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**A Polyhedral Study of Event-Based
Models for the Resource-Constrained
Project Scheduling Problem**

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A Polyhedral Study of Event-Based Models for the Resource-Constrained Project Scheduling Problem

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Abstract. We consider event-based Mixed-Integer Programming (MIP) formulations for the Resource-Constrained Project Scheduling Problem (RCPSP) that represent an alternative to the common time-indexed model (DDT) of Pritsker et al. (1969) when the scheduling time horizon is large or job processing times are subject to huge variations. In contrast to the time-indexed model, the size of event-based models does not depend on the time horizon. For the two event-based models OOE and SEE of Koné et al. (2011) we present new valid inequalities that dominate the original formulation. Furthermore, we introduce a new event-based mode: the *Interval Event-Based Model* (IEE). We deduce linear transformations between all three models that yield the strict domination order $IEE \succ SEE \succ OOE$ for their linear programming (LP) relaxations, meaning that the new IEE model has the strongest linear relaxation among the event-based models. In addition, we show that the popular DDT model can be retrieved from IEE by certain polyhedral operations, thus giving a unifying view on a complete branch of MIP models for the RCPSP. Finally, we analyze the computational performance of the presented models on test sets of the PSPLIB (Kolisch and Sprecher 1997).

1 Introduction

In the RCPSP we are given a set of jobs $\mathcal{J} = \{1, \dots, n\}$ with processing times $p_j \in \mathbb{Z}_{>0}$ for all $j \in \mathcal{J}$ that are processed without interruption. In addition, we are given a set of renewable resources \mathcal{R} where each resource $k \in \mathcal{R}$ is given a capacity $R_k \in \mathbb{Z}_{>0}$ and every job $j \in \mathcal{J}$ allocates $r_{jk} \in \mathbb{Z}_{\geq 0}$ units of resource k during its execution. Furthermore, there are precedence constraints $\mathcal{P} \subseteq \mathcal{J} \times \mathcal{J}$ where $(i, j) \in \mathcal{P}$ indicates that job i must finish before job j starts. The RCPSP aims to compute start times $S_j \geq 0$ for all jobs $j \in \mathcal{J}$ that respect the precedence constraints and such that the resource demand at any time does not exceed the available capacities. The objective is to minimize the project completion time $C_{max} = \max_{j \in \mathcal{J}}(S_j + p_j)$ where C_{max} is denoted as the makespan. Formally,

the RCPSP can also be stated as

$$\begin{aligned} \min \quad & C_{max} \\ S_j + p_j & \leq C_{max} \quad \forall j \in \mathcal{J} \end{aligned} \tag{1}$$

$$\sum_{j \in \mathcal{J}: S_j \leq t < S_j + p_j} r_{jk} \leq R_k \quad \forall t \in [0, T], k \in \mathcal{R} \tag{2}$$

$$\begin{aligned} S_i + p_i & \leq S_j \quad \forall (i, j) \in \mathcal{P} \\ S_j & \geq 0 \quad \forall j \in \mathcal{J} \end{aligned} \tag{3}$$

where T is an arbitrary upper bound on the makespan.

The RCPSP is a fundamental problem in discrete optimization since it contains and combines several combinatorially hard problems into one optimization problem such as partition, packing and graph coloring. It has extensive applications that are often located in project planning, production planning, supply chain management, logistics and healthcare, see Artigues et al. (2013), Cardoen (2010) and Weglarz (2012). The general RCPSP is studied since the late 1960s with a first work of Johnson (1967) but until today a large amount of research has been invested into this problem, its variants and extensions (Brucker et al. 1999; Hartmann and Briskorn 2010; Herroelen et al. 1998; Kolisch and Hartmann 2006, Stork 2001). Nevertheless, it is still one of the computationally most challenging problems and in general very hard to solve to optimality. One branch of research studies exact Mixed-Integer Programming formulations (MIP) for the RCPSP. These models can generally be subdivided into *time-indexed*, *disjunctive* and *event-based* models. In the following we will give a brief overview of the most common MIP formulations for each type.

Time-Indexed Model. One of the first mathematical programming formulations for the RCPSP was introduced by Pritsker et al. (1969) which is commonly denoted as the time-indexed model (DDT). This model considers a discrete time horizon $\mathcal{T} = \{0, \dots, T\}$ where decision variables determine for every job a feasible start time in \mathcal{T} such that additionally all resource and precedence constraints are satisfied. In its core, DDT is still very popular today for modeling the RCPSP and related problems because it provides a decently strong linear relaxation and can be solved quickly on instances of moderate size (Bianco and Caramia 2013; Naber and Kolisch 2014). Several special cases, extensions and improvements of DDT have been studied, for example in Artigues (2017), Christofides et al. (1984), Hardin et al. (2008), Mingozzi et al. (1998), Möhring et al. (2003), Naber and Kolisch (2014), Zhu et al. (2006). The main drawback of DDT is however that the number of variables and constraints depends on T . Hence, if T is large then DDT becomes computationally intractable. This motivates to study compact models whose size is strongly polynomial in the number of jobs.

Disjunctive Models. Alternatively, the RCPSP can be modeled by considering the start time variables $S_j \geq 0$ explicitly. In this case, resource conflicts are

settled by decision variables that enforce additional precedence disjunctions of either $S_i + p_i \leq S_j$ or $S_j + p_j \leq S_i$ for two jobs i, j . In particular, Alvarez et al. (1993) study an exponential model that requires such a disjunction for at least two jobs in every incompatible job subset, that is a set of jobs that cannot all be scheduled in parallel. Artigues et al. (2003) overcome the exponential number of inequalities by a continuous resource flow extension that implies the precedences between the jobs and, in contrast, leads to a compact formulation. In both models, disjunctions are formulated by so-called 'big-M' inequalities that lead to very weak linear relaxations in general. Therefore, disjunctive models are generally not preferred for instances with many jobs.

Event-Based Models. Another possibility of modeling the RCPSP in a compact manner was given by Koné et al. (2011) who propose an event-based modeling concept. Events are considered as variable but sequential time points where jobs can start and finish. Every job is assigned to a start and end event while the resource and precedence constraints must be satisfied at every event. Since every job starts and ends exactly once, at most $2n$ events (reducible to n events) need to be considered which leads to a compact formulation. Moreover, event-based models do not involve similarly large constants as the disjunctive models. However, all known event-based models still suffer from weak linear relaxations but they represent the overall best alternative to DDT when the time horizon becomes large. Koné et al. examine two different event-based models OOE and SEE. In this article, we will revisit OOE and SEE, study their polyhedral relationships and propose possible enhancements.

Other Exact Solving Methods. Several other exact solving approaches exist for the RCPSP that are mainly based on branch-and-bound, see for example Brucker et al. (1998), Christofides et al. (1987), Demeulemeester and Herroelen (1997), Dorndorf et al. (2000), Heilmann (2003), De Reyck and Herroelen (1998), Sprecher and Drexl (1998), Zhu et al. (2006) that mainly differ in their branching decisions, branching order, dominance rules, domain propagation and considered lower bounds. More recently, constraint programming (CP) is used to solve the RCPSP, see for example Baptiste et al. (2012), Schutt et al. (2013), Tesch (2016) and Vilim (2011) whose techniques combine branch-and-bound with strong domain propagation and dynamic adding of logical constraints.

Contribution. For the models OOE and SEE of Koné et al. (2011) we propose stronger valid inequalities and discuss their impact on the LP-relaxations. Moreover, we state the new *Interval Event-Based Model* (IEE) that generalizes the modeling ideas of OOE and SEE. In particular, we reveal linear transformations between all three models OOE, SEE and IEE from which we deduce the strict domination order $IEE \succ SEE \succ OOE$ of their linear programming relaxations. That means IEE has the strongest linear relaxation while SEE dominates OOE. In general, however, we show that the integrality gap of IEE (thus for OOE and SEE) is unbounded. We also investigate the relationship between the event-based models and the time-indexed model DDT of Pristker et al. (1969) by showing

that DDT can be constructed from IEE by expansion and projection of its induced polyhedron. From our constructions we conclude that DDT has a stronger LP-relaxation than IEE. In total, this yields a unifying view on the whole class of event-based and time-indexed models for the RCPSP. Finally, we propose additional preprocessing steps to improve the computational performance of all studied models on RCPSP instances of the well-established PSPLIB test library (Kolisch and Sprecher 1997). With this work, we hope to shed some light on the polyhedral properties and relationships between some of the many formulations for the RCPSP that exist until now.

Outline. The paper is organized as follows. Sections 2-4 introduce the models OOE, SEE, IEE and examine new valid inequalities as well as their mutual polyhedral relationships. Additionally, Section 4 introduces the DDT model and explores the polyhedral relationship to IEE. In Section 5, further preprocessing steps are proposed to improve the computational performance of the models that is finally analyzed in our computational results of Section 6. We conclude with some final remarks in Section 7.

2 On-Off Event-Based Model

Koné et al. (2011) propose the event-based modeling concept for the RCPSP. There we are given a discrete set of events $\mathcal{E} = \{1, \dots, n\}$ where each event $e \in \mathcal{E}$ represents a time variable $t_e \geq 0$ at which jobs can start while all events appear sequentially, that is $t_e \leq t_{e+1}$ for all $e \in \mathcal{E}$. The dummy event $n + 1$ models the makespan where t_{n+1} equals the makespan value. For every job we have to assign a start and an end event. If job j starts at event e then $S_j = t_e$ holds and if j ends at event f then $S_j + p_j \leq t_f$ holds. The latter implication allows us to consider only start events. However, modeling the assignment of start and end events to jobs can be done in several ways which leads to different event-based formulations. In this section, we consider the first modeling variant that uses on-off assignments.

The *On-Off Event-Based Model* (OOE) of Koné et al. (2011) determines whether a job is active at an event (on) or not (off). In particular, a job j is *active* at event e , if it is processed during the time interval $[t_e, t_{e+1})$. Therefore, we introduce decision variables $u_{je} \in \{0, 1\}$ with $u_{je} = 1$ if and only if job j is active at event e and $u_{je} = 0$ otherwise. For convenience, let us also define $u_{j0} = u_{jn+1} = 0$ for all $j \in \mathcal{J}$. Moreover, let

$$\mathcal{A} = \{(e, f) \in \mathcal{E} \times \mathcal{E} \cup \{n + 1\} : e < f\}$$

denote the set of consecutive event pairs that is used throughout the rest of the paper. The OOE model states as follows:

$$\begin{aligned} \min \quad & t_{n+1} \\ \sum_{e \in \mathcal{E}} u_{je} & \geq 1 \quad \forall j \in \mathcal{J} \end{aligned} \quad (4)$$

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot u_{je} \leq R_k \quad \forall k \in \mathcal{R}, e \in \mathcal{E} \quad (5)$$

$$\begin{aligned} t_e + p_j \cdot (u_{je} - u_{je-1} + u_{jf-1} - u_{jf} - 1) & \leq t_f \\ \forall j \in \mathcal{J}, (e, f) \in \mathcal{A} \end{aligned} \quad (6)$$

$$\begin{aligned} \sum_{e' < e} u_{je'} & \leq (e-1) \cdot (1 - u_{je} + u_{je-1}) \\ \forall j \in \mathcal{J}, e \in \mathcal{E} \end{aligned} \quad (7)$$

$$\begin{aligned} \sum_{e' \geq e} u_{je'} & \leq (n-e+1) \cdot (1 + u_{je} - u_{je-1}) \\ \forall j \in \mathcal{J}, e \in \mathcal{E} \end{aligned} \quad (8)$$

$$\sum_{e' \leq e} u_{je'} \leq e \cdot (1 - u_{ie}) \quad \forall (i, j) \in \mathcal{P}, e \in \mathcal{E} \quad (9)$$

$$t_e \leq t_{e+1} \quad \forall e \in \mathcal{E} \quad (10)$$

$$t_e \geq 0 \quad \forall e \in \mathcal{E} \cup \{n+1\}$$

$$u_{je} \in \{0, 1\} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}.$$

The objective function minimizes the makespan. Inequalities (4) ensure that every job $j \in \mathcal{J}$ is active for at least one event. Moreover, inequalities (5) indicate that the total resource consumption at every event must not exceed the resource capacities. Note that job j starts at event e if and only if $u_{je} - u_{je-1} = 1$ and j ends at event f if and only if $u_{jf-1} - u_{jf} = 1$. Hence, inequalities (6) imply that if job j starts at event e and ends at event f then $t_e + p_j \leq t_f$ must hold. Inequalities (7) and (8) model the non-preemption of the jobs, that means if job j starts at event e then it cannot be active at an event $e' \in \{1, \dots, e-1\}$. Similarly, if job j ends at event f then it cannot be active at an event $e' \in \{f, \dots, n\}$. Inequalities (9) require for every precedence pair $(i, j) \in \mathcal{P}$, that if job i is active at event e then job j cannot be active at an event $e' \in \{1, \dots, e\}$. Finally, inequalities (10) state that all events appear sequentially. In total, OOE has n^2 binary variables and $\mathcal{O}(n \cdot (|\mathcal{R}| + |\mathcal{A}| + |\mathcal{P}|))$ constraints where $|\mathcal{A}| = \binom{n+1}{2}$.

In the following we will introduce stronger inequalities for OOE and analyze their impact on the LP lower bound.

2.1 Valid Inequalities

We will state stronger *non-preemptive*, *start-end time* and *precedence* inequalities.

Non-Preemptive Inequalities Consider the following stronger version of inequalities (7) and (8) that model non-preemption.

Lemma 1. *The following inequalities dominate the non-preemptive inequalities (7) and (8):*

$$\begin{aligned} u_{je} - u_{jf} + u_{jg} &\leq 1 \\ \forall j \in \mathcal{J}, e, f, g \in \mathcal{E} : e < f < g. \end{aligned} \quad (11)$$

Proof. Inequalities (11) indicate that if job j is active at two events e and g with $e < g$ then j must also be active at all events f with $e < f < g$, so inequalities (11) are valid. Now consider a fixed job $j \in \mathcal{J}$ and event $g \in \mathcal{E}$ and let $f = g - 1$. Summing up inequalities (11) for $e' = 1, \dots, g - 2$ and adding the trivial inequality $u_{jg} = u_{jg-1} - u_{jg-1} + u_{jg} \leq 1$ yields

$$\sum_{e' < g} u_{je'} + (g - 1) \cdot (-u_{jg-1} + u_{jg}) \leq (g - 1)$$

which is equivalent to (7). Furthermore, for a fixed event $f \in \mathcal{E}$ let $e = f - 1$ then summing up inequalities (11) for all $g = f + 1, \dots, n$ and adding the trivial inequality $u_{jf-1} = u_{jf} - u_{jf} + u_{jf-1} \leq 1$ yields

$$\sum_{g \geq f} u_{jg} + (n - f + 1) \cdot (-u_{jf} + u_{jf-1}) \leq (n - f + 1)$$

which is equivalent to (8). This shows the lemma. \square

While there are $2n^2$ inequalities of the form (7) and (8) there are $n \cdot \binom{n}{3}$ inequalities of the stronger form (11), so the stronger form implies a factor of $\mathcal{O}(n^2)$ additional constraints. However, one can see that even the stronger inequalities (11) are weak in a polyhedral sense which already indicates that non-preemption in OOE allows only weak linear relaxations.

Recently, Nattaf et al. (2017) propose the following generalization of inequalities (11) by considering all event subsets of odd cardinality:

$$\sum_{q=0}^{2l} (-1)^q \cdot u_{je_q} \leq 1 \quad \forall j \in \mathcal{J}, \{e_0, \dots, e_{2l}\} \subseteq \mathcal{E}. \quad (12)$$

Notably, they showed that inequalities (12) yield a complete description of the integer polytope that restricts to the non-preemptive inequalities (7), (8) and that the associated separation problem can be solved in strongly polynomial time. Unfortunately, the efficient separation of inequalities (12) does not directly transfer to improve solving OOE because the separation procedure would need to be called at every node of the MIP search tree to solve the LP subproblems. Hence, the number of generated inequalities during MIP solving can increase drastically and, in turn, this would require a higher-level inequality management of dynamically adding and deleting constraints to keep the LP subproblems at a

manageable size. Moreover, we will show that adding inequalities (12) does not affect the LP lower bound, see Section 2.2. Therefore we propose the contrary, that is to add as few non-preemptive inequalities as possible, for example by taking only inequalities (7) that are already sufficient to model non-preemption.

Start Time Inequalities Next, we present a stronger version of the start time inequalities (6).

Lemma 2. *The following inequalities dominate the start time inequalities (6):*

$$\begin{aligned} t_e + p_j \cdot (u_{jf} - u_{je} - u_{jg}) &\leq t_g \\ \forall j \in \mathcal{J}, e, f, g \in \mathcal{E} \cup \{0, n+1\} : e < f < g. \end{aligned} \quad (13)$$

Proof. Inequalities (13) state that if job j is active at event f but not at events e and g with $e < f < g$ then $t_e + p_j \leq t_g$ must hold which is valid. Since $u_{jg-1} \leq 1$ we obtain

$$\begin{aligned} p_j \cdot (u_{jg-1} - u_{jf-1} - u_{jg} + u_{jf} - 1) \\ \leq p_j \cdot (u_{jf} - u_{jf-1} - u_{jg}) \stackrel{(13)}{\leq} t_g - t_f \end{aligned}$$

for all jobs $j \in \mathcal{J}$ and events $f, g \in \mathcal{E}$ with $f < g$. The last inequality follows for $e = f - 1$. \square

Again, the stronger version has a factor of $\mathcal{O}(n)$ more inequalities and we observe that the stronger inequalities are not very strong in a polyhedral sense. Since the start time inequalities mainly affect the dual bound (t_{n+1} is determined by these inequalities), it suggests that the linear relaxation of OOE is weak in general. A more detailed analysis on the dual bound of OOE is given in Section 2.2.

Precedence Inequalities In the following we propose a stronger alternative for the precedence inequalities (9).

Lemma 3. *The following inequalities dominate the precedence inequalities (9):*

$$u_{je'} + u_{ie} \leq 1 \quad \forall (i, j) \in \mathcal{P}, e', e \in \mathcal{E} : e' \leq e. \quad (14)$$

Proof. Inequalities (9) state that for every precedence pair $(i, j) \in \mathcal{P}$ a conflicting assignment of job j to an event earlier or equal to an active event of job i is forbidden which is valid. For some fixed event $e \in \mathcal{E}$ summing up inequalities (14) for all $e' = 1, \dots, e$ we get $\sum_{e'=1}^e u_{je'} + e \cdot u_{ie} \leq e$ which is equivalent to (9). This shows the lemma. \square

The stable set structure of these inequalities suggests a generalization by considering whole paths in the precedence graph. Therefore, define the precedence digraph as $\mathcal{G} = (\mathcal{J}, \mathcal{P})$ and let $\pi = \{j_1, \dots, j_m\}$ be a directed path in \mathcal{G} .

Moreover, let $\mathcal{E}' = \{e_1, \dots, e_m\} \subseteq \mathcal{E}$ be a subset of events with $e_q \geq e_{q+1}$ for all $1 \leq q < m$. By similar arguments, consider the inequality

$$\sum_{q=1}^m u_{j_q e_q} \leq 1 \quad \forall \{e_1, \dots, e_m\} \subseteq \mathcal{E} : e_q \geq e_{q+1}, 1 \leq q < m \quad (15)$$

which excludes any invalid assignment of jobs in π to events in $\{e_1, \dots, e_m\}$. Similar to Nattaf et al. (2017), we are able to state a separation algorithm for inequalities (15). In order to compute a maximally violated inequality of (15) for a given feasible LP solution $u^* \in [0, 1]^{n^2}$ of OOE, we need to find a path $\pi = \{j_1, \dots, j_m\}$ in \mathcal{G} and an event set $\{e_1, \dots, e_m\} \subseteq \mathcal{E}$ with $e_{q+1} \geq e_q$ for $q = 1, \dots, m-1$ such that $\sum_{q=1}^m u_{j_q e_q}^* > 1$. We can do it as follows. Create a node for every pair (j, e) of jobs $j \in \mathcal{J}$ and events $e \in \mathcal{E}$. In addition, create arcs from node (i, f) to node (j, e) of length u_{if}^* if and only if $(i, j) \in \mathcal{P}$ and $f > e$. Furthermore, create a source node and add an arc of length u_{je}^* from the source to every other node (j, e) . Similarly, create a sink node and add an arc of length zero from every node (j, e) to the sink node. A maximally violated inequality of (15) can be found by computing a longest path from the source node to the sink node whose nodes (j, e) correspond to the variables u_{je}^* in the maximally violated inequality. Since the underlying graph is acyclic, the algorithm is linear in the number of edges which is $\mathcal{O}(|\mathcal{P}| \cdot n^2)$.

Again, inequalities (15) face the same problem as inequalities (12) because the separation algorithm would generally need to be called at every node of the MIP search tree to compute the LP subproblems. In this case, the number of inequalities can grow rigorously which would require an optimized inequality management to keep the model at an appropriate size. Therefore, our computational results do not include these inequalities but mainly because they do not contribute to LP lower bound, see the next section.

2.2 LP-Relaxation

In this section, we examine the quality of the LP-relaxation of OOE.

Proposition 1. *The LP-relaxation of OOE has an optimal objective value of zero, even if the stronger inequalities (12), (13) and (15) are included.*

Proof. We will construct a feasible LP solution of OOE including inequalities (12), (13) and (15) with objective value zero. Define the solution by $u_{je} = \frac{1}{n}$ for all $j \in \mathcal{J}, e \in \mathcal{E}$ and $t_e = 0$ for all $e \in \mathcal{E} \cup \{n+1\}$. All other variables are zero. We show LP-feasibility for each constraint separately. For the assignment constraints (4) we have

$$\sum_{e \in \mathcal{E}} u_{je} = 1 \geq 1$$

for every job $j \in \mathcal{J}$. The resource constraints (5) satisfy

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot u_{je} = \sum_{j \in \mathcal{J}} r_{jk} \cdot \frac{1}{n} \leq R_k$$

for all $k \in \mathcal{R}$ and $e \in \mathcal{E}$ where the last inequality holds because otherwise $r_{jk} > R_k$ for at least one job $j \in \mathcal{J}$ and resource $k \in \mathcal{R}$ which is infeasible. Moreover, for the stronger start time inequalities (13) we get

$$p_j \cdot (z_{jf} - z_{je} - z_{jg}) \leq 0 = t_g - t_e$$

for every $j \in \mathcal{J}$ and $e, f, g \in \mathcal{E} \cup \{0, n+1\}$ with $e < f < g$. The stronger non-preemptive inequalities (12) satisfy

$$\sum_{q=0}^{2l} (-1)^q \cdot u_{je_q} = \frac{1}{n} \leq 1$$

for every $j \in \mathcal{J}$ and event set $\{e_0, \dots, e_{2l}\} \subseteq \mathcal{E}$. Finally, for the stronger precedence constraints (15) we obtain

$$\sum_{q=1}^m u_{j_q e_q} = \frac{m}{n} \leq 1$$

for any path $\pi = \{j_1, \dots, j_m\}$ in the precedence graph \mathcal{G} and any subset $\{e_1, \dots, e_m\} \subseteq \mathcal{E}$ with $e_q \geq e_{q+1}$ for all $q = 1, \dots, m-1$. Hence, the solution is feasible and has objective value $t_{n+1} = 0$ which shows the proposition. \square

Corollary 1. *The integrality gap of OOE is unbounded.*

Despite our strengthening efforts, we deduce that it is not possible to remarkably improve the LP bound of OOE which indicates that OOE is weak from a polyhedral perspective. Furthermore, it indicates that adding the stronger inequalities will not substantially improve upon solving OOE because they increase the number of some inequalities by additional factors that are polynomial in n while the LP value does not improve. We believe that the benefit of OOE lies in its small model size, not in the strength of the formulation. Therefore, our computational results consider the sparsest possible variant of OOE given by inequalities (4)-(7) and (9). Compared to the other models, OOE has the smallest number of variables, so we can still hope to achieve good computational results by applying a modern MIP solver who performs clever branching and cutting plane generation, see Section 6

3 Start-End Event-Based Model

In this section, we examine the *Start-End Event-Based Model (SEE)* of Koné et al. (2011). We will state it in a reduced form because a set of continuous variables

of the original formulation can be omitted. Again, we are given variables $t_e \geq 0$ for each event $e \in \mathcal{E} \cup \{n+1\}$ where t_{n+1} denotes the makespan. The SEE model considers decision variables $x_{je} \in \{0, 1\}$ with $x_{je} = 1$ if and only if job j starts at event e and $x_{je} = 0$ otherwise. Equivalently, there are decision variables $y_{jf} \in \{0, 1\}$ with $y_{jf} = 1$ if and only if job j ends at event f and $y_{jf} = 0$ otherwise. Let $\mathcal{E}^+ = \{2, \dots, n+1\}$ denote the shifted event set at which jobs are allowed to finish. Then SEE can be formulated as:

$$\begin{aligned} \min \quad & t_{n+1} \\ \sum_{e \in \mathcal{E}} x_{je} &= 1 \quad \forall j \in \mathcal{J} \end{aligned} \tag{16}$$

$$\sum_{e \in \mathcal{E}^+} y_{je} = 1 \quad \forall j \in \mathcal{J} \tag{17}$$

$$\sum_{e' \geq e} x_{je'} + \sum_{e' \leq e} y_{je'} \leq 1 \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \tag{18}$$

$$\begin{aligned} \sum_{j \in \mathcal{J}} r_{jk} \cdot \left(\sum_{e' \leq e} x_{je'} - \sum_{e' \leq e} y_{je'} \right) &\leq R_k \\ \forall k \in \mathcal{R}, e \in \mathcal{E} \end{aligned} \tag{19}$$

$$t_e + p_j \cdot (x_{je} + y_{jf} - 1) \leq t_f \quad \forall j \in \mathcal{J}, (e, f) \in \mathcal{A} \tag{20}$$

$$\sum_{e' \leq e} x_{je'} + \sum_{e' > e} y_{ie'} \leq 1 \quad \forall (i, j) \in \mathcal{P}, e \in \mathcal{E} \tag{21}$$

$$t_e \leq t_{e+1} \quad \forall e \in \mathcal{E} \tag{22}$$

$$t_e \geq 0 \quad \forall e \in \mathcal{E} \cup \{n+1\}$$

$$x_{je} \in \{0, 1\} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}$$

$$y_{je} \in \{0, 1\} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}^+.$$

The objective function minimizes the makespan. Equations (16) and (17) state that every job must start and end at exactly one event while the start event must be prior to the end event (18). Moreover, inequalities (19) ensure that the resource consumption of all jobs that are active at one event must not exceed the resource capacities. Next, inequalities (20) require that if job j starts at event e and ends at event f then $t_e + p_j \leq t_f$ must hold. By inequalities (21), for each precedence pair $(i, j) \in \mathcal{P}$ the end event of job i must be prior to the start event of job j . In addition, the event times must be non-decreasing by (22). In total, SEE has $2n^2$ binary variables and $\mathcal{O}(n \cdot (|\mathcal{R}| + |\mathcal{A}| + |\mathcal{P}|))$ constraints.

In the next sections, we propose stronger valid inequalities for SEE and analyze the quality of its LP-relaxation.

3.1 Start Time Inequalities

In the following we present a stronger variant of the start time inequalities (20).

Lemma 4. *The following inequalities dominate the start time inequalities (20):*

$$t_e + p_j \cdot \left(\sum_{e' \geq e} x_{je'} + \sum_{f' \leq f} y_{jf'} - 1 \right) \leq t_f$$

$$\forall j \in \mathcal{J}, (e, f) \in \mathcal{A}. \quad (23)$$

Proof. Inequalities (23) state that if job j starts at an event $e' \in \{e, \dots, n\}$ and ends at an event $f' \in \{2, \dots, f\}$ with $e < f$ then $t_e + p_j \leq t_f$ must hold. Thus, the inequality is valid. Obviously, inequalities (23) are stronger than inequalities (20) which proves the lemma. \square

Since this strengthening comes with no expense of the model size, we will always assume the stronger version of SEE throughout the rest of the paper.

3.2 LP-Relaxation

In the following we will analyze the quality of the linear relaxation of SEE. The next result holds for the special case of no precedence constraints, so it also applies for the general RCPSP.

Proposition 2. *If $\mathcal{P} = \emptyset$ then the LP-relaxation of SEE has an optimal objective value of $p_{max} = \max_{j \in \mathcal{J}} p_j$.*

Proof. Considering inequalities (23) for $(e, f) = (1, n + 1)$ imply $t_{n+1} \geq p_j$ for all $j \in \mathcal{J}$, hence p_{max} is a lower bound on the optimal LP value. In the following we will construct a feasible LP solution that yields p_{max} also as upper bound. Hence, let a fractional solution of SEE be given by $x_{je} = y_{e+1} = \frac{1}{n}$ for all $j \in \mathcal{J}$, $e \in \mathcal{E}$ and $t_e = p_{max} \cdot \frac{e-1}{n}$ for all $e \in \mathcal{E}$. All other variables are zero. We will verify LP-feasibility for each constraint separately. Inequalities (16) and (17) are satisfied since $\sum_{e \in \mathcal{E}} x_{je} = \sum_{f \in \mathcal{E}^+} y_{jf} = \sum_{e \in \mathcal{E}} \frac{1}{n} = \sum_{f \in \mathcal{E}^+} \frac{1}{n} = 1$ for every $j \in \mathcal{J}$. For inequalities (18) we obtain

$$\sum_{e' \geq e} x_{je'} + \sum_{e' \leq e} y_{je'} = \frac{n - e + 1}{n} + \frac{e - 1}{n} = 1 \leq 1$$

for all $j \in \mathcal{J}$ and $e \in \mathcal{E}$. Moreover, for inequalities (19) we have

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot \left(\sum_{e' \leq e} x_{je'} - \sum_{e' \leq e} y_{je'} \right) = \sum_{j \in \mathcal{J}} r_{jk} \cdot \frac{1}{n} \leq R_k$$

for all $k \in \mathcal{R}$ and $e \in \mathcal{E}$ where the last inequality holds because otherwise $r_{jk} > R_k$ for some job $j \in \mathcal{J}$ and resource $k \in \mathcal{R}$ what is infeasible. Next, the start time inequalities (23) imply

$$p_j \cdot \left(\sum_{e' \geq e} x_{je'} + \sum_{f' \leq f} y_{jf'} - 1 \right)$$

$$= \frac{p_j \cdot (f - e)}{n} \leq \frac{p_{max} \cdot (f - e)}{n} = t_f - t_e$$

for all $j \in \mathcal{J}$ and $(e, f) \in \mathcal{A}$. Consequently, all constraints of SEE are satisfied for the given solution that has an objective value of $t_{n+1} = p_{max}$. Consequently, the optimal LP value has a lower and upper bound of p_{max} , which proves the proposition. \square

Corollary 2. *The integrality gap of SEE can be arbitrarily large.*

Note that if we apply the same LP solution from the proof of Proposition 2 to the precedence inequalities (21), we get

$$\sum_{e' \leq e} x_{je'} + \sum_{e' > e} y_{ie'} = \frac{e + n - e + 1}{n} = \frac{n + 1}{n} > 1$$

for every $(i, j) \in \mathcal{P}$ and $e \in \mathcal{E}$ which is not LP feasible. Loosely speaking, this suggests that the precedence constraints must be considered differently from the rest of the model because they allow a stronger modeling.

Since the LP bound of OOE equals zero, it follows that SEE has a strictly stronger LP bound than OOE. It remains to check whether SEE dominates OOE also on the whole LP-relaxation. This question is addressed in the next section.

3.3 Relationship to OOE

In this section, we study the polyhedral relationship between SEE and OOE. According to Section 2.1, define

$$P(OOE) = \{(t, u) \in \mathbb{R}_{\geq 0}^{n+1} \times [0, 1]^{n^2} : \\ (t, u) \text{ satisfies (4), (5), (10), (12), (13), (15)}\}$$

as the polyhedron of the linear relaxation of OOE including all stronger inequalities. Analogously, define

$$P(SEE) = \{(t, x, y) \in \mathbb{R}_{\geq 0}^{n+1} \times [0, 1]^{2n^2} : \\ (t, x, y) \text{ satisfies (16)-(19) and (21)-(23)}\}$$

as the polyhedron of the linear relaxation of SEE including the stronger start time inequalities. In the following we will study the polyhedral relationship between $P(OOE)$ and $P(SEE)$.

A first important observation is that we can express the u_{je} variables of OOE in terms of the x_{je} and y_{je} variables of SEE by the linear transformation:

$$u_{je} = \sum_{e' \leq e} x_{je'} - \sum_{e' \leq e} y_{je'} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (24)$$

that says that job j is active at event e if and only if j starts at an event $e' \in \{1, \dots, e\}$ and ends at an event $f' \in \{e + 1, \dots, n + 1\}$ what is certainly true.

Let $\Phi : P(SEE) \rightarrow P(OOE)$ with $(t, x, y) \mapsto (t, u)$ denote the linear transformation that is given by equations (24) and the identity map for the t_e -variables. We prove the following.

Theorem 1. $\Phi(P(SEE)) \subsetneq P(OOE)$

Proof. We will show that every inequality of $P(OOE)$ is implied by an inequality of $P(SEE)$ under the transformation Φ . First, for inequalities (4) we get

$$\begin{aligned}
\sum_{e \in \mathcal{E}} u_{je} &= \sum_{e \in \mathcal{E}} \left(\sum_{e' \leq e} x_{je'} - \sum_{e' \leq e} y_{je'} \right) \\
&= \sum_{e \in \mathcal{E}} \left(\sum_{e' < e} x_{je'} - \sum_{e' \leq e} y_{je'} \right) + \sum_{e \in \mathcal{E}} x_{je} \\
&\stackrel{(16)}{=} \sum_{e \in \mathcal{E}} \left(1 - \sum_{e' \geq e} x_{je'} - \sum_{e' \leq e} y_{je'} \right) + \sum_{e \in \mathcal{E}} x_{je} \\
&\stackrel{(18)}{\geq} \sum_{e \in \mathcal{E}} x_{je} \stackrel{(16)}{=} 1
\end{aligned}$$

for every $j \in \mathcal{J}$. Moreover, for inequalities (5) we have

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot u_{je} = \sum_{j \in \mathcal{J}} r_{jk} \cdot \left(\sum_{e' \leq e} x_{je'} - \sum_{e' \leq e} y_{je'} \right) \stackrel{(19)}{\leq} R_k$$

for all $k \in \mathcal{R}$ and $e \in \mathcal{E}$. Inequalities (10) are the same as for $P(SEE)$ and for the stronger non-preemptive inequalities (12) it holds

$$\begin{aligned}
\sum_{q=0}^{2l} (-1)^q \cdot u_{je_q} &= \sum_{q=0}^{2l} (-1)^q \cdot \left(\sum_{e' \leq e_q} x_{je'} - \sum_{e' \leq e_q} y_{je'} \right) \\
&\stackrel{(18)}{\leq} \sum_{q=0}^{2l} (-1)^q = 1
\end{aligned}$$

for all $j \in \mathcal{J}$ and all subsets $\{e_0, \dots, e_{2l}\} \subseteq \mathcal{E}$ of odd cardinality. Next, inequalities (13) translate to

$$\begin{aligned}
& p_j \cdot (u_{jf} - u_{je} - u_{ig}) \\
&= p_j \cdot \left(\sum_{e' \leq f} x_{je'} - \sum_{e' \leq f} y_{je'} - \sum_{e' \leq e} x_{je'} + \sum_{e' \leq e} y_{je'} \right. \\
&\quad \left. - \sum_{e' \leq g} x_{je'} + \sum_{e' \leq g} y_{je'} \right) \\
&\leq p_j \cdot \left(- \sum_{e' \leq e} x_{je'} + \sum_{e' \leq g} y_{je'} \right) \\
&\stackrel{(16)}{\leq} p_j \cdot \left(\sum_{e' \geq e} x_{je'} + \sum_{e' \leq g} y_{je'} - 1 \right) \stackrel{(23)}{\leq} t_g - t_e
\end{aligned}$$

for all $j \in \mathcal{J}$ and $e, f, g \in \mathcal{E} \cup \{0, n+1\}$ with $e < f < g$. Furthermore, let $\pi = (j_1, \dots, j_m)$ be a path in the precedence digraph $\mathcal{G} = (\mathcal{J}, \mathcal{P})$ and let $\{e_1, \dots, e_m\} \subseteq \mathcal{E}$ be a subset of events with $e_q \geq e_{q+1}$ for $q = 1, \dots, m-1$. For the stronger precedence inequalities (15) we get

$$\begin{aligned}
\sum_{q=1}^m u_{j_q e_q} &= \sum_{q=1}^m \left(\sum_{e' \leq e_q} x_{j_q e'} - \sum_{e' \leq e_q} y_{j_q e'} \right) \\
&= \sum_{q=1}^{m-1} \left(\sum_{e' \leq e_q} x_{j_{q+1} e'} - \sum_{e' \leq e_q} y_{j_q e'} \right) + x_{j_1 e_1} - y_{j_m e_m} \\
&\stackrel{(17)}{=} \sum_{q=1}^{m-1} \left(\sum_{e' \leq e_q} x_{j_{q+1} e'} + \sum_{e' > e_q} y_{j_q e'} - 1 \right) + x_{j_1 e_1} - y_{j_m e_m} \\
&\stackrel{(21)}{\leq} x_{j_1 e_1} - y_{j_m e_m} \leq 1.
\end{aligned}$$

Consequently, every inequality of $P(OOE)$ is implied by inequalities of $P(SEE)$ under the linear transformation Φ . By Propositions 1 and 2, the LP value of OOE is strictly smaller than the LP value of SEE . Since both LP values are determined by t_{n+1} and Φ maps t_{n+1} under identity, we conclude that the strict inclusion $\Phi(P(SEE)) \subsetneq P(OOE)$ holds. This completes the proof. \square

It follows that SEE yields a strictly stronger formulation than OOE at the expense of doubling the number of variables. In the next section, we will consider a sparse reformulation of SEE that has useful properties, again by applying a linear transformation.

3.4 Unimodular Reformulation

In this section, we study a further linear transformation of SEE that yields an equivalent model but the obtained constraint matrix is much sparser what can be exploited by modern MIP solvers. Consider the linear transformation of SEE that is given by the equations

$$\tilde{x}_{je} = \sum_{e' \leq e} x_{je'} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (25)$$

$$\tilde{y}_{je} = \sum_{e' \leq e} y_{je'} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}^+ \quad (26)$$

which indicates that $\tilde{x}_{je} = 1$ if and only if job j starts not later than event e and $\tilde{y}_{je} = 1$ if and only if job j ends not later than event e . For convenience, assume again that $\tilde{x}_{j0} = \tilde{y}_{j1} = 0$. The transformed model uses the variables \tilde{x} and \tilde{y} and we will denote it as the *Revised Start-End Event-Based Model* (RSEE) which reads as follows:

$$\begin{aligned} \min \quad & t_{n+1} \\ \tilde{x}_{jn} &= 1 \quad \forall j \in \mathcal{J} \end{aligned} \quad (27)$$

$$\tilde{y}_{jn+1} = 1 \quad \forall j \in \mathcal{J} \quad (28)$$

$$\tilde{x}_{je} \leq \tilde{x}_{je+1} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} : e < n \quad (29)$$

$$\tilde{y}_{je} \leq \tilde{y}_{je+1} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}^+ : e < n + 1 \quad (30)$$

$$\tilde{y}_{je+1} \leq \tilde{x}_{je} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (31)$$

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot (\tilde{x}_{je} - \tilde{y}_{je}) \leq R_k \quad \forall k \in \mathcal{R}, e \in \mathcal{E} \quad (32)$$

$$t_e + p_j \cdot (\tilde{y}_{jf} - \tilde{x}_{je-1}) \leq t_f \quad \forall j \in \mathcal{J}, (e, f) \in \mathcal{A} \quad (33)$$

$$\tilde{x}_{je} \leq \tilde{y}_{ie} \quad \forall (i, j) \in \mathcal{P}, e \in \mathcal{E} \quad (34)$$

$$t_e \leq t_{e+1} \quad \forall e \in \mathcal{E} \quad (35)$$

$$t_e \geq 0 \quad \forall e \in \mathcal{E} \cup \{n + 1\}$$

$$\tilde{x}_{je} \in \{0, 1\} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}$$

$$\tilde{y}_{je} \in \{0, 1\} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}^+.$$

The objective function minimizes the makespan. Equations (27) and (28) say that every job starts or ends until event n or $n + 1$ respectively. Inequalities (29) and (30) express that if job j starts/finishes until event e then it also starts/finishes until event $e + 1$. Moreover, inequalities (31) require that if job j finishes until event $e + 1$ then it must also start until event e . In addition, inequalities (32) are the resource constraints where a job j is active at event e if and only if j starts until e but does not finish until e . The time constraints (33) indicate that if job j finishes until event f but does not start until event $e - 1$ then $t_e + p_j \leq t_f$ must hold. Finally, the precedence constraints (34) require for each precedence pair $(i, j) \in \mathcal{P}$ that if job j starts until event e then job i must also finish until e .

We observe that the induced constraint matrix of RSEE has much less non-zero coefficients than SEE since most rows contain only $\mathcal{O}(1)$ entries. This property is a rather technical improvement than a theoretical one because it primarily improves the representation of the constraint set what affects the performance of the used MIP solver. Furthermore, the constraint matrix of RSEE includes substructures that are well-suited for variable propagations, see as follows.

A matrix $A \in \mathbb{R}^{p \times q}$ is called *totally unimodular*, if every non-singular square submatrix A' of A has determinant $\det(A') \in \{-1, 1\}$. The importance of totally unimodular matrices in integer programming is omnipresent, see Schrijver (2002) for an overview. We will show some implications of total unimodularity in certain substructures of RSEE.

Lemma 5. *The transformation matrix of equations (25) and (26) is totally unimodular.*

Proof. Equations (25) and (26) can be written in matrix form as $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$. Since the columns of A can easily be arranged such that A has only consecutive ones in each row, we get a sufficient condition for A being totally unimodular. \square

Lemma 5 says that the transformation (25), (26) is unimodular which implies that feasible integer solutions of SEE are mapped to feasible integer solutions of RSEE. This shows the equivalence of integer solutions of SEE and RSEE. Moreover, there are similar substructures within RSEE.

Lemma 6. *The constraint matrix defined by inequalities (27)-(31) and (34) is totally unimodular.*

Proof. The constraint matrix associated with inequalities (27)-(31) and (34) corresponds to a network matrix that is known to be totally unimodular, see Schrijver (2002). \square

It follows that if we consider RSEE only with inequalities (29)-(31) and (34) then we already obtain integer solutions. An interesting direction will be to combine the integral property with Lagrangian relaxation and flow computations in the underlying job-event network, similar to Möhring et al. (2003) for the time-indexed model, to compute or approximate feasible integer solutions of the original formulation. The complexity will be to choose proper penalty weights on the involved variables.

Our computational results reveal that the sparse formulation RSEE indeed improves upon the solution quality compared to SEE, see Section 6.

4 Interval Event-Based Model

In this section, we present a new event-based model for the RCPSP, the *Interval Event-Based Model* (IEE). This model considers decision variables $z_{jef} \in \{0, 1\}$ for all $j \in \mathcal{J}$ and $(e, f) \in \mathcal{A}$ where $z_{jef} = 1$ if and only if job j starts at event

e and ends at event f , otherwise $z_{jef} = 1$. This can also be interpreted as an assignment of job j to the event interval $[t_e, t_f]$ in which j must be processed. Hence, IEE incorporates two decisions of SEE that determine a start and an end event for each job into a single decision variable. The complete IEE model states as:

$$\begin{aligned} \min \quad & t_{n+1} \\ \sum_{(e,f) \in \mathcal{A}} z_{jef} &= 1 \quad \forall j \in \mathcal{J} \end{aligned} \quad (36)$$

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot \sum_{e' \leq e < f'} z_{je'f'} \leq R_k \quad \forall k \in \mathcal{R}, e \in \mathcal{E} \quad (37)$$

$$t_e + p_j \cdot \sum_{e \leq e' < f' \leq f} z_{je'f'} \leq t_f \quad \forall j \in \mathcal{J}, (e, f) \in \mathcal{A} \quad (38)$$

$$\begin{aligned} \sum_{(e',f') \in \mathcal{A}: f' > e} z_{ie'f'} + \sum_{(e',f') \in \mathcal{A}: e' \leq e} z_{je'f'} &\leq 1 \\ \forall (i, j) \in \mathcal{P}, e \in \mathcal{E} \end{aligned} \quad (39)$$

$$\begin{aligned} t_e &\geq 0 \quad \forall e \in \mathcal{E} \cup \{n+1\} \\ z_{jef} &\in \{0, 1\} \quad \forall j \in \mathcal{J}, (e, f) \in \mathcal{A}. \end{aligned}$$

The objective function is to minimize the makespan. Equations (36) ensure that every job has exactly one start and end event. Inequalities (37) require that the resource constraints are satisfied at every event. In addition, inequalities (38) indicate that if job j is scheduled between events e and f then $t_e + p_j \leq t_f$ must hold. Next, inequalities (39) forbid for every precedence pair $(i, j) \in \mathcal{P}$ that job i ends at an event later than the start event of job j . In total, IEE has $n \cdot \binom{n+1}{2}$ binary variables and $\mathcal{O}(n \cdot (|\mathcal{R}| + |\mathcal{A}| + |\mathcal{P}|))$ inequalities.

Compared to OOE and SEE, the IEE model requires a factor of $\mathcal{O}(n)$ additional decision variables. In turn, the number of inequalities is slightly smaller in practice and they do not involve any linearization techniques as used in inequalities (13) and (23) for OOE and SEE.

In the next sections, we examine the LP-relaxation of IEE and its polyhedral relationship to OOE, SEE and the time-indexed model DDT.

4.1 LP-Relaxation

In this section we study the quality of the LP-relaxation of IEE. Similar to SEE, the following result is stated for the case of no precedence constraints, therefore it also applies for the general RCPSP.

Proposition 3. *If $\mathcal{P} = \emptyset$ then the LP-relaxation of IEE has an optimal objective value of $p_{max} = \max_{j \in \mathcal{J}} p_j$.*

Proof. Considering inequalities (38) for $(e, f) = (1, n+1)$ we get $t_{n+1} \geq p_j$ for all $j \in \mathcal{J}$. Hence, p_{max} is a lower bound on the optimal LP value. Next, we

construct a feasible LP solution that yields p_{max} also as upper bound. Let a fractional solution of IEE be given by $z_{jef} = \frac{1}{n}$ for all $j \in \mathcal{J}$ and all $(e, f) \in \mathcal{A}$ with $f = e + 1$. Moreover, let $t_e = p_{max} \cdot \frac{e-1}{n}$ for all $e \in \mathcal{E} \cup \{n+1\}$. All other variables are zero. To show LP-feasibility, we check each inequality separately. For inequalities (36) we get

$$\sum_{(e,f) \in \mathcal{A}} z_{jef} = \sum_{e \in \mathcal{E}} \frac{1}{n} = 1$$

for all $j \in \mathcal{J}$. Moreover, for inequalities (38) we get

$$\begin{aligned} & p_j \cdot \sum_{e \leq e' < f' \leq f} z_{je'f'} \\ &= p_j \cdot \frac{f-e}{n} \leq p_{max} \cdot \frac{f-e}{n} = t_f - t_e \end{aligned}$$

for all $j \in \mathcal{J}$ and $(e, f) \in \mathcal{A}$. Inequalities (37) yield

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot \sum_{e' \leq e < f'} z_{je'f'} = \sum_{j \in \mathcal{J}} \frac{r_{jk}}{n} \leq R_k$$

for all $k \in \mathcal{R}$ and $e \in \mathcal{E}$ where the last inequality holds because otherwise there exists a job $j \in \mathcal{J}$ with $r_{jk} > R_k$ which would imply an infeasible problem. Since $t_{n+1} = p_{max}$, the optimal LP value has a lower and upper bound of p_{max} . This shows the proposition. \square

Corollary 3. *The integrality gap of IEE can be arbitrarily large.*

If we insert the same LP solution into the precedence inequalities of IEE we get a similar result as for SEE which indicates that the precedence constraints are slightly stronger in a polyhedral context. Moreover, Proposition 3 shows that the optimal LP value of IEE does not improve upon SEE. Therefore, it remains to verify whether their LP-relaxations are equivalent or not. We will address this question in the next section.

4.2 Relationship to SEE

In the following we compare the LP-relaxations of IEE and SEE. For this, recall the polyhedron $P(SEE)$ as defined in Section 3.3 and define analogously

$$P(IEE) = \{ (t, z) \in \mathbb{R}_{\geq 0}^{n+1} \times [0, 1]^{n \cdot |\mathcal{A}|} : \\ (t, z) \text{ satisfies (36)-(39)} \}$$

as the polyhedron of the linear relaxation of IEE. Again, we deduce a linear transformation

$$x_{je} = \sum_{f > e} z_{jef} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (40)$$

$$y_{jf} = \sum_{e < f} z_{jef} \quad \forall j \in \mathcal{J}, f \in \mathcal{E}^+ \quad (41)$$

between SEE and IEE that reformulates the assignment of jobs to start and end events. Moreover, let $\Phi : P(IEE) \rightarrow P(SEE)$ with $(t, z) \mapsto (t, x, y)$ denote the linear transformation given by (40), (41) and the identity map for the t_e -variables. First, we prove the weak inclusion under the transformation Φ .

Theorem 2. $\Phi(P(IEE)) \subseteq P(SEE)$

Proof. We will show that under the transformation Φ , every inequality of $P(SEE)$ is dominated by an inequality of $P(IEE)$. For inequalities (16) and (17) we get

$$\begin{aligned} \sum_{e \in \mathcal{E}} x_{je} &= \sum_{e \in \mathcal{E}} \sum_{f > e} z_{jef} \stackrel{(36)}{=} 1 \\ \sum_{f \in \mathcal{E}^+} y_{jf} &= \sum_{f \in \mathcal{E}^+} \sum_{e < f} z_{jef} \stackrel{(36)}{=} 1 \end{aligned}$$

for every $j \in \mathcal{J}$. Moreover, for inequalities (18) we have

$$\begin{aligned} \sum_{e' \geq e} x_{je'} + \sum_{f' \leq e} y_{jf'} &= \sum_{e' \geq e} \sum_{f' > e'} z_{je'f'} + \sum_{f' \leq e} \sum_{e' < f'} z_{je'f'} \\ &\leq \sum_{(e', f') \in \mathcal{A}} z_{je'f'} \stackrel{(36)}{=} 1 \end{aligned}$$

for all $j \in \mathcal{J}$ and $e \in \mathcal{E}$. Next, inequalities (23) imply

$$\begin{aligned} & p_j \cdot \left(\sum_{e' \geq e} x_{je'} + \sum_{f' \leq f} y_{jf'} - 1 \right) \\ & \stackrel{(36)}{=} p_j \cdot \left(\sum_{e' \geq e} \sum_{f' > e'} z_{je'f'} + \sum_{f' \leq f} \sum_{e' < f'} z_{je'f'} - \sum_{(e', f') \in \mathcal{A}} z_{je'f'} \right) \\ & = p_j \cdot \left(\sum_{e \leq e' < f' \leq f} z_{je'f'} - \sum_{e' < e < f < f'} z_{je'f'} \right) \\ & \leq p_j \cdot \sum_{e \leq e' < f' \leq f} z_{je'f'} \stackrel{(38)}{\leq} t_f - t_e \end{aligned}$$

for all $j \in \mathcal{J}$ and $(e, f) \in \mathcal{A}$. Furthermore, for inequalities (19) we get

$$\begin{aligned} & \sum_{j \in \mathcal{J}} r_{jk} \cdot \left(\sum_{e' \leq e} x_{je'} - \sum_{f' \leq e} y_{jf'} \right) \\ & = \sum_{j \in \mathcal{J}} r_{jk} \cdot \left(\sum_{e' \leq e} \sum_{f' > e'} z_{je'f'} - \sum_{f' \leq e} \sum_{e' < f'} z_{je'f'} \right) \\ & = \sum_{j \in \mathcal{J}} r_{jk} \cdot \sum_{e' \leq e < f'} z_{je'f'} \stackrel{(37)}{\leq} R_k \end{aligned}$$

for all resources $k \in \mathcal{R}$ and $e \in \mathcal{E}$. Finally, for inequalities (21) we obtain

$$\begin{aligned}
& \sum_{e' \leq e} x_{je} + \sum_{f' > e} y_{if'} \\
&= \sum_{e' \leq e} \sum_{f' > e'} z_{je'f'} + \sum_{f' > e} \sum_{e' < f'} z_{ie'f'} \\
&= \sum_{(e', f') \in \mathcal{A}: e' \leq e} z_{je'f'} + \sum_{(e', f') \in \mathcal{A}: f' > e} z_{ie'f'} \stackrel{(39)}{\leq} 1
\end{aligned}$$

for every $(i, j) \in \mathcal{P}$ and $e \in \mathcal{E}$. Therefore, every inequality of $P(SEE)$ is implied by inequalities of $P(IEE)$ with respect to Φ and this completes the proof. \square

We can now combine the linear transformation Φ_1 given by (40), (41) from IEE to SEE and the linear transformation Φ_2 given by (24) from SEE to OOE in order to get a nested linear transformation $\tilde{\Phi} = \Phi_2 \circ \Phi_1$ from IEE to OOE. It can be written as

$$u_{je} = \sum_{e' \leq e < f'} z_{je'f'} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (42)$$

and states that job j is active at event e if and only if j starts at an event earlier or equal to e and ends at an event later than e . By combining Theorems 1 and 2 it also implies the following.

Corollary 4. $\tilde{\Phi}(P(IEE)) \subsetneq P(OOE)$

Back to SEE, we have not distinguished yet whether $\Phi(P(IEE)) = P(SEE)$ holds what can be assumed since the LP values of SEE and IEE are equal for $\mathcal{P} = \emptyset$ by the same arguments. However, we will give a counterexample.

Proposition 4. $\Phi(P(IEE)) \neq P(SEE)$

Proof. Consider two jobs $\mathcal{J} = \{1, 2\}$ with $p_j = 1$ and one resource with capacity $R = 3$ and resource demands $r_j = 2$ for $j = 1, 2$. Thus, the event sets are given by $\mathcal{E} = \{1, 2\}$ and $\mathcal{E}^+ = \{2, 3\}$. In addition, a feasible LP solution of SEE is given by $x_{1,1} = y_{1,2} = 1$, $x_{2,1} = x_{2,2} = y_{2,2} = y_{2,3} = 0.5$ and $t_1 = 0$, $t_2 = t_3 = 1$. All other variables are zero. We now need to find variable values z_{jef} of IEE such that $\Phi(t, z) = (t, x, y)$. Since $t_2 = t_3$, we have $z_{2,2,3} = 0$ by inequalities (38). In turn, this implies $z_{2,1,2} = z_{2,1,3} = 0.5$. But since $z_{1,1,2} = 1$ the resource constraints (37) of IEE at event $e = 1$ are violated because $2 \cdot (z_{1,1,2} + z_{1,1,3}) + 2 \cdot (z_{2,1,2} + z_{2,1,3}) = 4 > 3$. It follows that for the constructed solution (t, x, y) of SEE there exists no solution (t, z) of IEE such that $\Phi(t, z) = (t, x, y)$. Consequently, $\Phi(P(IEE)) \neq P(SEE)$ which proves the proposition.

Combining Theorem 2 and Proposition 4 we get the following corollary.

Corollary 5. $\Phi(P(IEE)) \subsetneq P(SEE)$

Consequently, IEE strictly dominates SEE in the polyhedral setting which makes IEE the strongest event-based formulation. This leads to the strict domination order of $IEE \succ SEE \succ OOE$ of their respective linear programming relaxations which completes our polyhedral study of the event-based models. Moving a model layer more general again, it remains to study the polyhedral relationship of the event-based models to the time-indexed model DDT of Pritsker et al. (1969). We will study their connection in the next section.

4.3 Relationship to DDT

In this section, we will first introduce the time-indexed model DDT of Pritsker et al. (1969) and after that, we will investigate its polyhedral relationship to IEE.

Time-Indexed Model (DDT). Let $\mathcal{T} = \{0, \dots, T\}$ be a discrete time horizon where T is an upper bound on the makespan. For every job $j \in \mathcal{J}$ and time $t \in \mathcal{T}$ there is a decision variable $x_{jt} \in \{0, 1\}$ with $x_{jt} = 1$ if and only if job j starts at time t and $x_{jt} = 0$ otherwise. For convenience in writing, we assume that for all $j \in \mathcal{J}$ we have $x_{jt} = 0$ for all $t < 0$ (even if not contained in the model) and $x_{jt} = 0$ for all $t > T - p_j$. Then the DDT model states as:

$$\sum_{t \in \mathcal{T}} x_{jt} = 1 \quad \forall j \in \mathcal{J} \quad (43)$$

$$\sum_{j \in \mathcal{J}} \sum_{t' = t - p_j + 1}^t r_{jk} \cdot x_{jt'} \leq R_k \quad \forall k \in \mathcal{R}, t \in \mathcal{T} \quad (44)$$

$$\sum_{t' \geq t - p_i + 1} x_{it'} + \sum_{t' \leq t} x_{jt'} \leq 1 \quad \forall (i, j) \in \mathcal{P}, t \in \mathcal{T} \quad (45)$$

$$x_{jt} \in \{0, 1\} \quad \forall j \in \mathcal{J}, t \in \mathcal{T}.$$

Equalities (43) say that every job gets assigned exactly one start time. Inequalities (44) ensure that the resource capacities are never exceeded and inequalities (45) model the precedence constraints that are due to Christofides et al. (1987). In total, DDT has $\mathcal{O}(n \cdot T)$ variables and $\mathcal{O}(n + T \cdot (|\mathcal{R}| + |\mathcal{P}|))$ constraints. Since the number of variables and constraints scales with T , DDT quickly becomes intractable when T gets large. This explains the relevance for stronger event-based models.

Note that we do not state the makespan objective for DDT because it requires to introduce additional variables and precedence relations but here we are only interested in the basic polyhedral structure of DDT. Define the polytope of the linear relaxation of DDT by

$$P(\text{DDT}) = \{x \in [0, 1]^{n \cdot (T+1)} : x \text{ satisfies (43)-(45)}\}.$$

For our polyhedral comparison between DDT and IEE we will assume that $|\mathcal{E}| \leq |\mathcal{T}|$. In addition, let $\text{IEE}(\mathcal{E})$ denote the IEE model according to the event set $\mathcal{E} = \{1, \dots, n\}$ as used in the paper before. The idea is to expand the event

set and to consider $IEE(\mathcal{T})$ for the discrete time horizon \mathcal{T} . In particular, we can show that an expansion of the event set yields equivalent solutions.

Proposition 5. *Let $\mathcal{E}, \mathcal{E}'$ be two event sets with $|\mathcal{E}| = n$ and $|\mathcal{E}| \leq |\mathcal{E}'|$. Any fractional solution of $P(IEE(\mathcal{E}'))$ can be converted into an equivalent fractional solution of $P(IEE(\mathcal{E}))$ and vice versa.*

Proof. Assume any solution $(t, z) \in P(IEE(\mathcal{E}))$. Since $|\mathcal{E}| \leq |\mathcal{E}'|$ we also have $(t, z) \in P(IEE(\mathcal{E}'))$ by proper indexing of \mathcal{E}' . Conversely, let $(t, z) \in P(IEE(\mathcal{E}'))$. According to inequalities (38) we assume without loss of generality that $t_f = \max_{j \in \mathcal{J}, e < f} (t_e + p_j \cdot z_{jef})$ holds for all $f \in \mathcal{E}^+$. We give a strongly polynomial algorithm to convert (t, z) into a solution (t^*, z^*) of $P(IEE(\mathcal{E}))$. Initially, let $z_{jef}^* = 0$ for all $j \in \mathcal{J}$ and $(e, f) \in \mathcal{A}$. The idea is to collapse redundant events in \mathcal{E}' until we arrive at an equivalent solution for the smaller event set \mathcal{E} . For this, let us consider all intervals $(t_{e'}, t_{f'}]$ for which there exists $z_{j e' f'} > 0$. From these intervals we construct the induced interval graph $\mathcal{G}_I = (V, E)$ with nodes $V = \{(j, e', f') \in \mathcal{J} \times \mathcal{A} : z_{j e' f'} > 0\}$ and edges $E = \{(i, e', f'), (j, e'', f'') \in V \times V : (t_{e'}, t_{f'}] \cap (t_{e''}, t_{f''}] \neq \emptyset\}$. Next, we collect all the maximal cliques C_1, \dots, C_m in \mathcal{G}_I that are uniquely ordered by their appearance from left to right in the interval graph. In particular, for interval graphs we have $m \leq n$. If (j, e', f') is exclusively contained in the cliques C_e, \dots, C_f then we increase $z_{j e' f'}^*$ by $z_{j e' f'}$. This is done for all job intervals. For the remaining variables we set $t_1^* = 0$ and $t_f^* = \max_{j \in \mathcal{J}, e < f} (t_e^* + p_j \cdot z_{jef}^*)$ for all $f \in \mathcal{E}^+$. The resulting solution satisfies $(t^*, z^*) \in P(IEE(\mathcal{E}))$ because none of the constraints (36)-(39) is violated due to collapsing redundant events and by constraints (38) we get that all t_f^* variables are feasible. This shows the proposition. \square

Now consider the expanded model $IEE(\mathcal{T})$ according to the time horizon $\mathcal{T} = \{0, \dots, T\}$. If we take the re-indexed event set \mathcal{T} then we may rewrite the variables of IEE as $z_{j,t,t'}$ where $j \in \mathcal{J}$ and $(t, t') \in \mathcal{A}$ with $t, t' \in \mathcal{T}$. Define the subset of variables $N = \{z_{j,t,t'} : j \in \mathcal{J}, (t, t') \in \mathcal{A} : t' - t = p_j\}$ and the polytopes

$$\begin{aligned} P_{(z, \bar{z})}(IEE(\mathcal{T})) &= \{(z, \bar{z}) : (t, z, \bar{z}) \in P(IEE(\mathcal{T})), \\ &\quad z \in N, \bar{z} \notin N\} \\ P_{(z, 0)}(IEE(\mathcal{T})) &= \{(z, \bar{z}) : (t, z, \bar{z}) \in P(IEE(\mathcal{T})), \\ &\quad z \in N, \bar{z} \notin N, \bar{z} = 0\} \end{aligned}$$

By definition, we directly get the following corollary.

Corollary 6. $P_{(z, 0)}(IEE(\mathcal{T})) \subset P_{(z, \bar{z})}(IEE(\mathcal{T}))$

For the connection to DDT, consider the further projection onto the non-zero variables of $P_{(z, 0)}(IEE(\mathcal{T}))$ given by

$$\begin{aligned} P_z(IEE(\mathcal{T})) &= \{z : (t, z, \bar{z}) \in P(IEE(\mathcal{T})), \\ &\quad z \in N, \bar{z} \notin N, \bar{z} = 0\}. \end{aligned}$$

We now apply the transformation

$$x_{jt} = z_{j,t,t+p_j} \quad \forall j \in \mathcal{J}, (t, t') \in \mathcal{A} : t' - t = p_j$$

in order to get a relation between $P_z(IEE(\mathcal{T}))$ and $P(DDT)$. It turns out that both polytopes are equal.

Theorem 3. $P_z(IEE(\mathcal{T})) = P(DDT)$

Proof. We show that the inequalities that describe $P(DDT)$ are equivalent to those that describe $P_z(IEE(\mathcal{T}))$ or $P_{(z,0)}(IEE(\mathcal{T}))$ respectively. Since for all $j \in \mathcal{J}$ we have $x_{jt} = 0$ for all $t < 0$ and $t > T - p_j$, we also assume that $z_{j,t,t+p_j} = 0$ for all $t < 0$ and $t > T - p_j$ even if the variables are not contained in $IEE(\mathcal{T})$. This will not affect the proof. First, equations (36) and (43) are equivalent since

$$\sum_{t \in \mathcal{T}} x_{jt} = \sum_{t \in \mathcal{T}} z_{j,t,t+p_j} = \sum_{(t,t') \in \mathcal{A}} z_{j,t,t'} \stackrel{(36),(43)}{=} 1$$

for all $j \in \mathcal{J}$. Furthermore, inequalities (37) and (44) are equal because

$$\begin{aligned} & \sum_{j \in \mathcal{J}} r_{jk} \cdot \sum_{t'=t-p_j+1}^t x_{jt'} = \sum_{j \in \mathcal{J}} r_{jk} \cdot \sum_{t'=t-p_j+1}^t z_{j,t',t'+p_j} \\ &= \sum_{j \in \mathcal{J}} r_{jk} \cdot \sum_{(t',t'') \in \mathcal{A}: t' \leq t < t''} z_{j,t',t''} \stackrel{(37),(44)}{\leq} R_k \end{aligned}$$

for all $k \in \mathcal{R}$ and $t \in \mathcal{T}$. Finally, for inequalities (39) and (45) we get

$$\begin{aligned} & \sum_{t' \geq t-p_i+1} x_{it'} + \sum_{t' \leq t} x_{jt'} \\ &= \sum_{t' \geq t-p_i+1} z_{i,t',t'+p_i} + \sum_{t' \leq t} z_{j,t',t'+p_j} \\ &= \sum_{(t',t'') \in \mathcal{A}: t'' > t} z_{i,t',t''} + \sum_{(t',t'') \in \mathcal{A}: t' \leq t} z_{j,t',t''} \stackrel{(39),(45)}{\leq} 1 \end{aligned}$$

for all $(i, j) \in \mathcal{P}$ and $t \in \mathcal{T}$. In particular, inequalities (38) do not affect the polytope $P_z(IEE(\mathcal{T}))$ since for any vector $z \in P_z(IEE(\mathcal{T}))$ we can always find feasible values for the t_e -variables that satisfy these inequalities. Consequently, both polytopes are described by the same set of inequalities which proves the theorem. \square

Hence, lifting $P(DDT)$ (by a zero vector) into the variable space of $IEE(\mathcal{T})$ we get the following consequence of Corollary 6 and Theorem 3.

Corollary 7. *DDT is stronger than IEE(\mathcal{T}).*

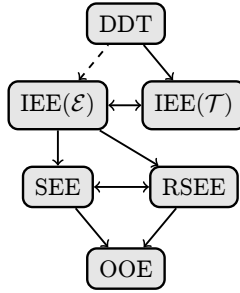


Fig. 1. Model hierarchy of event-based models and time indexed model for the RCPSP. Stronger to weaker models from top to bottom.

Since by Proposition 5 the solutions of $IEE(\mathcal{T})$ and $IEE(\mathcal{E})$ are equivalent we further deduce the following.

Corollary 8. *DDT is stronger than $IEE(\mathcal{E})$.*

Note that Corollary 8 must be regarded with caution because the polytopes $P(DDT)$ and $P_z(IEE(\mathcal{E}))$ exist in different subspaces which makes both formulations incomparable in general. However, since solutions to $P_z(IEE(\mathcal{E}))$ and $P_z(IEE(\mathcal{T}))$ can be made equivalent by adding or deleting redundant events, see Proposition 5, we deduce that $P(DDT)$ yields a stronger polytope than $P_z(IEE(\mathcal{E}))$. Furthermore, we have to remark that this holds only for the projection of the decision variables. Since in DDT, the objective function is modeled via decision variables x_{jt} , see Pritsker et al. (1969), and in IEE via the continuous t_{n+1} variable we get that their objective functions are expressed in disjoint subspaces. In general, this makes it hard to compare the LP bounds of IEE and DDT.

To summarize our results of the last sections, we have shown that DDT can be obtained from IEE by expansion and restriction of its variable space from which we derive a complete model hierarchy of the event-based models that also draw the connection to the time-indexed model, see Figure 1.

5 Preprocessing

Before we turn to our computational results, we study preprocessing techniques that further reduce the model sizes and improve the basic formulations of the three models OOE (u -variables), SEE (x, y -variables) and IEE (z -variables).

5.1 Reducing the Number of Events

In general, not all events $\mathcal{E} = \{1, \dots, n\}$ must have an allocated job in a feasible integer solution to OOE, SEE or IEE. Thus, one approach to reduce the size of the event-based models is to decrease the number of events. Therefore, we ask

if there exists an optimal solution to the RCPSP that uses at most k events or, more generally, if there exists *any* solution to the RCPSP with at most k events. Let us denote the latter decision problem as k -EP. We will show that k -EP is already NP-complete by performing a reduction from the *bin-packing problem* (BPP).

Bin Packing Problem. In the BPP we are given a set of items \mathcal{N} where each item $j \in \mathcal{N}$ has a weight $w_j \geq 0$. Moreover, we are given a set of bins \mathcal{B} where each bin $b \in \mathcal{B}$ has the same capacity C . The decision problem k -BPP of the BPP asks if there exists an assignment of all items to at most k bins such that the total item weight in every bin does not exceed the capacity C . The k -BPP is a well-known NP-complete problem and contains the *partition problem* as a special case, see Garey and Johnson (2002).

Proposition 6. *k -EP is NP-complete.*

Proof. Given a solution to the RCPSP, the number of used events can be retrieved in polynomial time, so k -EP is in NP. Assume an instance of k -BPP and convert it into an instance of k -EP by setting $\mathcal{J} = \mathcal{N}$, $p_j = 1$ for all $j \in \mathcal{J}$ and consider one resource with capacity $R_1 = C$ and $r_{j1} = w_j$ for all $j \in \mathcal{J}$. Using this construction, there exists a feasible bin packing of size at most k if and only if there exists a feasible RCPSP schedule that uses at most k events, or has makespan at most k respectively. Hence, k -BPP yields a 'yes' instance if and only if k -EP yields a 'yes' instance which shows the proposition. \square

Proposition 6 reveals that reducing the number of events is a non-trivial issue. In particular, the actual question of deciding whether there exists an *optimal* schedule with at most k start events is NP-hard because checking whether a given schedule is optimal cannot be done in polynomial time, unless $P = NP$. Note that this does not prevent from finding polynomial-time certificates for a smaller number of events. Since we focus on exact solutions to the RCPSP we will use the complete event set $\mathcal{E} = \{1, \dots, n\}$ for our computational results.

5.2 Eliminating Assignments from Precedence Constraints

As pointed out by Koné et al. (2011), the feasible event set of a job can be reduced by integrating the precedence constraints. Let n_j^- be the number of predecessors and n_j^+ be the number of successors of job j including all transitive precedence relations in the precedence graph $\mathcal{G} = (\mathcal{J}, \mathcal{P})$. Since $\mathcal{E} = \{1, \dots, n\}$ we can assume that every predecessor or successor of a job j starts at its own event, so we exclude any assignment of job j to a start event $e \in \{1, \dots, n_j^-\}$ and an end event $f \in \{n - n_j^+ + 1, \dots, n + 1\}$. Hence, we perform the reductions

$$\begin{aligned} u_{je} &= 0 & \forall j \in \mathcal{J}, e \in \mathcal{E} : e \leq n_j^-, e \geq n - n_j^+ \\ x_{je} &= 0 & \forall j \in \mathcal{J}, e \in \mathcal{E} : e \leq n_j^-, e \geq n - n_j^+ \\ y_{jf} &= 0 & \forall j \in \mathcal{J}, f \in \mathcal{E}^+ : f - 1 \leq n_j^-, f \geq n - n_j^+ + 1 \\ z_{jef} &= 0 & \forall j \in \mathcal{J}, (e, f) \in \mathcal{A} : e \leq n_j^-, f \geq n - n_j^+ + 1 \end{aligned}$$

for each of the models OOE, SEE and IEE respectively. Many current MIP solvers can handle the above equations efficiently by deleting all implied redundant variables and constraints in their preprocessing phase.

5.3 Time Bound Inequalities

In all three models OOE, SEE and IEE the event time inequalities (6), (23), (38) contain the largest number of inequalities but they are also responsible for the generally weak LP bounds. In the following we present two strengthening approaches.

Integrating Time Windows For every job $j \in \mathcal{J}$ we compute a time window $[E_j, L_j]$ in which j must be scheduled. This is done as follows. We first compute an upper bound T on the makespan by applying a list scheduling algorithm, see Kolisch and Hartmann (1999), and perform constraint propagation on the eligible time windows as done in Brucker et al. (2000). In particular, we perform precedence propagations and energetic reasoning propagations, see Baptiste et al. (1999) and Tesch (2016). Thus, the earliest start times respect at least the inequalities $E_i + p_j \leq E_j$ for all $(i, j) \in \mathcal{P}$. Using the time windows $[E_j, L_j]$, we derive inequalities that basically require $t_e \in [E_j, L_j]$ if job j starts at event e . In particular, for OOE we get the inequalities

$$E_j \cdot u_{je} \leq t_e \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (46)$$

$$t_e \leq T + (L_j - T) \cdot u_{je} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (47)$$

which state that $E_j \leq t_e \leq L_j$ if job j is active at event e . Similarly, for SEE we obtain the inequalities

$$E_j \cdot \sum_{e' \leq e} x_{je'} \leq t_e \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (48)$$

$$(E_j + p_j) \cdot \sum_{f' \leq f} y_{jf'} \leq t_f \quad \forall j \in \mathcal{J}, e \in \mathcal{E}^+ \quad (49)$$

$$t_e \leq T + (L_j - p_j - T) \cdot \sum_{e' \geq e} x_{je'} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (50)$$

$$t_f \leq T + (L_j - T) \cdot \sum_{f' \geq f} y_{jf'} \quad \forall j \in \mathcal{J}, e \in \mathcal{E}^+ \quad (51)$$

which imply that $E_j \leq t_e \leq L_j - p_j$ and $E_j + p_j \leq t_f \leq L_j$ if job j starts at event e and ends at event f . Moreover, by applying the transformations (40)

and (41) from SEE to IEE we get equivalent inequalities

$$E_j \cdot \sum_{e' \leq e} \sum_{f' > e'} z_{je'f'} \leq t_e \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (52)$$

$$(E_j + p_j) \cdot \sum_{f' \leq f} \sum_{e' < f'} z_{je'f'} \leq t_f \quad \forall j \in \mathcal{J}, f \in \mathcal{E}^+ \quad (53)$$

$$t_e \leq T + (L_j - p_j - T) \cdot \sum_{e' \geq e} \sum_{f' > e'} z_{je'f'} \quad \forall j \in \mathcal{J}, e \in \mathcal{E} \quad (54)$$

$$t_f \leq T + (L_j - T) \cdot \sum_{f' \geq f} \sum_{e' < f'} z_{je'f'} \quad \forall j \in \mathcal{J}, f \in \mathcal{E}^+. \quad (55)$$

for IEE that imply the same conditions as for SEE. The basic idea of these inequalities was already proposed by Koné et al. (2011) but our inequalities dominate the originally proposed ones. Adding these inequalities we get the following LP bound for SEE and IEE.

Lemma 7. *Let L_P denote the length of the longest path in the precedence graph $\mathcal{G} = (\mathcal{J}, \mathcal{P})$. Adding the time bound inequalities for SEE and IEE yields in both models an optimal LP value of at least L_P .*

Proof. From $L_P \leq \max_{j \in \mathcal{J}} (E_j + p_j)$ after time window preprocessing and the fact that the added inequalities imply $E_j + p_j \leq t_{n+1}$ for all $j \in \mathcal{J}$ we get the desired results. \square

In the next section, we add further inequalities that achieve an LP bound equal to an energetic lower bound for the RCPSP.

Energetic Time Bounds Assume $\mathcal{J}_{ef} \subseteq \mathcal{J}$ contains all jobs that start not earlier than event e and finish not later than event f . In this case, the energetic inequality $\sum_{j \in \mathcal{J}_{ef}} r_{jk} \cdot p_j \leq R_k \cdot (t_f - t_e)$ must hold which means that the consumed energy in the interval $[t_e, t_f]$ must not exceed the available energy. Hence, we add the following inequalities to SEE

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot p_j \cdot \left(\sum_{e' \geq e} x_{je'} + \sum_{f' \leq f} y_{jf'} - 1 \right) \leq R_k \cdot (t_f - t_e) \quad \forall k \in \mathcal{R}, (e, f) \in \mathcal{A} \quad (56)$$

while for IEE we get by (40) and (41) equivalently

$$\sum_{j \in \mathcal{J}} r_{jk} \cdot p_j \cdot \sum_{e \leq e' < f' \leq f} z_{je'f'} \leq R_k \cdot (t_f - t_e) \quad \forall k \in \mathcal{R}, (e, f) \in \mathcal{A}. \quad (57)$$

Due to the weak modeling possibilities, we do not achieve similarly strong inequalities for OOE, so we omit them.

Lemma 8. *Let $B = \max_{k \in \mathcal{R}} \left(\sum_{j \in \mathcal{J}} r_{jk} \cdot p_j \right) / R_k$ denote the energetic lower bound for the RCPSP. Adding the energetic inequalities for SEE and IEE yields an optimal LP value of at least B in both models.*

Proof. For SEE and IEE consider the energetic inequality with $(e, f) = (1, n+1)$ which implies $\sum_{j \in \mathcal{J}} r_{jk} \cdot p_j \leq R_k \cdot t_{n+1}$ for every $k \in \mathcal{R}$. This proves the result. \square

Our computational experience showed that adding all energetic inequalities does not substantially improve the solving performance of SEE (RSEE respectively) and IEE. Therefore, we add inequalities (56) and (57) only for all $k \in \mathcal{R}$ and $(e, n+1) \in \mathcal{A}$ in order to focus on the LP bound that is determined by t_{n+1} .

5.4 Maximal Interval Event Length

The IEE model considers binary variables z_{jef} for every job $j \in \mathcal{J}$ and every pair of events $(e, f) \in \mathcal{A}$. Hence, it has a factor of $\mathcal{O}(n)$ more binary variables compared to OOE and SEE which constitutes a potential bottleneck for solving IEE. However, in most integer solutions of IEE where $z_{jef} = 1$ for some job $j \in \mathcal{J}$ the distance $f - e$ between the start event e and the end event f is rather small. Therefore, the idea is to compute an upper bound $\delta_j \geq f - e$ for every job $j \in \mathcal{J}$ such that there exists no feasible schedule with $z_{jef} = 1$ and $f - e > \delta_j$. In this case, all variables z_{jef} with $f - e > \delta_j$ can be eliminated.

Hence, consider a fixed job $j \in \mathcal{J}$. We denote δ_j as the maximum number of different jobs that can start while job j is active. Thus, assume job j is processed in the time interval $[S_j, S_j + p_j]$. To compute δ_j , we maximize the number of jobs that can simultaneously start in the time interval $[S_j, S_j + p_j - 1]$ (left knapsack) and at time $S_j + p_j - 1$ (right knapsack). In order to keep the computations simple, the left knapsack takes energetic bounds, that means for every resource $k \in \mathcal{R}$ it has capacity $(R_k - r_{jk}) \cdot (p_j - 1)$ and every job $i \neq j$ has weight $r_{ik} \cdot p_i$. In turn, the right knapsack takes one-dimensional bounds, that means for every resource $k \in \mathcal{R}$ it has capacity $R_k - r_{jk}$ and every job $i \neq j$ has weight r_{ik} . Moreover, we allow only jobs i with $p_i \leq p_j - 1$ to be assigned to the left knapsack. Otherwise, if $p_i > p_j - 1$ then job i is active at time $S_j + p_i - 1$ and, without loss of generality, we can shift i to start at time $S_j + p_j - 1$ such that it is only contained in the right knapsack. Let \mathcal{J}_j^P and \mathcal{J}_j^S be the set of predecessors and successors of job j according to the precedence graph including all transitive precedence relations. The set of jobs that can be assigned to the left and right knapsack is given by

$$\begin{aligned} \mathcal{J}_j^L &= \{i \in \mathcal{J} : i \neq j, i \notin \mathcal{J}_j^P \cup \mathcal{J}_j^S, p_i \leq p_j - 1\}, \\ \mathcal{J}_j^R &= \{i \in \mathcal{J} : i \neq j, i \notin \mathcal{J}_j^P \cup \mathcal{J}_j^S\} \end{aligned}$$

where $\mathcal{J}_j^L \subseteq \mathcal{J}_j^R$, see Figure 2 for an example.

We model the combined knapsack problem as an integer program with binary variables $v_i^L, v_i^R \in \{0, 1\}$ that are equal to one, if job i is assigned to the left or

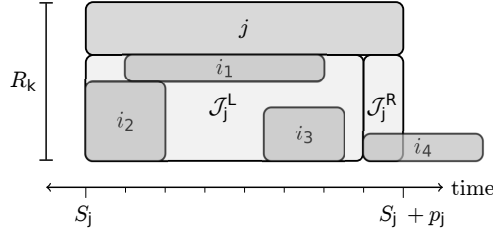


Fig. 2. Maximum number of jobs that can start while job j is active: the left knapsack contains jobs $i_1, i_2, i_3 \in \mathcal{J}_j^L$ using energetic bounds and the right knapsack contains the job $i_4 \in \mathcal{J}_j^R$ using a one-dimensional knapsack bound.

the right knapsack respectively. Thus, we solve

$$\delta_j = \max \sum_{i \in \mathcal{J}_j^L} v_i^L + \sum_{i \in \mathcal{J}_j^R} v_i^R$$

$$v_i^L + v_i^R \leq 1 \quad \forall i \in \mathcal{J}_j^L \quad (58)$$

$$\sum_{i \in \mathcal{J}_j^L} r_{ik} \cdot p_i \cdot v_i^L \leq (R_k - r_{jk}) \cdot (p_j - 1) \quad \forall k \in \mathcal{R} \quad (59)$$

$$\sum_{i \in \mathcal{J}_j^R} r_{ik} \cdot v_i^R \leq R_k - r_{jk} \quad \forall k \in \mathcal{R} \quad (60)$$

$$v_i^L + v_k^L \leq 1$$

$$\forall (i, k) : i \in \mathcal{J}_j^L, k \in \mathcal{J}_j^L \cap \mathcal{J}_i^S : p_i + p_k > p_j - 1 \quad (61)$$

$$v_i^R + v_k^R \leq 1 \quad \forall (i, k) : i \in \mathcal{J}_j^R, k \in \mathcal{J}_j^R \cap \mathcal{J}_i^S \quad (62)$$

$$v_i^L \in \{0, 1\} \quad \forall i \in \mathcal{J}_j^L$$

$$v_i^R \in \{0, 1\} \quad \forall i \in \mathcal{J}_j^R.$$

The objective function maximizes the number of assigned jobs while every job can be assigned to at most one knapsack by inequality (58). As introduced, the left and right knapsack inequalities are given by (59) and (60). Furthermore, inequalities (61) and (62) forbid the assignments of two invalid precedence-constrained jobs to the same knapsack. Even though the stated problem is theoretically hard (contains the knapsack problem), it can be solved very quickly by current MIP solvers. After computing the values δ_j for every job $j \in \mathcal{J}$, we eliminate all variables z_{jef} with $j \in \mathcal{J}$ and $f - e > \delta_j$. On practical instances, we observe that this reduction approach generally more than halves the number of variables of IEE. However, note that the same approach cannot be used to reduce the set \mathcal{A} since this would cause inconsistencies between the t_e variables due to the time constraints (6), (23) and (38). In the next section, we will give more detailed informations on the obtained computational results.

6 Computational Results

In this section, we analyze the computational performance of the presented models on the J30 and J60 test sets of the PSPLIB (Kolisch and Sprecher 1997) where each consists of 480 instances with 30 jobs and 60 jobs respectively. Both test sets consider four resources of different capacities, resource demands and individual precedence constraints. In previous works, the instances have been distinguished by different parameters such as: order strength, network complexity, resource factor, resource strength, disjunction ratio and process range, see Artigues et al. (2013) and Koné et al. (2011).

We implemented the models DDT, OOE, SEE, RSEE and IEE using the C++ interface of the commercial MIP solver Gurobi 7.5.1 in default settings. The tests are performed on an Intel Xeon E5-2680 CPU with 2.7 Ghz using 8 cores for each instance. The time limit of each instances was set to 600 seconds.

For each of the test sets J30 and J60 we compute two charts. The first chart displays the number of instances where the optimality gap is below or equal to the given value on the x -axis. The *optimality gap* is defined as $1 - \frac{lb}{ub}$ where lb is the computed dual bound and ub the computed primal bound after solving each instance. Thus, the first chart shows the real performance of exact solving the RCPSP. The second chart shows the number of instances in which the min-primal-dual gap is below or equal to the given value on the x -axis. The *min-primal-dual* gap is defined as $1 - \max\left(\frac{ub^*}{ub}, \frac{lb}{lb^*}\right)$ where ub and lb are defined as before while ub^* and lb^* are the best known upper and lower bound of the considered instance. Hence, the second chart displays the approximation quality to the lower or the upper bound. This is because we observed that in the beginning of the solving process the MIP solver often decides to improve either the primal or dual bound while the other bound is disregarded during the remaining solving process. Hence, on many instances one criterion outperforms the other one, especially on the J60 instances.

Before turning to our computational results, we mention that DDT should be considered separately because event-based models are supposed to apply for the case where the time horizon is large. On the J30 and J60 instances however the time horizon has moderate size, so DDT has small size and therefore performs quite well compared to the event-based models. However, one can easily scale the time horizon and processing times of any instance by a large factor such that DDT will always be outperformed by the event-based models as shown in Koné et al. (2011). Nevertheless, to allow a comparison on commonly known RCPSP instances we will also show the results of DDT.

For all event-based models we consider the full event set $\mathcal{E} = \{1, \dots, n\}$ and apply all preprocessing steps of Section 5. For DDT, we restrict the decision variables of each job $j \in \mathcal{J}$ to the time windows $[E_j, L_j]$ as given in Section 5.3.

On the J30 test set, see Figure 3, all event-based models show an almost equivalent performance for the optimality gap. The reason for this is mainly due to the strong influence of the time window preprocessing on the J30 instances that equals the solving behavior of the event-based models. Similarly,

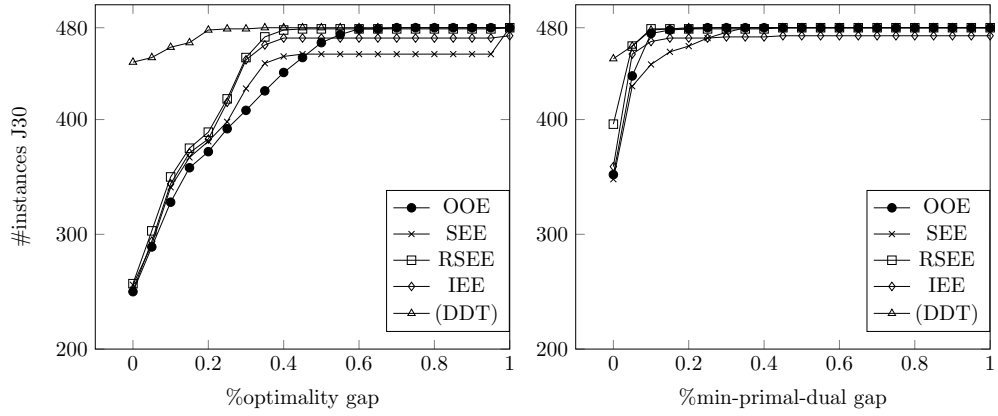


Fig. 3. J30 instances: optimality gap (left) and min-primal-dual gap (right).

DDT highly benefits from the time window preprocessing because the resulting number of variables and constraints is sufficiently small such that it outperforms the event-based models. Considering the min-primal-dual gap, we see that RSEE dominates all other event-based models. That means RSEE is superior to all other event-based models in the primal or the dual bound. Furthermore, this shows that the sparse representation of RSEE has a positive impact on the MIP solving performance. Moreover, RSEE closes the gap to DDT which shows that the event-based models can compete with DDT in the primal or dual bound approximation on instances where DDT should be highly superior.

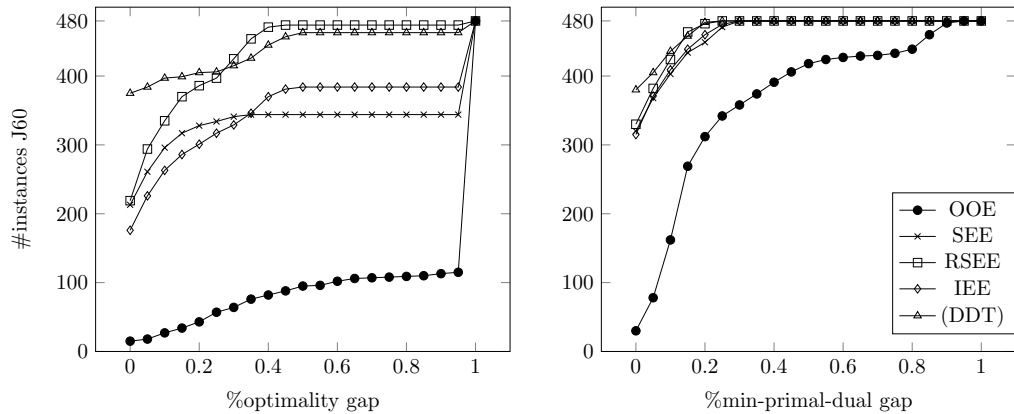


Fig. 4. J60 instances: optimality gap (left) and min-primal-dual gap (right).

On the J60 test set, see Figure 4, the influence of the time window preprocessing decreases and the more model-specific solving performance comes to light. For the optimality gap we observe again that RSEE solves as many instances to optimality as its polyhedrally equivalent counterpart SEE but highly dominates all other event-based models on the overall scale. This shows that the sparsity of RSEE has a high impact on the solving performance. Moreover, it replaces DDT as the best model from an optimality gap of 30% where DDT is not able to make progress on a couple of hard instances. Despite the stronger formulation and additional preprocessing, the new IEE model underlies SEE in direct comparison. The main reason is the still large number of variables and its seemingly complex polyhedral structure that causes expensive LP solving. In contrast to the other models, IEE barely exited the root node of the search tree such which lets us believe that IEE has remaining potential if the LP subproblems can be solved more efficiently. Reversely, OOE reflects its proven theoretical strength and is clearly inferior to all other models. For the min-primal-dual criterion, we observe again that RSEE dominates all other event-based models while SEE and IEE perform almost identically. The models RSEE, SEE and IEE are even almost as strong as DDT in the primal or dual bound. Again, OOE has the worst performance since it is not able to close both primal and dual gap because of its weak linear relaxation. Remarkably, the event-based models SEE, RSEE and IEE achieve the best known dual or primal bound on about 62% of the instances.

In the following we will give a quick summary of each model:

- **OOE**: fast on small instances; poor performance on large instances; weak linear relaxation
- **SEE**: decent overall performance; small model size; decent LP bound
- **RSEE**: best event-based model; benefits sparse constraint matrix; decent LP bound
- **IEE**: strongest event-based model in theory; average performance; large number of variables; expensive LP solving
- **(DDT)**: best model when time horizon is small; good LP bounds; inferior to all event-based models for large time horizon

Comparing our results to Koné et al. (2011), we are able to considerably improve upon the computational performance of all event-based models. In Koné et al. (2011), OOE was declared as the best performing model while SEE was considerably outperformed. Our theoretical and computational results show the opposite. While in Koné et al. (2011), SEE solved 2.9% of all J30 instances to optimality, we achieve 53.5% and 82.5% where lower or upper bound is optimal (RSEE). We believe that the main reason is the stronger event time inequalities (23) that have high impact on the dual bound during MIP solving. For the J60 test set we can even solve about 44% of the instances to optimality and for RSEE almost about 95% of all J60 instances are solved within 35% of optimality. Hence, being able to approach the J60 test set with compact formulations constitutes a clear improvement. Naturally, one has to incorporate current developments in MIP solving and computation power but our main improvements are of theoretical nature.

7 Conclusion

We studied the class of event-based models for the RCPSP and gave a complete characterization of their mutual polyhedral relationships and their connection to the common time-indexed model of Pritsker et al. (1969). Our proposed improvements made it possible to approach more difficult test sets of the PSPLIB using event-based models. For the future it will be of interest to further improve the solving performance of event-based models, for example by incorporating more complex on-top algorithms, such as Lagrangian relaxation, in order to use integer substructures (as stated for RSEE) for solving relaxed subproblems by fast combinatorial algorithms.

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