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**Using Bilevel Optimization to find
Severe Transport Situations in
Gas Transmission Networks**

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Using Bilevel Optimization to find Severe Transport Situations in Gas Transmission Networks

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Abstract

In the context of gas transmission in decoupled entry-exit systems, many approaches to determine the network capacity are based on the evaluation of realistic and severe transport situations. In this paper, we review the Reference Point Method, which is an algorithm used in practice to generate a set of scenarios using the so-called transport moment as a measure for severity. We introduce a new algorithm for finding severe transport situations that considers an actual routing of the flow through the network and is designed to handle issues arising from cyclic structures in a more dynamical manner. Further, in order to better approximate the physics of gas, an alternative, potential based flow formulation is proposed. The report concludes with a case study based on data from the benchmark library GasLib.

1 Introduction

Natural gas is one of the most important sources of energy and has to be transported over long distances using systems of pipelines. These networks are owned and operated by *Transmission System Operators* (TSOs) using technical elements like compressors or valves in order to realize the transport. Traditionally, both trading and transport of gas were provided by the same company, which enabled long-term planning based on reliable forecasts for supply and demand. Nevertheless, recent regulation towards a market liberalization in the EU has led to the decoupling of trading and transport. Now the TSOs sell transport capacity that is booked by the traders at entry and exit points independently. Within these booked capacities, the traders may then nominate amounts of gas that they want to insert or withdraw from the network in the short term, that is hours or days. The TSOs have to ensure that transport can be realized in all *balanced* situations, meaning that the amount of gas inserted at the entries is equal to the amount withdrawn at the exits during a certain time horizon.

Different strategies to estimate the transport capacity of gas networks have been developed, usually based on the validation of a finite set of scenarios.

An approach that was developed within the research cooperation “ForNe” between the German TSO Open Grid Europe GmbH and several research institutions is documented in the book “Evaluating Gas Network Capacities” [KHPS15]. Historical data for gas demand at exits are used to estimate distributions, which are then used to sample realistic scenarios. Each scenario is completed by supply values that are chosen to be *extreme*, meaning that the scenario is balanced and maximal w.r.t. a random preference order on the entries. Afterwards, the numerous resulting transport situations are individually checked for feasibility in an automated fashion using optimization methods.

On the other hand, the TSO Gasunie from the Netherlands describes an approach [SHD⁺15] that first identifies a small set of “stress tests” for their network, which is then validated with simulation tools, in particular the “Multi Case Approach” (MCA) [vdH04]. The stress tests should be maximally *severe*, which is quantified using the *transport moment*, a measure of a flow amount transported over long distances, and cover all relevant directions of flow. The method to generate a “complete” set of stress tests is based on distances relative to a reference point, and it is summarized in more detail in Section 3. Afterwards, the set of stress tests is reduced using a similarity measure based on node distances.

Inspired by the second approach, we present a new algorithm which has the goal to find additional severe transport situations in Section 4. It differs in the selection of entry-exit pairs where in- and outflow are simultaneously increased, as well as in the definition of the severity measure. The proposed measure, called *minimum transport moment*, is independent of the chosen reference point and based on an actual routing of the flow. In addition, in Section 5 we discuss the problem of finding a *maximally severe* transport situation w.r.t. the minimum transport moment, with a formulation as a bilevel optimization problem.

In Section 6, we discuss a variation of the minimum transport moment, which is based on a flow formulation that is linearly constrained by node potentials, accounting for the pressure loss due to friction in pipelines.

Finally, a case study using network data with corresponding transport situations from the benchmark instance `gaslib-582` [HJO⁺15] is presented in Section 7.

2 Definitions and Notation

In order to present the mathematical concepts in this paper, we are going to model gas networks as connected, directed flow networks $G = (V, A)$ with an anti-parallel arc set $A \subseteq V \times V$. For each arc $a \in A$ we are given a length value $\ell_a \in \mathbb{R}_{\geq 0}$ and for each pair of anti-parallel arcs $uv, vu \in A$ it holds that $\ell_{uv} = \ell_{vu}$.

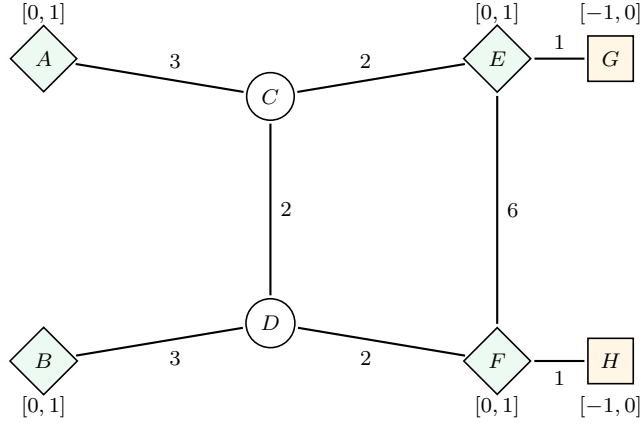


Figure 1: Example network with diamond-shaped entries A,B,E, and F, square-shaped exits G and H, and circle-shaped inner nodes C and D. Arc lengths as well as the feasible supply and demand intervals are depicted.

Further, let $V^+ \subseteq V$ and $V^- \subseteq V$ be subsets of the node set V representing the *entries* and the *exits* of the network, respectively. W.l.o.g. we assume that $V^+ \cap V^- = \emptyset$. All remaining nodes are *inner nodes* and we denote them by $V^0 := V \setminus (V^+ \cup V^-)$.

For each entry $v \in V^+$ a lower and an upper bound $\underline{b}_v, \bar{b}_v \in \mathbb{R}_{\geq 0}$ with $\underline{b}_v \leq \bar{b}_v$ on its *supply* is given. The supply is the amount of flow that may enter the network at v . Analogously, for each exit $w \in V^-$ we are given a lower and an upper bound $\underline{b}_w, \bar{b}_w \in \mathbb{R}_{\leq 0}$ with $\underline{b}_w \leq \bar{b}_w$ on its *demand*, that is the amount of flow that may be withdrawn at w . For all inner nodes we assume that flow conservation holds. Therefore, for all $u \in V^0$ we define $\underline{b}_u = \bar{b}_u = 0$. An example network is shown in Figure 1.

A *transport situation*, synonymously called *supply and demand vector* in the following, is a vector $b \in \mathbb{R}^V$ with $b_u \in [\underline{b}_u, \bar{b}_u]$ for all $u \in V$. A transport situation is called *balanced* if $\sum_{u \in V} b_u = 0$. Given a transport situation b , the *capacity* of an entry is defined as the difference between its upper bound and its current supply, i.e., $\bar{b}_v - b_v$ for $v \in V^+$. For an exit $w \in V^-$, the capacity is defined as the difference between its current demand and its lower bound, i.e., $b_w - \underline{b}_w$.

For the algorithms discussed in this paper the lengths of shortest paths towards entry and exit nodes play an important role. Therefore, let $d_{vw} \in \mathbb{R}_{\geq 0}$ denote the length of a shortest path between the nodes v and w with respect to the arc lengths ℓ_a .

3 The Reference Point Method

The *Reference Point Method*, as presented in [SHD⁺15], is an algorithm that attempts to find transport situations that maximize the *transport moment* $T = QD$, which is defined as the product of the *system throughput* Q , i.e., the total flow from the entries to the exits, and the so-called *mean transportation distance* D . To determine D , a reference point P has to be chosen and distances to all entry and exit nodes are calculated. D is then defined as the difference of the *centers of gravity* of the entry and exit nodes, which are the supply- and demand-weighted average distances towards P .

In the following, we introduce the version of the Reference Point Method that, according to [SHD⁺15], is used in practice to determine *stress tests*. These are transport situations which are extreme in terms of transport effort but still realistic w.r.t. capacity contracts.

First of all, a node $u \in V^+ \cup V^-$ is chosen as reference point and the lengths of shortest paths between it and all entries and exits are determined, i.e., d_{uv} for all $v \in V^+$ and d_{uw} for all $w \in V^-$, using for example Dijkstra's algorithm [Dij59]. Note that for the correctness of the algorithm $\underline{b}_v = 0$ for all $v \in V^+$ and $\bar{b}_w = 0$ for all $w \in V^-$ is assumed in the following.

The Reference Point Method starts with the zero flow transport situation, i.e., it assumes that all supply and demand values are equal to zero. Next, the supply at an entry with non-zero capacity which is closest to u as well as the demand at an exit with non-zero capacity farthest away from u are increased as long as their capacity bounds allow for it. This procedure is repeated until an increase of the flow leads to a decrease in T , as the centers of gravity shift, that is the average distances between entries and exits are reduced. At this point, a stress test is found and the algorithm terminates returning the current transport situation.

The transport moment can be described as a linear function in supply and demand variables b_v and b_w for all $v \in V^+$ and $w \in V^-$, with the constraints that they do not exceed their bounds and that the transport situation is balanced. Consequently, given a reference point u , the algorithm solves the linear program:

$$\begin{aligned}
 & \underset{b}{\text{maximize}} && - \sum_{v \in V^+} d_{uv} b_v - \sum_{w \in V^-} d_{uw} b_w \\
 & \text{subject to} && \sum_{v \in V^+} b_v + \sum_{w \in V^-} b_w = 0 \\
 & && b_v \in [0, \bar{b}_v] \quad \forall v \in V^+ \\
 & && b_w \in [\underline{b}_w, 0] \quad \forall w \in V^-
 \end{aligned} \tag{1}$$

Note that the b_w variables for exits $w \in V^-$ take on negative values.

The stress tests generated by applying the Reference Point Method to each $u \in V^+ \cup V^-$ are called *global stress tests*. For the network in Figure 1 the four resulting global stress tests can be found in Figure 2.

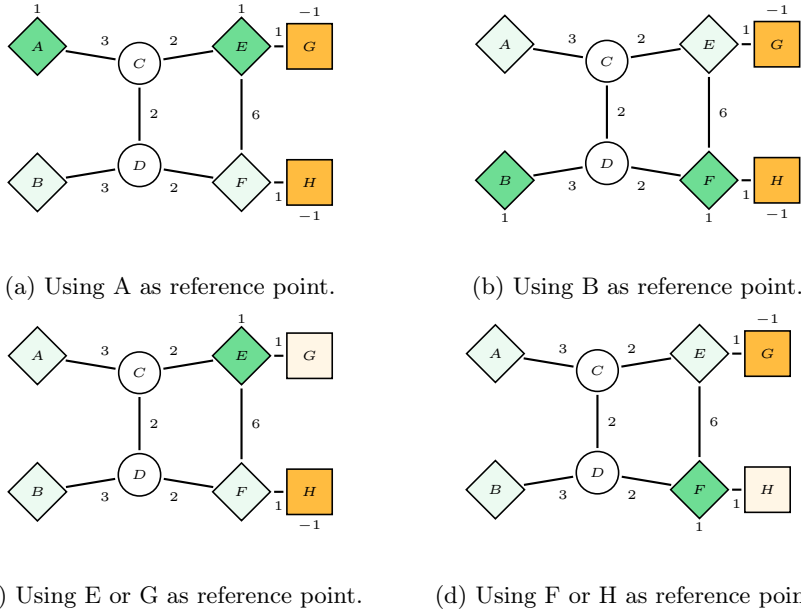


Figure 2: Global stress tests for the example network from Figure 1. All non-zero supply or demand values are written above or below highlighted nodes.

As in the example, especially if multiple entries and exits are located close to each other, it may happen that the global stress tests, which are derived when using them as reference points, are identical. A set of reference points leading to the same global stress test is called a *cluster*. There are two clusters in our example: $\{E, G\}$ and $\{F, H\}$. By maximizing the transport moment within single clusters, additional, so-called *local stress tests* are derived. For more details, we refer to [SHD⁺15].

Due to the fixation of the distances towards the reference point u and the presence of a cycle (ring structure) in the example network, the transport situation shown in Figure 3 is not found by the algorithm. Although [SHD⁺15] describes a procedure based on clusters to handle these problems, it is not clear how this particular result can be derived.

Given a supply and demand vector, consider the value of a Minimum Cost Flow, where the arc lengths are used as weights. The transport situation in Figure 3 maximizes this value among all possible demand and supply vectors. Therefore, as an addition to the Reference Point Method, we propose a new algorithm called *Greedy Minimum Cost Flow Method* in the next section, which attempts to find transport situations that are severe with respect to this value.

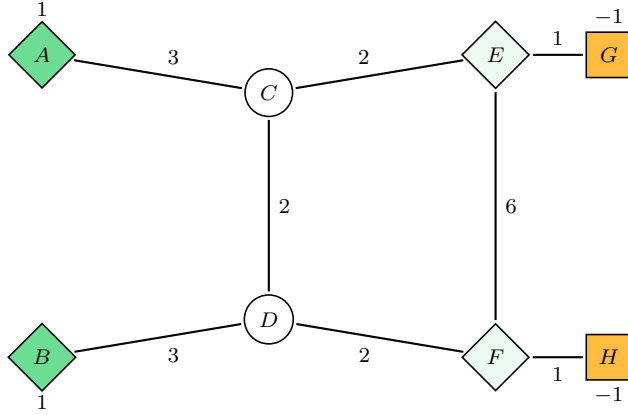


Figure 3: Transport situation not found by the Reference Point Method.

4 The Greedy Minimum Cost Flow Method

In this section we present a new algorithm for finding severe transport situations in gas networks. It is inspired by the Reference Point Method algorithm from Section 3, but differs in three main aspects:

First and most importantly, the severity measure for transport situations which we propose is based on actual *routing* of the flow through the network. Recall that ℓ_a denotes the length of an arc $a \in A$ and let f_a denote the (non-negative) flow on it. We define the *minimum transport moment* as $T = \sum_{a \in A} \ell_a f_a$. Moreover, while we want to identify transport situations that are severe with respect to this measure, we assume that, for any given transport situation, the flow is routed through the network in a way that it minimizes T . This assumption is based on the rationale that this is a natural goal of the TSO when trying to minimize transportation cost.

Hence, finding the minimum transport moment for a given transport situation means solving a *Minimum Cost Flow Problem* (MCF), where the lengths ℓ_a are used as weights on the arcs, see for example [AMO93]. This is readily formulated as a linear program: Given a fixed vector of supply and demand $b \in \mathbb{R}^V$, the MCF problem can be formulated as

$$\begin{aligned}
 & \underset{f}{\text{minimize}} && \sum_{a \in A} \ell_a f_a \\
 & \text{subject to} && \sum_{a \in \delta^+(u)} f_a - \sum_{a \in \delta^-(u)} f_a = b_u \quad \forall u \in V \\
 & && f_a \geq 0 \quad \forall a \in A
 \end{aligned} \tag{2}$$

Note, that this problem has a solution for balanced transport situations only.

Second, instead of using a fixed distance for each node when choosing the next opposing entry or exit, namely the length of a shortest path towards the chosen reference point, we consider the actual lengths of shortest paths between it and the possible candidates.

Third, our algorithm does not only consider flow going from the close-by entries towards far-away exits when determining the next node-pair whose supply and demand values should be simultaneously increased. The roles of entries and exits are also switched, i.e., we look for close-by exits and a far-away entries instead. Therefore, we introduce two versions of the algorithm: An entry-directed and an exit-directed one.

Our algorithm, which we call *Greedy Minimum Cost Flow Method* (Greedy-MCF), follows several steps, which are similar to the Reference Point Method. We describe the entry-directed version here, but switching the roles of entries and exits, as mentioned above, yields the exit-directed version.

1. Choose a reference point $u \in V^+$, set $v := u$, and start with the zero transport situation, i.e., $b := \mathbf{0} \in \mathbb{R}^V$.
2. Choose an opposing exit $w \in V^-$ having non-zero capacity with maximum distance towards v .
3. Increase the supply at v and the demand at w simultaneously until one of the two nodes hits a bound.
4. Determine the minimum transport moment T w.r.t. b by solving (2).
5. If T has decreased, revert the last change and return b . Otherwise, if there is no more entry or exit capacity left, stop and return b . Otherwise, assign v an entry with non-zero capacity that is closest to u and go to 2.

Note that like the Reference Point Method, the Greedy-MCF assumes that $\underline{b}_v = 0$ holds for all $v \in V^+$ and $\bar{b}_w = 0$ holds for all $w \in V^-$ in order to generate balanced solutions that are conform with the bounds.

The role of u is similar to the reference point of Section 3: It is used to describe the direction from which flow is supposed to enter the network, in the entry-directed case, or leave the network, in the exit-directed case. But the distances that are used to determine the next opposing node are independent of it. By using the actual distances towards the current v and by evaluating (2) in each loop of the algorithm, we avoid problems arising from cyclic structures, in particular flow augmentations that might relax the overall transport situation and lead to a decrease in the minimum transport momentum. We illustrate the algorithm with two examples, based on the 8-shaped network shown in Figure 4.

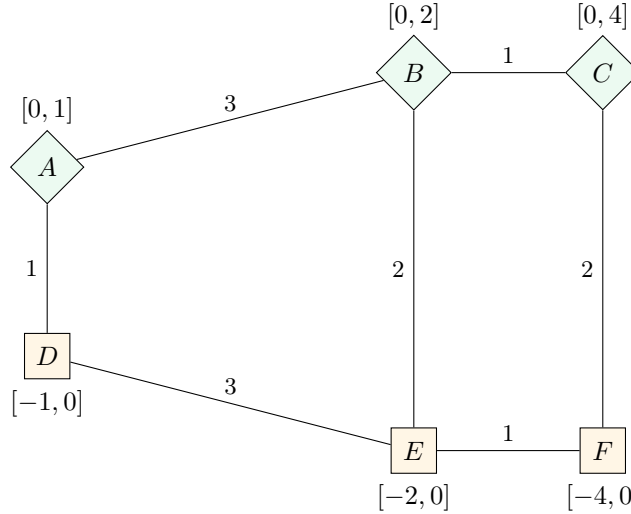


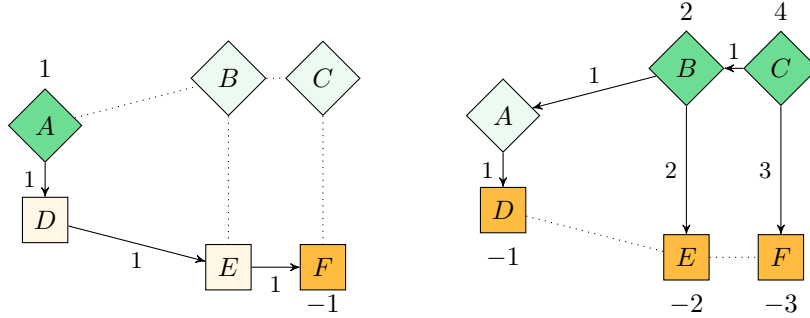
Figure 4: 8-shaped network with arc lengths and supply/demand intervals.

Example 1

First of all, we choose entry A as reference point. The exit which is the furthest away from A is F . We increase supply and demand at both nodes by one unit and hit the capacity bound of A . This gives us $b = (1, 0, 0, 0, 0, -1)$ and $T = 5$ (one flow unit along the path A, D, E, F). Next, consider entry B because it is closer to A than C . We choose D as opposing exit, since $d_{BD} = 4$ is greater than $d_{BE} = 2$ and $d_{BF} = 3$. Hence, we get $b = (1, 1, 0, -1, 0, -1)$ with $T = 4$. This is a smaller minimum transport moment than we had before, so we stop and terminate with $b = (1, 0, 0, 0, 0, -1)$. The final transport situation is shown in Figure 5a.

Example 2

Next, we choose B as reference point and D as the opposing exit, leading to $b = (0, 1, 0, -1, 0, 0)$ and $T = 4$. B still has non-zero capacity, hence we choose F as the next opposing exit and increase both nodes by one flow unit, and end up with $b = (0, 2, 0, -1, 0, -1)$ and $T = 7$. Next, we consider entry C and exit E as the next entry-exit pair and increase both by two flow units resulting in $b = (0, 2, 2, -1, -2, -1)$ and $T = 11$. The last exit with unused capacity is F , hence we increase C and F by two flow units. This results in $b = (0, 2, 4, -1, -2, -3)$ with $T = 15$. The last possible step would be to increase A and F by one flow unit, but this would yield a minimum transport momentum of $T = 13$. Therefore we stop and terminate with $b = (0, 2, 4, -1, -2, -3)$ and $T = 15$. The final transport situation is shown in Figure 5b.



(a) Result of Greedy-MCF starting at A. (b) Result of Greedy-MCF starting at B.

Figure 5: Example Greedy-MCF solutions for the 8-shaped network in Figure 4.

5 Finding the Maximum Minimum Transport Moment using a Bilevel Formulation

Let us now discuss the problem of finding a transport situation that leads to a maximum minimum transport moment. This setting can be described as a Stackelberg game in the context of game theory [VS52]. The two parties involved are the traders as *leader* and the TSO as *follower*. In the first stage, the leader defines the supply and demand vector b , which should be balanced and respect the capacity bounds. In the second stage, the follower chooses how the flow f is routed through the network, in order to implement the transport situation given by b . While the TSO tries to *minimize* the transport moment, the (presumed antagonistic) traders try to *maximize* it. This description of a Stackelberg game can be formulated in a canonical way as a bilevel optimization problem [CMS07].

This problem we call the *Uncapacitated Maximum Minimum Cost Flow Problem* (UMMCF) and it can be formalized as follows:

$$\begin{aligned}
 & \underset{b}{\text{maximize}} && \sum_{a \in A} \ell_a f_a \\
 & \text{subject to} && b_u \in [\underline{b}_u, \bar{b}_u] \quad \forall u \in V \\
 & && \underset{f}{\text{minimize}} && \sum_{a \in A} \ell_a f_a \\
 & \text{subject to} && \sum_{a \in \delta^+(u)} f_a - \sum_{a \in \delta^-(u)} f_a = b_u \quad \forall u \in V \\
 & && f_a \geq 0 \quad \forall a \in A
 \end{aligned} \tag{3}$$

It has the form of a *linear* bilevel programming problem (both levels feature continuous variables and linear constraints). In practice, this class of problems is solved with the so-called KKT reformulation [CMS07].

Note that the optimal solution for UMMCF with respect to the introductory example network in Figure 1 is the transport situation in Figure 3, which is not found by the Reference Point Method. This solution is also generated by the Greedy-MCF when using the exit-directed version and any exit as reference point. But in general, the Greedy-MCF does not generate an optimal solution for UMMCF, as the case study in Section 7 shows.

As mentioned in Section 3, the Reference Point Method is designed to solve the linear program (1). It proceeds by increasing the supply and demand of each entry or exit until it hits a bound, before the next one is considered. Therefore, in each global stress test there is at most one node $u \in V^+ \cup V^-$ whose supply or demand value b_u is not equal to one of its bounds.

A similar result can be shown for UMMCF, justifying that the same augmentation procedure is used in the Greedy-MCF. In the following, a solution (b, f) is called *bound-close* if all supply and demand values are equal to one of their bounds except for at most one, i.e., $b_u \neq \underline{b}_u$ and $b_u \neq \bar{b}_u$ for at most one node $u \in V^+ \cup V^-$. Lemma 1 now shows that for each UMMCF instance there exists a bound-close solution, which is optimal.

Lemma 1

There exists an optimal and bound-close solution for UMMCF.

Proof. Let (b, f) be any optimal solution for UMMCF. If it is not bound-close, there exist two nodes $v, w \in V^+ \cup V^-$ whose demand or supply values are both not equal to one of their bounds, i.e., $b_v \neq \underline{b}_v$ and $b_v \neq \bar{b}_v$, as well as $b_w \neq \underline{b}_w$ and $b_w \neq \bar{b}_w$.

The basic idea of the proof is to derive two new optimal solutions by in- and decreasing in- and outflow in case that $v \in V^+$ and $w \in V^-$, or by shifting in- or outflow in case that $v, w \in V^+$ or $v, w \in V^-$, respectively. Both solutions are constructed in such a way that either b_v or b_w is set to one of its bounds.

Let $\Delta_1 := \min\{b_v - \underline{b}_v, \bar{b}_w - b_w\} > 0$ and $\Delta_2 := \min\{\bar{b}_v - b_v, b_w - \underline{b}_w\} > 0$. Consider the demand and supply vectors \tilde{b} and \hat{b} defined as

$$\tilde{b}_u := \begin{cases} b_v - \Delta_1 & \text{if } u = v \\ b_w + \Delta_1 & \text{if } u = w \\ b_u & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{b}_u := \begin{cases} b_v + \Delta_2 & \text{if } u = v \\ b_w - \Delta_2 & \text{if } u = w \\ b_u & \text{otherwise.} \end{cases}$$

By construction, \tilde{b} and \hat{b} respect all bounds and are balanced. Additionally, for \tilde{b} either $\tilde{b}_v = \underline{b}_v$ or $\tilde{b}_w = \bar{b}_w$ holds, and analogously for \hat{b} we have that $\hat{b}_v = \bar{b}_v$ or $\hat{b}_w = \underline{b}_w$. An example for the case $v \in V^+$ and $w \in V^-$ is sketched in Figure 6. Next, let \tilde{f} and \hat{f} be optimal solutions to the MCF problem induced by \tilde{b} and \hat{b} , respectively. Since G is connected and we do not consider capacities on the arcs, such a solution always exists. By construction (\tilde{b}, \tilde{f}) and (\hat{b}, \hat{f}) are feasible solutions and $c(b, f) \geq c(\tilde{b}, \tilde{f})$ and $c(b, f) \geq c(\hat{b}, \hat{f})$ holds because (b, f) is optimal. This observation we denote by $(*)$ in the following.

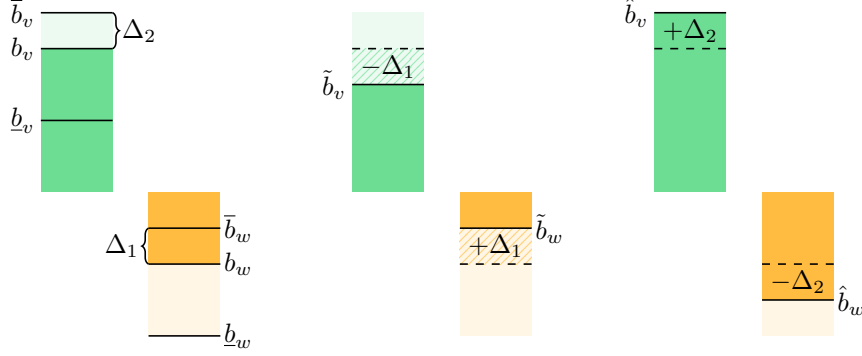


Figure 6: Derived transport situations \tilde{b} and \hat{b} in case that $v \in V^+$ and $w \in V^-$.

Next, for $\lambda := \frac{\Delta_2}{\Delta_1 + \Delta_2} \in (0, 1)$ it holds that $\lambda \tilde{b} + (1 - \lambda) \hat{b} = b$. This is obviously true for all nodes $u \in (V^+ \cup V^-) \setminus \{v, w\}$. For v it holds that

$$\begin{aligned}
 \lambda \tilde{b}_v + (1 - \lambda) \hat{b}_v &= \frac{\Delta_2}{\Delta_1 + \Delta_2} (b_v - \Delta_1) + \left(1 - \frac{\Delta_2}{\Delta_1 + \Delta_2}\right) (b_v + \Delta_2) \\
 &= \frac{\Delta_2}{\Delta_1 + \Delta_2} (b_v - \Delta_1) + \frac{\Delta_1}{\Delta_1 + \Delta_2} (b_v + \Delta_2) \\
 &= \frac{b_v \Delta_2 - \Delta_1 \Delta_2 + b_v \Delta_1 + \Delta_1 \Delta_2}{\Delta_1 + \Delta_2} = \frac{b_v (\Delta_1 + \Delta_2)}{\Delta_1 + \Delta_2} = b_v,
 \end{aligned}$$

and for w an analogous argument can be made. Further, since $\lambda \in (0, 1)$, $\lambda \tilde{f} + (1 - \lambda) \hat{f}$ is a feasible flow for the supply and demand vector b , because

$$(b, \lambda \tilde{f} + (1 - \lambda) \hat{f}) = (\lambda \tilde{b} + (1 - \lambda) \hat{b}, \lambda \tilde{f} + (1 - \lambda) \hat{f}) = \lambda (\tilde{b}, \tilde{f}) + (1 - \lambda) (\hat{b}, \hat{f})$$

respects all bounds and linear constraints of (3). Additionally, since (b, f) is optimal, it follows that $c(b, f) = c(\tilde{b}, \tilde{f}) = c(\hat{b}, \hat{f})$, because

$$\begin{aligned}
 c(b, f) &\leq c(b, \lambda \tilde{f} + (1 - \lambda) \hat{f}) \\
 &= \sum_{a \in A} \ell_a (\lambda \tilde{f}_a + (1 - \lambda) \hat{f}_a) \\
 &= \lambda \sum_{a \in A} \ell_a \tilde{f}_a + (1 - \lambda) \sum_{a \in A} \ell_a \hat{f}_a \\
 &= \lambda c(\tilde{b}, \tilde{f}) + (1 - \lambda) c(\hat{b}, \hat{f}) \\
 &\stackrel{(*)}{\leq} \lambda c(b, f) + (1 - \lambda) c(b, f) = c(b, f).
 \end{aligned}$$

Therefore, (\tilde{b}, \tilde{f}) and (\hat{b}, \hat{f}) are optimal solutions with the supply or demand value of either v or w being equal to one of their bounds.

Iteratively applying the algorithmic procedure to node pairs $v, w \in V^+ \cup V^-$, whose demand or supply values are not at their bounds, we derive an optimal and bound-close solution after at most $n - 1$ iterations. \square

6 The Potential Transport Moment

In this section, we propose an alternative way to distribute the flow on the arcs, as a basis for the transport moment computation. While the flow on the arcs was determined to lead to the minimum transport moment in Section 4 and 5, this point of view may be overly simplistic. This is because only shortest paths in the network are considered, while alternative routes are not used at all.

Gas flow through pipelines incurs a friction induced pressure drop, which is often modeled by the Weymouth equation in the stationary case [vdH04, KHPS15]:

$$p_u^2 - p_v^2 = \ell_{uv} C_{uv} f_{uv} |f_{uv}|, \quad (4)$$

where, p denotes the pressure at the nodes and the coefficient C represents pipe parameters such as diameter and roughness. These equations imply that the flow f cannot run in directed cycles, but is instead distributed among several paths from entries to exits, according to the “resistance” given by the coefficient ℓC . Furthermore, in passive networks, consisting only of pipelines with no active elements such as compressors, the flow distribution f is uniquely defined by the supply and demand vector b , see for example [CCHL78, FH13].

Instead of using the Weymouth equation (4), we employ a similar, but *linear* equation. Furthermore, we only use the pipe length for the resistance coefficient, in correspondence to the definition of the minimum transport moment:

$$\pi_u - \pi_v = \ell_{uv} (f_{uv} - f_{vu}). \quad (5)$$

Note, that only a single equation is added for each pair of anti-parallel arcs $uv, vu \in A$. Moreover, these equations still satisfy the conditions that lead to a unique flow distribution f , which means that no objective needs to be defined for the TSO when routing the flow.

As a consequence, we can formulate a single-level optimization problem that finds the most severe transport situation with respect to the *Potential Transport Moment* (PTM):

$$\begin{aligned}
& \underset{b, f, \pi}{\text{maximize}} && \sum_a \ell_a f_a \\
& \text{subject to} && \sum_{vw} f_{vw} - \sum_{uv} f_{uv} = b_v && \forall v \in V \\
& && \pi_u - \pi_v = \ell_{uv}(f_{uv} - f_{vu}) && \forall uv, vu \in A \\
& && f_{uv} f_{vu} = 0 && \forall uv, vu \in A \\
& && f_a \geq 0 && \forall a \in A \\
& && b_u \in [\underline{b}_u, \bar{b}_u] && \forall u \in V \\
& && \pi_v \in \mathbb{R} && \forall v \in V
\end{aligned} \tag{6}$$

Note that the complementarity constraint $f_{uv}f_{vu} = 0$ is needed, because otherwise flow would be sent in a cycle from u to v and back. In this case the problem would be unbounded. The complementarity can be seen as a disjunction $(f_{uv} = 0) \vee (f_{vu} = 0)$ that can be modeled in a MIP setting using auxiliary binary variables $x_{uv} \in \{0, 1\}$ for each anti-parallel arc pair $uv, vu \in A$ as

$$\begin{aligned}
f_{uv} &\leq x_{uv}M \\
f_{vu} &\leq (1 - x_{uv})M,
\end{aligned} \tag{7}$$

where M can be any bound valid for all f_{uv} , for example $M := \sum_{v \in V^+} \bar{b}_v$.

In general, optimal solutions for UMMCF and PTM differ in the resulting transport situations and arc flows. To see this, consider the flow network depicted in Figure 7. It is easy to verify that

$$b_u := \begin{cases} 1 & \text{if } u \in \{E_1, E_2\} \\ -1 & \text{if } u \in \{X_1, X_2\} \end{cases} \text{ and } f_a := \begin{cases} 1 & \text{if } a \in \{E_1X_1, E_2X_2\} \\ 0 & \text{otherwise.} \end{cases}$$

is the unique optimal solution for the corresponding UMMCF instance with a minimum transport moment of 2. On the other hand, fixing supplies and demands to b in the corresponding PTM formulation results in the solution (b, π, f) with

$$\pi_u := \begin{cases} 1 & \text{if } u = \{E_1, E_2\} \\ 0 & \text{if } u = \{X_1, X_2\}, \end{cases}$$

the same flow vector f and the same objective value of 2. Nevertheless, an optimal solution (b^*, π^*, f^*) for the PTM instance with value $\frac{24}{11}$ is given by

$$b_u^* := \begin{cases} 1 & \text{if } u = E_1 \\ -1 & \text{if } u = X_2 \\ 0 & \text{if } u \in \{X_1, E_2\} \end{cases} \quad \pi_u^* := \begin{cases} \frac{12}{11} & \text{if } u = E_1 \\ \frac{8}{11} & \text{if } u = E_2 \\ \frac{9}{11} & \text{if } u = X_1 \\ 0 & \text{if } u = X_2 \end{cases}$$

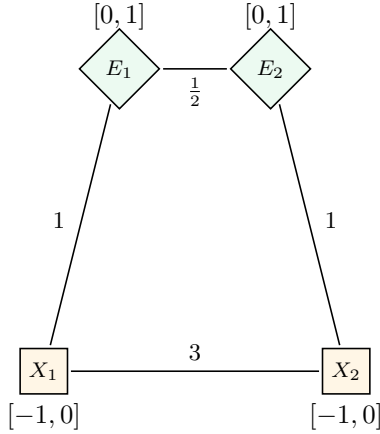


Figure 7: Flow network where optimal solutions for UMMCF and PTM differ.

$$f_a^* := \begin{cases} \frac{8}{11} & \text{if } a \in \{E_1E_2, E_2X_2\} \\ \frac{3}{11} & \text{if } a \in \{E_1X_1, X_1X_2\}. \end{cases}$$

7 Case Study: gaslib-582

We now present a short computational study based on the data from the `gaslib-582` network from the GasLib benchmark library [HJO⁺15]. The network topology and parameters are based on real data of a part of the German pipeline system, but slightly perturbed. In addition, it contains a collection of 4227 balanced transport situations, that were created with the methods described in Chapter 14 of [KHPS15] and should cover realistic scenarios well. For each of these situations, we know the feasibility *status*, with respect to the MILP model in Chapter 6 of [KHPS15]. The status can be *feasible*, if a solution to the transport problem was found, *infeasible*, if it could be proven that no such solution exists, or *no solution*, meaning that no solution was found within the time limit, but that it might still exist. In this sense, the infeasible situations are more severe than the feasible situations, taking into account the whole model including compressors, pressure bounds etc.

We have computed the minimum transport moment (2) for each of the situations and show a histogram in Figure 8 grouped by status. The solution values range from approximately 198847 up to 1122826. It is quite obvious that feasible situations have a lower minimum transport moment than infeasible situations, but there is also some overlap where situations which have the same value may be feasible or infeasible.

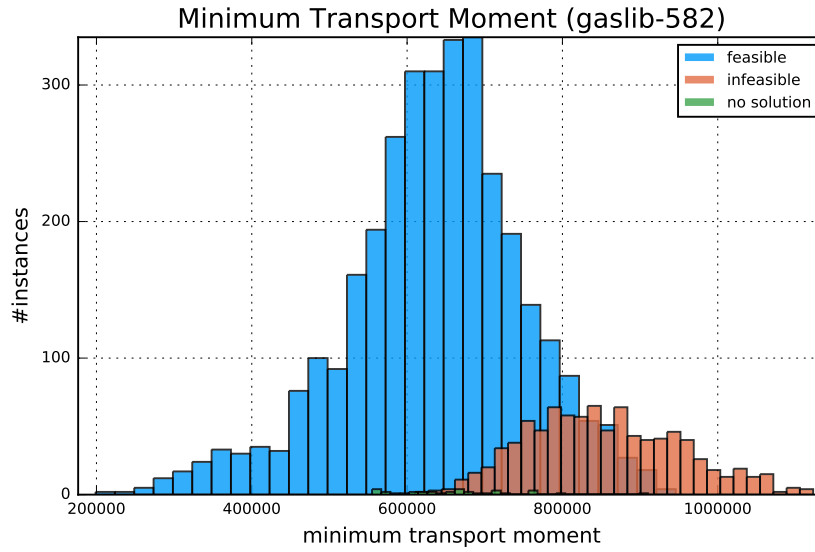


Figure 8: Histogram of minimum transport moment for scenarios in gaslib-582, grouped by feasibility status.

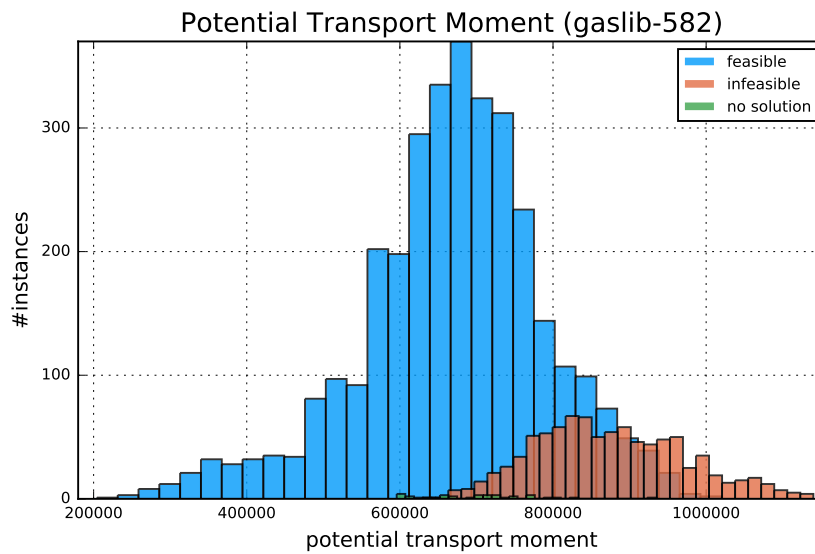


Figure 9: Histogram of potential transport moment for scenarios in gaslib-582, grouped by feasibility status.

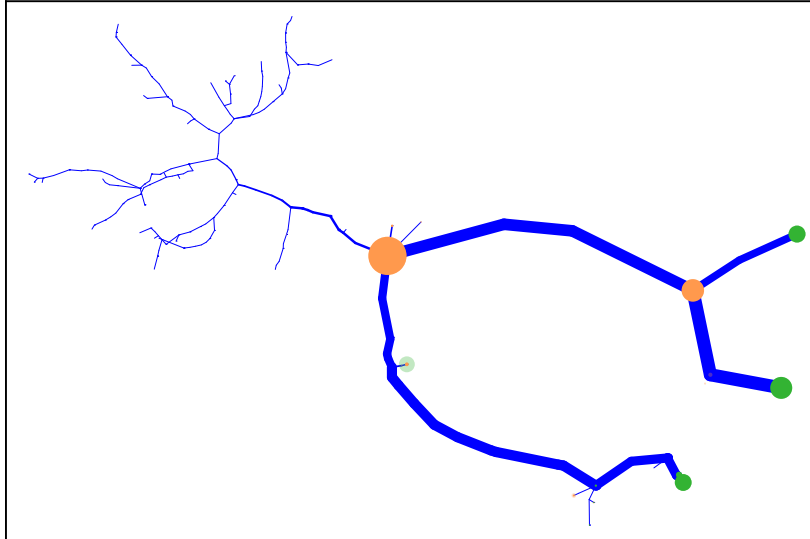


Figure 10: Optimal solution of UMMCF on gaslib-582.

Further, we have also computed the potential transport moment (6) for all scenarios. Here, the values range from approximately 204260 up to 1140021, and the histogram in Figure 9 looks almost identical to the one corresponding to the minimum transport moment.

In order to decide whether the benchmark library contains really severe transport situations, we solved the UMMCF problem as well as the PTM problem for the `gaslib-582` data. Unfortunately, no capacity bounds on the nodes are provided by GasLib. So we assumed bounds given by extremal values that occur in the 4227 situations.

Solving the PTM yields an optimal solution with a value of approximately 1456905, almost 30% larger than the maximum value in the data set. Using the demand and supply values from this solution we derive a minimum transport momentum of approximately 1375486. Solving UMMCF yields an optimal solution with value 1406674, which is about 23% larger than the maximum value from the data set. The Greedy-MCF generates a transport situation with value 1379907 and does therefore not find an optimal solution for UMMCF.

Both optimal solutions are visualized in Figures 10 and 11, respectively. Green and orange disks represent entry and exit nodes, respectively, and solid colors show the used capacity while transparent colors show the available but unused capacities. The flow on arcs is drawn in blue. Not many nodes are visible in these drawings. This is because although the network features 31 entries and 129 exits, only 85 have nonzero in- or outflow and often a relatively small amount. The difference between the two optimal solutions is depicted in Figure 12. Only the flow, demand, and supply values differing by more than 10^{-6} are depicted.

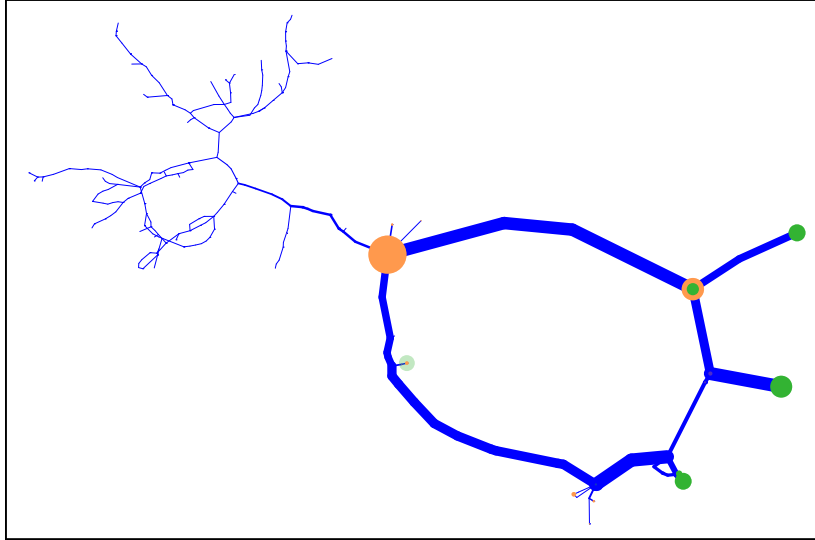


Figure 11: Optimal solution of PTM on gaslib-582.

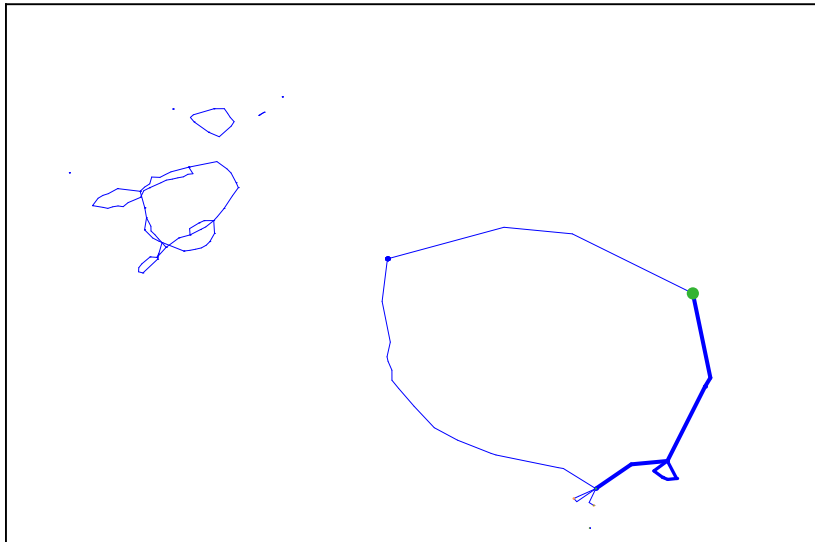


Figure 12: Difference between the optimal solutions of UMMCF and PTM.

8 Outlook

We have defined the UMMCF problem, with the application of severe transport situations in gas transmission in mind, and investigated the Greedy-MCF heuristic that fails to find an optimal solution in general. Further research could use the special structure of the network and reduce the model complexity with respect to the complementarity constraints when trying to solve UMMCF for even bigger networks. Maybe some valid bounds for the dual variables in the KKT reformulation could be derived and used for a formulation that uses binary variables and big- M constraints instead of SOS1 constraints.

Going back to the application, the use of transport moment as a measure of severity of a flow situation has itself some limitations. First, the physical details of how gas flow is distributed among the pipelines and the resulting pressure drop are grossly simplified. Here the potential transport moment can be a first step towards a more detailed view. Its close relation to the minimum transport moment should be investigated further. Additionally, the impact and capacity of active network elements such as valves and compressors are ignored completely, even though they play a very important role in the actual operation. In particular with compressors, there is often a lower bound on flow throughput required when using them. With this knowledge in mind, we cannot claim in general that if a specific (severe) flow situation is feasible, all scaled down situations are feasible, too.

A more accurate replacement for the transport moment might be the total cost of compression required to realize the transport of flow and satisfy all pressure bounds. This cost is related to the used power, a highly nonlinear and non-convex function of pressure ratio and flow. Due to this nonconvexity, the KKT reformulation in the context of a Bilevel Optimization formulation would not be directly applicable. Further, a model including the relevant pressure bounds might become infeasible.

Indicators for the severity of situations are also investigated in [ST16], with the goal of predicting the feasibility status of given, fixed situations, when solving expensive optimization problems.

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