



ANDREAS GRIEWANK, TOM STREUBEL, RICHARD HASENFELDER  
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The work for the article has been partially conducted within the Research Campus MODAL funded by the German Federal Ministry of Education and Research (BMBF) (fund number 05M14ZAM).

Zuse Institute Berlin  
Takustr. 7  
14195 Berlin  
Germany

Telephone: +49 30-84185-0  
Telefax: +49 30-84185-125

E-mail: [bibliothek@zib.de](mailto:bibliothek@zib.de)  
URL: <http://www.zib.de>

ZIB-Report (Print) ISSN 1438-0064  
ZIB-Report (Internet) ISSN 2192-7782

# Piecewise linear secant approximation via Algorithmic Piecewise Differentiation\*

Andreas Griewank<sup>1</sup>, Tom Streubel<sup>2,3</sup>, Richard Hasenfelder<sup>3</sup>, and Manuel Radons<sup>3</sup>

<sup>1</sup>School of Mathematical Sciences and Information Technology, Ecuador

<sup>2</sup>Zuse Institute Berlin, Germany

<sup>3</sup>Humboldt University of Berlin, Germany

*griewank@yachaytech.edu.ec, streubel@zib.de, hasenfel@math.hu-berlin.de, radons@math.hu-berlin.de*

## Abstract

It is shown how piecewise differentiable functions  $F : \mathbb{R}^n \mapsto \mathbb{R}^m$  that are defined by evaluation programs can be approximated locally by a piecewise linear model based on a pair of sample points  $\tilde{x}$  and  $\hat{x}$ . We show that the discrepancy between function and model at any point  $x$  is of the bilinear order  $O(\|x - \tilde{x}\|\|x - \hat{x}\|)$ . This is a little surprising since  $x \in \mathbb{R}^n$  may vary over the whole Euclidean space, and we utilize only two function samples  $\tilde{F} = F(\tilde{x})$  and  $\hat{F} = F(\hat{x})$ , as well as the intermediates computed during their evaluation. As an application of the piecewise linearization procedure we devise a generalized Newton's method based on successive piecewise linearization and prove for it sufficient conditions for convergence and convergence rates. We conclude with the derivation of formulas for the numerically stable implementation of the aforedeveloped piecewise linearization methods.

**Keywords** Automatic differentiation, Computational graph, Lipschitz continuity, Generalized Hermite interpolation, ADOL-C

## 1 Introduction

In this paper we refine and extend the theory of piecewise linearizations of piecewise differentiable functions  $F : \mathbb{R}^n \mapsto \mathbb{R}^m$  that was introduced in [4]. Throughout we assume that any such function is defined by an evaluation procedure consisting of a sequence of elemental functions  $v_i = \varphi_i(v_j)_{j \prec i}$ , where  $\varphi$  is either smooth or the absolute value function. The data dependence relation  $\prec$  generates a partial ordering, which yields a directed acyclic graph. In other words, we assume that we have a straight line program without any loops or jumps in the control flow. The data dependence relation  $\prec$  generates a partial ordering, which yields a directed acyclic graph.

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\*Our notion of linearity includes nonhomogeneous functions, where the adjective *affine* or perhaps *polyhedral* would be more precise. However, such mathematical terminology might be less appealing to computational practitioners and to the best of our knowledge there are no good nouns corresponding to *linearity* and *linearization* for the adjectives *affine* and *polyhedral*.

In the above reference a piecewise linearization  $\Delta F(x; \Delta x)$  of  $F$  about the point  $\hat{x}$  is constructed such that, for arbitrary  $x \in \mathbb{R}^n$  with  $\hat{F} = F(\hat{x})$ , it holds

$$F(x) = \hat{F} + \Delta F(\hat{x}; x - \hat{x}) + \mathcal{O}(\|x - \hat{x}\|^2).$$

Since  $\Delta F(\hat{x}; \Delta x)$  is constructed by replacing smooth elements by their tangent lines at  $\hat{x}$ , we will refer to it as **piecewise tangent linearization**. An important property of the piecewise linearizations is that they vary continuously with the sample point or points at which they are developed.

As a generalization of  $\Delta F(\hat{x}; \Delta x)$  we construct a **piecewise secant linearization**  $\Delta F(\check{x}, \hat{x}; \Delta x)$  of  $F$  at the pair  $\check{x}$  and  $\hat{x}$  such that for the midpoints  $\hat{x} = (\check{x} + \hat{x})/2$  and  $\hat{F} = (\check{F} + \hat{F})/2$  with  $\check{F} = F(\check{x})$  and  $\hat{F} = F(\hat{x})$  it holds

$$F(x) = \hat{F} + \Delta F(\check{x}, \hat{x}; x - \hat{x}) + \mathcal{O}(\|x - \check{x}\| \|x - \hat{x}\|).$$

The new  $\Delta F(\check{x}, \hat{x}; \Delta x)$  reduces to  $\Delta F(\hat{x}; \Delta x)$  when the two sample points  $\check{x}$  and  $\hat{x}$  coalesce at  $\hat{x}$ .

Our results in this article are twofold:

- Firstly, we present approximation properties and Lipschitz continuity estimates. Here the two major points are that the piecewise linearization is a second order approximation to the underlying function. Moreover, we do not only prove the existence of Lipschitz constants, but provide explicit estimates.
- Secondly, we develop an application for the piecewise linearization. Namely, we introduce, for each linearization mode, a Newton's method based on successive piecewise linearization and give sufficient conditions for convergence, as well as statements on the convergence rates.

In an appended section we moreover provide formulas and pseudocode for singularity free and thus numerically stable implementations of the piecewise secant linearizations.

One significant advantage of the piecewise linearization-approach is that, once the piecewise linearization is generated, we also know where its kinks are located. This means that we are liberated of the complications of event handling that the nondifferentiabilities of piecewise smooth functions usually bring about, which was one of the motivations for the development of the techniques presented in [4]. In said reference it is proved that the variations of the piecewise linear model are Lipschitz continuous with respect to perturbations of the development point. The results mentioned in the first bullet point represent a significant improvement over those given in [4] in so far as they are not only sharper, but also provide explicit bounds for the Lipschitz constants for the variations of the model with respect to perturbations of the base point.

## Content and structure

To provide some more background, we will, in the second section, elaborate on the setting outlined above both from the mathematical and the implementation perspective. In Section 3 we will introduce the piecewise linearization framework that is investigated thereafter. In Section 4 we derive the approximation and stability properties of said framework. In the subsequent section the generalized Newton's methods are derived for both linearization modes.

In the final Section 5 we show the secant linearization can be computed in a division free, centered form such that it reduces continuously to the tangent mode when the reference points  $\tilde{x}$  and  $\hat{x}$  coalesce. We also briefly discuss the implementation in an ADOL-C type fashion.

## 2 Preliminary remarks

Let  $\tilde{\Phi}$  be a library of elemental functions that conforms to the condition of elemental differentiability as described in [5]. In the following we will consider piecewise differentiable functions that are defined by an evaluation procedure consisting of a sequence of such elemental functions  $v_i = \varphi_i(v_j)_{j \prec i}$ , where the  $\varphi_i$  are contained in a library

$$\Phi := \tilde{\Phi} \cup \{\mathbf{abs}()\}.$$

We remark that the inclusion of the absolute value function into the library means that we can also evaluate  $\min()$  and  $\max()$  calls via the identities

$$\max(u, w) = (u + w + \mathbf{abs}(u - w))/2, \quad \min(u, w) = (u + w - \mathbf{abs}(u - w))/2. \quad (1)$$

There is a slight implicit restriction, namely, we assume that whenever  $\min$  or  $\max$  are evaluated both their arguments have well defined finite values so that the same is true for their sum and difference. On the other hand, the expression  $\min(1, 1/\mathbf{abs}(u))$  makes perfect sense in IEEE arithmetic [9], but rewriting it as above leads to a *NaN* at  $u = 0$ . While this restriction may appear quite technical, it imposes the requirement that all relevant quantities are well defined at least in some open neighborhood, which is exactly in the nature of piecewise differentiability.

Piecewise differentiability has been extensively studied for example by Kummer [10] and Scholtes [14]. In accordance with the categorization in [?] we will refer to procedures containing only  $C^{1,1}(\mathcal{D}_i)$  functions  $\varphi_i$  and the absolute as  $\mathbf{abs}()$  as Level-1 nonsmooth. Allowing the Euclidean norm or even jumps gets us into the realm of Level-2 or even Level-3 nonsmoothness.

Obviously, in the Level-1 nonsmooth case the overall composite function may still be *composite differentiable* if the programmer has used  $\mathbf{abs}()$  wisely, for example only to first scale and later unscale variables to improve numerical stability. Then these operations will not effect the theoretical function values and their differentiability properties. Moreover, we will see that our piecewise linearization approach will, in fact, yield the correct derivatives of such composite differentiable functions. We consider this a very important achievement for AD tools.

The function  $\Delta F(\hat{x}; \Delta x)$  is incremental in that  $\Delta F(x; 0) = 0$ , but like general piecewise linear continuous functions, it is only locally and positively homogeneous so that

$$\Delta F(\hat{x}; \alpha \Delta x) = \alpha \Delta F(\hat{x}; \Delta x) \quad \text{for } 0 < \alpha \|\Delta x\| < \rho(\hat{x}).$$

Here the bound  $\rho(x)$  is positive everywhere, but generally not continuous with respect to  $\hat{x}$ .

In [4] it has been shown that the Jacobians of the linear pieces of  $\Delta F(\hat{x}; \alpha \Delta x)$  in the ball of radius  $\rho(\hat{x})$  about  $\hat{x}$  are conically active generalized Jacobians of the underlying nonlinear function  $F(x)$ . We will not elaborate on this connection here, because even in the smooth case secant approximations need not correspond to exact Jacobians. In contrast, the generalized derivative sets in the sense of Clarke [3] are for piecewise differentiable functions almost everywhere just singletons, containing the classical Jacobian matrix. In floating point arithmetic the user or client algorithm will then quite likely never 'see' the nonsmoothness or gain any useful information about it.

### 3 Piecewise linearization by tangents and secants

Suppose the vector function  $F : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  in question is evaluated by a sequence of assignments

$$v_i = v_j \circ v_k \quad \text{or} \quad v_i = \varphi_i(v_j) \quad \text{for} \quad i = 1 \dots l.$$

Here  $\circ \in \{+, -, *\}$  is a polynomial arithmetic operation and

$$\varphi_i \in \Phi \equiv \{\text{rec, sqrt, sin, cos, exp, log, \dots, abs, \dots}\}$$

a univariate function. The user or reader may extend the library by other locally Lipschitz-continuously differentiable functions like the analysis favorites

$$\varphi(u) \equiv |u| > 0 ? u^p \cdot \sin(1/u) : 0 \quad \text{for} \quad p \geq 3.$$

But then he or she is responsible for supplying an evaluation procedure for both the elemental function  $\varphi$  and its derivative  $\varphi'$ , which cannot be based mechanically on the chain rule.

Following the notation from [5] we partition the sequence of scalar variables  $v_i$  into the vector triple

$$(x, z, y) = (v_{1-n}, \dots, v_{-1}, v_0, \dots, v_{l-m}, v_{l-m+1}, \dots, v_l) \in \mathbb{R}^{n+l}$$

such that  $x \in \mathbb{R}^n$  is the vector of independent variables,  $y \in \mathbb{R}^m$  the vector of dependent variables and  $z \in \mathbb{R}^{l-m}$  the (internal) vector of intermediates.

Some of the elemental functions like the reciprocal  $\text{rec}(u) \equiv 1/u$ , the square root and the logarithm are not globally defined. As mentioned above, we will assume that the input variables  $x$  are restricted to an open domain  $\mathcal{D} \subset \mathbb{R}^n$  such that all resulting intermediate values  $v_i = v_i(x)$  are well defined.

Throughout we will assume that the evaluation procedure for  $F$  involves exactly  $s \geq 0$  calls to **abs**(), including min and max rewritten or at least reinterpreted as discussed above. Starting from  $\hat{x}$  and an increment  $\Delta x = x - \hat{x}$ , we will now construct for each intermediate  $v_i$  an approximation

$$v_i(\hat{x} + \Delta x) - \hat{v}_i \approx \Delta v_i \equiv \Delta v_i(\Delta x).$$

Here the incremental function  $\Delta v_i(\Delta x)$  is continuous and piecewise linear, with  $\hat{x}$  or  $\check{x}$  and  $\hat{x}$  considered constant in the tangent and secant case, respectively. Hence, we will often list  $\Delta x$ , but only rarely use  $\hat{x}$ ,  $\check{x}$  and  $\hat{x}$  as arguments of the  $\Delta v_i$  in proofs.

#### Defining relations for tangent approximation

We use the reference values  $\hat{v}_i = v_i(\hat{x})$  and, assuming that all  $\varphi_i$  other than the absolute value function are differentiable within the domain of interest, we may use the tangent linearizations

$$\Delta v_i = \Delta v_j \pm \Delta v_k \quad \text{for} \quad v_i = v_j \pm v_k, \quad (2)$$

$$\Delta v_i = \hat{v}_j * \Delta v_k + \Delta v_j * \hat{v}_k \quad \text{for} \quad v_i = v_j * v_k, \quad (3)$$

$$\Delta v_i = \hat{c}_{ij} * \Delta v_j \quad \text{for} \quad v_i = \varphi_i(v_j) \neq \mathbf{abs}(). \quad (4)$$

Here  $\hat{c}_{ij} \equiv \varphi'_i(\hat{v}_j)$ , which will be different for the secant linearization.

If no absolute value or other nonsmooth elemental occurs, the function  $y = F(x)$  is, at the current point, differentiable and by the chain rule we have the relation

$$\Delta y = \Delta F(\hat{x}; \Delta x) \equiv F'(\hat{x})\Delta x,$$

where  $F'(x) \in \mathbb{R}^{m \times n}$  is the Jacobian matrix. Thus we observe the obvious fact that smooth differentiation is equivalent to linearizing all elemental functions.

Now, let us move to the piecewise differentiable scenario, where the absolute value function does occur  $s > 0$  times. We then may obtain a piecewise linearization of the vector function  $F(\hat{x} + \Delta x) - F(\hat{x})$  by incrementing

$$\Delta v_i = \mathbf{abs}(\hat{v}_j + \Delta v_j) - \hat{v}_i \quad \text{when} \quad v_i = \mathbf{abs}(v_j) . \quad (5)$$

Here  $\hat{v}_i = \mathbf{abs}(\hat{v}_j)$ , which will be slightly different for the secant linearization. In other words, we keep the piecewise linear function  $\mathbf{abs}()$  unchanged so that the resulting  $\Delta y$  represents, for each fixed  $x \in \mathcal{D}$ , the piecewise linear and continuous *increment function*

$$\Delta y = \Delta y(\Delta x) = \Delta F(\hat{x}; \Delta x) : \mathbb{R}^n \rightarrow \mathbb{R}^m .$$

### Defining relations for secant approximation

In the tangent approximation the reference point was always the evaluation point  $\hat{x}$  and the resulting values  $\hat{v}_i = v_i(\hat{x})$ . Now we will make reference to the midpoints

$$\hat{v}_i \equiv (\check{v}_i + \hat{v}_i)/2 \quad \text{of} \quad \check{v}_i \equiv v_i(\check{x}) \quad \text{and} \quad \hat{v}_i \equiv v_i(\hat{x}) . \quad (6)$$

So we have really the functional dependence  $\hat{v}_i = \hat{v}_i(\check{x}, \hat{x})$ , which is at least Lipschitz continuous under our assumptions. Now an intriguing observation is that the recurrences (2) and (3) for arithmetic operations can stay just the same, and the recurrence (4) for nonlinear univariates is still formally valid, except that the tangent slope  $\varphi'(\hat{v}_j)$  must be replaced by the secant slope

$$c_{ij} \equiv \begin{cases} \varphi'_i(\hat{v}_j) & \text{if } \check{v}_j = \hat{v}_j \\ (\hat{v}_i - \check{v}_i)/(\hat{v}_j - \check{v}_j) & \text{otherwise} \end{cases} . \quad (7)$$

Theoretically, some  $\check{v}_i$  and  $\hat{v}_i$  may coincide, even if the underlying sample points  $\check{x}$  and  $\hat{x}$  are not selected identically, in which case the secant based model would reduce to the tangent based model. While exact coincidence of any pair  $\check{v}_i$  and  $\hat{v}_i$  is rather unlikely, taking the divided difference over small increments is likely to generate numerical cancellation. Therefore we will develop a division free centered form in Section 6. Finally, the nonsmooth rule (5) can stay unchanged except that we now set

$$\hat{v}_i \equiv \frac{1}{2}(\check{v}_i + \hat{v}_i) = \frac{1}{2}[\mathbf{abs}(\check{v}_j) + \mathbf{abs}(\hat{v}_j)] . \quad (8)$$

Hence, it is immediately clear that the new secant approximation reduces to the old tangent approximation when  $\check{x} = \hat{x}$ . In general, we will denote the mapping between the input increments  $\Delta x \in \mathbb{R}^n$  and the resulting values  $\Delta y \in \mathbb{R}^m$  by

$$\Delta y = \Delta y(\Delta x) = \Delta F(\check{x}, \hat{x}; \Delta x) : \mathbb{R}^n \rightarrow \mathbb{R}^m .$$

Its piecewise linear structure is very much the same as that of the tangent based model, which is described in detail in [4]. Here we emphasize its quality in approximating the underlying nonlinear and nonsmooth  $F$ .

Note that in the tangent case both functions  $F_i$  are linearized as tangents at the central point  $\hat{x}$ . Like at almost all points  $\hat{x}$ , generalized differentiation in the usual sense would reduce

to conventional differentiation and just return the single tangent slope  $F'_2(\hat{x})$  without yielding any information about the nearby kink in the function.

In contrast to the tangent model, the secant model is not a priori unique in that it depends quite strongly on the procedural representation of the vector function  $F$  and not just its values, i.e. its properties as a mathematical map. For example, one can easily check that applying the above secant modeling rules to  $f(x) = \log(\exp(x))$  does not yield the same approximation  $\Delta f(\check{x}, \hat{x})$  as the one for  $f(x) = x$ . On the other hand the natural secant linearization rule for the product  $v = u \cdot w$  is equivalent to that obtained by applying the Appolonius identity

$$u \cdot w = \frac{1}{4} [(u + w)^2 + (u - w)^2].$$

Of course, the same is true for the tangent linearization and we may assume without loss of generality that we only have three kinds of elemental functions, the addition, the modulus and smooth univariate functions. That reduction greatly simplifies the thoretical analysis but might not be numerically optimal for actual implementations.

## 4 Approximation and stability properties

In contrast to the presentation in our previous papers we will now also use the nonincremental forms

$$\diamond_{\hat{x}} F(x) \equiv F(\hat{x}) + \Delta F(\hat{x}; x - \hat{x})$$

and

$$\diamond_{\hat{x}}^{\check{x}} F(x) \equiv \frac{1}{2}(F(\check{x}) + F(\hat{x})) + \Delta F(\check{x}, \hat{x}; x - \hat{x}).$$

In terms of this notation we find for the square as a univariate nonlinear function

$$\begin{aligned} \diamond_{\hat{x}}^{\check{x}} x^2 &= \frac{1}{2} [\check{x}^2 + \hat{x}^2] + \frac{\hat{x}^2 - \check{x}^2}{\hat{x} - \check{x}} \left[ x - \frac{1}{2}(\check{x} + \hat{x}) \right] \\ &= \frac{1}{2} [\check{x}^2 + \hat{x}^2] + (\hat{x} + \check{x}) \left( x - \frac{1}{2}[\check{x} + \hat{x}] \right) \\ &= (\hat{x} + \check{x})x - \hat{x}\check{x} = 2\hat{x}(x - \hat{x}) + \hat{x}^2. \end{aligned}$$

**Lemma 4.1.** *Plugging the secant approximation for the square into the Appolonius identity, we obtain for the general multiplication*

$$\begin{aligned} \diamond_{(\hat{u}, \hat{w})}^{(\check{u}, \check{w})}(uw) &= \frac{1}{4} \left[ \diamond_{(\hat{u}, \hat{w})}^{(\check{u}, \check{w})}(u + w)^2 - \diamond_{(\hat{u}, \hat{w})}^{(\check{u}, \check{w})}(u - w)^2 \right] \\ &= \hat{u}\hat{w} + \hat{w}(u - \hat{u}) + \hat{u}(w - \hat{w}). \end{aligned}$$

*Proof.*

$$\begin{aligned} \diamond_{(\hat{u}, \hat{w})}^{(\check{u}, \check{w})}(uw) &= \frac{1}{4} \left[ \diamond_{(\hat{u}, \hat{w})}^{(\check{u}, \check{w})}(u + w)^2 - \diamond_{(\hat{u}, \hat{w})}^{(\check{u}, \check{w})}(u - w)^2 \right] \\ &= \frac{1}{4} [(\check{u} + \check{w} + \hat{u} + \hat{w})(u + w) - (\check{u} + \check{w})(\hat{u} + \hat{w}) \\ &\quad - (\check{u} - \check{w} + \hat{u} - \hat{w})(u - w) + (\check{u} - \check{w})(\hat{u} - \hat{w})] \\ &= \frac{1}{4} (\hat{w} + \check{w})(\hat{u} + \check{u}) + (\hat{w} + \check{w})(u - \frac{1}{2}(\hat{u} + \check{u})) + (\hat{u} + \check{u})(w - \frac{1}{2}(\hat{w} + \check{w})) \\ &= \hat{u}\hat{w} + \hat{w}(u - \hat{u}) + \hat{u}(w - \hat{w}). \end{aligned}$$

□



From here on we will denote by  $\|\cdot\| \equiv \|\cdot\|_\infty$  the infinity norm. Due to the norm equivalence in finite dimensional spaces all inequalities to be derived take the same form in other norms, provided the constants are adjusted accordingly. The infinity norm is particularly convenient, since we can then prove the following result for the general vector case  $m > 1$  by considering the absolute values of the individual components  $f = F_i$  for  $i = 1 \dots m$ .

**Proposition 4.2.** *Suppose  $x, \tilde{x}, \hat{x}, \tilde{y}, \hat{y}, \tilde{z}, \hat{z} \in \mathbb{R}^n$  are restricted to a sufficiently small closed convex neighborhood  $K \in \mathbb{R}^n$  where the evaluation procedure for  $F : \mathbb{R}^n \mapsto \mathbb{R}^m$  is well defined. Then there are Lipschitz constants  $\beta_F$  and  $\gamma_F$  such that we have*

(i) *Lipschitz continuity of function and secant models*

$$\max(\|F(x) - F(\tilde{x})\|, \|\diamond_{\hat{x}} F(x) - \diamond_{\hat{x}} F(\tilde{x})\|, \|\diamond_{\tilde{x}} F(x) - \diamond_{\tilde{x}} F(\tilde{x})\|) \leq \beta_F \|x - \tilde{x}\|$$

The constant  $\beta_F$  can be defined by the recurrences  $\beta_v = \beta_u + \beta_w$  if  $v = u + w$ ,  $\beta_v = \beta_u$  if  $v = |u|$  and

$$\beta_v = \beta_u L_K(\varphi) \quad \text{if } v = \varphi(u) \quad \text{with } L_K(\varphi) \equiv \max_{x \in K} |\varphi'(u(x))|$$

(ii) *Error between function and secant or tangent model*

$$\|F(x) - \diamond_{\hat{x}} F(x)\| \leq \frac{1}{2} \gamma_F \|x - \hat{x}\| \|x - \tilde{x}\|$$

$$\|F(x) - \diamond_{\tilde{x}} F(x)\| \leq \frac{1}{2} \gamma_F \|x - \tilde{x}\|^2$$

The constant  $\gamma_F$  can be defined using the recurrences  $\gamma_v = \gamma_u + \gamma_w$  if  $v = u + w$ ,  $\gamma_v = \gamma_u$  if  $v = |u|$  and

$$\gamma_v = L_K(\varphi) \gamma_u + L_K(\varphi') \beta_u^2 \quad \text{if } v = \varphi(u) \quad \text{with } L_K(\varphi') \equiv \max_{x \in K} |\varphi''(u(x))|$$

(iii) *Lipschitz continuity of secant and tangent model*

$$\|\diamond_{\hat{z}} F(x) - \diamond_{\hat{y}} F(x)\| \leq \gamma_F \max[\|\hat{z} - \hat{y}\| \max(\|x - \tilde{y}\|, \|x - \tilde{z}\|), \|\tilde{z} - \tilde{y}\| \max(\|x - \hat{y}\|, \|x - \hat{z}\|)]$$

$$\|\diamond_{\tilde{z}} F(x) - \diamond_{\tilde{y}} F(x)\| \leq \gamma_F \|\tilde{z} - \tilde{y}\| \max(\|x - \tilde{y}\|, \|x - \tilde{z}\|)$$

(iv) *Lipschitz continuity of the incremental part: Abbreviating  $\Delta y = \hat{y} - \tilde{y}$  and  $\Delta z = \hat{z} - \tilde{z}$  we obtain in the secant case*

$$\begin{aligned} & \|\Delta_{\hat{z}} F(\Delta x) - \Delta_{\hat{y}} F(\Delta x)\| \\ & \leq 2\beta_F \|\tilde{z} - \tilde{y}\| + \frac{1}{2} \gamma_F (\|\tilde{z} - \tilde{y}\| + \max(\|\Delta y\|, \|\Delta z\|))^2 \\ & \quad + \gamma_F (\|\tilde{z} - \tilde{y}\| + \frac{1}{2} (\|\Delta y\| + \|\Delta z\|)) \|\Delta x\| \end{aligned}$$

which reduces in the tangent case to

$$\|\Delta_{\tilde{z}} F(\Delta x) - \Delta_{\tilde{y}} F(\Delta x)\| \leq 2\beta_F \|\tilde{z} - \tilde{y}\| + \frac{1}{2} \gamma_F \|\tilde{z} - \tilde{y}\|^2 + \gamma_F \|\tilde{z} - \tilde{y}\| \|\Delta x\|$$

Here the second bounds applying to the tangent model are always specializations of the previous ones for the secant model.

*Proof.* Since otherwise the bounds can be applied componentwise we may assume without loss of generality that  $F$  is a scalar function  $f$  and the norm in the range just the absolute value  $|\cdot|$ . The proof proceeds by induction on the intermediate quantities  $v$  in the computational graph of  $f$ . We will define the constants  $\beta_v$  and  $\gamma_v$  recursively on the basis of the Lipschitz constants of the elemental functions and their derivatives. For the minimal nodes representing the independent variables all assertions are trivially true with the constants  $\beta_{x_i} = 1$  and  $\gamma_{x_i} = 0$ . For smooth univariates  $v = \varphi(u)$  the assertion (i) follows by the chain rule and the induction hypothesis for the argument  $u$ . The absolute value function  $v = \text{abs}(u)$  naturally maintains the Lipschitz constant and the addition/subtraction summates them. For the multiplication  $v = u \cdot w$  we can use the Appolonius identity as shown above. By definition for fixed  $(\check{x}, \hat{x})$  the Lipschitz continuity of  $\diamond_{\check{x}}^{\hat{x}} v(x)$  with respect to  $x$  is equivalent to that of  $\Delta v$  with respect to  $\Delta x = x - \frac{1}{2}(\check{x} + \hat{x})$  and the latter follows immediately from the propagation rules. Thus we have established (i) for all cases.

Now let us consider the approximation property (ii). For additions  $v = u + w$  we may also set  $\gamma_v = \gamma_u + \gamma_w$  and then have by the triangle inequality

$$\begin{aligned} |v(x) - \diamond_{\check{x}}^{\hat{x}} v(x)| &\leq |u(x) - \diamond_{\check{x}}^{\hat{x}} u(x)| + |w(x) - \diamond_{\check{x}}^{\hat{x}} w(x)| \\ &\leq \frac{1}{2}\gamma_u(\|x - \hat{x}\|\|x - \check{x}\|) + \frac{1}{2}\gamma_w(\|x - \hat{x}\|\|x - \check{x}\|) = \frac{1}{2}\gamma_v(\|x - \hat{x}\|\|x - \check{x}\|). \end{aligned}$$

For the absolute value function  $v = \text{abs}(u)$  we may also set  $\gamma_v = \gamma_u$ , since

$$|v(x) - \diamond_{\check{x}}^{\hat{x}} v(x)| = ||u| - |\diamond_{\check{x}}^{\hat{x}} u(x)|| \leq |u - \diamond_{\check{x}}^{\hat{x}} u(x)| \leq \frac{1}{2}\gamma_u(\|x - \hat{x}\|\|x - \check{x}\|).$$

For the univariate functions  $v = \varphi(u)$  we have with  $\tilde{u} \equiv \diamond_{\check{x}}^{\hat{x}} u(x)$

$$|v(x) - \diamond_{\check{x}}^{\hat{x}} v(x)| \leq |\varphi(u(x)) - \varphi(\tilde{u})| + |\varphi(\tilde{u}) - \diamond_{\tilde{u}}^{\hat{u}} \varphi(\tilde{u})|.$$

By the mean value theorem and the induction hypothesis, the first term is bounded by

$$L_K(\varphi)|u(x) - \diamond_{\check{x}}^{\hat{x}} u(x)| \leq \frac{1}{2}L_K(\varphi)\gamma_u(\|x - \hat{x}\|\|x - \check{x}\|).$$

The second term represents the error in the Hermite interpolation of  $\varphi$  between  $\tilde{u}$  and  $\hat{u}$ . With  $L_K(\varphi')$  a Lipschitz constant of  $\varphi'$  on  $u(K)$  it is bounded by

$$\frac{1}{2}L_K(\varphi')|\tilde{u} - \hat{u}| \leq \frac{1}{2}L_K(\varphi')\beta_u^2(\|x - \hat{x}\|\|x - \check{x}\|),$$

where the last bound follows from the fact that according to (i) the approximation  $\diamond_{\check{x}}^{\hat{x}} u(x)$  has the Lipschitz constant  $\beta_u$  and takes at  $\check{x}$  and  $\hat{x}$  the values  $\tilde{u}$  and  $\hat{u}$ . Hence, we have shown that (ii) holds indeed with

$$\gamma_v = L_K(\varphi)\gamma_u + L_K(\varphi')\beta_u^2. \quad (9)$$

Next we want to prove Lipschitz continuity of the model as stated in (iii). Again, we find for additions and the abs function that the assertion is almost trivial with the constants  $\gamma$  either being summated or just passed on. The challenge is once more the induction through the nonlinear univariates  $v = \varphi(u)$ . To limit the notational complexity we will connect the two point pairs at the  $u$  level by straight lines setting

$$\begin{aligned} \tilde{u} \equiv \tilde{u}(t) \equiv u(\tilde{y})(1-t) + tu(\tilde{z}) &\Rightarrow \partial\tilde{u}(t)/\partial t = \Delta\tilde{u} \equiv u(\tilde{z}) - u(\tilde{y}), \\ \hat{u} \equiv \hat{u}(t) \equiv u(\hat{y})(1-t) + tu(\hat{z}) &\Rightarrow \partial\hat{u}(t)/\partial t = \Delta\hat{u} \equiv u(\hat{z}) - u(\hat{y}), \\ \tilde{u} \equiv \tilde{u}(t) \equiv \diamond_{\tilde{y}}^{\tilde{z}} u(x)(1-t) + t\diamond_{\hat{y}}^{\hat{z}} u(x) &\Rightarrow \partial\tilde{u}/\partial t = \Delta\tilde{u} \equiv \diamond_{\tilde{z}}^{\tilde{z}} u(x) - \diamond_{\tilde{y}}^{\tilde{y}} u(x). \end{aligned}$$

Here  $x$  is fixed and we assume as induction hypothesis that

$$\|\Delta\tilde{u}\| \leq \gamma_u \max[\|\hat{z} - \hat{y}\| \max(\|x - \hat{y}\|, \|x - \hat{z}\|), \|\hat{z} - \hat{y}\| \max(\|x - \hat{y}\|, \|x - \hat{z}\|)]. \quad (10)$$

By the mean value theorem we find that for some  $t \in [0, 1]$

$$\begin{aligned} \Delta\tilde{v} &\equiv \diamond_{\hat{z}}^{\tilde{z}} v(x) - \diamond_{\hat{y}}^{\tilde{y}} v(x) \\ &= \frac{\partial}{\partial t} \left\{ \frac{1}{2} [\varphi(\tilde{u}) + \varphi(\hat{u})] + \frac{\varphi(\hat{u}) - \varphi(\tilde{u})}{\hat{u} - \tilde{u}} \left[ \tilde{u} - \frac{1}{2} (\tilde{u} + \hat{u}) \right] \right\} \\ &= \frac{1}{2} [\varphi'(\tilde{u})\Delta\tilde{u} + \varphi'(\hat{u})\Delta\hat{u}] + \frac{\varphi(\hat{u}) - \varphi(\tilde{u})}{\hat{u} - \tilde{u}} [\Delta\tilde{u} - \frac{1}{2} (\Delta\tilde{u} + \Delta\hat{u})] \\ &\quad + \left\{ \frac{\varphi'(\hat{u})\Delta\hat{u} - \varphi'(\tilde{u})\Delta\tilde{u}}{\hat{u} - \tilde{u}} - \frac{(\varphi(\hat{u}) - \varphi(\tilde{u}))(\Delta\hat{u} - \Delta\tilde{u})}{(\hat{u} - \tilde{u})^2} \right\} \left[ \tilde{u} - \frac{1}{2} (\tilde{u} + \hat{u}) \right] \\ &= \frac{1}{2} [\varphi'(\tilde{u})\Delta\tilde{u} + \varphi'(\hat{u})\Delta\hat{u}] + \varphi'(\bar{u}) \left[ \Delta\tilde{u} - \frac{1}{2} (\Delta\tilde{u} + \Delta\hat{u}) \right] \\ &\quad + \left\{ \frac{\varphi'(\hat{u})\Delta\hat{u} - \varphi'(\tilde{u})\Delta\tilde{u} - \varphi'(\bar{u})(\Delta\hat{u} - \Delta\tilde{u})}{\hat{u} - \tilde{u}} \right\} \left[ \tilde{u} - \frac{1}{2} (\tilde{u} + \hat{u}) \right] \\ &= \frac{1}{2} \left\{ \frac{(\varphi'(\tilde{u}) - \varphi'(\bar{u}))\Delta\tilde{u} + (\varphi'(\hat{u}) - \varphi'(\bar{u}))\Delta\hat{u}}{\hat{u} - \tilde{u}} \right\} (\hat{u} - \tilde{u}) + \varphi'(\bar{u})\Delta\tilde{u} \\ &\quad + \frac{1}{2} \left\{ \frac{(\varphi'(\hat{u}) - \varphi'(\bar{u}))\Delta\hat{u} - (\varphi'(\tilde{u}) - \varphi'(\bar{u}))\Delta\tilde{u}}{\hat{u} - \tilde{u}} \right\} [(\tilde{u} - \tilde{u}) + (\tilde{u} - \hat{u})] \\ &= \frac{\varphi'(\hat{u}) - \varphi'(\bar{u})}{\hat{u} - \tilde{u}} \Delta\hat{u}(\tilde{u} - \tilde{u}) - \frac{\varphi'(\tilde{u}) - \varphi'(\bar{u})}{\hat{u} - \tilde{u}} \Delta\tilde{u}(\tilde{u} - \hat{u}) + \varphi'(\bar{u})\Delta\tilde{u}, \end{aligned}$$

where in the last three lines  $\bar{u}$  is a mean value between  $\tilde{u}$  and  $\hat{u}$  at which the difference quotient of  $\varphi$  over the intervening interval is equal to its derivative. That means the two quotients are bounded according to

$$\begin{aligned} \left| \frac{\varphi'(\hat{u}) - \varphi'(\bar{u})}{\hat{u} - \tilde{u}} \right| + \left| \frac{\varphi'(\tilde{u}) - \varphi'(\bar{u})}{\hat{u} - \tilde{u}} \right| &\leq \left| \frac{\varphi'(\hat{u}) - \varphi'(\bar{u})}{\hat{u} - \bar{u}} \right| \left| \frac{\hat{u} - \bar{u}}{\hat{u} - \tilde{u}} \right| + \left| \frac{\varphi'(\tilde{u}) - \varphi'(\bar{u})}{\tilde{u} - \bar{u}} \right| \left| \frac{\tilde{u} - \bar{u}}{\hat{u} - \tilde{u}} \right| \\ &\leq L_K(\varphi') \left( \left| \frac{\hat{u} - \bar{u}}{\hat{u} - \tilde{u}} \right| + \left| \frac{\tilde{u} - \bar{u}}{\hat{u} - \tilde{u}} \right| \right) = L_K(\varphi'), \end{aligned}$$

where the last equality follows from  $\bar{u}$  being between  $\tilde{u}$  and  $\hat{u}$ . Hence, we find that

$$|\Delta\tilde{v}| \leq L_K(\varphi') \max(|\Delta\hat{u}||\tilde{u} - \tilde{u}|, |\Delta\tilde{u}||\tilde{u} - \hat{u}|) + L_K(\varphi)|\Delta\tilde{u}|. \quad (11)$$

The middle factors are easily bounded by

$$|\Delta\hat{u}| = |u(\hat{z}) - u(\hat{y})| \leq \beta_u \|\hat{z} - \hat{y}\| \quad \text{and} \quad |\Delta\tilde{u}| = |u(\hat{z}) - u(\tilde{y})| \leq \beta_u \|\hat{z} - \tilde{y}\|.$$

That leaves us with the second factors, which are linear in  $t$  such that

$$\begin{aligned} |\tilde{u}(t) - \hat{u}(t)| &\leq \max(|\tilde{u}(0) - \hat{u}(0)|, |\tilde{u}(1) - \hat{u}(1)|) \\ &= \max(|\diamond_{\hat{y}}^{\tilde{y}} u(x) - u(\hat{y})|, |\diamond_{\hat{z}}^{\tilde{z}} u(x) - u(\hat{z})|) \\ &= \max(|\diamond_{\hat{y}}^{\tilde{y}} u(x) - \diamond_{\hat{y}}^{\tilde{y}} u(\hat{y})|, |\diamond_{\hat{z}}^{\tilde{z}} u(x) - \diamond_{\hat{z}}^{\tilde{z}} u(\hat{z})|) \\ &\leq \beta_u \max(\|x - \hat{y}\|, \|x - \hat{z}\|), \end{aligned}$$

where the last inequality follows from (i). Analogously we can derive

$$|\tilde{u}(t) - \hat{u}(t)| \leq \beta_u \max(\|x - \hat{y}\|, \|x - \hat{z}\|).$$

Substituting this into (11) we get

$$|\Delta \tilde{v}| \leq \gamma_v \max(\|\hat{z} - \hat{y}\| \|x - \check{y}\|, \|\hat{z} - \hat{y}\| \|x - \check{z}\|, \|\check{z} - \check{y}\| \|x - \hat{y}\|, \|\check{z} - \check{y}\| \|x - \hat{z}\|),$$

with  $\gamma_v \equiv L_K(\varphi)\gamma_u + L_K(\varphi')\beta_u^2$ , which completes the proof of (iii). Finally, we have to prove (iv), which gives a bound on the increment part only. With the results already proved above and a few triangle inequalities one gets the following:

$$\begin{aligned} & \|\Delta_{\hat{z}}^{\hat{z}} F(\Delta x) - \Delta_{\hat{y}}^{\hat{y}} F(\Delta x)\| \\ &= \|\diamond_{\hat{z}}^{\hat{z}} F(\hat{z} + \Delta x) - F(\hat{z}) - (\diamond_{\hat{y}}^{\hat{y}} F(\hat{y} + \Delta x) - F(\hat{y}))\| \\ &= \|\diamond_{\hat{z}}^{\hat{z}} F(\xi) - F(\hat{z}) + \diamond_{\hat{z}}^{\hat{z}} F(\hat{z} + \Delta x) - \diamond_{\hat{z}}^{\hat{z}} F(\xi) \\ &\quad - \diamond_{\hat{y}}^{\hat{y}} F(\xi) + F(\hat{y}) - \diamond_{\hat{y}}^{\hat{y}} F(\hat{y} + \Delta x) + \diamond_{\hat{y}}^{\hat{y}} F(\xi)\|, \text{ where } \xi := \frac{1}{2}(\hat{z} + \hat{y}) + \Delta x \\ &\leq \beta_F \|\hat{z} - \hat{y}\| + \beta_F \|\hat{z} + \Delta x - \frac{1}{2}(\hat{z} + \hat{y}) - \Delta x\| + \beta_F \|\hat{y} + \Delta x - \frac{1}{2}(\hat{z} + \hat{y}) - \Delta x\| + \\ &\quad + \gamma_F \max(\|\hat{z} - \hat{y}\| \max(\|\xi - \check{z}\|, \|\xi - \check{y}\|), \|\check{z} - \check{y}\| \max(\|\xi - \hat{z}\|, \|\xi - \hat{y}\|)) \\ &= 2\beta_F \|\hat{z} - \hat{y}\| + \gamma_F \max(\|\hat{z} - \hat{y}\| \max(\|\xi - \check{z}\|, \|\xi - \check{y}\|), \|\check{z} - \check{y}\| \max(\|\xi - \hat{z}\|, \|\xi - \hat{y}\|)). \end{aligned}$$

Now, since for example

$$\|\frac{1}{2}(\hat{y} + \hat{z}) - \check{z}\| = \|\frac{1}{2}(\hat{y} - \hat{z}) - \check{z} + \frac{1}{2}(\hat{z} + \hat{z})\| \leq \frac{1}{2}(\|\hat{y} - \hat{z}\| + \|\Delta z\|),$$

both inner maxima can be bounded by the same expression, namely

$$\|\Delta x\| + \frac{1}{2}(\|\hat{z} - \hat{y}\| + \max(\|\Delta z\|, \|\Delta y\|)),$$

so that we obtain the upper bound

$$\begin{aligned} & \|\Delta_{\hat{z}}^{\hat{z}} F(\Delta x) - \Delta_{\hat{y}}^{\hat{y}} F(\Delta x)\| \\ &\leq 2\beta_F \|\hat{z} - \hat{y}\| + \gamma_F \max(\|\hat{z} - \hat{y}\|, \|\check{z} - \check{y}\|) \left[ \|\Delta x\| + \frac{1}{2}(\|\hat{z} - \hat{y}\| + \max(\|\Delta z\|, \|\Delta y\|)) \right]. \end{aligned}$$

Finally, we can also bound

$$\|\check{y} - \check{z}\| = \|\check{y} - \hat{y} - \check{z} + \hat{z} + \hat{y} - \hat{z}\| \leq \|\hat{y} - \hat{z}\| + \frac{1}{2}(\|\Delta y\| + \|\Delta x\|),$$

which yields the assertion after some elementary modifications. From the secant result we can easily get the bound for the tangent model by setting  $\hat{z} = \check{z} = \hat{z}$  and  $\hat{y} = \check{y} = \hat{y}$ .  $\square$

As one can see by setting  $\check{y} = x = \check{z}$  the assertion (ii) almost follows from (ii), except that a factor of 2 is lost in the constants. The proposition also states that the values of  $F$  at  $\check{x}$  and  $\hat{x}$  are reproduced exactly by our approximation as one would expect from a secant approximation. This property clearly nails down the piecewise linearization rules (5) and (4) with (7) for all univariate functions. Also, there is no doubt that addition and subtraction should be linearized according to (2) and that multiplications  $v_i = c v_j$  by constants  $c$  should yield the differentiated version  $\Delta v_i = c \Delta v_j$ , which is a special case of (3). For general multiplications  $v_i = v_j * v_k$  the two values  $\hat{v}_i = \hat{v}_j * \hat{v}_k$  and  $\check{v}_i = \check{v}_j * \check{v}_k$  could also be interpolated by linear functions other than the one defined by (3). However, we currently see no possible gain in that flexibility, and maintaining the usual product rule form seems rather attractive.

## 5 Generalized Newton methods by piecewise linearization

We will proceed by proposing and analyzing a possible application of the tangent and secant approximations developed in the previous sections. For this we present generalized Newtons methods based on the piecewise linear approximations, both for the tangent and the secant mode. The merit of these methods is the fact that they impose no strong differentiability requirements but require only piecewise differentiability at the root in question. The tangent mode piecewise linearization based Newton is defined as follows:

**Definition 1** (Solvability of tangent mode Newton). Let  $F \in \text{span}(\Phi_{\text{abs}})$  and  $x^*$  an isolated root of  $F$  in an open neighborhood  $\mathcal{U} \subset \mathcal{D}$ . If there is a  $\rho > 0$  such that for all  $x \in \bar{B}_\rho(x^*)$  it holds that  $(\diamond_x F)^{-1}(0) \cap \bar{B}_\rho(x^*)$  is nonempty, then successive piecewise tangent linearization is solvable near  $x^*$  and the generalized Newton iteration defined by the recurrence:

$$x_{j+1} \in \text{argmin}\{\|x - x_j\| \mid x \in (\diamond_{x_j} F)^{-1}(0)\}$$

is well defined.

Given a Newton iteration that fulfills the above definition and the openness of the piecewise linear model, we can prove that the iteration converges with second order.

**Proposition 5.1** (Quadratic convergence of tangent mode Newton). *Let  $F \in \text{span}(\Phi_{\text{abs}})$  and  $x^*$  a root of  $F$  in an open neighborhood  $\mathcal{U} \subset \mathcal{D}$ . If there is a  $\rho > 0$  such that the above Newton iteration is solvable near  $x^*$  and if the piecewise linear model  $\diamond_{x^*} F$  developed in  $x^*$  is strong metrically regular in  $x^*$ , i.e.*

$$\exists c > 0 \forall x \in \bar{B}_\rho(x^*) : \|x - x^*\| \leq c \|\diamond_{x^*} F(x)\|,$$

*which is equivalent to coherent orientation or openness, then the Newton iteration converges with second order.*

*Proof.* By Proposition 4.2 (iii) it holds that:

$$\begin{aligned} & \|F(x^*) + \Delta F(x^*; x_{j+1} - x^*) - F(x_j) - \Delta F(x_j; x_{j+1} - x_j)\| \\ & \leq L \|x_j - x^*\| \max(\|x_{j+1} - x_j\|, \|x_{j+1} - x^*\|) \\ & \leq L \|x_j - x^*\| (\|x_{j+1} - x_j\| + \|x_{j+1} - x^*\|), \end{aligned}$$

where  $F(x^*) = 0$  and  $x_{j+1}$  chosen such that  $F(x_j) + \Delta F(x_j; x_{j+1} - x_j) = 0$  and thus

$$\|\Delta F(x^*; x_{j+1} - x^*)\| \leq L \|x_j - x^*\| (\|x_{j+1} - x_j\| + \|x_{j+1} - x^*\|).$$

With the strong metric regularity of the model developed in  $x^*$  it follows that

$$c^{-1} \|x_{j+1} - x^*\| \leq L \|x_j - x^*\| (\|x_{j+1} - x_j\| + \|x_{j+1} - x^*\|).$$

Let  $\rho$  be such that  $cL\rho \leq \frac{1}{4}$  and  $x_j \in \bar{B}_\rho(x^*)$ , then

$$\begin{aligned} & \|x_{j+1} - x^*\| \leq \frac{1}{4} (\|x_{j+1} - x_j\| + \|x_{j+1} - x^*\|) \\ & \implies \|x_{j+1} - x^*\| \leq \frac{1}{4} (\|x_j - x^*\| + 2\|x_{j+1} - x^*\|) \\ & \implies \frac{1}{2} \|x_{j+1} - x^*\| \leq \frac{1}{4} \|x_j - x^*\| \Leftrightarrow \|x_{j+1} - x^*\| \leq \frac{1}{2} \|x_j - x^*\|. \end{aligned} \tag{12}$$

With line (12) we have proved convergence, while the following proves quadratic convergence:

$$\begin{aligned}
& \|x_{j+1} - x^*\| \leq cL\|x_j - x^*\|(\|x_j - x^*\| + 2\|x_{j+1} - x^*\|) \\
\implies & \|x_{j+1} - x^*\| \leq cL\|x_j - x^*\|(\|x_j - x^*\| + \|x_j - x^*\|) \\
\implies & \|x_{j+1} - x^*\| \leq 2cL\|x_j - x^*\|^2.
\end{aligned}$$

□

In a very similar fashion we can state and prove the convergence of the generalized secant method, albeit its order of convergence is somewhat smaller.

**Definition 2** (Solvability of secant mode iteration). Let  $F \in \text{span}(\Phi_{\text{abs}})$  and  $x^*$  a root of  $F$  in an open neighborhood  $\mathcal{U} \subset \mathcal{D}$ . If there is a  $\rho > 0$  such that for all  $\tilde{x}, \hat{x} \in \bar{B}_\rho(x^*)$  it holds that  $(\diamond_{\tilde{x}}^{\hat{x}} F)^{-1}(0) \cap \bar{B}_\rho(x^*)$  is nonempty, then successive piecewise secant linearization is solvable near  $x^*$  and the Newton-like iteration defined by the recurrence:

$$x_{j+1} \in \text{argmin}\{\|x - x_j\| \mid x \in (\diamond_{x_{j-1}}^{x_j} F)^{-1}(0)\}$$

is well defined.

**Proposition 5.2** (Superlinear convergence of secant mode iteration). Let  $F \in \text{span}(\Phi_{\text{abs}})$  and  $x^*$  a root of  $F$  in an open neighborhood  $\mathcal{U} \subset \mathcal{D}$ . If there is a  $\rho > 0$  such that the above iteration is solvable near  $x^*$  and if the piecewise linear model  $\diamond_{x^*} F$  developed in  $x^*$  is strong metrically regular in  $x^*$ , i.e.

$$\exists c > 0 \forall x \in \bar{B}_\rho(x^*) : \|x - x^*\| \leq c\|\diamond_{x^*} F(x)\|,$$

which is equivalent to coherent orientation or openness, then the Newton-like iteration converges with the superlinear order  $(1 + \sqrt{5})/2 = 1.618\dots$

*Proof.* By Proposition 4.2 (iii) it holds:

$$\begin{aligned}
& \left\| \frac{1}{2}(F(x^*) + F(x^*)) + \Delta F(x^*, x^*; x_{j+1} - \frac{x^* + x^*}{2}) - \frac{1}{2}(F(x_{j-1}) + F(x_j)) - \right. \\
& \quad \left. - \Delta F(x_{j-1}, x_j; x_{j+1} - \frac{x_{j-1} + x_j}{2}) \right\| \\
& \leq L \max(\|x_{j-1} - x^*\| \max(\|x_{j+1} - x_j\|, \|x_{j+1} - x^*\|), \\
& \quad \|x_j - x^*\| \max(\|x_{j+1} - x_{j-1}\|, \|x_{j+1} - x^*\|)),
\end{aligned}$$

where  $F(x^*) = 0$  and  $x_{j+1}$  chosen such that  $\frac{1}{2}(F(x_{j-1}) + F(x_j)) + \Delta F(x_{j-1}, x_j; x_{j+1} - \frac{x_{j-1} + x_j}{2}) = 0$  and thus

$$\begin{aligned}
\|\diamond_{x^*} F f(x_{j+1})\| & \leq L \max(\|x_{j-1} - x^*\| \max(\|x_{j+1} - x_j\|, \|x_{j+1} - x^*\|), \\
& \quad \|x_j - x^*\| \max(\|x_{j+1} - x_{j-1}\|, \|x_{j+1} - x^*\|)).
\end{aligned}$$

With the strong metric regularity of the model developed in  $x^*$  it follows that

$$\begin{aligned}
c^{-1}\|x_{j+1} - x^*\| & \leq L \max(\|x_{j-1} - x^*\| \max(\|x_{j+1} - x_j\|, \|x_{j+1} - x^*\|), \\
& \quad + \|x_j - x^*\| \max(\|x_{j+1} - x_{j-1}\|, \|x_{j+1} - x^*\|)).
\end{aligned}$$

Let  $\rho \leq \frac{1}{3}$  and  $x_{j-1}, x_j \in \bar{B}_\rho(x^*)$ , then

$$\begin{aligned}
\|x_{j+1} - x^*\| &\leq \frac{1}{3} \max[\max(\|x_{j+1} - x_{j-1}\|, \|x_{j+1} - x^*\|), \max(\|x_{j+1} - x_j\|, \|x_{j+1} - x^*\|)] \\
&= \frac{1}{3} \max(\max(\|x_{j+1} - x_{j-1}\|, \|x_{j+1} - x_j\|), \|x_{j+1} - x^*\|) \\
&\leq \frac{1}{3} \max(\max(\|x_{j+1} - x^*\| + \|x_{j-1} - x^*\|, \|x_{j+1} - x^*\| + \|x_j - x^*\|), \|x_{j+1} - x^*\|) \\
&= \frac{1}{3} \max(\|x_{j+1} - x^*\| + \max(\|x_{j-1} - x^*\|, \|x_j - x^*\|), \|x_{j+1} - x^*\|) \\
&= \frac{1}{3} \|x_{j+1} - x^*\| + \frac{1}{3} \max(\|x_{j-1} - x^*\|, \|x_j - x^*\|) \\
\implies \frac{2}{3} \|x_{j+1} - x^*\| &\leq \frac{1}{3} \max(\|x_{j-1} - x^*\|, \|x_j - x^*\|) \\
\implies \|x_{j+1} - x^*\| &\leq \frac{1}{2} \max(\|x_{j-1} - x^*\|, \|x_j - x^*\|).
\end{aligned}$$

With the last line we have proven convergence and the following proves superlinear convergence:

$$\begin{aligned}
\|x_{j+1} - x^*\| &\leq L \max(\|x_{j-1} - x^*\| \max(\|x_{j+1} - x_j\|, \|x_{j+1} - x^*\|), \\
&\quad \|x_j - x^*\| \max(\|x_{j+1} - x_{j-1}\|, \|x_{j+1} - x^*\|)) \\
&\leq cL \max(\|x_{j-1} - x^*\| \max(\|x_j - x^*\| + \|x_{j+1} - x^*\|, \|x_{j+1} - x^*\|), \\
&\quad \|x_j - x^*\| \max(\|x_{j-1} - x^*\| + \|x_{j+1} - x^*\|, \|x_{j+1} - x^*\|)) \\
&= cL \max(\|x_{j-1} - x^*\| (\|x_j - x^*\| + \|x_{j+1} - x^*\|), \\
&\quad \|x_j - x^*\| (\|x_{j-1} - x^*\| + \|x_{j+1} - x^*\|)) \\
&= cL \|x_{j-1} - x^*\| \|x_j - x^*\| + cL \|x_{j+1} - x^*\| \max(\|x_{j-1} - x^*\|, \|x_j - x^*\|) \\
&\leq cL \|x_{j-1} - x^*\| \|x_j - x^*\| + cL \|x_{j+1} - x^*\| \rho \\
\implies \|x_{j+1} - x^*\| &\leq \frac{cL}{(1 - cL\rho)} \|x_{j-1} - x^*\| \|x_j - x^*\|.
\end{aligned}$$

Irrespective of the number of variables  $n$  this error recursion is identical to the one for the univariate secant method, whose order is well known [11] to be equal to the golden ratio  $(1 + \sqrt{5})/2$ .  $\square$

So far we have simply assumed that the local model problems  $\mathring{F} + \Delta F(\mathring{x}; \Delta x) = 0$  or  $\mathring{F} + \Delta F(\mathring{x}, \hat{x}; \Delta x) = 0$  are solvable at all reference points or pairs near the root  $x^*$ . Using the fixed point theorem of Brouwer we obtain:

**Proposition 5.3** (Sufficient condition for solvability). *Let  $F \in \text{span}(\Phi_{abs})$  and  $x^*$  a root of  $F$  in an open neighborhood  $\mathcal{U} \subset \mathcal{D}$ . If the piecewise linear model  $\diamond_{x^*} F$  developed in  $x^*$  is homeomorphic, then there exist radii  $\rho > 0$  and  $\varepsilon > 0$  such that every model  $\mathring{F} + \Delta F(\mathring{x}, \hat{x}; \Delta x)$  developed at two points  $\mathring{x}, \hat{x} \in \bar{B}_\rho(x^*)$  has at least one root  $\Delta \bar{x} \in \bar{B}_\varepsilon(0)$ . This holds in particular for the tangent mode, where  $\hat{x} = \mathring{x}$ .*

*Proof.* The assertion holds by the fixed point theorem of Brouwer. Let  $L > 0$  be the Lipschitz constant of  $\Delta^{-1}F(x^*; \Delta y) = \Delta x$ , the inverse function to  $\Delta y = \Delta F(x^*; \Delta x)$ . Furthermore, let  $\rho > 0$ ,  $\hat{\varepsilon} > 0$  and  $\mu > 0$  be chosen such that:

$$\begin{aligned}
\forall \mathring{x}, \hat{x} \in \bar{B}_\rho(x^*) \forall x \equiv \mathring{x} + \Delta x \in \bar{B}_{\hat{\varepsilon}}(\mathring{x}) : \\
\|F(x) - \frac{F(\mathring{x}) + F(\hat{x})}{2} - \Delta F(\mathring{x}, \hat{x}; \Delta x)\| \leq \mu \|x - \mathring{x}\| \|x - \hat{x}\|.
\end{aligned}$$

One should note that  $\mu$  has to be chosen uniformly for the whole ball  $\bar{B}_\rho(x^*)$ , but this is possible due to the strong Lipschitz continuity of the piecewise linear models and the local Lipschitz

continuity of  $F$ . Now let the following auxiliary function  $g_{\hat{x}}$  be parametrized in  $\hat{x} \in \bar{B}_\rho(x^*)$ :

$$g_{\hat{x}}(\Delta x) \equiv \Delta x - \Delta^{-1}F(x^*; \mathring{F} + \Delta F(\tilde{x}, \hat{x}; \Delta x)).$$

In the following we will show that  $g_{\hat{x}}$  has a fixed point, because for a fixed point it holds that:

$$\begin{aligned} \Delta x = g_{\hat{x}}(\Delta x) &\iff \Delta^{-1}F(x^*; \mathring{F} + \Delta F(\tilde{x}, \hat{x}; \Delta x)) = 0 \\ &\iff \mathring{F} + \Delta F(\tilde{x}, \hat{x}; \Delta x) = 0. \end{aligned}$$

The last line shows that a fixed point has to be a root of the model  $\mathring{F} + \Delta F(\tilde{x}, \hat{x}; \Delta x)$ . As a composition of continuous functions  $g_{\hat{x}}$  is also continuous. Due to the fixed point theorem of Brouwer it is sufficient to show that there exists  $\varepsilon > 0$  such that  $g_{\hat{x}}$  maps the compact ball  $\bar{B}_\varepsilon(\hat{x})$  into itself.

$$\|g_{\hat{x}}(\Delta x)\| = \|\Delta^{-1}F(x^*; \Delta F(x^*; \Delta x)) - \Delta^{-1}F(x^*; \mathring{F} + \Delta F(\tilde{x}, \hat{x}; \Delta x))\|$$

Since  $\Delta F(x^*; \Delta x)$  is homeomorphic, the inverse mapping is Lipschitz continuous.

$$\begin{aligned} \|g_{\hat{x}}(\Delta x)\| &\leq L\|\Delta F(x^*; \Delta x) - \mathring{F} - \Delta F(\tilde{x}, \hat{x}; \Delta x)\| \\ &= L\|F(x^*) + \Delta F(x^*; \Delta x) - F(x) + \mathcal{O}(\|x - \tilde{x}\|\|x - \hat{x}\|)\| \\ &= L\|F(x^* + \Delta x) - F(x) + \mathcal{O}(\|\Delta x\|^2) + \mathcal{O}(\|x - \tilde{x}\|\|x - \hat{x}\|)\| \\ &\leq L[\|x^* - \hat{x}\| + \mathcal{O}(\|\Delta x\|^2) + \mathcal{O}(\|x - \tilde{x}\|\|x - \hat{x}\|)] \\ &\leq L(\rho + \tilde{\mu} \max\{\|x - \tilde{x}\|, \|x - \hat{x}\|\}^2), \end{aligned}$$

where  $\tilde{\mu}$  is the maximum of the  $\mu$  in tangent and secant mode. Let  $\max\{\|x - \tilde{x}\|, \|x - \hat{x}\|\} \leq \min(\hat{\varepsilon}, \varepsilon)$  with  $\varepsilon^{-1} > \max(1, L\tilde{\mu})$ , then additionally it has to hold for  $\rho$  that:

$$\begin{aligned} L(\rho + \tilde{\mu}\varepsilon^2) &\leq \varepsilon \\ \iff L\rho &\leq \varepsilon - L\tilde{\mu}\varepsilon^2 \iff \rho \leq \varepsilon \left( \frac{1}{L} - \tilde{\mu}\varepsilon \right), \end{aligned}$$

which completes the proof.  $\square$

The inner iterations of both Theorems 5.1 and 5.2 require the solution of piecewise linear equations, which are surjective as perturbations of  $F(x_*, \text{Delta}x)$ , which is assumed homeomorphic. However, they may possess several solutions  $\Delta x_j$ , of which we must find one of minimal norm. So far, the solvers that we know and surveyed in [4, 6, 15, 12] require at least coherent orientation to guarantee convergence. Hence, the actual implementation of successive piecewise linearization needs further study.

## 6 Singularity Free Implementation

In contrast to the tangent mode of piecewise linearization, the secant mode involves two points of evaluation  $\tilde{v}_i, \hat{v}_i \in \mathbb{R}^n$ . These define a line segment  $[\tilde{v}_i, \hat{v}_i] \equiv \{\lambda\tilde{v}_i + (1 - \lambda)\hat{v}_i \mid \lambda \in [0, 1]\}$  for any intermediate operation. The formal definition of the secant slope given in equation (7) may cause numerically unstable divisions when the denominator gets small during the transition from secant to tangent mode, e.g. when the secant mode Newton iteration scheme converges.



However, in this section we will provide *singularity free* closed form expressions. Therefore an exception handling at  $\check{v}_i = \hat{v}_i$  will no longer be necessary. To that end we move from the line segment representation to a midpoint-radius based representation. Now let  $v = \varphi(u)$ , where  $\varphi \in \{\sin, \exp, \dots\}$  is some elementary operation and

$$(\check{v}_i, \hat{v}_i) = (\varphi(\check{v}_j), \varphi(\hat{v}_j)) \mapsto (\check{v}_i, \delta v_i), \quad \text{where} \quad \check{v}_i = \frac{\hat{v}_i + \check{v}_i}{2} \quad \text{and} \quad \delta v_i = \frac{\hat{v}_i - \check{v}_i}{2}.$$

We adopted the concept of representing intervals via midpoint and radius from interval arithmetic (described in detail e.g. by Siegfried Rump in [13], or by Götz Alefeld and Jürgen Herzberger in [1]). However, in the present setting the radius  $\delta v_i \in \mathbb{R}$  is allowed to get negative as well, since the corresponding line segment  $[\check{v}_i, \hat{v}_i]$  is undirected.

Now one can rewrite the secant slope of differentiable functions to the new representation

$$c_{ij} \equiv \frac{\hat{v}_i - \check{v}_i}{\hat{v}_j - \check{v}_j} = \frac{\delta v_i}{\delta v_j}.$$

Using some algebraic manipulations one can find individual formulas for the aforementioned propagation rules of the secant mode:

binary operation	$\check{v}_i$	$\delta v_i$	$c_{ij}$	$c_{ik}$
$v_i = v_j + v_k$	$\check{v}_j + \check{v}_k$	$\delta v_j + \delta v_k$	1	1
$v_i = v_j - v_k$	$\check{v}_j - \check{v}_k$	$\delta v_j - \delta v_k$	1	-1
$v_i = v_j \cdot v_k$	$\check{v}_j \check{v}_k + \delta v_j \delta v_k$	$\delta v_j \check{v}_k + \check{v}_j \delta v_k$	$\check{v}_k$	$\check{v}_j$
$v_i = \frac{v_j}{v_k}$	$\frac{\check{v}_j \check{v}_k - \delta v_j \delta v_k}{\check{v}_k^2 - \delta v_k^2}$	$\frac{\delta v_j \check{v}_k - \check{v}_j \delta v_k}{\check{v}_k^2 - \delta v_k^2}$	$\frac{1}{\check{v}_k}$	$-\frac{\check{v}_j}{\check{v}_k^2 - \delta v_k^2}$

Alternatively, we could represent  $v_i = v_j/v_k$  as an application of a multiplication on the reciprocal  $1/v_k$ . Furthermore, we can represent the multiplication by the Appolonius identity as above. Moreover, for unary operations we get:

unary operation	$\check{v}_i$	$\delta v_i$	$c_{ij}$
$v_i = \sin(v_j)$	$\sin(\check{v}_j) \cos(\delta v_j)$	$\cos(\check{v}_j) \sin(\delta v_j)$	$\cos(\check{v}_j) \text{sinc}(\delta v_j)$
$v_i = \cos(v_j)$	$\cos(\check{v}_j) \cos(\delta v_j)$	$-\sin(\check{v}_j) \sin(\delta v_j)$	$-\sin(\check{v}_j) \text{sinc}(\delta v_j)$
$v_i = \exp(v_j)$	$\exp(\check{v}_j) \cosh(\delta v_j)$	$\exp(\check{v}_j) \sinh(\delta v_j)$	$\exp(\check{v}_j) \text{sinhc}(\delta v_j)$
$v_i = \log(v_j)$	$\frac{1}{2} \log(\check{v}_j^2 + \delta v_j^2)$	$\text{artanh}\left(\frac{\delta v_j}{\check{v}_j}\right)$	$\frac{1}{\check{v}_j} \text{artanhc}\left(\frac{\delta v_j}{\check{v}_j}\right)$

Note that by, e.g., [17]  $\text{sinc}(x)$  and  $\text{sinhc}(x)$  (hyperbolic  $\text{sinc}(x)$ ) have regular Taylor expansions

$$\sin(x) = x \cdot \text{sinc}(x) = x \cdot \sum_{i=0}^{\infty} \frac{(-x^2)^i}{(2i+1)!}, \quad \text{sinc}(x) \equiv x \cdot \text{sinhc}(x) \equiv x \cdot \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i+1)!}.$$

We want to define  $\text{artanhc}(x)$  similar to  $\text{sinhc}(x)$  via its Taylor expansion:

$$\tanh^{-1}(x) = \text{artanh}(x) = x \cdot \text{artanhc}(x) = x \cdot \sum_{i=0}^{\infty} \frac{x^{2i}}{2i+1}$$

and it can be implemented in a similar fashion as  $\operatorname{sinhc}(x)$  from the boost c++ libraries (see [2]). For Root functions ( $v_i = \sqrt[n]{v_j}$ ), general Powers ( $v_i = v_j^c$ , or in a binary fashion  $v_i = v_j^{v_k}$ ) and monomials ( $v_i = v_j^n$ ) one can use the identity

$$v_i = v_j^c = \exp(c \cdot \log(v_j)) \quad \text{or} \quad v_i = v_j^{v_k} = \exp(v_k \cdot \log(v_j))$$

and apply the rules above. Of course, the base  $v_j > 0$  has to be positive, but there is a less restrictive alternative for monomials:

monomials	$\hat{v}_i$	$\delta v_i$	$c_{ij}$
$v_i = v_j^n$ ( $n$ natural number)	$\sum_{\substack{k \leq n \\ \text{even}}} \binom{n}{k} \hat{v}_j^{n-k} \delta v_j^k$	$\sum_{\substack{k \leq n \\ \text{odd}}} \binom{n}{k} \hat{v}_j^{n-k} \delta v_j^k$	$\sum_{\substack{k \leq n \\ \text{odd}}} \binom{n}{k} \hat{v}_j^{n-k} \delta v_j^{k-1}$
$v_i = v_j^2$ (special case: square)	$\hat{v}_j^2 + \delta v_j^2$	$2\hat{v}_j \delta v_j$	$2\hat{v}_j$

*Remark 1* (General approximation for unary operations). Of course one can find a lot more singularity free formulas for the secant slopes of other operations. Using a Taylor expansion approach one can provide general approximation formulas for the triplet  $\hat{v}_i, \delta v_i$  and  $c_{ij}$  by:

$$\hat{v}_i = \frac{1}{2} \left( \sum_{k \geq 0} \frac{\varphi^{(k)}(\hat{v}_j)}{k!} \delta v_j^k + \sum_{k \geq 0} \frac{\varphi^{(k)}(\hat{v}_j)}{k!} (-\delta v_j)^k \right) = \sum_{\substack{k \geq 0 \\ k \text{ even}}} \frac{\varphi^{(k)}(\hat{v}_j)}{k!} \delta v_j^k,$$

$$\delta v_j = \frac{\varphi(\hat{v}_j + \delta v_j) - \varphi(\hat{v}_j - \delta v_j)}{2} = \sum_{\substack{k > 0 \\ k \text{ odd}}} \frac{\varphi^{(k)}(\hat{v}_j)}{k!} \delta v_j^k, \quad c_{ij} = \sum_{\substack{k > 0 \\ k \text{ odd}}} \frac{\varphi^{(k)}(\hat{v}_j)}{k!} \delta v_j^{k-1}.$$

## C++ Implementation Strategies for Singularity free Formulas

The following is a simple modification of common AD tools, such as ADOL-C [16]. In [5, Section 6.1] the forward mode of directional derivative propagation is described. We can adopt this concept and modify the `adouble`-constructor as follows:

```

1 #include <math.h>
2 #include <boost/math/special_functions/sinc.hpp>
3 using namespace std;
4 using namespace boost::math;
5 class adouble{ public:
6     double val;
7     double mid, rad;
8     double increment; };

```

In the following we need `sin` from the standard math package and the `sinc`-function from the boost libraries. The original constructor consists of two components: The value `val` and the directional derivative `dot`. We replace `dot` by the increment `increment` and extend the constructor for the midpoint `mid` ( $\hat{v}_i$ ) and the radius `rad` ( $\delta v_i$ ). Following [5, Section 6.1] we want to demonstrate the implementation of `sin` and the multiplication as representatives for a unary and a binary operation:

```

9  adouble sin(adouble u){
10     adouble v;
11     v.val = sin(u.val);
12     v.mid = sin(u.mid) * cos(u.rad);
13     v.rad = cos(u.mid) * sin(u.rad);
14     v.increment = cos(u.mid) * sinc_pi(u.rad) * u.increment;
15     return v; }
16  adouble operator* (adouble u, adouble w){
17     adouble v;
18     v.val = u.val * w.val;
19     v.mid = u.mid * w.mid + u.rad * w.rad;
20     v.rad = u.rad * w.mid + u.mid * w.rad;
21     v.increment = w.mid * u.increment + u.mid * w.increment;
22     return v; }

```

And, of course, the *abs*-function is of particular interest:

```

23  adouble abs(adouble u){
24     adouble v;
25     v.val = abs(u.val);
26     v.mid = 0.5*(abs(u.mid + u.rad) + abs(u.mid - u.rad));
27     v.rad = 0.5*(abs(u.mid + u.rad) - abs(u.mid - u.rad));
28     v.increment = abs(u.mid + u.increment) - v.mid;
29     return v; }

```

## 7 Numerical example

Consider the function

$$f(x) = \begin{bmatrix} \cos(\varphi(\angle x) - \angle x) & -\sin(\varphi(\angle x) - \angle x) \\ \sin(\varphi(\angle x) - \angle x) & \cos(\varphi(\angle x) - \angle x) \end{bmatrix} \cdot x - c,$$

where  $c = [1.001, 10.01]^\top$ , and  $\angle x \in [0, 2\pi[$  is the angle of  $x$  in polar coordinate representation. Moreover,  $\varphi$ , which is defined by

$$\varphi(\psi) \equiv \psi + \frac{8}{5\pi}\psi^2 - \frac{8}{5\pi^2}\psi^3 + \frac{2}{5\pi^3}\psi^4$$

maps  $[0, 2\pi[$  strictly monotonically onto itself. Hence,  $f(x)$  is bijective. Furthermore, the function is differentiable everywhere except at the origin. There it is, just as the Euclidean norm, locally Lipschitz continuous and not even piecewise differentiable in the sense of [14].

We investigated the behavior of the tangent and secant mode Newton on  $f$  both with and without noise. That is, we investigated  $f$  and  $\tilde{f}$ , where

$$\tilde{f}(x) = f(x) + \frac{\sin(5000 \cdot [x_0 + x_1])}{10^4}.$$

The secant mode was started with the initial values  $\tilde{x} = [-3.7, -2.05]^\top$  and  $\hat{x} = [7.0, 8.0]^\top$ . The tangent mode was started with the mean value of said points.

We recall the well known formula for approximating the convergence rate numerically:

$$\gamma \approx \frac{\log \left| \frac{x_{n+1} - x_n}{x_n - x_{n-1}} \right|}{\log \left| \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} \right|}.$$

The first table shows both modes' residuals in the iterations without noise.

iteration	residual with tangent mode	res. with secant mode
0	13.3919956235	5.81435555868
1	5.65630249881	19.6157765738
2	3.39957297287	4.99712831052
3	0.00920821188601	1.2635478817
4	$2.65403135735e - 06$	0.0882985228824
5	$2.13162820728e - 13$	0.00192152171278
6	$8.881784197e - 15$	$5.19844852542e - 06$
7		$3.0607116841e - 10$
8		$8.881784197e - 16$
$\gamma$	$2.07814399547 \approx 2$	$1.64753467681 \approx \frac{1+\sqrt{5}}{2}$

The next table shows the residuals of the iterations with noise.

iteration	residual with tangent mode	res. with secant mode
0	13.3920216719	5.81445152008
1	60.308012713	19.6157077846
2	81.8554424857	4.99709320593
3	8.54532016753	1.26352999658
4	5.69986744799	0.0883961689187
5	1.92933721639	0.00178884970844
6	0.425650504358	0.00013247524387
7	0.127098157087	$3.54530493811e - 05$
8	0.0253173333077	$2.27749388494e - 06$
9	0.00248354296218	$3.65673925216e - 08$
10	0.000950425306504	$3.71373256312e - 11$
11	$6.728080753e - 05$	$4.95408564432e - 15$
12	$1.87922375892e - 06$	
13	$1.57356337999e - 09$	
14	$2.89904818901e - 15$	
$\gamma$	$2.01181918489 \approx 2$	$1.66153582947 \approx \frac{1+\sqrt{5}}{2}$

Both methods attain their theoretical convergence rates in both iterations. However, we observe that the secant mode fares better with the problem with noise as it cuts through the latter, while the tangent Newton is thrown off for several iteration steps.

## 8 Final Remarks

The framework developed in the present paper, as well as in [4, 6, 15, 12, 7], aims at presenting a viable approach to piecewise differentiability as it may occur, e.g., in nonsmooth nonlinear systems or ODEs with nonsmooth right hand side. The piecewise linearizations, which were first introduced in [4] can be obtained in an automated fashion by an adaptation of AD, such as ADOL-C [16].

The generalized Newton iterations introduced in Section 5 are intended as an alternative to semismooth Newton [8]. The quadratic convergence rate of the tangent version is in line with

that of semismooth Newton. For one step of semismooth Newton a limiting Jacobian has to be calculated, which is, in general, nonunique. The piecewise linear Newton's methods solves a piecewise linear system in each step. The latter is NP-hard in general, but may be solvable in practice with essentially the same effort as an ordinary linear system. Hence, it is likely highly situation dependent, which approach yields better performance.

It is our hope that, in combination with the formulas for numerically stable implementation of the secant linearization, the generalized Newton's methods can be developed into robust and stable workhorse algorithms, which might even outperform the quadratic methods on selected problems. For example on problems with oscillating noise we observed that the secant method required fewer Newton steps to converge, as it cuts through the oscillations.

As a last remark, we conjecture that the explicit Lipschitz constants in Proposition 4.2 will allow to calculate explicit convergence radii for the piecewise linear Newton's methods, which would then be a significant improvement on the mere existence results in the present work and for the semismooth Newton's method.

## Acknowledgements

The work for the article has been partially conducted within the Research Campus MODAL funded by the German Federal Ministry of Education and Research (BMBF) (fund number 05M14ZAM).

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