \setminus Z'} |b_{w_i}| = \sum_{z_i \in Z \setminus Z'} s(z_i)$ hold. Furthermore, as b must be balanced, because otherwise no feasible solution f can exist, it follows that

$$\sum_{z_i \in Z'} b_{u_i} = \sum_{z_i \in Z'} s(z_i) = \sum_{z \in Z} \frac{s(z)}{2}$$

and therefore Z' is a feasible partition. □

Deciding whether or not there exists a feasible partition is an NP-complete problem, see for example SP12 in [6]. Therefore, UMMCF is NP-hard, since any polynomial-time algorithm applied to I_Z deciding whether or not I_Z admits a solution (b,f) with $c(b,f) = \sum_{z \in Z} s(z)$ could decide whether or not I contains a feasible partition by Lemma 8.

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