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# **Rank-Perfect and Weakly Rank-Perfect Graphs<sup>1</sup>**

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# Rank-Perfect and Weakly Rank-Perfect Graphs

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## Abstract

An edge  $e$  of a perfect graph  $G$  is critical if  $G - e$  is imperfect. We would like to decide whether  $G - e$  is still “almost perfect” or already “very imperfect”. Via relaxations of the stable set polytope of a graph, we define two superclasses of perfect graphs: rank-perfect and weakly rank-perfect graphs. Membership in those two classes indicates how far an imperfect graph is away from being perfect. We study the cases, when a critical edge is removed from the line graph of a bipartite graph or from the complement of such a graph.

## 1 Introduction

The present paper aims at finding some kind of a stage of imperfectness. BERGE proposed to call a graph  $G = (V, E)$  **perfect** if, for each of its (node-induced) subgraphs  $G' \subseteq G$ , the chromatic number  $\chi(G')$  equals the clique number  $\omega(G')$ . That is, for all  $G' \subseteq G$ , we need as many stable sets to cover all nodes of  $G'$  as the maximum clique of  $G'$  has nodes (a set  $V' \subseteq V$  is a clique (stable set) of  $G$  if the nodes in  $V'$  are mutually (non-)adjacent; a maximum clique (stable set) contains a maximal number of nodes).

BERGE [1] conjectured two characterizations of perfect graphs. His first conjecture was that a graph  $G$  is perfect iff the clique covering number  $\bar{\chi}(G')$  equals the stability number  $\alpha(G') \forall G' \subseteq G$  (i.e., that we need as many cliques to cover all nodes of  $G'$  as a maximum stable set of  $G'$  has nodes). Since complementation transforms stable sets into cliques and colorings into clique coverings, we have  $\alpha(G) = \omega(\bar{G})$  and  $\chi(G) = \bar{\chi}(\bar{G})$  where  $\bar{G}$  denotes the complement of  $G$ . Hence, BERGE conjectured and LOVÁSZ [8] proved that a graph  $G$  is perfect if and only if its complement  $\bar{G}$  is. The second BERGE

conjecture concerns a characterization of perfect graphs via forbidden subgraphs. It is a simple observation that chordless odd cycles  $C_{2k+1}$  with  $k \geq 2$ , termed **odd holes**, and their complements  $\overline{C_{2k+1}}$ , called **odd antiholes**, are imperfect. Clearly, each graph containing an odd hole or an odd antihole as subgraph is imperfect as well. BERGE conjectured in [1] (*Strong Perfect Graph Conjecture*): a graph is perfect iff it contains neither odd holes nor odd antiholes as subgraphs. PADBERG [11] introduced the notion of **minimally imperfect graphs** which are imperfect graphs such that all of their proper induced subgraphs are perfect. Using this term, the Strong Perfect Graph Conjecture reads: odd holes and odd antiholes are the only minimally imperfect graphs. In order to give a characterization of minimally imperfect graphs (and thereby to verify or falsify the Strong Perfect Graph Conjecture), many fascinating structures of such graphs have been discovered, see [8, 11].

The motivation to ask for measuring imperfectness was the following: we studied in [14, 15, 16] **critical edges** in perfect graphs  $G$ , i.e., edges  $e$  the deletion of which yields an imperfect graph  $G - e$ . In order to decide whether the resulting graph  $G - e$  is still “almost perfect” or already “very imperfect”, some kind of stage of imperfectness is required. For that, we introduce two superclasses of perfect graphs by relaxing perfectness with respect to the polyhedral characterization via the stable set polytope, see [16] for more details.

The **stable set polytope**  $\text{STAB}(G)$  of a graph  $G$  is defined as the convex hull of the incidence vectors of all stable sets of  $G$ .  $\text{STAB}(G)$  has, as every polytope, a representation as the intersection of finitely many half-spaces, but the complete system of facet-defining inequalities is unknown for most graphs. An example is the class of **nonnegativity constraints**

$$x_i \geq 0 \tag{0}$$

for all nodes  $i$  of  $G$ . **Clique constraints**

$$\sum_{i \in Q} x_i \leq 1 \tag{1}$$

with  $Q \subseteq G$  clique are valid with respect to  $\text{STAB}(G)$  for every graph  $G$  (since a clique and a stable set have at most one node in common). The stable set polytope of every graph is contained in the polytope given by the nonnegativity constraints and clique constraints. Perfect graphs are precisely those graphs for which nonnegativity and clique constraints suffice to describe their stable set polytope [2]. Following a suggestion of GRÖTSCHEL,

LOVÁSZ, and SCHRIJVER [5] one may relax the notion of perfectness by generalizing clique constraints to other classes of inequalities valid for the stable set polytope and then by investigating all graphs such that their stable set polytope is entirely described by nonnegativity constraints and the inequalities in question. A natural way to generalize clique constraints is to consider all 0/1-inequalities, i.e., to investigate the **rank constraints**

$$\sum_{i \in G'} x_i \leq \alpha(G') \quad (2)$$

associated with arbitrary induced subgraphs  $G' \subseteq G$ . Every rank constraint is obviously valid for the stable set polytope and defines in some cases also a facet (see next section for examples of graphs  $G$  where the full rank constraint associated with the whole graph  $G$  is facet-defining). If a rank constraint is associated with a *proper* subgraph  $G' \subset G$ , then it does not yield a facet of  $\text{STAB}(G)$  in general, even if  $\sum_{i \in G'} x_i \leq \alpha(G')$  is facet-defining for  $\text{STAB}(G')$ . In the latter case, we can determine a facet

$$\sum_{i \in G'} x_i + \sum_{i \in G - G'} a_i x_i \leq \alpha(G') \quad (3)$$

of the stable set polytope of the whole graph  $G$  by computing appropriate coefficients  $a_i$  for all nodes  $i$  in  $G - G'$  via sequential lifting [10] (see Section 2). We call the obtained facets (3) **weak rank constraints** if the base rank constraint associated with  $G'$  is facet-defining for  $\text{STAB}(G')$ . Clearly, every facet-defining rank constraint is a weak rank constraint with  $a_i = 0$  for  $i \in G - G'$ . All remaining nontrivial facets have the form

$$\sum_{i \in G} a_i x_i \leq \alpha(G, a) \quad (4)$$

where we interpret the vector  $a = (a_1, \dots, a_n)$  to be a node weighting of  $G$  associating the weight  $a_i$  to  $i \in G$  and denote the weighted graph by  $(G, a)$ . Furthermore,  $\alpha(G, a) = \max\{\sum_{i \in S} a_i : S \subseteq G \text{ stable set}\}$  stands for the weighted stability number. The stable set polytope of every graph is entirely described by all trivial facets (0) and all nontrivial facets (4). With the above inequality classes, we analogously define three further polytopes associated with  $G$ :  $\text{WSTAB}(G)$  as the polytope described by facets of type (0) and (3),  $\text{RSTAB}(G)$  defined by the inequality classes (0) and (2), and  $\text{QSTAB}(G)$  to be the polytope with (0) and (1) as facet-defining system. Then, by the relations of the inequality classes above, we immediately see the following inclusion relations of these polytopes:

$$\text{STAB}(G) \subseteq \text{WSTAB}(G) \subseteq \text{RSTAB}(G) \subseteq \text{QSTAB}(G).$$

Each of the polytopes  $\text{QSTAB}(G)$ ,  $\text{RSTAB}(G)$ ,  $\text{WSTAB}(G)$  is a relaxation of  $\text{STAB}(G)$ . Very often, these inclusions are proper. In a few important cases, however, equality holds. Most notably, perfect graphs are characterized in [2] to be precisely the graphs  $G$  with

$$\text{QSTAB}(G) = \text{STAB}(G).$$

Let us call all graphs  $G$  with

$$\text{RSTAB}(G) = \text{STAB}(G)$$

**rank-perfect.** Every perfect graph is obviously an example of a rank-perfect graph. PADBERG [11] showed that the stable set polytope of minimally imperfect graphs  $G$  admits, besides facets of type (0) and (1), only the full rank facet associated with  $G$  itself. Thus, minimally imperfect graphs are rank-perfect. Line graphs are further graphs which are known to be rank-perfect due to EDMONDS and PULLEYBLANK [4] (the **line graph**  $L(F)$  of a graph  $F$  is obtained by taking the edges of  $F$  as nodes of  $L(F)$  and connecting two nodes in  $L(F)$  iff the corresponding edges of  $F$  are incident), see Section 2 and Section 3 for more details. As a natural generalization of rank-perfect graphs, we analogously define graphs  $G$  with

$$\text{WSTAB}(G) = \text{STAB}(G)$$

to be **weakly rank-perfect**. Two graph classes are known to consist of weakly rank-perfect graphs due to SHEPHERD [12]: near-bipartite graphs and complements of line graphs. A graph  $G$  is **near-bipartite** if  $G - N(v)$  can be partitioned into two stable sets for all nodes  $v$  of  $G$ , (i.e. if removing all neighbors of an arbitrary node leaves the graph bipartite). The class of near-bipartite graphs contains all complements of line graphs (the non-neighbors of a node  $v$  in  $\overline{L(F)}$  correspond to the edges incident to the edge  $v$  in  $F$ , hence to two cliques in  $L(F)$  and to two stable sets in  $\overline{L(F)}$ ), see Section 2 and Section 4.

The classes of rank-perfect and weakly rank-perfect graphs constitute two superclasses of perfect graphs and can be used to decide how close or far a graph is from being perfect. Asking how imperfect a graph is that arises by deleting a critical edge  $e$  from a perfect graph  $G$  means, therefore, to figure out whether  $G - e$  is rank-perfect or weakly rank-perfect. In Section 3 and Section 4, we investigate the cases when  $G$  is the line graph of a bipartite graph and the complement of such a graph, respectively.

## 2 Rank Constraints and Sequential Lifting

Determining the system of facet-defining inequalities of the stable set polytope is very difficult in general. Thus one often tries to find classes of valid inequalities for  $\text{STAB}(G)$  and to investigate when those valid inequalities yield facets of  $\text{STAB}(G)$ . One natural class of valid nontrivial inequalities are rank constraints (2) associated with induced subgraphs  $G' \subseteq G$ . For convenience, we often write (2) as  $x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1})$ ,  $x(G', \mathbb{1}) \leq \alpha(G')$ , or just  $x(G') \leq \alpha(G')$  where  $\mathbb{1} = (1, \dots, 1)$ . The goal is to find out, for which subgraphs  $G' \subseteq G$ , the associated rank constraint  $x(G') \leq \alpha(G')$  yields a facet of  $\text{STAB}(G)$ . A first step towards this goal is to identify those graphs  $G$  for which their stable set polytope has the **full rank constraint**

$$x(G, \mathbb{1}) \leq \alpha(G, \mathbb{1}). \quad (2a)$$

as a facet. PADBERG showed this if  $G$  is a clique [10] or minimally imperfect [11]. Further examples of graphs with the full rank facet are **prime antiwebs**  $\overline{W}_{\alpha-1}^n$  [13]: an antiweb  $\overline{W}_{\alpha-1}^n$  has nodes  $1, \dots, n$  and edges  $ij$  for  $|i - j| > \alpha - 1 \pmod{n}$  and is prime if  $n$  and  $\alpha$  are relatively prime. Note that  $\overline{W}_1^{2\alpha+1}$  is an odd antihole and  $\overline{W}_{\alpha-1}^{2\alpha+1}$  an odd hole. Moreover, EDMONDS and PULLEYBLANK [4] established via matching theory that line graphs of 2-connected hypomatchable graphs have the full rank facet:  $H$  is called **hypomatchable** if, for all nodes  $v$  of  $H$ , the subgraph  $H - v$  admits a matching meeting all nodes. (Note: matchings of  $H$  correspond to stable sets of its line graph  $L(H)$  since the line operator transforms non-incident edges of  $H$  to non-adjacent nodes of  $L(H)$ ).

For some cases, a sufficient condition is known when a rank constraint  $x(G') \leq \alpha(G')$  associated with a *proper* subgraph  $G' \subset G$  yields a facet of the stable set polytope of the whole graph  $G$ . PADBERG [10] showed that clique constraints  $x(Q, \mathbb{1}) \leq 1$  are facet-inducing for  $\text{STAB}(G)$  iff  $Q$  is an (inclusion-wise) *maximal* clique of  $G$ . The result of EDMONDS and PULLEYBLANK [4] implies that a rank constraint

$$x(L(H), \mathbb{1}) \leq \frac{|H| - 1}{2} \quad (2b)$$

associated with the line graph of a 2-connected hypomatchable graph  $H \subseteq F$  is a facet of  $\text{STAB}(L(F))$  iff  $H$  is an *induced* subgraph of  $F$ . In general, a rank constraint associated with a proper subgraph  $G' \subset G$  does not need to provide a facet of  $\text{STAB}(G)$ , even if  $\text{STAB}(G')$  admits the full rank facet. This is the case for, e.g., **odd hole constraints**

$$x(C_{2k+1}, \mathbb{1}) \leq k \quad (2c)$$

with  $C_{2k+1} \subset G$  and for **odd antihole constraints**

$$x(\overline{C}_{2k+1}, \mathbb{1}) \leq 2 \quad (2d)$$

with  $\overline{C}_{2k+1} \subset G$ . Figure 1(a) shows a graph with an induced  $C_5$  (note a  $C_5$  is both an odd hole and an odd antihole) but the rank constraint associated with this  $C_5$  does not induce a facet of the stable set polytope of the whole graph.

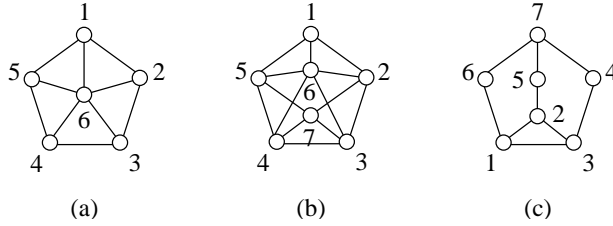


Figure 1

However, rank constraints  $x(G') \leq \alpha(G')$  with  $G' \subset G$  may be strengthened to a facet of  $\text{STAB}(G)$  using **sequential lifting** [10], i.e., by determining appropriate lifting coefficients  $a_i$  for all nodes  $i$  in  $G - G'$  such that the right hand side  $\alpha(G')$  of the inequality is still satisfied and that there are  $|G|$  many stable sets of weight  $\alpha(G')$  the incidence vectors of which are linearly independent. Every inequality of the form

$$x(G', \mathbb{1}) + x(G - G', a) \leq \alpha(G', \mathbb{1}) \quad (3)$$

is a weak rank constraint if it is obtained by lifting a base rank constraint  $x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1})$  which is facet-defining for  $\text{STAB}(G')$ , i.e., if  $G'$  has the full rank facet. The graph  $G$  depicted in Figure 1(a) yields a weak rank constraint basing on an odd hole constraint by using a lifting coefficient  $\neq 0, 1$  (thus  $G$  is not rank-perfect in particular).  $G$  consists of an odd hole (nodes 1, ..., 5) and a central node adjacent to all nodes of the odd hole (such graphs are termed **odd wheels**). The  $C_5$  yields 5 stable sets of weight 2 the incidence vectors of which are linearly independent. In order to construct the remaining stable set of weight 2 containing the central node 6, we have to choose lifting coefficient  $a_6 = 2$ . The resulting facet  $x(C_5, \mathbb{1}) + 2x_6 \leq 2$  of  $\text{STAB}(G)$  is a special weak rank constraint, called **odd wheel constraint**

$$x(C_{2k+1}, \mathbb{1}) + kx_c \leq k \quad (3a)$$

where  $c$  is the central node adjacent to all nodes of the odd hole  $C_{2k+1}$  and  $k \geq 2$ . SHEPHERD [12] studied a more general weak rank constraint

$$\sum_{i \leq k} \frac{1}{\alpha(\overline{W}_i)} x(\overline{W}_i) + x(Q, \mathbb{1}) \leq 1 \quad (3b)$$

associated with the complete join of prime antiwebs  $\overline{W}_1, \dots, \overline{W}_k$  and a clique  $Q$  and showed that the stable set polytope of near-bipartite graphs have facets of type (3b) as only nontrivial facets. (The **complete join** of two disjoint graphs  $G_1$  and  $G_2$  is obtained by joining every node of  $G_1$  and every node of  $G_2$  by an edge, e.g., every odd wheel is the complete join of an odd hole and a single node.) SHEPHERD [12] showed further that odd antiholes are the only prime antiwebs that occur in complements of line graphs. Thus the only nontrivial facets of their stable set polytope are weak rank constraints associated with the complete join of odd antiholes  $A_1, \dots, A_k$  and a clique  $Q$ .

$$\sum_{i \leq k} x(A_i) + 2x(Q) \leq 2 \quad (3c)$$

Note that the weak rank facet obtained by lifting may depend on the order in which the nodes are lifted. Hence, lifting a base rank constraint may result in *several* weak rank constraints. The graph  $G$  in Figure 1(b), e.g., contains the 5-wheel from Figure 1(a) as induced subgraph and the associated odd wheel constraint  $x(C_5, \mathbb{1}) + 2x_6 \leq 2$  is also a facet of  $\text{STAB}(G)$ . Furthermore, there is another way to lift the rank constraint associated with the  $C_5$  to a facet of  $\text{STAB}(G)$ , namely, by choosing  $a_6 = 1$  and  $a_7 = 1$  (i.e.,  $\text{STAB}(G)$  also admits the full rank facet).

Finally,  $\text{STAB}(G)$  may admit nontrivial facets which are not weak rank constraints. The stable set polytope of the graph  $G$  in Figure 1(c), e.g., has the facet  $\sum_{i \leq 6} x_i + 2x_7 \leq 3$  which is not a weak rank constraint: among the nodes of  $G$  with coefficient 1, there is no subgraph  $G'$  such that  $x(G') \leq 3$  is a facet of  $\text{STAB}(G')$ . (That means, there is no facet-inducing structure of a proper subgraph  $G' \subset G$  which we could lift to a facet of  $\text{STAB}(G)$ .) In particular, the graph  $G$  in Figure 1(c) is an example of a graph which is not weakly rank-perfect. (Checking the stable set polytopes of small imperfect graphs yields that  $G$  and  $\overline{G}$  are the only two not weakly rank-perfect graphs up to seven nodes.)

### 3 Critical Edges in Line Graphs of Bipartite Graphs

Let  $xy$  stand for a critical edge in the line graph  $G$  of a bipartite graph  $F$ . Then  $G$  is perfect by KÖNIG [6] but  $G - xy$  not anymore. We like to know whether removing only one critical edge leaves the graph still “almost perfect” or makes it already “very imperfect”. In order to decide how imperfect  $G - xy$  is we formulate the following problem.



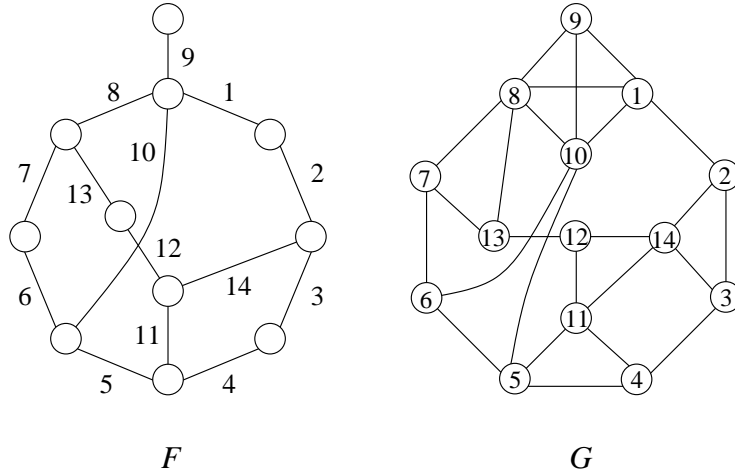


Figure 2

**Problem 3.1** *Let  $G$  be the line graph of a bipartite graph and  $xy$  be a critical edge of  $G$ . Is  $G - xy$  rank-perfect or weakly rank-perfect?*

For treating Problem 3.1, we have to study the facets of  $\text{STAB}(G - xy)$ . The graphs  $G - xy$  are no line graphs anymore in general (if we delete the edge  $xy = v_1v_8$  from the line graph  $G$  shown in Figure 2, then there is a claw  $v_1v_5v_8, v_{10}$  in  $G - xy$ ), so we cannot just apply the result of EDMONDS and PULLEYBLANK [4]. All nontrivial facets of the stable set polytope different from clique constraints have imperfect support graphs. Hence the knowledge of the minimally imperfect subgraphs of  $G - xy$  may help solving Problem 3.1. We describe a structure in the bipartite graph  $F$  that leads to critical edges  $xy$  in the line graph  $G$  of  $F$  and to odd holes in  $G - xy$ . We say that two incident edges  $x$  and  $y$  form an **H-pair** in  $F$  if there is an edge  $e_{x,y}$  incident to the common node of  $x$  and  $y$  and if there is a (not necessarily induced) even cycle  $C_{x,y}$  containing  $x$  and  $y$  but only one endnode of  $e_{x,y}$  (see Figure 3(a)). Then  $L(C_{x,y})$  is an even hole and the node in  $L(F)$  corresponding to  $e_{x,y}$  has precisely two neighbors on  $L(C_{x,y})$ , namely,  $x$  and  $y$  (see Figure 3(b)). Thus removing the edge  $xy$  from  $L(F)$  yields an odd hole and  $xy$  is a critical edge of  $L(F)$  in particular. (Note that H-pair comes from *hole*.) From results in [14] follows that, conversely, the edges  $x$  and  $y$  must form an H-pair in  $F$  if  $xy$  is a critical edge of the line graph.

**Theorem 3.2** (WAGLER [14]) *Let  $G$  be the line graph of a bipartite graph  $F$ . An edge  $xy$  of  $G$  is critical iff  $x$  and  $y$  form an H-pair in  $F$ . The only minimally imperfect subgraphs of  $G - xy$  are odd holes.*

In [15] is shown how to lift a rank constraint associated with a *single* odd hole in  $G - xy$  to a facet of  $\text{STAB}(G - xy)$ . But there also appear line graphs

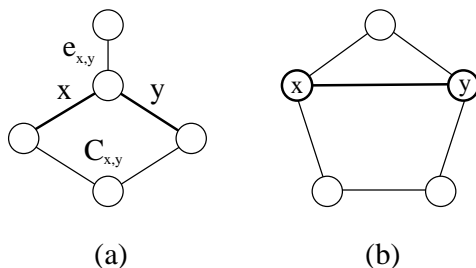


Figure 3

of hypomatchable graphs in the *union* of the odd holes in  $G - xy$ . That means, we have to consider more general facets of  $\text{STAB}(G - xy)$  than lifted odd hole constraints. For that, we try to “translate” the result of EDMONDS and PULLEYBLANK [4] to the case of the studied graphs  $G - xy$ . We ask how to find the graphs  $L(H) \subseteq G - xy$  with  $H$  hypomatchable and how to lift the rank constraint associated with  $L(H)$  to a facet of  $\text{STAB}(G - xy)$ .

Recall that a graph  $H$  is called hypomatchable if it does not admit a perfect matching but  $H - v$  does for all  $v \in V(H)$  (a matching is **perfect** if it meets all nodes of the graph). LOVÁSZ [9] proved the following characterization of hypomatchable graphs which we will frequently use in the sequel: A graph  $H$  is hypomatchable if and only if there is a sequence  $H_0, H_1, \dots, H_k = H$  of graphs such that  $H_0$  is an odd cycle and for  $1 \leq i \leq k$ ,  $H_i$  is obtained from  $H_{i-1}$  by adding an odd path  $E_i$  that joins two (not necessarily distinct) nodes of  $H_{i-1}$  and has all internal nodes outside  $H_{i-1}$ . The odd path  $E_i$  is called an **ear** and the sequence  $H_0, H_1, \dots, H_k = H$  an **ear decomposition** of  $H$ . Every ear decomposition  $H_0, H_1, \dots, H_k = H$  defines a sequence  $E_1, \dots, E_k$  of corresponding ears with  $H_i = H_{i-1} \cup E_i$  for  $1 \leq i \leq k$ . Hence, we may also characterize  $H$  by this ear sequence  $H_0, E_1, \dots, E_k$  and use both sequences to describe the same ear decomposition. Hypomatchable graphs have an odd number of nodes, are non-bipartite and connected, but not necessarily 2-connected. By definition, an ear may be an odd cycle attached to a single node of the previous graph. However, these degenerated ears can be avoided if  $H$  is 2-connected: CORNUÉJOLS and PULLEYBLANK proved in [3] that every 2-connected hypomatchable graph  $H$  admits an ear decomposition  $H_0, H_1, \dots, H_k = H$  with  $H_i$  2-connected for  $0 \leq i \leq k$ .

Turning to hypomatchable graphs  $H$  with  $L(H) \subseteq G - xy$ , we first observe that we are only interested in 2-connected graphs (otherwise the rank constraint associated with  $L(H)$  is never facet-defining by [4]) and have at

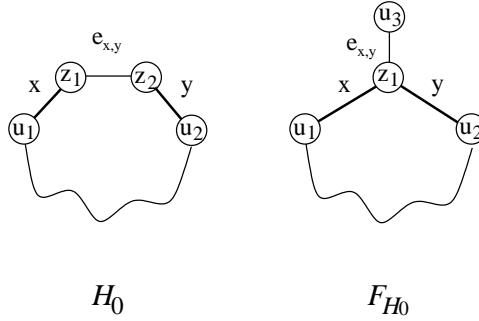


Figure 4

least five nodes (otherwise  $H$  is a triangle and the rank constraint associated with  $L(H)$  is a clique constraint). Then  $H$  admits an ear decomposition  $H_0, H_1, \dots, H_k = H$  such that  $H_i$  is 2-connected for  $0 \leq i \leq k$  by [3], i.e., the corresponding ears  $E_1, \dots, E_k$  have distinct endnodes. Since  $H_0$  is an odd cycle,  $H \not\subseteq F$  follows (recall:  $F$  is bipartite and does not contain any odd cycle). Thus we have to characterize those subgraphs  $F_H \subseteq F$  with  $L(F_H) - xy = L(H)$ . Clearly, we have  $x, y \in V(L(F_H)) = V(L(H))$  and  $x, y \in E(F_H) = E(H)$ . Moreover, the edges  $x$  and  $y$  are not incident in  $H$  but incident in  $F_H$ , while every pair of edges  $e, e'$  distinct from  $x, y$  is incident in  $H$  iff  $e, e'$  are incident in  $F_H$ . Furthermore, every odd cycle  $C$  of  $H$  contains  $x$  and  $y$  and has length  $\geq 5$  ( $x \notin E(C)$  or  $y \notin E(C)$  would imply  $C \subseteq F_H$  in contradiction to  $F_H$  bipartite, hence  $C$  contains the non-incident edges  $x$  and  $y$  and cannot be a triangle). In particular,  $H_0$  contains  $x$  and  $y$  and has length  $\geq 5$ . Denote by  $F_{H_i}$  the graph with  $L(F_{H_i}) - xy = L(H_i)$  for  $0 \leq i \leq k$ . Then we have the following.

**Lemma 3.3**  $H_0$  contains the path  $x = u_1z_1$ ,  $e_{x,y} = z_1z_2$ ,  $y = z_2u_2$ . Those edges correspond in  $F_{H_0}$  to a claw  $x = u_1z_1$ ,  $e_{x,y} = u_3z_1$ ,  $y = u_2z_1$ . All remaining edges of  $H_0$  form with  $x$  and  $y$  an even cycle in  $F_{H_0}$ .

**Remark.** The edges  $x = u_1z_1$  and  $y = z_2u_2$  are non-incident in  $H$  but incident in  $F_H$ . We assume  $z_1 \in V(F_H)$ ,  $z_2 \notin V(F_H)$ , and introduce a new node  $u_3 \in V(F_H)$  which is not contained in  $V(H)$ , see Figure 4. Note further that the graph  $F_{H_0}$  consists of an edge  $e_{x,y}$  and an even cycle  $C_{x,y}$  through  $x$  and  $y$  as in Figure 3(a), i.e.,  $x$  and  $y$  form an H-pair in  $F_H$ .

**Proof.** Since  $x$  and  $y$  are non-incident in  $H$  and belong to  $H_0$ , we may represent  $H_0$  as shown in Figure 5(a) with  $i, j > 0$ . The edges  $x = u_1z_1$  and  $y = zz'$  of  $H_0$  are incident in  $F_{H_0}$ , say  $z_1$  is their common endnode. For  $z_1 = z, u_2 = z'$  in  $F_{H_0}$  (see Figure 5(b)),  $e_1$  and  $f_1$  would become incident

in  $F_H$ , a contradiction. So we deduce  $z_1 = z'$  and  $u_2 = z$  (see Figure 5(c)). Then  $e_i$  is incident to  $x$  in  $F_{H_0}$ , thus  $i = 1$  follows and we insert a new node  $u_3$  in  $F_{H_0}$  in order to keep  $F_H$  loop-less, and get  $e_1 = e_{x,y}$  (see Figure 4). The edges  $y, f_1, \dots, f_j, x$  form a cycle in  $F_{H_0}$  and, therefore,  $j$  is even.  $\square$

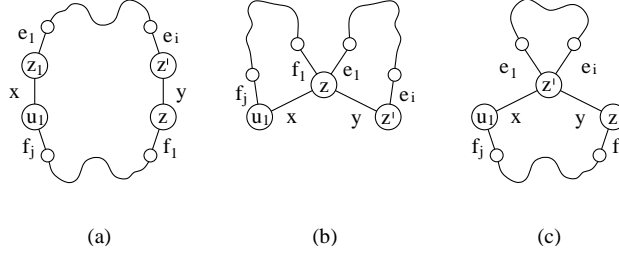


Figure 5

**Lemma 3.4** *For every ear  $E_i$  of  $H$  which is not an edge parallel to  $e_{x,y} = z_1z_2$  holds:  $E_i$  has neither  $z_1$  nor  $z_2$  as an endnode. There is no even path in  $H - \{z_1, z_2\}$  connecting the endnodes of  $E_i$ .*

**Proof.** Assume  $E_i$  with  $i \in \{1, \dots, k\}$  minimal is an ear of  $H$  with  $z_1$  or  $z_2$  as an endnode. W.l.o.g., let  $z_1$  and  $v$  be the endnodes of  $E_i$  in  $H$  and  $e_1$  be the edge of  $E_i$  incident to  $z_1$  in  $H$ . If  $v \neq z_2, u_2$  holds,  $e_1$  is incident to  $e_{x,y}$  and  $x$  but not to  $y$  in  $H$ . Keeping these incidences, the corresponding edge in  $F_H$  would be  $e_1 = u_3u_1$  forming a triangle with  $x$  and  $e_{x,y}$  in  $F_H$ , a contradiction. For  $v = u_2$ , the ear  $E_i$  forms with  $y$  and  $e_{x,y}$  an odd cycle in  $H$  not containing  $x$ , a contradiction. In the remaining case  $v = z_2$ , the incidences are preserved only if  $E_i = e_1$ , i.e., if  $E_i$  is an edge parallel to  $e_{x,y}$  in  $H$ . Finally, if there is an ear  $E_i$  with endnodes  $v, v'$  distinct from  $z_1, z_2$  in  $H$  and an even path  $P$  connecting  $v$  and  $v'$  in  $H - \{z_1, z_2\}$ , then  $E_i \cup P$  would be an odd cycle in  $H$  not containing  $x$  and  $y$ , a contradiction.  $\square$

Edges parallel to  $e_{x,y}$  in  $H$  have the same incidences as  $e_{x,y}$  in  $F_H$  and play the same role as  $e_{x,y}$  in  $F_H$ . For technical reasons, we treat the case of those edges by lifting the corresponding nodes of  $G - xy$  and consider only graphs  $H$  that do not admit any edge parallel to  $e_{x,y}$ . Then, by Lemma 3.4, none of the ears  $E_1, \dots, E_k$  of  $H$  is incident to  $e_{x,y}$  and  $H - e_{x,y}$  is bipartite. We deduce  $F_{H_i} = F_{H_0} \cup \bigcup_{1 \leq j \leq i} E_j$  for  $0 \leq i \leq k$  and call  $F_{H_0}, E_1, \dots, E_k$  a **pseudo-ear decomposition** of  $F_H$ . We are now prepared to lift the rank constraint associated with  $L(F_H) - xy = L(H)$  to a facet of  $\text{STAB}(G - xy)$ .

**Theorem 3.5** *Let  $G$  be the line graph of a bipartite graph  $F$  and  $xy$  a critical edge of  $G$ . Every subgraph  $F_H \subseteq F$  containing  $x$  and  $y$  induces in  $\text{STAB}(G - xy)$  the facet*

$$\sum_{v_i \in L(F_H) - xy} x_i + \sum_{v_i \in N_{x,y}} x_i \leq \alpha(L(F_H) - xy) \quad (3d)$$

*if  $F_H$  has a pseudo-ear decomposition and  $F_H - \{z_1, u_3\}$  is an induced subgraph of  $F$  where  $N_{x,y} = \{v_i \in (G - L(F_H)) - xy : v_i \in N(x) \cap N(y)\}$  and  $z_1, u_3$  are the endnodes of the only edge incident to  $x, y$  in  $F_H$ .*

**Remark.** Every  $F_H$  with a pseudo-ear decomposition corresponds to a 2-connected hypomatchable graph  $H$ . The condition that only induced hypomatchable subgraphs are facet-defining is covered by considering only those graphs  $F_H$  where  $F_H - \{z_1, u_3\}$  is an induced subgraph of  $F$  and by lifting the nodes in  $N_{x,y}$ . (The nodes in  $N_{x,y}$  correspond to edges parallel to  $e_{x,y}$  in  $H$  or to edges  $z_1 w$  of  $F$  with  $w \in V(F_H) - \{z_1, u_3\}$ .) Further, (3d) is the *unique* lifting of the base rank constraint associated with  $L(H) = L(F_H) - xy$  and is itself a rank constraint since only lifting coefficients equal to 1 and 0 are needed. Moreover, in the special case that  $L(H)$  is an odd hole (i.e., that  $H$  consists in the odd cycle  $H_0$  only), then (3d) is a lifted odd hole constraint as described in [15].

**Proof.** Let  $F$  be bipartite,  $G$  be the line graph of  $F$ , and  $xy$  be a critical edge of  $G$ . Consider a subgraph  $F_H \subseteq F$  fulfilling the conditions above with  $x = u_1 z_1, y = u_2 z_1 \in E(F_H)$ . Then there is a 2-connected hypomatchable graph  $H$  with  $L(H) = L(F_H) - xy$ .  $H$  admits an ear decomposition  $H_0, H_1, \dots, H_k = H$  such that  $H_i$  is 2-connected for  $0 \leq i \leq k$  by [3],  $H_0$  is an odd cycle of length  $\geq 5$  containing  $x, e_{x,y}, y$  by Lemma 3.3, and the ears  $E_1, \dots, E_k$  of  $H$  have distinct endnodes not incident to  $e_{x,y}$  by Lemma 3.4. In order to prove the assertion of the theorem, we essentially make use of the correspondence between stable sets in  $L(F_H) - xy$  and matchings in  $H$ . First, we show that the inequality (3d) is valid for  $\text{STAB}(G - xy)$  by the following claim.

**Claim 1.** *There is no maximum stable set  $S$  of  $L(F_H) - xy$  such that  $S \cup \{v_i\}$  is a stable set in  $G - xy$  for any  $v_i \in N_{x,y}$ .*

Assume in contrary there is a maximum stable set  $S$  of  $L(F_H) - xy$  and some  $v_i \in N_{x,y}$  adjacent to none of the nodes in  $S$ . Let  $e_i$  be the edge of  $F$  corresponding to the node  $v_i$  of  $G - xy$  and  $S$  correspond to a maximum matching

$M$  of  $H$ . Since  $F$  is bipartite,  $e_i$  is incident to the common endnode of  $x$ ,  $e_{x,y}$ , and  $y$  in  $F$ . Hence  $M$  must neither contain  $x$ ,  $e_{x,y}$ , nor  $y$ . So  $M$  does not meet the two nodes  $z_1$  and  $z_2$  of  $H$  by Lemma 3.4, a contradiction to  $M$  maximum.  $\diamond$

According to the definition of  $N_{x,y} \subseteq V(G - xy)$ , the corresponding edges in  $E(F)$  are all incident to  $z_1$ . Hence  $N_{x,y}$  induces a clique in  $G - xy$  and Claim 1 yields that (3d) is valid for  $\text{STAB}(G - xy)$ . In order to find  $|V(G)|$  many stable sets in  $G - xy$  that satisfy (3d) with equality and have affinely independent incidence vectors, we use the result of EDMONDS and PULLEY-BLANK [4] as starting point. (We explicitly give here a construction similarly to that done in [4], since we will need it for the proof of Claim 3.)

**Claim 2.** *There is a set  $\mathcal{M}$  of  $|E(H)|$  maximum matchings of  $H$  the incidence vectors of which are affinely independent.*

We know that  $H$  admits an ear decomposition  $H_0, H_1, \dots, H_k$  such that  $H_i$  is 2-connected for  $0 \leq i \leq k$ . By induction each  $H_i$  has  $|E(H_i)|$  maximum matchings whose incidence vectors are affinely independent: The assertion is obviously satisfied by the odd cycle  $H_0$ . Suppose the result is true for  $H_{i-1}$  and consider  $H_i$ . Let  $v$  and  $v'$  be the endnodes of the ear  $E_i$  (recall:  $v \neq v'$  by  $H_i$  2-connected). By induction there are  $|E(H_{i-1})|$  independent maximum matchings of  $H_{i-1}$ . Each can be extended to a maximum matching of  $H_i$  by adding the 2nd, 4th,  $\dots$  edges of  $E_i$ . Then  $|E(E_i)| - 1$  additional matchings can be obtained by considering each internal node  $w$  of  $E_i$ , if one constructs a matching of  $E_i$  which does not meet  $w$  and  $w' \in \{v, v'\}$  such that the path in  $E_i$  connecting  $w$  and  $w'$  has odd length. We extend this matching of  $E_i$  to a maximum matching of  $H_i$  by adding a matching of  $H_{i-1}$  which does not meet only the endnode of  $E_i$  distinct from  $w'$ . This is the only matching constructed so far not meeting  $w$ , so it is independent of all others. We get the last matching by starting with the 1st, 3rd,  $\dots$  edges of  $E_i$  and combining it with a perfect matching of  $H_{i-1} - v$  from which the edge incident to  $v'$  has been removed (if the ear  $E_i$  has length one, this is the only new matching to construct). The last matching contains one edge of  $H_{i-1}$  less than the others, so it is independent of all others. By this construction, we obtain the studied set  $\mathcal{M}$  of  $|E(H)|$  maximum matchings of  $H$  whose incidence vectors are affinely independent.  $\diamond$

For later convenience, let  $\mathcal{M}_v$  denote the set of matchings belonging to  $\mathcal{M}$  which do not meet  $v \in V(H)$ , i.e., which are perfect matchings of  $H - v$ . Since matchings of  $H$  correspond to stable sets of  $L(H) = L(F_H) - xy$ ,  $\mathcal{M}$

represents a set of  $|E(H)| = |E(F_H)| = n_H$  many stable sets, say  $S_1, \dots, S_{n_H}$  of  $G - xy$ . In order to show that (3d) is a facet of  $\text{STAB}(G - xy)$ , we construct the remaining stable sets  $S_{n_H+1}, \dots, S_n$  of  $G - xy$  with  $n = |E(F)| = |V(G)|$  such that  $S_1, \dots, S_n$  satisfy (3d) with equality and have affinely independent incidence vectors. For that, we partition the edge set  $E(F) - E(F_H)$  with respect to  $F_H$  as follows.

$$\begin{aligned} N_0 &:= \{e_i \in E(F) - E(F_H) : e_i = vv' \text{ with } v, v' \notin V(F_H)\} \\ N_1 &:= \{e_i \in E(F) - E(F_H) : e_i = vw \text{ with } v \notin V(F_H), w \in V(F_H) - z_1\} \\ N_2 &:= \{e_i \in E(F) - E(F_H) : e_i = u_3w \text{ with } w \in V(F_H) - z_1\} \\ N_{x,y} &:= \{e_i \in E(F) - E(F_H) : e_i = z_1w\} \end{aligned}$$

(Note that edges in  $E(F) - E(F_H)$  with both endnodes in  $V(F_H)$  are incident to either  $u_3$  or  $z_1$  by Lemma 3.4, hence those edges are contained in either  $N_2$  or  $N_{x,y}$ .) We find, for each  $e_i \in N_0 \cup N_1 \cup N_2$ , a matching  $M \in \mathcal{M}$  such that  $M \cup \{e_i\}$  corresponds to a stable set of  $G - xy$ . Clearly, there is no such matching for any  $e_i \in N_{x,y}$  by Claim 1. But for all  $e_i \in N_{x,y}$ , we are able to construct a matching  $M \in \mathcal{M}$  such that  $(M - \{e_{x,y}\}) \cup \{e_i\}$  corresponds to a stable set of  $G - xy$ . For that, we provide the following claim.

**Claim 3.** *Let  $w \in V(F_H) - \{z_1, u_3\}$ . If  $w$  has the same color as  $z_1$  in  $F_H$ , then there is a matching  $M \in \mathcal{M}_w$  with  $x, y \in M$ . Otherwise, there is  $M \in \mathcal{M}_w$  with  $e_{x,y} \in M$ .*

For the studied ear decomposition  $H_0, H_1, \dots, H_k = H$  of  $H$  we inductively construct a maximum matching  $M_j$  of that  $H_j$  for  $j \in \{0, \dots, k\}$  minimal with  $w \in V(H_j)$ . The assertion is satisfied for  $j = 0$  in both cases by Lemma 3.3. Assume the assertion is true for  $j = i - 1$  and consider the case  $j = i$ . Then  $w$  is an internal node of the ear  $E_i$ . Let  $v'$  be the endnode of  $E_i$  with the same color as  $w$  and let  $v$  be the other one (recall:  $E_i$  connects nodes of different colors in  $F_H$ ). Then by induction there is a perfect matching  $M_{i-1}$  of  $H_{i-1} - v'$  that fulfills the assertion. We extend  $M_{i-1}$  to  $M_i$  by a maximum matching of  $E_i$  that contains the edge of  $E_i$  incident to  $v'$  and does not meet  $w$  and  $v$ . We yield the studied matching  $M$  of  $H$  by extending  $M_i$  if we choose the 2nd, 4th,  $\dots$  edges of every ear  $E_{i+1}, \dots, E_k$ :  $M$  contains either  $x, y$  or  $e_{x,y}$ , it does not meet  $w$  but all other nodes of  $H$ , and by comparing the constructions used here and in Claim 2,  $M$  belongs to  $\mathcal{M}$ .  $\diamond$

Now, we are prepared to construct the remaining stable sets  $S_{n_H+1}, \dots, S_n$  of  $G - xy$  which we need, one for each  $e_i \in E(F) - E(F_H)$ .

$$S_i = \begin{cases} \{e_i\} \cup M & \text{if } e_i \in N_0 \text{ with } M \in \mathcal{M} \\ \{e_i\} \cup M & \text{if } e_i \in N_1 \text{ with } M \in \mathcal{M}_w \text{ for } w \neq u_3 \\ & \text{with } M \in \mathcal{M}_{z_1} \text{ for } w = u_3 \\ \{e_i\} \cup M & \text{if } e_i \in N_2 \text{ with } M \in \mathcal{M}_w, x, y \in M \\ \{e_i\} \cup (M - e_{x,y}) & \text{if } e_i \in N_{x,y} \text{ with } M \in \mathcal{M}, e_{x,y} \in M \text{ for } w \notin V(F_H) \\ & \text{with } M \in \mathcal{M}_w, e_{x,y} \in M \text{ for } w \in V(F_H) \end{cases}$$

(For each  $e_i = u_3 w \in N_2$ , the node  $w$  has the same color as  $z_1$  in  $F_H$ , hence Claim 3 yields the existence of the above matching in  $\mathcal{M}_w$ . For each  $e_i = z_1 w \in N_{x,y}$  with  $w \in V(F_H)$ , the node  $w$  is distinct from  $u_3$  and not contained in the same color class of  $F_H$  as  $z_1$ . Again, Claim 3 yields the existence of the studied matching in  $\mathcal{M}_w$ .)

The incidence vectors  $\chi^{S_{n_H+1}}, \dots, \chi^{S_n}$  are affinely independent, since  $\chi_i^{S_i}$  is the only non-zero entry of  $\chi_i^{S_1}, \dots, \chi_i^{S_n} \forall e_i \in E(F) - E(F_H)$  and  $\lambda_i = 0$  follows for  $n_H + 1 \leq i \leq n$  by

$$\sum_{1 \leq i \leq n} \lambda_i \chi^{S_i} = 0.$$

The vectors  $\chi^{S_1}, \dots, \chi^{S_{n_H}}$  are affinely independent by Claim 2. Finally, the stable sets  $S_1, \dots, S_n$  are chosen such that  $|S_i \cap (L(F_H) \cup N_{x,y})| = \alpha(L(F_H) - xy)$  holds for  $1 \leq i \leq n$ , i.e.,  $\chi^{S_1}, \dots, \chi^{S_n}$  satisfy (3d) with equality and so the constraint induces a facet of  $\text{STAB}(G - xy)$ .  $\square$

To illustrate Theorem 3.5 by an example, consider the bipartite graph  $F$  and its line graph  $G$  shown in Figure 2. The edges  $x = e_1$  and  $y = e_8$  form an H-pair in  $F$ , hence the edge  $xy = v_1 v_8$  is critical in  $G$ . In order to find all facets of type (3d) of  $\text{STAB}(G - xy)$ , we look for all graphs  $F_H \subseteq F$  that satisfy the conditions of Theorem 3.5. Those graphs  $F_H \subseteq F$  admit a pseudo-ear decomposition  $F_{H_0}, E_1, \dots, E_k$  where  $F_{H_0}$  consists of an even cycle  $C_{x,y}$  through  $x, y$  and an edge  $e_{x,y}$  incident to  $x, y$  (see Figure 4). In  $F$ , there are four cycles running through  $x$  and  $y$ .

$$\begin{aligned} C_{x,y}^{(1)} &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \\ C_{x,y}^{(2)} &= \{e_1, e_2, e_3, e_4, e_{11}, e_{12}, e_{13}, e_8\} \\ C_{x,y}^{(3)} &= \{e_1, e_2, e_{14}, e_{11}, e_5, e_6, e_7, e_8\} \\ C_{x,y}^{(4)} &= \{e_1, e_2, e_{14}, e_{12}, e_{13}, e_8\} \end{aligned}$$



(Note that  $e_{10}$  is a chord of  $C_{x,y}^{(1)}$  and  $C_{x,y}^{(3)}$  while  $C_{x,y}^{(2)}$  and  $C_{x,y}^{(4)}$  yield a graph  $F_{H_0}$  with both edges  $e_9$  and  $e_{10}$ .)  $F$  contains the following seven graphs  $F_H$  with a pseudo-ear decomposition  $F_{H_0}, E_1, \dots, E_k$ .

	$F_{H_0}$	$E_1$	$E_2$	$N_{x,y}$
(a)	$C_{x,y}^{(1)} \cup e_9$	$e_{11}, e_{12}, e_{13}$	$e_{14}$	$e_{10}$
(b)	$C_{x,y}^{(1)} \cup e_9$			$e_{10}$
(c)	$C_{x,y}^{(2)} \cup e_9$	$e_{14}$		$e_{10}$
(c')	$C_{x,y}^{(2)} \cup e_{10}$	$e_{14}$		$e_9$
(d)	$C_{x,y}^{(3)} \cup e_9$			$e_{10}$
(e)	$C_{x,y}^{(4)} \cup e_9$			$e_{10}$
(e')	$C_{x,y}^{(4)} \cup e_{10}$			$e_9$

The graphs  $F_H \cup N_{x,y}$  yield the same support in the cases (c),(c') and (e),(e') hence we get the following five facets of type (3d) in  $\text{STAB}(G - xy)$  (note: (a) is the full rank facet of  $G - xy$ ).

$$\begin{aligned}
\text{(a)} \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} \leq 5 \\
\text{(b)} \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \leq 4 \\
\text{(c)} \quad & x_1 + x_2 + x_3 + x_4 \qquad \qquad \qquad + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} \leq 4 \\
\text{(d)} \quad & x_1 + x_2 \qquad \qquad + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} \qquad \qquad + x_{14} \leq 4 \\
\text{(e)} \quad & x_1 + x_2 \qquad \qquad \qquad \qquad \qquad \qquad + x_8 + x_9 + x_{10} \qquad \qquad + x_{12} + x_{13} + x_{14} \leq 3
\end{aligned}$$

$\text{STAB}(G - xy)$  has, besides those five facets of type (3d), only nonnegativity facets (0) and clique facets (1), i.e.,  $G - xy$  is rank-perfect. All graphs  $G - xy$  arising from line graphs of bipartite graphs by removing a critical edge that occurred during the investigations of the author so far are rank-perfect in particular. However, it is unknown whether this is true in general and, therefore, Problem 3.1 is unsolved.

## 4 Critical Edges in Complements of Line Graphs of Bipartite Graphs

Let  $xy$  stand for a critical edge in the complement  $G$  of the line graph of a bipartite graph  $F$ . Then  $G$  is perfect by KÖNIG [7] but  $G - xy$  not anymore. Again, we like to know “how imperfect”  $G - xy$  is.

**Problem 4.1** *Let  $G$  be the complement of the line graph of a bipartite graph and  $xy$  be a critical edge of  $G$ . Is  $G - xy$  rank-perfect or weakly rank-perfect?*

To solve Problem 4.1, we again study the facets of  $\text{STAB}(G-xy)$ . The graphs  $G-xy$  are not near-bipartite in general (if we delete the edge  $xy = v_1v_7$  from the complement  $G$  of the line graph of  $F$  shown in Figure 6, then there is the triangle  $v_1v_6v_8$  among the non-neighbors of  $y = v_7$  in  $G-xy$ ). Thus we cannot apply the result of SHEPHERD [12].

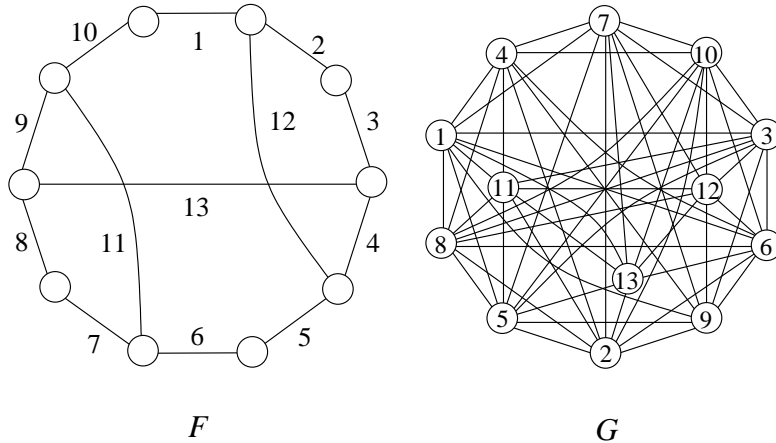


Figure 6

Since all nontrivial facets of the stable set polytope different from clique constraints have imperfect support graphs, the knowledge of the minimally imperfect subgraphs of  $G-xy$  may help again. We describe a structure in the bipartite graph  $F$  that leads to critical edges  $xy$  in  $G = \overline{L(F)}$  and to odd antiholes in  $G-xy$ . Two non-incident edges  $x$  and  $y$  are called an **A-pair** if they are the endedges of a (not necessarily induced) odd path  $P_{x,y}$  of length at least five (see Figure 7(a)). Then  $L(P_{x,y})$  is an even, chordless path of length at least four with endnodes  $x$  and  $y$  or, in other words,  $L(P_{x,y})$  is an odd hole where the edge between  $x$  and  $y$  is missing (see Figure 7(b)). Thus  $\overline{L(P_{x,y})}$  is an odd antihole with one additional edge, namely, the critical edge  $xy$  (see Figure 7(c)), and removing  $xy$  yields an odd antihole. (Note that A-pair comes from antihole.) From results in [14] follows that, conversely, the edges  $x$  and  $y$  must form an A-pair in  $F$  if  $xy$  is a critical edge of the complement of its line graph.

**Theorem 4.2** (WAGLER [14]) *Let  $G$  be the complement of the line graph of a bipartite graph  $F$ . An edge  $e = xy$  of  $G$  is critical iff  $x$  and  $y$  form an A-pair in  $F$ . The only minimally imperfect subgraphs of  $G-xy$  are odd antiholes.*

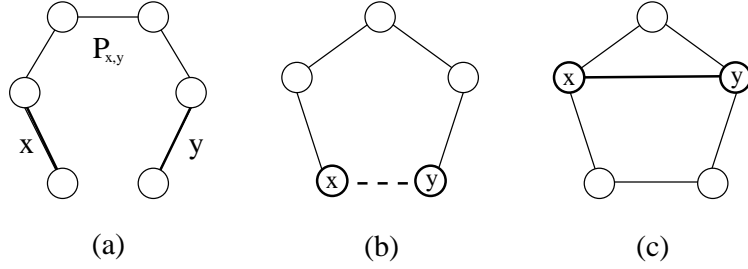


Figure 7

Note that the odd antiholes in  $G - xy$  are clearly not node-disjoint since they all contain the endnodes  $x$  and  $y$  of the critical edge. Due to the result of SHEPHERD [12], we expect that we have to lift constraints associated with the complete join of one odd antihole and a clique in  $G - xy$  to facets of  $\text{STAB}(G - xy)$ . Recall that odd antiholes in  $G - xy$  correspond to odd paths  $P_{x,y}$  in  $F$  with endedges  $x$  and  $y$ . Moreover, cliques of  $G$  come from matchings in  $F$ .

**Theorem 4.3** (WAGLER [15]) *Let  $G$  be the complement of the line graph of a bipartite graph  $F$  and  $P_{x,y} \subseteq F$  be a path of odd length  $\geq 5$  with  $x$  and  $y$  as endedges. Then*

$$\sum_{v_i \in \overline{L(F')}} x_i + 2 \sum_{v_i \in \overline{L(M)}} x_i \leq 2 \quad (3e)$$

*is a facet in  $\text{STAB}(G - xy)$  for every pair of (partial) subgraphs  $F', M \subseteq F$  such that*

- (i)  $F'$  is a chordless path or a cycle,
- (ii)  $F'$  contains  $P_{x,y}$  but no edge incident to both  $x$  and  $y$ ,
- (iii)  $M$  is a matching with nodes in  $F - V(F')$ ,
- (iv) every edge in  $F - E(F' \cup M)$  is incident to both  $x$  and  $y$ , to two incident edges of  $F'$ , or to one edge of  $M$ .

**Remark.** The facets of type (3e) can be interpreted as two kinds of liftings: On the one hand, they base on a constraint of type (3c) associated with the complete join of an odd antihole  $\overline{L(P_{x,y})} - xy$  and a clique  $\overline{L(M)}$ . All nodes of  $G - xy$  corresponding to the edges of  $F'$  not belonging to  $P_{x,y}$  have to be lifted with coefficient 1. On the other hand, we may also consider facets of type (3e) to base on odd antihole constraints (2d) associated with  $\overline{L(P_{x,y})} - xy$ . Then we lift the edges of  $F'$  not belonging to  $P_{x,y}$  with coefficient 1 and the edges of the matching  $M$  with coefficient 2.

In order to illustrate the assertion of Theorem 4.3, we show how the partial subgraphs  $F', M \subseteq F$  look like and, thereby, how to lift the corresponding odd antihole constraints to facets of type (3e). Consider the bipartite graph  $F$  and the complement  $G$  of its line graph shown in Figure 6. The edges  $x = e_1$  and  $y = e_7$  form an A-pair in  $F$ , hence  $xy = v_1v_7$  is a critical edge of  $G$  by Theorem 4.2. There are five paths  $P_{x,y}$  of odd length  $\geq 5$  with endedges  $x$  and  $y$  in  $F$ :  $P_{x,y}^{(1)} = e_1e_2e_3e_{13}e_9e_{11}e_7$ ,  $P_{x,y}^{(2)} = e_1e_2e_3e_4e_5e_6e_7$ ,  $P_{x,y}^{(3)} = e_1e_{10}e_9e_8e_7$ ,  $P_{x,y}^{(4)} = e_1e_{12}e_5e_6e_7$ , and  $P_{x,y}^{(5)} = e_1e_{12}e_4e_{13}e_9e_{11}e_7$ . The following subgraphs  $F', M \subseteq F$  satisfy the conditions of Theorem 4.3.

	$F'$	$M$
(a)	$P_{x,y}^{(1)}$	$\{e_5\}$
(b)	$P_{x,y}^{(2)}$	$\{e_9\}$
(c)	$P_{x,y}^{(2)} \cup P_{x,y}^{(3)}$	
(d)	$P_{x,y}^{(3)}$	$\{e_3, e_5\}$
(e)	$\{e_3, e_2\} \cup P_{x,y}^{(3)}$	$\{e_5\}$
(f)	$\{e_2\} \cup P_{x,y}^{(3)} \cup \{e_6\}$	$\{e_4\}$
(g)	$\{e_3, e_4, e_{12}\} \cup P_{x,y}^{(3)} \cup \{e_6\}$	$\{e_4\}$
(h)	$P_{x,y}^{(3)} \cup P_{x,y}^{(4)}$	
(i)	$P_{x,y}^{(4)}$	$\{e_3, e_9\}$
(j)	$\{e_{10}\} \cup P_{x,y}^{(4)}$	$\{e_{13}\}$
(k)	$\{e_{10}\} \cup P_{x,y}^{(4)} \cup \{e_8, e_{13}, e_3\}$	
(l)	$\{e_3, e_{13}, e_9, e_{10}\} \cup P_{x,y}^{(4)}$	
(m)	$P_{x,y}^{(5)}$	

( $P_{x,y}^{(1)}$  and  $P_{x,y}^{(5)}$  admit a unique lifting each (see cases (a) and (m)) while we obtain different liftings for the other three paths. In cases (c) and (h), the facet is a common lifting of two odd antiholes, the respective two paths  $P_{x,y}$  complement one another to a cycle.  $F'$  is sometimes identical with  $P_{x,y}$ , extends  $P_{x,y}$  on one end or on both ends.  $M$  is a maximal matching of  $F - F'$  in each case. Further  $M = \{e_{13}\}$  is also a maximal matching of  $F - P_{x,y}^{(4)}$  but condition (iv) is violated by the edge  $e_{13}$ , thus  $M = \{e_{13}\}$  and  $F' = P_{x,y}^{(4)}$  does not yield a facet. In particular, condition (iv) is stronger than the requirement that  $M$  is a maximal matching of  $F - V(F')$ .) The resulting facets of type (3e) in  $\text{STAB}(G - xy)$  are:

$$\begin{array}{llllllll}
\text{(a)} & x_1 + x_2 + x_3 & + 2x_5 & + x_7 & + x_9 & + x_{11} & + x_{13} & \leq 2 \\
\text{(b)} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & & + 2x_9 & & & & \leq 2 \\
\text{(c)} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} & & & & & & \leq 2 \\
\text{(d)} & x_1 & + 2x_3 & + 2x_5 & + x_7 + x_8 + x_9 + x_{10} & & & \leq 2 \\
\text{(e)} & x_1 + x_2 + x_3 & + 2x_5 & + x_7 + x_8 + x_9 + x_{10} & & & & \leq 2 \\
\text{(f)} & x_1 + x_2 & + 2x_4 & + x_6 + x_7 + x_8 + x_9 + x_{10} & & & & \leq 2 \\
\text{(g)} & x_1 & + x_3 + x_4 & + x_6 + x_7 + x_8 + x_9 + x_{10} & + x_{12} & & & \leq 2 \\
\text{(h)} & x_1 & + 2x_3 & + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} & + x_{12} & & & \leq 2 \\
\text{(i)} & x_1 & + 2x_3 & + x_5 + x_6 + x_7 & + 2x_9 & + x_{12} & & \leq 2 \\
\text{(j)} & x_1 & & + x_5 + x_6 + x_7 & + x_{10} & + x_{12} + 2x_{13} & & \leq 2 \\
\text{(k)} & x_1 & + x_3 & + x_5 + x_6 + x_7 + x_8 & + x_{10} & + x_{12} + x_{13} & & \leq 2 \\
\text{(l)} & x_1 & + x_3 & + x_5 + x_6 + x_7 & + x_9 + x_{10} & + x_{12} + x_{13} & & \leq 2 \\
\text{(m)} & x_1 & & + x_4 & + x_7 & + x_9 & + x_{11} + x_{12} + x_{13} & \leq 2
\end{array}$$

$\text{STAB}(G - xy)$  has, besides the weak rank facets of type (3e) above, only nonnegativity facets (0) and clique facets (1). That this is true for all graphs  $G - xy$  arising from the complement  $G$  of the line graph of a bipartite graph by removing a critical edge  $xy$  shows the next theorem (this result can also be found in [16]).

**Theorem 4.4** *Let  $G_F$  be the complement of the line graph of a bipartite graph  $F$  and  $e$  an edge of  $G_F$ . A description of  $\text{STAB}(G_F - e)$  is given by the nonnegativity constraints (0), the clique constraints (1), and the lifted odd antihole constraints (3e).*

**Proof.** Consider a bipartite graph  $F$ . Denote by  $G_F$  the complement of the line graph of  $F$  throughout the proof.  $G_F$  is perfect by [7]. If  $e$  is not a critical edge of  $G_F$ , then  $G_F - e$  is perfect by definition, i.e., the constraints (0) and (1) suffice to describe  $\text{STAB}(G_F - e)$ . Thus let  $e$  be a critical edge and  $x(G_F, a) \leq \alpha(G_F, a) = b$  be a facet of  $\text{STAB}(G_F - e)$  different from (0) and (1). We have to show that it is a lifted odd antihole facet of  $\text{STAB}(G_F - e)$  of type (3e). There are  $|G_F|$  maximum weighted stable sets of  $(G_F, a)$  whose incidence vectors are linearly independent. Let  $B$  denote the non-singular matrix with those vectors as rows. Consider the subgraph  $G_{F^*}$  of  $G_F$  induced by  $\{v_i \in G_F : a_i > 0\}$  and let  $B^*$  be the matrix with all columns of  $B$  that correspond to nodes in  $G_{F^*}$  (i.e., to all nodes with non-zero weight). The columns of  $B^*$  are linearly independent since  $B$  has full column rank and, therefore,  $B^*$  has  $|G_{F^*}|$  linearly independent rows. Furthermore,  $\alpha(G_F, a) = \alpha(G_{F^*}, a^*)$  holds where  $a^*$  is a restriction of  $a$  to  $G_{F^*}$ . In other words, it suffices to exhibit that

$$\sum_{v_i \in G_{F^*}} a_i x_i \leq b \tag{*}$$

is a facet of type (3e) of  $\text{STAB}(G_{F^*} - e)$ . Let  $F^*$  be the partial subgraph of  $F$  with  $V(F^*) = \{w \in V(F) : w \text{ incident to an } e_i \in E(F) \text{ with } a_i > 0\}$  and  $E(F^*) = \{e_i \in E(F) : a_i > 0\}$  (i.e.,  $G_{F^*} = \overline{L(F^*)}$ ) and let  $n^* = |G_{F^*}| = |E(F^*)|$ . Since  $(*)$  is particularly a facet of  $\text{STAB}(G_{F^*} - e)$ , there have to be maximum weighted stable sets  $S_1, \dots, S_{n^*}$  in  $(G_{F^*} - e, a)$  with  $\alpha(S_i, a) = b$ , whose incidence vectors  $\chi^{S_1}, \dots, \chi^{S_{n^*}}$  are linearly independent. If we count, in the sequel, the numbers of maximal stable sets, maximum weighted stable sets, or maximum weighted stable sets with linearly independent incidence vectors in any subgraph  $G'_F$  of  $G_F$ , let us use the following abbreviations for convenience:

$$\begin{aligned} \text{MSS}(G'_F) &: \text{number of maximal stable sets in } G'_F \\ \text{MWSS}(G'_F, a) &: \text{number of maximum weighted stable sets in } (G'_F, a) \\ \text{MWSS}_{\text{ind}}(G'_F, a) &: \text{number of maximum weighted stable sets in } (G'_F, a) \\ &\quad \text{whose incidence vectors are linearly independent} \end{aligned}$$

(we still use  $a$  as node weighting restricted to  $G'_F \subseteq G_F$ ). Then  $\text{MSS}(G'_F) \geq \text{MWSS}(G'_F, a) \geq \text{MWSS}_{\text{ind}}(G'_F, a)$  holds and  $\text{MWSS}_{\text{ind}}(G_{F^*} - e, a) \geq n^*$ , since  $(*)$  is a facet of  $\text{STAB}(G_{F^*} - e)$  by assumption. Note that stable sets  $S \subseteq G_{F^*}$  correspond to edge sets  $E_S \subseteq E(F^*)$  with the property that there is a node in  $V(F^*)$  all edges in  $E_S$  are incident to. Consequently, if  $F_i$  and  $F_j$  are different components of  $F^*$  and  $S$  is a stable set in  $G_{F_i \cup F_j}$ , we either have  $S \subseteq G_{F_i}$  or  $S \subseteq G_{F_j}$ . Hence, in order to determine  $\text{MSS}(G_{F_i \cup F_j})$ ,  $\text{MWSS}(G_{F_i \cup F_j}, a)$ , or  $\text{MWSS}_{\text{ind}}(G_{F_i \cup F_j}, a)$ , we often count the corresponding numbers with respect to  $G_{F_i}$  and  $G_{F_j}$  separately. Furthermore, we write  $v_i$  for the node of  $G_F$  corresponding to  $e_i \in E(F)$  and use

$$\begin{aligned} S_w &= \{v_i \in V(G_{F^*}) : e_i \in E(F^*) \text{ is incident to } w \in V(F^*)\} \\ V_2(F) &= \{w \in V(F) : d_F(w) \geq 2\} \end{aligned}$$

to denote a stable set in  $G_{F^*}$  corresponding to the set of all edges in  $F^*$  incident to  $w \in V(F^*)$  and to denote the set of all nodes in  $F$  with degree at least 2, respectively. Finally, let  $e = xy$ , then we have  $x, y \in V(G_{F^*})$  and  $x, y \in E(F^*)$ , since  $(*)$  is neither of type (0) nor of type (1). Furthermore, by Theorem 4.2 follows that  $x$  and  $y$  form an A-pair in  $F^*$  and, therefore, are non-incident edges of  $F^*$ . The proof that  $(*)$  is a facet of type (3e) of  $\text{STAB}(G_F - e)$  is organized in the following claims.

**Claim 1.**  $|V_2(F_i)| = \text{MSS}(G_{F_i}) \geq \text{MWSS}(G_{F_i}, a)$  is satisfied by every component  $F_i$  of  $F^*$  with  $|E(F_i)| > 1$ .

**Claim 2.** Every component  $F_i$  of  $F^*$  with  $\text{MWSS}(G_{F_i}, a) = |G_{F_i}|$  is either isomorphic to a single edge or to an even hole.

**Claim 3.** *The edges  $x$  and  $y$  are contained in the same component  $F'$  of  $F^*$ . Furthermore,  $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \geq |G_{F'}|$  and  $\text{MWSS}_{\text{ind}}(G_{F^* - F'} - xy, a) \leq |G_{F^* - F'}|$  hold.*

**Claim 4.**  *$F'$  is either isomorphic to a chordless path or to an even hole of length  $\geq 6$ ; we obtain  $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) = |G_{F'}|$ .*

**Claim 5.** *In  $F'$ , every path between  $x$  and  $y$  has an odd number of edges  $\geq 3$ ; all edges of  $F'$  have weight  $b/2$ .*

**Claim 6.** *The subgraph  $M' \subseteq F^*$  induced by  $V(F^*) - V(F')$  is a matching; all edges of  $M'$  have weight  $b$ .*

*Proof of Claim 1.* Let  $F_i$  be a component of  $F^*$  with  $|E(F_i)| > 1$ . Suppose the edge  $e_0 = ww'$  of  $F_i$  is the only edge incident to  $w \in V(F_i)$ . Then the node  $v_0 \in V(G_{F_i})$  is contained in the stable sets  $S_w, S_{w'}$  of  $G_{F_i}$ . Since  $F_i$  is connected and we have  $|E(F_i)| > 1$  by precondition,  $d_{F_i}(w') \geq 2$  and, therefore,  $S_w = \{v_0\} \subset S_{w'}$  must hold. Thus  $S_w$  cannot be a maximal stable set of  $G_{F_i}$  for all  $w \in V(F_i) - V_2(F_i)$  and, as a consequence,

$$|V_2(F_i)| = \text{MSS}(G_{F_i}).$$

$\text{MSS}(G_{F_i}) \geq \text{MWSS}(G_{F_i}, a)$  obviously holds since  $a$  is a nonnegative node weighting.  $\diamond$

*Proof of Claim 2.* Suppose  $F_i$  to be a component of  $F^*$  with  $\text{MWSS}(G_{F_i}, a) = |G_{F_i}|$ . In the case  $|E(F_i)| = 1$ ,  $F_i$  is isomorphic to a single edge. Now, assume  $|E(F_i)| > 1$ . By  $\text{MWSS}(G_{F_i}, a) = |G_{F_i}|$  and Claim 1, we get  $|V_2(F_i)| \geq \text{MWSS}(G_{F_i}, a) = |G_{F_i}| = |E(F_i)|$ . On the other hand,  $|E(F_i)| \geq |V_2(F_i)|$  obviously holds by  $|E(F_i)| > 1$ . Thus  $|E(F_i)| = |V_2(F_i)|$  follows and  $F_i$  has to be a chordless cycle. Since  $F^*$  is bipartite,  $F_i$  is isomorphic to an even hole.  $\diamond$

*Proof of Claim 3.* Let  $F_1, \dots, F_k$  be the components of  $F^*$ . In the case  $|E(F_i)| = 1$ ,  $\text{MWSS}(G_{F_i}, a) \leq |G_{F_i}| = 1$  obviously holds. If  $|E(F_i)| > 1$ , then  $|G_{F_i}| = |E(F_i)| \geq |V_2(F_i)| \geq \text{MWSS}(G_{F_i}, a)$  follows on account of  $|E(F_i)| \geq |V_2(F_i)|$  and Claim 1, i.e.,  $|G_{F_i}| \geq \text{MWSS}(G_{F_i}, a)$  holds. Assume  $x \in E(F_j)$  and  $y \in E(F_l)$  for  $j \neq l$ . Then  $\{x, y\}$  is the only stable set of  $G_{F^*} - xy$  containing  $x$  and  $y$ , thus we have  $\text{MWSS}(G_{F^*}, a) \geq \text{MWSS}_{\text{ind}}(G_{F^*}, a) \geq n^* - 1$ .

$$\sum_{1 \leq i \leq k} \text{MWSS}(G_{F_i}, a) \geq \sum_{1 \leq i \leq k} \text{MWSS}_{\text{ind}}(G_{F_i}, a) \geq \sum_{1 \leq i \leq k} |E(F_i)| - 1 = \sum_{1 \leq i \leq k} |G_{F_i}| - 1$$

holds particularly and, by  $|G_{F_i}| \geq \text{MWSS}(G_{F_i}, a)$  for  $1 \leq i \leq k$ , we have equality for at least  $k - 1$  components. Without loss of generality, let  $|G_{F_i}| = \text{MWSS}(G_{F_i}, a)$  for  $F_1, \dots, F_{k-1}$ . Then  $F_1, \dots, F_{k-1}$  are either isomorphic to single edges or to even holes by Claim 2.  $\text{MWSS}_{\text{ind}}(\overline{C}_{2t}, a) \leq 2t - 1$  implies  $F_1, \dots, F_{k-1}$  have to be single edges. Without loss of generality, let  $j \in \{1, \dots, k - 1\}$  (i.e., let  $F_j = \{x\}$ ). Then we certainly have

$$\sum_{1 \leq i \leq k-1} \text{MWSS}(G_{F_i}, a) = k - 1$$

but one of these maximum weighted stable sets of  $G_{F^*}$  is  $\{x\}$  which cannot be a maximum weighted stable set of  $G_{F^*} - xy$ . Therefore it must also hold  $|G_{F_k}| = \text{MWSS}(G_{F_k}, a)$ . If  $|E(F_k)| = 1$ , the maximum weighted stable sets of  $G_{F^*}$  corresponding to  $F_j$  and  $F_l$  are  $\{x\}$  and  $\{y\}$ , respectively. Thus  $\{x\}$  and  $\{y\}$  are contained in precisely one maximal stable set of  $G_{F^*} - xy$ , namely  $\{x, y\}$ . Then  $\text{MWSS}(G_{F^*} - xy, a) \leq |G_{F^*}| - 1$  would contradict the precondition that  $(*)$  is a facet of  $\text{STAB}(G_{F^*} - xy)$ . Hence we have  $|E(F_k)| > 1$ . But now,  $F_k$  has to be isomorphic to an even hole by Claim 2 and  $\text{MWSS}_{\text{ind}}(\overline{C}_{2t}, a) < 2t$  leads to a contradiction. Thus  $x$  and  $y$  appear in the same component, say  $F'$ , of  $F^*$ . We have  $\text{MWSS}(G_{F^* - F'} - xy, a) \leq |G_{F^* - F'}|$  and  $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \geq |G_{F'}|$  since  $(*)$  is a facet of  $\text{STAB}(G_{F^*} - xy)$ .  $\diamond$

*Proof of Claim 4.* Consider the component  $F' \subseteq F^*$  containing  $x$  and  $y$ . Decompose  $F'$  into a set  $\mathcal{C} = \{C_1, \dots, C_k\}$  of node disjoint holes and a set  $\mathcal{T} = \{T_1, \dots, T_{k'}\}$  of node disjoint trees such that the subgraph of  $F'$  induced by  $V(F') - V(\mathcal{C})$  admits precisely  $T_1, \dots, T_{k'}$  as its components. Denote by

$l^{\mathcal{C}}$  : the number of edges in  $E(F') - E(\mathcal{C})$  with both endnodes in  $V(\mathcal{C})$ ,  
 $l_i^{\mathcal{C}}$  : the number of edges linking the tree  $T_i$  with  $\mathcal{C}$ .

Let  $T_i^{\mathcal{C}}$  stand for the union of the tree  $T_i$  and the  $l_i^{\mathcal{C}}$  edges connecting  $T_i$  with  $\mathcal{C}$  and

$l_i$  : the number of nodes of degree 1 in  $T_i$  adjacent to some node in  $\mathcal{C}$ ,  
 $l'_i$  : the number of nodes of degree 1 in  $T_i$  not linked to any node in  $\mathcal{C}$ .

In particular,  $l_i^{\mathcal{C}} \geq l_i \geq 0$  holds for  $1 \leq i \leq k'$  by definition. Moreover, we observe

$$|E(F')| = \sum_{1 \leq j \leq k} |E(C_j)| + l^{\mathcal{C}} + \sum_{1 \leq i \leq k'} (|E(T_i)| + l_i^{\mathcal{C}}). \quad (\text{i})$$

We know from Claim 3 that  $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \geq |G_{F'}| = |E(F')|$  holds. In order to estimate  $\text{MWSS}_{\text{ind}}(G_{F'}, a)$  with respect to  $|E(F')|$ , first recall



that every maximal stable set of  $G_{F'}$  is a stable set  $S_w$  with  $w \in V_2(F')$ . Because the sets  $\mathcal{C}$  and  $\mathcal{T}$  define a node partition of  $F'$ , let us count the number of maximal stable sets of  $G_{F'}$  corresponding to nodes in  $\mathcal{C}$  and  $\mathcal{T}$  separately. Consider  $C_j \in \mathcal{C}$  and let  $C'_j$  stand for  $C_j$  together with all edges incident to  $C_j$  (i.e., together with all edges  $ww' \in E(F')$  with  $w \in C_j$ ,  $w' \in (\mathcal{C} - C_j) \cup \mathcal{T}$ ). Then  $V_2(C'_j) = V_2(C_j)$  implies  $\text{MSS}(G_{C'_j}) = |C_j|$  by Claim 1. Let  $M$  be the matrix with the incidence vectors of the  $|C_j|$  maximal stable sets of  $G_{C'_j}$  as rows. Choose the columns of  $M$  such that the first  $|E(C_j)|$  columns correspond to the edges of  $C_j$ . Then the remaining columns of  $M$ , corresponding to the edges incident to  $C_j$ , do not play any role for determining the rank of  $M$ . The maximal stable set matrix of the even antihole  $G_{C_j}$  (and so  $M$ ) has rank  $|E(C_j)| - 1$ , therefore

$$\text{MWSS}_{\text{ind}}(G_{\mathcal{C}}, a) \leq \sum_{1 \leq j \leq k} (|E(C_j)| - 1) - l^{\mathcal{C}} = |E(F[\mathcal{C}])| - k - l^{\mathcal{C}} \quad (\text{ii})$$

follows. (Note:  $E(F[\mathcal{C}])$  contains the edges of all holes in  $\mathcal{C}$  and the  $l^{\mathcal{C}}$  additional edges with both endnodes in  $V(\mathcal{C})$ .) Particularly, the edges connecting some hole in  $\mathcal{C}$  and some tree in  $\mathcal{T}$  do not belong to  $E(F[\mathcal{C}])$ . Turn now to the remaining part of  $F'$  consisting of  $T_1^{\mathcal{C}}, \dots, T_{k'}^{\mathcal{C}}$ . First note that  $T_i^{\mathcal{C}}$  is not necessarily a tree for  $1 \leq i \leq k'$  since two edges may link  $T_i$  with the same node  $w \in \mathcal{C}$ . But the maximal stable set  $S_w$  of  $G_{F'}$  is already included in  $\text{MWSS}_{\text{ind}}(G_{\mathcal{C}}, a)$  (see (ii)), so only maximal stable sets  $S_{w'}$  of  $G_{F'}$  with  $w' \in \mathcal{T}$  are left to count. Hence, in order to estimate  $\text{MWSS}_{\text{ind}}(G_{T_i^{\mathcal{C}}}, a)$  with respect to  $|E(T_i^{\mathcal{C}})|$ , consider  $T_i^{\mathcal{C}}$  as a tree for  $1 \leq i \leq k'$ . We have

$$|V_2(T_i^{\mathcal{C}})| = |V(T_i)| - l'_i \leq |V(T_i)| < |V(T_i)| + 1 + l_i^{\mathcal{C}} = |E(T_i)| + l_i^{\mathcal{C}} \quad (\text{iii})$$

by the assumption for  $T_i^{\mathcal{C}}$ , by  $l'_i \geq 0$ , by  $l_i^{\mathcal{C}} \geq 0$ , and since  $T_i$  is a tree. The estimation (iii) implies that  $T_i^{\mathcal{C}}$  cannot yield  $|E(T_i)| + l_i^{\mathcal{C}}$  maximum weighted stable sets in  $G_{F'}$ .

$$\text{MWSS}_{\text{ind}}(G_{T_i^{\mathcal{C}}}, a) \leq \text{MWSS}(G_{T_i^{\mathcal{C}}}, a) \leq |E(T_i)| + l_i^{\mathcal{C}} - 1 \quad (\text{iv})$$

follows for  $1 \leq i \leq k'$ . So we combine (ii) and (iv) to

$$\text{MWSS}_{\text{ind}}(G_{F'}, a) \leq |G_{F'}| - (k + l^{\mathcal{C}} + k'). \quad (\text{v})$$

Since  $F'$  is bipartite, there are at most two edges  $z, z'$  of  $F'$  which are incident to both  $x$  and  $y$ . Thus only if there is a  $C_4$  with edges  $\{x, z, y, z'\}$  in  $F'$ , there can be two stable sets in  $G_{F'} - xy$  containing  $x$  and  $y$ . Therefore  $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) - \text{MWSS}_{\text{ind}}(G_{F'}, a) \leq 2$  must hold and we get  $k +$

$l^c + k' \leq 2$  by Claim 3 and (v). Provided  $k + l^c + k' = 2$ ,  $F'$  would be disconnected for  $k = 2$  or  $k' = 2$ . Hence assume  $k, k' = 1$  and refrain from using indices for the only hole  $C \in \mathcal{C}$  and the only tree  $T \in \mathcal{T}$ . By  $\text{MWSS}_{\text{bind}}(G_{F'} - xy, a) - \text{MWSS}_{\text{bind}}(G_{F'}, a) = 2$ ,  $F'$  contains a  $C_4$  with edges  $x, z, y, z'$ . Without loss of generality, let  $E(C) = \{x, z, y, z'\}$ . Since  $\{x, z\}$ ,  $\{z, y\}$ ,  $\{y, z'\}$ , and  $\{z', x\}$  are not maximal stable sets in  $G_{F'} - xy$ , at least three nodes of  $C$  are linked to  $T$  (then the edges incident to these three nodes may yield maximum weighted stable sets) and  $l^c \geq 3$  follows, a contradiction to  $k + l^c + k' = 2$ . So  $k + l^c + k' = 1$  follows, thus either  $k = 1$  and  $l^c, k' = 0$  or  $k' = 1$  and  $l^c, k = 0$ . In the case of  $k = 1$ ,  $F'$  is isomorphic to an even hole. If  $F'$  were a  $C_4$  with edges  $\{x, z, y, z'\}$ ,  $G_{F'} - xy$  would have at most the maximum weighted stable sets  $\{x, z, y\}$  and  $\{x, z', y\}$  in contradiction to Claim 3. Hence  $F'$  is an even hole of length  $\geq 6$  and  $\text{MWSS}_{\text{bind}}(G_{F'} - xy, a) \leq \text{MWSS}_{\text{bind}}(G_{F'}, a) + 1 = |G_{F'}| - k + 1 = |G_{F'}|$  holds by (v). In the case  $k' = 1$ ,  $F'$  is isomorphic to a tree. So  $F'$  cannot contain a  $C_4$  with edges  $\{x, z, y, z'\}$  in particular and  $\text{MWSS}_{\text{bind}}(G_{F'} - xy, a) \leq \text{MWSS}_{\text{bind}}(G_{F'}, a) + 1$  follows. Let  $l'$  stand for the number of nodes with degree 1 in  $F'$ , then

$$|E(F')| + 1 - l' = |F'| - l' = |V_2(F')| \geq \text{MWSS}(G_{F'}, a) \geq |E(F')| - 1$$

holds since  $F'$  is a tree, by the definition of  $l'$ , Claim 1, and Claim 3. From  $l' \leq 2$  follows  $l' = 2$ . I.e.,  $F'$  is a chordless path and we obtain  $\text{MWSS}_{\text{bind}}(G_{F'} - xy, a) \leq |G_{F'}|$  by (iv). Therefore,  $\text{MWSS}_{\text{bind}}(G_{F'} - xy, a) \leq |G_{F'}|$  is true in both cases, if  $F'$  is an even hole of length  $\geq 6$  or a chordless path, and the assertion of the claim follows by Claim 3.  $\diamond$

*Proof of Claim 5.* Consider the component  $F' \subseteq F^*$  containing  $x$  and  $y$ . By Claim 4,  $F'$  is either an even hole of length  $\geq 6$  or a chordless path and there are, in particular, no two edges incident to both  $x$  and  $y$  in  $F'$ . Let  $E(F') = \{e_1, \dots, e_m\}$ , where  $e_i$  and  $e_{i+1}$  are incident edges for  $1 \leq i \leq m-1$  (and  $e_m$  and  $e_1$  if  $F'$  is an even hole). Let  $S_1, \dots, S_{m-1}$  with  $S_i = \{v_i, v_{i+1}\}$  be these maximal stable sets that yield  $|G_{F'}| - 1$  maximum weighted stable sets with linearly independent incidence vectors in  $G_{F'} - xy$ . Furthermore, let  $S_{xy}$  denote a stable set of  $G_{F'} - xy$  containing  $x$  and  $y$  (recall that there is only one such stable set since  $F'$  does not contain two edges incident to both  $x$  and  $y$ ). Then, the incidence vectors of  $S_1, \dots, S_{m-1}$  and  $S_{xy}$  have to be linearly independent. Without loss of generality, let  $x = e_j$  with  $1 \leq j < m-1$  and  $y = e_l$  with  $j+1 < l \leq m$  (actually,  $y \neq e_m$  if  $x = e_1$  and  $F'$  even hole). Assume  $l = j + 2k + 1$  with  $k > 0$ , then  $S_j, \dots, S_{l-1}, S_{xy}$  would be the maximal stable sets of an even antihole in  $G_{F'} - xy$ , i.e., their incidence vectors  $\chi^{S_j}, \dots, \chi^{S_{l-1}}, \chi^{S_{xy}}$  could not be linearly independent. Thus we have  $l = j + 2k$  with  $k > 0$ . For  $k = 1$ ,  $S_j = \{x, v_{j+1}\}$  and  $S_{j+1} = \{v_{j+1}, y\}$  are

certainly maximal stable sets in  $G_{F'}$ , but not in  $G_{F'} - xy$  (since  $\{x, v_{j+1}, y\}$  is a stable set in  $G_{F'} - xy$ ). Thus, for  $k = 1$  we have less maximal stable sets in  $G_{F'} - xy$  than nodes, a contradiction to Claim 4. Hence  $k > 1$  yields that every path between  $x$  and  $y$  has an odd number of edges  $\geq 3$  and, therefore,  $S_{xy} = \{x, y\}$  must hold. Now turn to the edge weights in  $F'$ . Since  $S_1, \dots, S_{m-1}, S_{xy}$  have to be maximum weighted stable sets in  $G_{F'} - xy$ , we obtain  $a_i + a_{i+1} = b$  for  $1 \leq i \leq m - 1$  and  $a_j + a_l = b$ . Furthermore,  $a_i = a_{i'}$  holds whenever  $i$  and  $i'$  have the same parity. In particular, we have  $a_j = a_l = b/2$  by  $a(S_{xy}) = b$  and  $l = j + 2k$ , so we finally obtain  $a_i = b/2$  for  $1 \leq i \leq m$ .  $\diamond$

*Proof of Claim 6.* Consider the subgraph  $M' \subseteq F^*$  induced by  $V(F^*) - V(F')$ . Claim 3 and Claim 4 imply  $\text{MWSS}_{\text{ind}}(G_{M'} - xy, a) = |G_{M'}|$ . Since  $M'$  is defined to be the union of all components  $F_i$  of  $F^*$  neither containing  $x$  nor  $y$ ,  $\text{MWSS}_{\text{ind}}(G_{M'} - xy, a) = \text{MWSS}_{\text{ind}}(G_{M'}, a)$  follows. Thus we infer  $\text{MWSS}_{\text{ind}}(G_{F_i}, a) = |G_{F_i}|$  for all  $F_i \subseteq M'$ . By Claim 2, every  $F_i$  has to be isomorphic to a single edge, i.e.,  $M'$  is a matching in  $F^*$ . Therefore, every  $G_{F_i}$  consists of a single node  $v_i$ . Since  $G_{F_i}$  have to yield a maximum weighted stable set of  $G_{F^*}$  for all  $F_i \subseteq M'$  by  $\text{MWSS}_{\text{ind}}(G_{M'} - xy, a) = |G_{M'}|$ ,  $\{v_i\}$  must be this maximum weighted stable set. We obtain that every node  $v_i \in G_{M'}$  and every edge  $e_i \in M'$ , respectively, has the weight  $b$ .  $\diamond$

I.e., (\*) has the form

$$\sum_{v_i \in G_{F'}} \frac{b}{2} x_i + \sum_{v_i \in G_{M'}} b x_i \leq b$$

and with scaling  $\frac{b}{2} = 1$ , we have shown  $a_i = 1 \forall v_i \in G_{F'}$  and  $a_i = 2 \forall v_i \in G_{M'}$ . For all remaining nodes  $v_i$  we have  $a_i = 0$ , since the above inequality is a facet of  $\text{STAB}(G_F - xy)$  by assumption. Thus every facet of  $\text{STAB}(G_F - e)$  different from (0) and (1) is indeed a facet of type (3e).  $\square$

Since the facets of type (3e) are weak rank constraints, the following corollary is an immediate consequence of Theorem 4.4 and solves Problem 4.1.

**Corollary 4.5** *All graphs, arising from complements of line graphs of bipartite graphs by deletion of an arbitrary edge, are weakly rank-perfect.*

That means: deleting an edge from the complement of the line graph of a bipartite graph creates a graph in the same stage of imperfectness as general complements of line graphs.

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