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Rational Solutions of Underdetermined Polynomial Equations

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Abstract

In this paper we report on an application of computer algebra in which mathematical puzzles are generated of a type that had been widely used in mathematics contests by a large number of participants worldwide.

The algorithmic aspect of our work provides a method to compute rational solutions of single polynomial equations that are typically large with $10^2 \ldots 10^5$ terms and that are heavily underdetermined.

It was possible to obtain this functionality by adding a number of new modules for a new type of splitting of equations to the existing package Crack that is normally used to solve polynomial algebraic and differential systems of equations.

1 Motivation

Mathematical puzzles of the form

$$\begin{array}{ccccc} \texttt{ab} & \times & \texttt{cd} & = & \texttt{ebaf} \\ + & & \div & & - \\ \texttt{egf} & \times & \texttt{h} & = & \texttt{hdc} \\ = & & = & & = \\ \texttt{ejc} & \times & \texttt{k} & = & \texttt{bfj} \end{array}$$

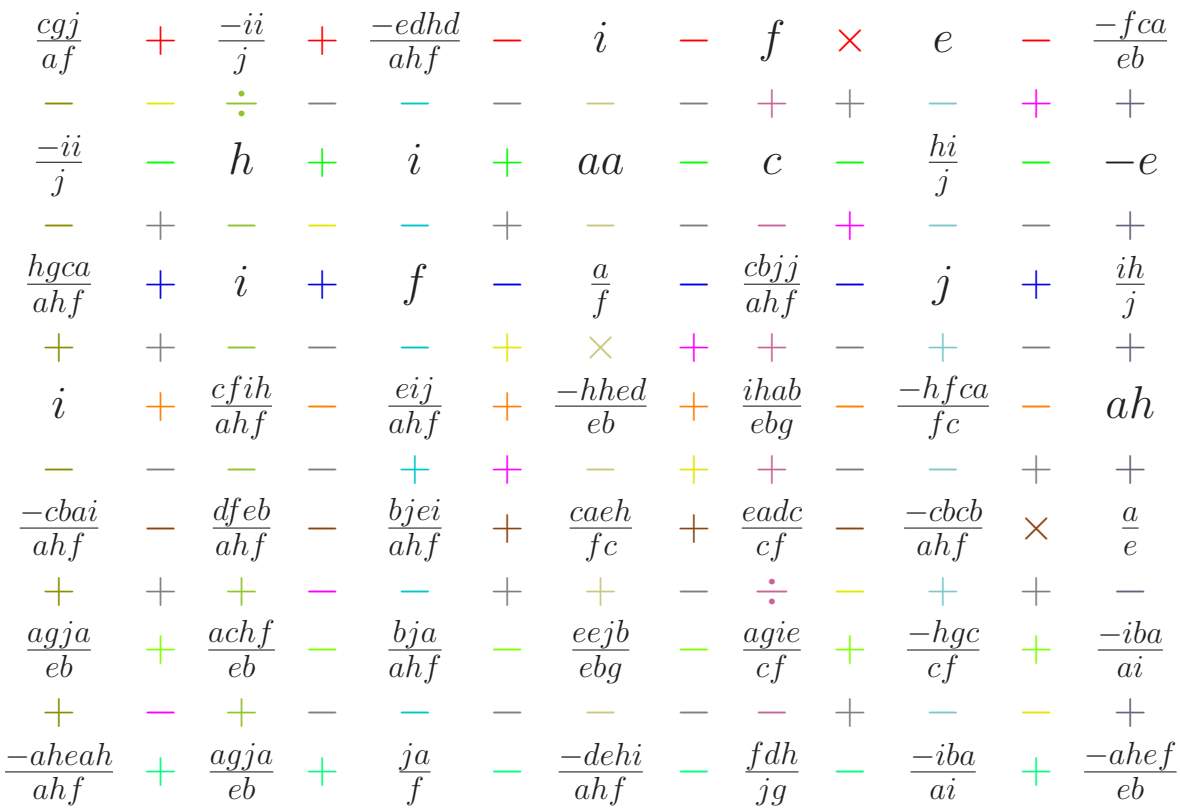
are known for a long time. In there each different letter represents a different digit. For example, `ab` represents a 2-digit integer with two different digits. The challenge is to find the value of each letter such that all 3 horizontal and 3 vertical equations are satisfied. A newer form

$$\begin{array}{ccccc} \texttt{ab} & \times & \texttt{cd} & = & \texttt{ebaf} \\ + & \times & \div & \div & - \\ \texttt{egf} & \times & \texttt{h} & = & \texttt{hdc} \\ = & = & = & = & = \\ \texttt{ejc} & \times & \texttt{k} & = & \texttt{bfj} \end{array}$$

with two additional diagonal relations has been created and first published by one author (TW) on the home page of Caribou Contests [1] in January 2010. Since then each day a new puzzle is shown together with the solution of the problem from the day before. Puzzles like the first one have been used in many Caribou Contests which are run six times a year. In first contests of the 2015/16 school year typically 16,000 students from 15 countries participated each time.

This type of problem is very popular among students. The NEAMC (North Eastern Asian Mathematics Competition) adopted them and now holds each year one event dedicated exclusively to such problems, taken them with permission from the Caribou website, 2015 for the first time with schools from 9 countries participating.

To make such puzzles even more interesting one can ask what it is that makes puzzles popular. What wide spread puzzles like sudoku and Rubik's cube have in common is that with the last step of their solution several conditions become suddenly fulfilled which gives a deep gratification to the problem solver. With this concept in mind the aim of our application is to create puzzles with even more conditions being suddenly satisfied when the puzzle is finally solved. In the following problem 36 conditions need to be fulfilled:



Not only the values of all 7 rows and 7 columns need to be zero, but also the values of all 11 diagonals running from the top left to the bottom right and the 11 diagonals running from the top right to the bottom left need to evaluate to zero. In these enlarged puzzles the usual order of the operations applies ($\times, \div$ before $+, -$).

In [2] a 7×7 problem is shown with all 49 numbers being integers. To create such a challenge one

1. fills the $(7+6) \times (7+6)$ matrix with randomly selected operators $+, -, \times, \div$, and with 7×7 variables (not sequences of letters representing digits like in the diagram above),
2. formulates the system of $7+7+11+11 = 36$ polynomial equations for the 49 variables,
3. computes rational solutions for this system with possibly several free parameters,
4. repeats the following steps:
   - replaces free parameters by random integer numbers, so that divisors and denominators do not become zero,
   - encodes digits by letters,
   - determines all solutions of the resulting puzzle

   until the created puzzle has only one solution.

The following section describes the computational problem. It is followed by a description of the basic idea for the key computational step, its variations and options and integration with standard techniques. A test to recognize the impossibility of rational solutions early speeds up the computation. After a description of our implementation we report on example computations. The paper closes with a discussion on computational aspects and a summary.

2 The Computational Problem

In the rest of this paper we are concerned with the generation of rational 7×7 puzzles. The system of 7+7+11+11=36 equations involves 49 unknowns, so 13 more unknowns than equations.

The case of 5×5 puzzles with only two diagonal conditions is very similar in that we have 5+5+1+1=12 conditions for 25 unknowns, i.e. also 13 more unknowns than equations.

The nonlinearity of the problem can be chosen in advance through the number of multiplications and divisions that occur in the puzzle. Around five $\times$ or $\div$ operations already create challenging algebraic problems that can not be solved in general and we do not know of computer algebra systems with modules that could find rational solutions of large underdetermined single equations or systems of polynomial equations.

If the puzzle contains only a limited number of multiplications and divisions then many conditions are either completely linear or they contain at least one linearly occuring variable, say u_i. In such an equation $0 = A_i u_i + B_i$ the A_i, B_i are polynomials in any other unknowns u_k except u_i. In this case progress can be made by considering the cases $A_i = 0$ and $A_i \neq 0$ i.e. by eliminating $u_i = -B_i/A_i$.

If no single u_i occurs linearly and if factorization over the rationals is not possible then the next tool of choice are steps towards computing a Gröbner bases, i.e. reductions and the computation of S-polynomials. For that one needs to define an ordering of variables. Lexicographical orderings are notoriously computationally expensive. Using instead a graded lexicographical ordering or total degree ordering keeps the total degree of generated polynomials low but only at the prize of increasing the degree evenly between the variables. This is expensive if there are many variables allowing a huge number of products of powers between them, like in our application. But even if the Gröbner basis can be computed, then mostly all equations are non-linear in all u_k and one has not made much progress towards rational solutions.

Often just one non-factorizable polynomial equation remains to be solved which is non-linear for all its variables and which is just one equation so Gröbner basis computations do not apply.

On the positive side, we have 13 more variables than equations, i.e. our system is heavily underdetermined. Furthermore, we do not need the general solution. Special rational solutions are useful for our purpose.

3 The Basic Idea

If the general solution is not required then additional equations can be added if the new system can then be solved with available techniques in terms of rational solutions. As long as the new system is still underdetermined, the danger of excluding all solutions is negligible. Our strategy will be to add equations by partitioning, i.e. splitting existing equations.

Let us assume a single equation is to be solved which is non-linear in all variables u_m, is not factorizable over the integers and is of degree d_i in u_i and therefore can be written as

$$0 = P(u_j) = \sum_{n=0}^{d_i} A_{in} u_i^n \tag{1}$$

where A_{in} is the coefficient of u_i^n and is a polynomial that may involve all unknowns u_k except u_i.

Method 1) The first approach is to split completely w.r.t. u_i and solve the system

$$0 = A_{in} \quad \text{for} \quad n = 0 \ldots d_i. \tag{2}$$

u_i becomes a free parameter in any obtained solution. These $d_i + 1$ new equations should typically involve up to $d_i + 1$ extra different unknowns to admit solutions.

The new system (2) can be investigated by standard techniques. For example, although the original equation (1) was non-linear in all variables, the equations of the new system (2) might be either linear in some u_p or be factorizable with one factor being linear in at least one u_q and thus the new system could be solvable simple by performing factorizations and substitutions.

Method 2) A less restrictive system derived from partial splitting requires at most $d_i - 1$ extra unknowns, i.e. 2 unknowns less:

$$0 = A_{in} \qquad \text{for} \quad n = 2 \dots d_i \tag{3}$$
$$0 = A_{i0} + A_{i1} u_i \tag{4}$$

where (4) can be used for a (potentially case generating) substitution of u_i. The value of u_i is rational as long as the values of the other u_k are rational.

Method 3) It is even less restrictive to split (1) only once into

$$0 = \sum_{n=2}^{d_i} A_{in} u_i^{n-2} \tag{5}$$
$$0 = A_{i0} + A_{i1} u_i. \tag{6}$$

Here u_i can also be eliminated completely (in the case $A_{i1} \neq 0$). With only one extra equation, only one extra u_k may be required for (5), (6) to admit a solution. For $d_i = 2$ methods 2) and 3) are identical.

An additional benefit of methods 2), 3) compared to method 1) is that u_i can be replaced also in all other equations that may have to be solved apart from (1).

4 Variations and Options

The three methods of equation splitting have several parameters and can be combined with standard solving techniques in different ways and with themselves recursively. For their optimal use one may have to strike a balance between

- finding many different solutions,
- finding solutions involving a maximum number of free parametric variables,
- managing computation times,
- managing computer memory,

all depending on the nature of the application. For example, one may need to solve one specific system and put maximal effort into this one investigation, or as in our case when generating puzzles, one may have many similar systems and could collect rational solutions for any one system and rather start investigating a new system before equations get too large.

The following are questions and answers concerning the three methods. Comments about their integration with other computational steps will be made in the next section.

A) *Does the order of splittings matter?*
The operations to split w.r.t. different variables commute, i.e. it does not matter whether to split first w.r.t. u_i and then u_j or in the opposite order as long as only splittings occur. The situation is different when between splittings also factorizations and substitutions are performed. Then the

order of splittings is relevant. The consequence is that if it is critical to find the maximal number of rational solutions, then one would have to consider all possible orders of splittings and has to perform all possible factorizations and substitutions in-between.

In our case having, for example, 3 equations left to solve after all substitutions and factorizations have been done, each involving 49 - 33 = 16 variables (from the 36 original equations, 33 have been used for substitutions so that 3 equations remain) then 3×16 different partial splittings can be done. These are too many to consider all permutations between them. Even to try all of them once as first splitting would be cumbersome. Therefore heuristics which splittings to try or which to try first are useful and are discussed next.

B) *If one has a system of equations, which equation shall be split with respect to which variable first/next?*
The following heuristics are applied in this order in automatic runs.

- The highest priority is to split an equation with the fewest number of variables for the following reason.

  If a system includes a (polynomial) equation involving only one variable then the variable can be eliminated using this equation or no rational solutions exist (see section 5.2 below). Equations with 2 variables are useful because splitting them gives one or more equations with only one variable which guarantee quick progress.

  Therefore equations with fewer variables lead quicker to progress through substitution or recognizing the non-existence of rational solutions.
- From the equations with the same lowest number of variables the advantage to split the shortest equation is that splitting creates even shorter equations which have the potential to shorten other long equations when substitutions are performed.
- d_i should be low to avoid imposing many restrictions in splitting methods 1) and 2) or avoid size explosion in method 3) when u_i is substituted in higher powers of u_i.
- A_{i0} and especially A_{i1} should be short to avoid length explosion arising from the substitution $u_i = -A_{i0}/A_{i1}$ of methods 2) and 3). Especially the size of A_{i1} is crucial because after substitution all equations are written as polynomials, i.e. all previously u_i independent terms will be multiplied with A_{i1} to the power of the maximum degree of u_i.
- A_{i0}, A_{i1} should involve as few unknowns as possible. After a substitution of u_i the degree of all involved variables will typically increase in other equations, making later partial splittings and substitutions of these variables more costly.

C) *Which of the three splitting methods shall be used and when?*
The answer depends partially on the degree of underdetermination. The more unknowns are available the more restrictive can be the splitting as performed in method 1. On the other hand, if the most general solution is to be found then method 3) is to be preferred.

A disadvantage of method 3) is that the substitution of u_i in the other equation (5) will most likely increase the degree of the other unknowns in (6) with the consequence that unknowns which occur linearly in (3) and allow the complete solution of all equations (3) might not occur linearly in (5) and thus not allow the solution of (5) purely by substitution. On the other hand, to solve (5) by substitution it is enough if only one unknown occurs linearly in it whereas to solve (3) $d_i - 1$ unknowns need to occur linearly in a way that this system can be solved.

The partial splittings introduced in section 3 are useless on their own. They need to be incorporated into a solver for polynomial systems.

5 Integration with Standard Techniques

Before discussing the options we want to describe the program environment to which one module for testing the rationality of solutions and 2 modules for partial separation (splitting) have been added.

5.1 The Package CRACK for Working with Systems of Equations

The computer algebra package CRACK [3], [4], is written in REDUCE [5]. It is a tool to investigate and solve systems of algebraic and differential equations. The following features are relevant for the successful integration of partial splitting methods.

- A system of equations to be solved is always considered together with a set of inequalities including OR-inequalities (lists of expressions of which at least one needs to be non-zero). Inequalities are actively collected, simplified and used to simplify equations and to avoid case distinctions.
- The computation investigates all cases and (sub-)^n cases that come up depending on the steps that are performed, like factorizations, case generating substitutions and adhoc case distinctions motivated by known solutions.
- About 50 modules can be applied with different parameters giving over 90 different calls ranging from substitutions, reductions, algebraic combinations for the shortening of equations, integrations and separations to applying the external packages Singular or DiffAlg to the whole system. In addition there is a large number of diagnostics commands available.
- A solution strategy is composed through a list of numbers (`proc_list`), each number representing a module. The modules are tried in the order they occur in the `proc_list` until one module is successful and then execution starts again at the beginning of the list. The list can be modified interactively but also through modules themselves and can easily be adapted to all types of problems: algebraic/differential, linear/nonlinear, medium/large.
- Equations are associated with property lists that also include results of earlier investigations to avoid duplicate checks.
- Most importantly, the program can be run fully automatically, semi automatically and fully interactively by allowing not only which step is to be done but also with which parameters and which equation(s).

5.2 A Rationality Test

Because the whole computation consists of a large tree of cases and (sub-)^n cases resulting from factorizations of equations and case generating substitutions, the finding of non-rational values of at least one variable in the solution would allow to stop the investigation of the current (sub-)case and to proceed with another (sub-)case. The following simple test of detecting non-rationality is implemented in a new module **89**.

If the system involves an equation in only one variable then this variable can either be eliminated using this equation or the system has no rational solution. If the equation is linear then it is used for substitution. If it is quadratic then if roots are rational then 2 cases are considered and substitutions are made otherwise no rational solutions exist. If the degree is higher than 2 then either the equation is factorizable or it has no rational solutions and computation also stops.

This test could be extended using the Hasse Principle to decide on the non-existence of rational solutions for polynomials of degree 2 in an arbitrary number of variables.

5.3 Our Implementation

To extend the package CRACK with partial separations we added 3 new modules.

 # 89 performs a rationality test as described above. If the system contains a non-linear equation in only one variable then

- if factorizability of this equation has not been completely checked yet then the factorization test of this equation is put on the `to_do_list`, else
- if factorizability is know then the splitting into 2 cases of one of the factors either being zero or being non-zero is put on the `to_do_list`, else
- the equation is not factorizable, investigation of the current case is terminated.

Any entry in the `to_do_list` is executed by the first module of the `proc_list`.

 # 90 implements method 2) as described in section 3. It is only applied if none of the equations is linear in any u_j. This is guaranteed in automatic mode because in `proc_list` it is placed after case-generating substitutions (module **21**). In interactive mode module **90** issues a hint if substitution is possible.

If no substitution is possible then all pairs (P, u_i) of equations $0 = P$ and variables u_i appearing in P are discarded if

- any one coefficient A_{in} of $u_i^n,\ n > 1$ can be shown to be non-zero based on the inequalities that are known, or
- if A_{in} and in the case that A_{in} is factorizable also all its factors are nonlinear in all their u_k.

For the remaining pairs all those which allow the solution of the system (3), (4) by the REDUCE `solve` command are listed so that in interactive mode the user can select a pair (P, u_i) and in automatic (batch-)mode a pair from the shortest equation P is selected. The corresponding equations (3), (4) are added to the current system of equations and execution proceeds with the start of the `proc_list`.

 # 91 implements method 3). This module is also only executed if no equation is linear in any one u_i. In addition only those pairs (P, u_i) are considered for which is $A_{i1} \neq 0$ so that u_i can be eliminated. From all those pairs the user can select a pair in interactive mode and in automatic mode one pair is selected according to weights discussed under **B)** in section 4. For the selected pair (P, u_i) an induced case splitting into the two cases $A_{i0} + u_i A_{i1} = 0$ and $A_{i0} + u_i A_{i1} \neq 0$ is put on the `to_do_list` to be executed next. In this way, if both cases are considered, no solution is lost. In practise the case $A_{i0} + u_i A_{i1} \neq 0$ will hardly ever be tried because of the many further case distinctions that would be necessary because $A_{i0} + u_i A_{i1} \neq 0$ itself does not provide any simplification. In the case $A_{i0} + u_i A_{i1} = 0$ the other equations (5) are derived automatically later when $A_{i1} = 0$ or $u_i = -A_{i0}/A_{i1}$ are considered.

The `proc_list` used for the computations in section 6 is (1 89 20 77 47 21 38) where the numbers encode the following modules which are continuously tried in this order. They perform, if possible,

1 any urgent steps listed on the `to_do_list` which initially is empty but can be filled through other modules (e.g. **89**) during the computation. Entries on this list consist of a module number and a list of equations that the module is to be applied to.

89 the rationality test described under 5.2

20 a substitution that does not generate a case distinction

77 the factorization of any one equation

47 the start of a case distinction of an equation that is factorized

21 a case-generating substitution

38 exit of automatic (batch-) mode and entering interactive mode because the earlier modules in this list can not make progress.

When execution is stopped because **38** was reached then one can save the worksheet and try interactively, for example, these modules

90 the partial splitting method 2) described above

91 the partial splitting method 3) described above

27 a polynomial reduction step according to some predefined ordering

30 a Gröbner step according to some predefined ordering

59 the system including a monomial ordering is exported and the package SINGULAR [6] is called to compute a Gröbner basis in a prescribed time limit which. In case of success the Gröbner basis is read back into CRACK for further solution.

The calculation described in section 6 below runs automatically as long as at least one module in `proc_list` can be applied. When the computation stops, module **90** is executed interactively by simply typing 90 <Enter>. Afterwards the computation continues fully automatically based on **proc_list** until the next time that module **90** is called interactively. The whole computation could be run fully automatically with the `proc_list` (1 89 20 77 47 21 90). The only reason that we execute module **90** manually is to record the number of equations still to be solved and their size and to report these data in the following section to give the reader an impression of the tkind of systems to which module **90** was applied.

Module **90** is programmed to select and perform one partial splitting and not to start a large case distinction and finally trto y all possible splittings as it would be done if module **91** would be applied.

The reason that we use module **90** and not **91** is that the whole tree of all possible cases of all splittings of all equations w.r.t. all u_i and that recursively would be far too large. Another reason is that we do not need too many solutions with the same operator setting like (7) but we rather want to repeat computations with other random operator settings to generate differently looking puzzles.

An optimal `proc_list` is found by following general principles and using experience. The list also dependents on the size and nature of the computational problem. Interactive experiments about the effect of different modules at different stages of the solution process help also to improve the list and to adapt it to a specific computational problem.

6 Example Computations

The following operator setting involves 3 multiplications and 2 divisions.

$$\begin{array}{cccccccccccccc}
u_1 & + & u_2 & + & u_3 & - & u_4 & - & u_5 & \times & u_6 & - & u_7 \\
- & - & / & - & - & - & - & - & + & + & - & + & + \\
u_8 & - & u_9 & + & u_{10} & + & u_{11} & - & u_{12} & - & u_{13} & - & u_{14} \\
- & + & - & - & - & + & - & - & - & + & - & - & + \\
u_{15} & + & u_{16} & + & u_{17} & - & u_{18} & - & u_{19} & - & u_{20} & + & u_{21} \\
+ & + & - & - & - & + & \times & + & + & - & + & - & + \\
u_{22} & + & u_{23} & - & u_{24} & + & u_{25} & + & u_{26} & - & u_{27} & - & u_{28} \\
- & - & - & - & + & + & - & + & + & - & - & + & + \\
u_{29} & - & u_{30} & - & u_{31} & + & u_{32} & + & u_{33} & - & u_{34} & \times & u_{35} \\
+ & + & + & - & - & + & + & - & / & - & + & + & - \\
u_{36} & + & u_{37} & - & u_{38} & - & u_{39} & - & u_{40} & + & u_{41} & + & u_{42} \\
+ & - & + & - & - & - & - & - & - & + & - & - & + \\
u_{43} & + & u_{44} & + & u_{45} & - & u_{46} & - & u_{47} & - & u_{48} & + & u_{49}
\end{array} \tag{7}$$

Using the above `proc_list` the modules **1, 89, 20, 77, 47, 21, 38** are tried in this order where module **47** starts a case distinction if an equation is known to be factorizable (found before by module **77**) and where module **21** starts a case distinctions if a case generating substitution is possible. When module **38** is executed and automatic execution stops the partial splitting module **90** is used for the first time in *case 1.1.2.2* which has one equation with 1027 terms in 12 variables left to be solved. Module **90** replaces it with one equation with 1014 terms and one factorizable equation with 7 terms. No rational solution results from this splitting

Later in *case 1.2* one equation with 1268 terms and 13 variables remains. Module **90** replaces it with one equation with 1253 terms and one factorizable equation with 15 terms. This system has 4 rational solutions resulting in cases:

- *1.2.1.2.1.2* with 10 free parameters and 3814 terms in numerators and denominators of the 49-10=39 expressions for the 39 computed variables.
- *1.2.1.2.2* with 11 free parameters and 27032 terms in the expressions for the 38 computed variables.
- *1.2.2.2.1.2.2* with 10 free parameters and 3973 terms in expressions for the 39 computed variables.
- *1.2.2.2.2* with 11 free parameters and 3973 terms in expressions for the 38 computed variables.

The next time module **90** is applied is in *case 2.1.2* having one equation with 605 terms and 13 variables left to be solved. This equation is replaced by one equation with 599 terms and one factorizable equation in 5 terms. This system has no rational solution.

Afterwards in *case 2.2* one equation with 118879 terms in 14 variables remains. Module **90** is applied and replaces this equation by 6 equations with 105, 649, 2922, 10816, 35007, 69380 terms. When a substitution is performed later, the system becomes too large to continue the computation on the 32 GB 64 Bit linux machine running the computer algebra system REDUCE.

The 7×7 puzzle in section 1 is obtained when in the second of the above four solutions the 11 parameters are replaced by some small integers and afterwards digits are replaced by letters. It is verified that this puzzle has a unique solution.

7 Comments on Solutions to the Application

A key parameter in the generation of 7×7 puzzles is the number of multiplications and divisions. The higher their number is, the more nonlinear is the algebraic system, the more terms are involved in rational solutions and the larger are the numbers in the puzzle.

To increase the fun part in solving large puzzles and to avoid the tedious part of replacing a letter by a digit everywhere and checking the numerical correctness of rows, columns and diagonals an interactive puzzle solving page [7] was created. On this page (at the bottom of 7×7) the manual solution of the above puzzle is shown.

To generate 7×7 *integer* puzzles instead of rational puzzles one could impose extra conditions $A_{i1} = 1$ and pay with the excess of available variables.

8 Summary

Apart from progress in the creation of highly rewarding mathematical puzzles a simple method with three variations has been suggested to find rational solutions of heavily underdetermined single equations or systems of polynomial equations. These methods separate equations partially. An important ingredient to a successful application is the integration with other conventional methods for solving polynomial systems. Such an integration with the package CRACK was easily possible due to the modular and interactive nature of CRACK. An example computation has been described together with a mathematical puzzle that has been generated.

References

[1] Caribou Contests: `cariboutests.com`

[2] Zarboni, A. and Wolf, T.: Caribou Math Contests: Not just a contest but a way of learning and enjoying math, Fields Notes, v. 15:2, p 6-7 (2015), `https://www.fields.utoronto.ca/aboutus/annual_reports/FieldsNotesSummer2015.pdf`

[3] Wolf, T: Applications of CRACK in the Classification of Integrable Systems, CRM Proceedings and Lecture Notes, 37 (2004) pp. 283-300.

[4] Wolf, T.: CRACK online tutorial: `https://lie.ac.brocku.ca/crack/demo/`
Last Version: `http://lie.ac.brocku.ca/twolf/crack/crack.last.version.tar.gz`

[5] Hearn, A.: The Computer Algebra System REDUCE
`http://reduce-algebra.sourceforge.net/` (2008).

[6] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: SINGULAR 4-0-2 — A computer algebra system for polynomial computations. `http://www.singular.uni-kl.de` (2015).

[7] Hill, M.: Interactive Calcrostic Puzzle:
`https://test.cariboutests.com/test_game/calcrostic.php`