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Resource-Constrained Project  
Scheduling Problem**

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# Improved Compact Models for the Resource-Constrained Project Scheduling Problem

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**Abstract.** In this article, we study compact Mixed-Integer Programming (MIP) models for the Resource-Constrained Project Scheduling Problem (RCPSP). Compared to the classical time-indexed formulation, the size of compact models is strongly polynomial in the number of jobs. In addition to two compact models from the literature, we propose a new compact model. We can show that all three compact models are equivalent by successive linear transformations. For their LP-relaxations, however, we state a full inclusion hierarchy where our new model dominates the previous models in terms of polyhedral strength. Moreover, we reveal a polyhedral relationship to the common time-indexed model. Furthermore, a general class of valid cutting planes for the compact models is introduced and finally all models are evaluated by computational experiments.

## 1 Introduction

In the *Resource-Constrained Project Scheduling Problem* (RCPSP) we are given a set of  $n$  non-preemptive jobs  $j \in \mathcal{J}$  with processing times  $p_j > 0$  and a set of resources  $k \in \mathcal{R}$  with capacity  $R_k \geq 0$  where each job  $j \in \mathcal{J}$  has a demand of  $r_{jk} \geq 0$  units of resource  $k \in \mathcal{R}$ . Furthermore, there are precedence relations  $\mathcal{P} \subset \mathcal{J} \times \mathcal{J}$  between the jobs where  $(i, j) \in \mathcal{P}$  indicates that job  $i$  must end before job  $j$  starts. In the RCPSP we want to compute starting times for all jobs that satisfy the precedence constraints and such that at any point in time the resource consumptions of all active jobs does not exceed the capacities. The objective is to minimize the *makespan* which equals the total project duration.

The most common MIP formulation for the RCPSP is the time-indexed model of Pritsker et al. [10]. In this model, every job is assigned to a starting time within a discrete scheduling horizon. Many variants and extensions of the time-indexed model have been studied during the last decades, see for example [4, 5, 9]. But since the model size grows quadratically with the scheduling horizon, time-indexed models are still computationally intractable for large time horizons. This motivates the study of *compact* MIP models for the RCPSP whose size is strongly polynomial in the number of jobs. Currently, mainly two types of compact models are

known. Artigues et al. [1] introduce a *flow-based* compact model where a resource flow determines the precedences between the jobs. Koné et al. [8] develop two *event-based* compact models that assign all jobs to a fixed position in the starting order of the jobs.

In this article, we introduce a new event-based compact model and we study the polyhedral relationship between our model, the two models of Koné et al. [8] and the time-indexed model of Pritsker et al. [10].

## 2 MIP Models

First, we briefly introduce the main modeling concepts of the time-indexed model of Pritsker et al. [10] and the two compact models of Koné et al. [8].

**Time-Indexed Model (DDT)** [10, 5]. The time-indexed case considers a discrete time horizon  $\mathcal{T} = \{0, \dots, T\}$ , discrete processing times  $p_j \in \mathbb{Z}_{>0}$  and decision variables  $x_{jt} \in \{0, 1\}$  that are one, if job  $j \in \mathcal{J}$  starts at time  $t \in \mathcal{T}$ . Resource constraints are applied at every time point  $t \in \mathcal{T}$ , therefore the model size grows quadratically with  $T$ .

**Event-Based Compact Models** [8]. In the compact models of Koné et al., we are given a set of *events*  $\mathcal{V}$  where an event denotes a time point where one or multiple jobs start. Every event  $v \in \mathcal{V}$  is therefore correlated to a variable  $t_v \geq 0$  that describes the start time of all jobs that start at event  $v$ . All events appear sequentially, that is  $t_v \leq t_{v+1}$  holds for all  $v \in \mathcal{V}$ . Since  $n$  events are sufficient, the makespan is modeled by the variable  $t_{n+1} \geq 0$ .

**(i) On-/Off Event-Based Model (OOE).** This model uses *activity variables*  $u_{jv} \in \{0, 1\}$  that are one, if job  $j \in \mathcal{J}$  is executed during the event interval  $[t_v, t_{v+1})$  with  $v \in \mathcal{V}$ .

**(ii) Start-/End Event-Based Model (SEE).** The model considers variables  $y_{jv} \in \{0, 1\}$  and  $\bar{y}_{jw} \in \{0, 1\}$  that are one, if job  $j \in \mathcal{J}$  starts at event  $v \in \mathcal{V}$  or ends at event  $w \in \mathcal{V}' = \{v + 1 \mid v \in \mathcal{V}\}$  respectively.

In [12], stronger inequalities for OOE and SEE are introduced.

### 2.1 A New Compact Model

We now introduce a new event-based model, the *Disaggregated Position Model* (DP), which considers decision variables  $z_{jvw} \in \{0, 1\}$  that are one, if job  $j \in \mathcal{J}$  starts at event  $v \in \mathcal{V}$  and ends at event  $w \in \mathcal{V}'$ . The model states

$$\begin{aligned} \min \quad & t_{n+1} & (1) \\ \sum_{(v,w) \in \mathcal{A}} \quad & z_{jvw} = 1 \quad \forall j \in \mathcal{J} & (2) \end{aligned}$$

$$p_j \cdot \sum_{v \leq v' < w' \leq w} z_{jv'w'} \leq t_w - t_v \quad \forall j \in \mathcal{J}, (v, w) \in \mathcal{A} \quad (3)$$

$$\sum_{j \in \mathcal{J}} \sum_{v \leq v' < w} r_{jk} \cdot z_{jvw} \leq R_k \quad \forall v' \in \mathcal{V}, k \in \mathcal{R} \quad (4)$$

$$\sum_{(v, w) \in \mathcal{A}: w > v'} z_{ivw} + \sum_{(v, w) \in \mathcal{A}: v \leq v'} z_{jvw} \leq 1 \quad \forall (i, j) \in \mathcal{P}, v' \in \mathcal{V} \quad (5)$$

$$t_v \geq 0 \quad \forall v \in \mathcal{V} \cup \{n+1\}$$

$$z_{jvw} \in \{0, 1\} \quad \forall j \in \mathcal{J}, (v, w) \in \mathcal{A}$$

where the objective (1) is to minimize the makespan. By inequalities (2) every job starts and ends at events  $(v, w) \in \mathcal{A} = \{(v, w) \in \mathcal{V} \times \mathcal{V}' \mid v < w\}$ . Inequalities (3) determine the time lag  $t_w - t_v \geq 0$  between two events  $(v, w) \in \mathcal{A}$  that is the maximum duration  $p_j$  of a job  $j \in \mathcal{J}$  that is scheduled between events  $v$  and  $w$ . Inequalities (4) ensure that the resource consumptions of all active jobs at event  $v' \in \mathcal{V}$  do not exceed the capacities. For each precedence relation  $(i, j) \in \mathcal{P}$  inequalities (5) forbid that job  $j$  has a start event earlier than the end event of job  $i$ . In contrast to OOE and SEE, the DP model involves no big-M parameters or linearized expressions. In a preprocessing step, we can exclude variables  $z_{jvw}$  where  $w - v$  is large.

## 2.2 Linear Transformations

Between the compact models there hold the following linear transformations

$$u_{jv} = \sum_{v' \leq v} y_{jv'} - \sum_{v' \leq v} \bar{y}_{jv'} \quad \forall j \in \mathcal{J}, v \in \mathcal{V} \quad (6)$$

$$y_{jv} = \sum_{v < w} z_{jvw} \quad \forall j \in \mathcal{J}, v \in \mathcal{V} \quad (7)$$

$$\bar{y}_{jw} = \sum_{v < w} z_{jvw} \quad \forall j \in \mathcal{J}, w \in \mathcal{V}' \quad (8)$$

$$u_{jv} = \sum_{v' \leq v < w'} z_{jv'w'} \quad \forall j \in \mathcal{J}, v \in \mathcal{V} \quad (9)$$

where (6) links OOE with SEE, (7)-(8) links SEE with DP, and (9) links OOE with DP by applying both transformations consecutively. Denote by  $\Phi_1$  and  $\Phi_2$  the linear transformations (9) and (6) respectively. Furthermore, given a MIP model  $M$ , let  $P^M$  be the associated polyhedron of its LP-relaxation and let  $P_I^M$  be the integer hull of  $P^M$ , see [11].

**Theorem 1.** *For the linear transformations  $\Phi_1$  and  $\Phi_2$  it holds*

$$\Phi_1(P_I^{DP}) = \Phi_2(P_I^{SEE}) = P_I^{OOE} \quad \text{and} \quad \Phi_1(P^{DP}) \subset \Phi_2(P^{SEE}) \subset P^{OOE}.$$

**Theorem 2.** *Assume an instance of the DP model and expand it by setting  $\mathcal{V} = \mathcal{T} = \{0, \dots, T\}$ . Then take the restriction  $z_{jvw} = 0$  for all  $(v, w) \in \mathcal{A}$  with  $w - v \neq p_j$  and project it to the variables  $z_{jvw}$  with  $w - v = p_j$ . The resulting model  $DP'$  satisfies  $P^{DP'} = P^{DDT}$ .*

Theorem 1 states the equivalence between integer solutions of the models OOE, SEE and DP. According to their LP-relaxations, however, our new DP model dominates the compact models OOE and SEE of Koné et al. [8]. It further reveals that SEE is stronger than OOE. Moreover, Theorem 2 states a relationship between the compact models and the time-indexed model DDT. In other words, DDT is obtained from DP by subsequent expansion, restriction and projection of the corresponding polyhedron  $P^{DP}$ . Complete proofs of Theorems 1 and 2 can be found in [12].

### 3 Primal-Dual Cutting Planes

Let  $\mu_{jv} \geq 0$  be the duration of job  $j \in \mathcal{J}$  in the event interval  $[t_v, t_{v+1}]$  for an event  $v \in \mathcal{V}$ . We couple the  $\mu_{jv}$  variables into the compact models by adding the inequalities

$$\sum_{v \in \mathcal{V}} \mu_{jv} \geq p_j \quad \forall j \in \mathcal{J} \quad (10)$$

$$\mu_{jv} \leq p_j \cdot u_{jv} \quad \forall j \in \mathcal{J}, v \in \mathcal{V} \quad (11)$$

which indicate that every job has its processing time distributed over the event intervals  $v \in \mathcal{V}$  (10) but only at events where the job is active (11). Since the variables  $u_{jv}$  in inequality (11) belong only to OOE, the transformations (6) and (9) yield equivalent inequalities for SEE and DP.

Let a job subset  $F \subseteq \mathcal{J}$  be called *feasible*, if all jobs in  $F$  can be scheduled in parallel. Further let  $\mathcal{F} \subseteq 2^{\mathcal{J}}$  denote the set of all feasible job subsets.

**Theorem 3.** *Given coefficients  $\delta_j \geq 0$  for all  $j \in \mathcal{J}$  with  $\sum_{j \in F} \delta_j \leq 1$  for all feasible subsets  $F \in \mathcal{F}$  then the following inequalities are valid:*

$$\sum_{j \in \mathcal{J}} \delta_j \mu_{jv} \leq t_{v+1} - t_v \quad \forall v \in \mathcal{V}. \quad (12)$$

The inequalities (12) yield strong valid cutting planes on the proposed MIP extension. They further generalize many valid inequalities that were originally proposed for the LP model of Carlier & Néron [3], for example: *energetic reasoning cuts*, *(lifted) cover cuts*, *clique cuts* and *redundant function cuts*, see also [6]. As shown in [12], the cutting planes (12) originate from a primal-dual relation between the two linear programming models of Brucker & Knust [2] and Carlier & Néron [3]. Cutting planes of the form (12) can be computed by solving a linear program with generally exponentially many inequalities (for every  $F \in \mathcal{F}$ ). Therefore, we generate only a constant number of such cuts in a preprocessing step.

### 4 Computational Results

Our models were tested on 480 instances of the PSPLIB [7] where each instance considers 30 jobs, 4 resources and various precedence graphs.

Computations are performed on a 3.5 GHz Intel Xeon CPU, 16 GB RAM using CPLEX version 12.6. We implemented the time-indexed model DDT and the compact models OOE, SEE and DP. More specifically, we use a transformed but equivalent version of SEE [12] with a sparser constraint matrix what can be exploited by modern MIP solvers. All compact models include the extensions (10)-(11) for which we generated  $n = 30$  primal-dual cuts (12) according to randomized objective coefficients. The time limit of each instance is 300 seconds. Our experimental results are illustrated in Table 1.

The columns *opt* and *ub=opt* show the number of instances where the optimal solution was found, provably and non-provably respectively. Moreover, the columns *#vars* and *#cons* represent the average number of variables and constraints in the models. Columns  $\Delta lb$  and  $\Delta ub$  show the total difference of the computed lower- and upper bounds compared to the weakest model.

Among the compact models, the revised SEE model reveals the best performance mainly due to its sparse constraint matrix. The OOE model performs well in the primal but weak in the dual because it has a few number of binary variables but a weak LP-relaxation. Even though it constitutes the theoretically strongest compact model, DP shows weaker results than all other models. The main reasons are the huge number of binary variables and highly fractional LP-relaxations that consume a lot of computation time. More sophisticated preprocessing techniques might overcome this complexity in the future. In comparison to DDT, the compact models are slightly inferior on most instances. Interestingly, on a subset of generally hard instances the revised SEE model strictly outperforms DDT in the primal and dual because the extension cuts (10)-(12) yield drastically stronger lower bounds compared to DDT and to the original models of Koné et al. [8]. Even a small number of randomized cuts yields reasonably strong dual bounds [12].

Since the time horizon of any problem instance can be scaled arbitrarily, the compact models will always dominate DDT at a certain scaling factor.

**Table 1.** Comparison of the MIP models

<b>model</b>	<b>opt</b>	<b>ub=opt</b>	<b>#vars</b>	<b>#cons</b>	<b><math>\Delta lb</math></b>	<b><math>\Delta ub</math></b>
DDT	422	428	4980	9368	491	-377
OOE	265	392	900	9240	186	-600
SEE	295	407	1830	8123	415	-592
DP	219	324	13950	5370	0	0

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