

SEBASTIAN SCHENKER, RALF BORNDÖRFER, MARTIN SKUTELLA

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Zuse Institute Berlin  
Takustr. 7  
14195 Berlin  
Germany

Telephone: +49 30-84185-0  
Telefax: +49 30-84185-125

E-mail: [bibliothek@zib.de](mailto:bibliothek@zib.de)  
URL: <http://www.zib.de>

ZIB-Report (Print) ISSN 1438-0064  
ZIB-Report (Internet) ISSN 2192-7782

# A novel partitioning of the set of non-dominated points

Sebastian Schenker<sup>1,2</sup>, Ralf Borndörfer<sup>1</sup>, and Martin Skutella<sup>2</sup>

<sup>1</sup>Zuse Institute Berlin, Takustr. 7, 14195 Berlin, Germany,  
{schenker, borndoerfer}@zib.de

<sup>2</sup>TU Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany  
{schenker, skutella}@math.tu-berlin.de

**Abstract.** We consider a novel partitioning of the set of non-dominated points for general multi-objective integer programs with  $k$  objectives. The set of non-dominated points is partitioned into a set of non-dominated points whose efficient solutions are also efficient for some restricted subproblem with one less objective; the second partition comprises the non-dominated points whose efficient solutions are inefficient for any of the restricted subproblems. We show that the first partition has the nice property that it yields finite rectangular boxes in which the points of the second partition are located.

**Mathematics Subject Classification:** 90C10, 90C29

## 1 Introduction

We consider *general* multi-objective (aka multi-criteria) integer programming problems where one is given  $k$  objectives  $c_1, \dots, c_k \in \mathbb{R}^n$  and seeks to minimize all objectives  $c_i$ , for  $i = 1, \dots, k$ , simultaneously over all vectors  $x \in \mathbb{Z}^n$  subject to a set of linear inequality constraints (see Fig. 1). In particular, let  $M$  be some finite index set and suppose that for every  $i \in M$  we are given an  $n$ -dimensional vector  $a_i$  and a scalar  $b_i$ . We then consider the multi-objective problem

$$\begin{aligned} \min & (c_1^T x, \dots, c_k^T x) \\ \text{s.t.} & a_i^T x \leq b_i, \quad i \in M, \\ & x \in \mathbb{Z}^n. \end{aligned} \tag{MOP}$$

In the following let  $[k] := \{1, \dots, k\}$  with  $k \in \mathbb{N}$ . A feasible point  $y^* = (c_1^T x^*, \dots, c_k^T x^*)$  and its corresponding feasible solution  $x^*$  is *efficient* if there is no other feasible point

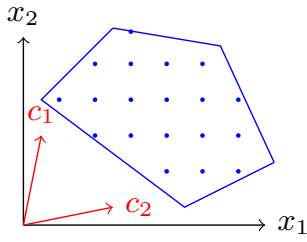


Figure 1: Feasible space of a bi-criteria integer minimization problem.

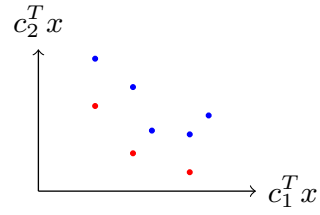


Figure 2: Image in objective space with non-dominated points (red) and dominated points (blue).

$y$  with  $y_i \leq y_i^*$  for  $i = 1, \dots, k$  and  $y_j < y_j^*$  for at least one  $j \in [k]$ . For a set of dominated and non-dominated points of a bi-objective integer program see Fig. 2. The challenge given a multi-objective integer program lies then in computing the set of all non-dominated points which will be denoted by  $\mathcal{Y}_N$ . Efficient solutions and non-dominated points, respectively, that are optimal for a weighted sum of the original objectives are called *supported*. For a more detailed introduction to multi-criteria optimisation we refer the reader to [4].

In what follows, we also consider the  $k - 1$  objective subproblem of (MOP)

$$\begin{aligned} \min & (c_1^T x, \dots, c_{i-1}^T x, c_{i+1}^T x, \dots, c_k^T x) \\ \text{s.t.} & a_i^T x \leq b_i, \quad i \in M, \\ & x \in \mathbb{Z}^n, \end{aligned} \tag{MOP- $i$ }$$

for some  $i \in [k]$ .

Determining whether a feasible point is non-dominated is NP-hard [7, 9] and even for problems with 2 objectives the set of non-dominated points can already be exponential in the input size [5]. However, the amazing progress in solver technology during the last decade [1] makes it nowadays possible to solve formerly intractable problems. This development has sparked a new interest in tackling multi-objective integer programs [6, 2, 3].

## 2 Partitioning the set of non-dominated points

A well-known partitioning of the set of non-dominated points is given by the set of supported non-dominated points and the set of unsupported non-dominated points. For problems with two objectives neighboring supported non-dominated points yield triangles in which potential unsupported non-dominated points are located. Hence, in the bi-objective case, a valid and often used approach is to compute the set of supported non-dominated points in a first phase and the set of unsupported points in a second phase [8].

It is not clear how to extend this approach to problems with more than two objectives because, in general, the supported non-dominated points do not give upper bounds on

the values of unsupported non-dominated points anymore (see Example 2.1).

**Example 2.1.** *The 3-objective integer program*

$$\begin{aligned}
& \min \left( \sum_{i=1}^4 \sum_{j=1}^4 c_{1_{ij}} x_{ij}, \sum_{i=1}^4 \sum_{j=1}^4 c_{2_{ij}} x_{ij}, \sum_{i=1}^4 \sum_{j=1}^4 c_{3_{ij}} x_{ij} \right) \\
& \text{s.t.} \\
& \quad \sum_{i=1}^4 x_{ij} = 1 \quad \forall j \in [4] \\
& \quad \sum_{j=1}^4 x_{ij} = 1 \quad \forall i \in [4] \\
& \quad \sum_{i=1}^4 \sum_{j=1}^4 c_{2_{ij}} x_{ij} \leq 15 \\
& \quad x_{ij} \in \{0, 1\}
\end{aligned}$$

with

$$c_1 = \begin{pmatrix} 3, 6, 4, 5 \\ 2, 3, 5, 4 \\ 3, 5, 4, 2 \\ 4, 5, 3, 6 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 2, 3, 5, 4 \\ 5, 3, 4, 3 \\ 5, 2, 6, 4 \\ 4, 5, 2, 5 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 4, 2, 4, 2 \\ 4, 2, 4, 6 \\ 4, 2, 6, 3 \\ 2, 4, 5, 3 \end{pmatrix}$$

has the three supported non-dominated points  $y^1 = [11, 11, 14]$ ,  $y^2 = [15, 9, 17]$ ,  $y^3 = [19, 14, 10]$ . It also has (among others) the unsupported non-dominated point  $y^4 = [17, 15, 11]$  whose second coordinate is not bounded from above by any of the second coordinates of  $y^1, y^2, y^3$  (see also Fig. 3)

Hence, approaches for three (or more) objectives generally do not distinguish/partition the set of non-dominated points into subsets with different corresponding properties. They usually compute arbitrary non-dominated points consecutively and discard dominated areas of the solution space until the solution space cannot contain any potential non-dominated point anymore.

In what follows, we present a novel partitioning of the set of non-dominated points: the first partition corresponds to the set of non-dominated points of (MOP) whose corresponding efficient solutions are also efficient for (MOP- $j$ ) for some  $i \in [k]$  and the second partition comprises the non-dominated points of (MOP) whose corresponding efficient solutions are inefficient for (MOP- $j$ ) for all  $i \in [k]$ . We will show that the first set of non-dominated points will bound the objective values of the non-dominated points in the second partition.

**Definition 2.1.** *Let  $i \in [k]$ . Then we define  $\mathcal{N}_{-i} \subseteq \mathcal{Y}_N$  to be the set of non-dominated points of (MOP) whose corresponding efficient solutions are also efficient for (MOP- $j$ ).*

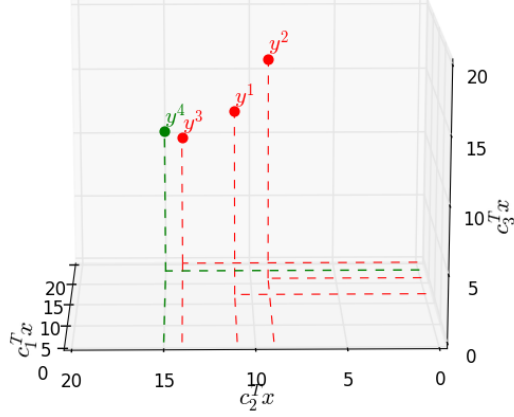


Figure 3: Unsupported non-dominated point that is not inside any potential box given by supported non-dominated points

**Remark 2.1.** Equivalently,  $\mathcal{N}_{-i}$  can be defined to be the set of non-dominated points of  $\mathcal{Y}_N$  which are also non-dominated for the projection of  $\mathcal{Y}_N$  onto the  $1, \dots, i-1, i+1, \dots, k$ -th coordinates.

Note that for  $i, j \in [k]$  with  $i \neq j$ , in general,  $\mathcal{N}_{-i} \cap \mathcal{N}_{-j} \neq \emptyset$  holds.

**Example 2.2.** Let  $\mathcal{N} := \{s = [11, 11, 14], t = [13, 16, 11], u = [19, 14, 10], v = [14, 14, 13]\}$  be the set of all non-dominated points for some 3-objective integer program. Then  $\mathcal{N}_{-1} = \{s, u\}$ ,  $\mathcal{N}_{-2} = \{s, t, u\}$  and  $\mathcal{N}_{-3} = \{s\}$ .

The first partition of non-dominated points will be given by the union of  $\mathcal{N}_{-i}$  over all  $i \in [k]$ . To simplify notation for what follows, we denote it by  $\mathcal{N} := \bigcup_{i=1}^k \mathcal{N}_{-i}$ . We now define the second partition.

**Definition 2.2.** Let  $\bar{\mathcal{N}} = \mathcal{Y}_N \setminus \mathcal{N}$  be the set of non-dominated points of (MOP) whose corresponding efficient solutions are inefficient for (MOP- $i$ ) for  $i = 1, \dots, k$ .

**Example 2.3.**  $\bar{\mathcal{N}} = \{v\}$  in the previous example with  $\mathcal{Y}_N := \{s = [11, 11, 14], t = [13, 16, 11], u = [19, 14, 10], v = [14, 14, 13]\}$ .

**Theorem 2.1.** Let  $\bar{y} \in \bar{\mathcal{N}}$ . Then there are  $y^1 \in \mathcal{N}_{-1}, \dots, y^k \in \mathcal{N}_{-k}$  such that

$$\max_{j \in [k] \setminus \{i\}} \{y_i^j\} \leq \bar{y}_i < y_i^i$$

for  $i = 1, \dots, k$ .

*Proof.* Assume  $\bar{y} \in \bar{\mathcal{N}}$  and let  $\bar{x}$  be the corresponding efficient solution for (MOP). By assumption,  $\bar{x}$  is inefficient for (MOP<sub>-i</sub>) for  $i = 1, \dots, k$ . Hence, for  $i = 1, \dots, k$ , there is  $y^i \in \mathcal{N}_{-i}$  with  $y^i \neq \bar{y}$  such that  $y_j^i \leq \bar{y}_j$  for  $j \in [k] \setminus \{i\}$ . Since  $\bar{y}$  is a non-dominated point of (MOP),  $y_i^i > \bar{y}_i$  must hold. Combining these inequalities we get

$$\begin{aligned} \max\{y_1^2, \dots, y_2^k\} &\leq \bar{y}_1 < y_1^1 \\ \max\{y_2^1, y_2^3, \dots, y_2^k\} &\leq \bar{y}_2 < y_2^2 \\ &\vdots \\ \max\{y_k^1, \dots, y_k^{k-1}\} &\leq \bar{y}_k < y_k^k \end{aligned}$$

□

Theorem 2.1 basically tells us that the non-dominated points in  $\mathcal{N}$  give us a lower bound as well as an upper bound on the objective values of the points in  $\bar{\mathcal{N}}$ .

**Example 2.4.** For  $v = [14, 14, 13]$  it holds that

$$\begin{aligned} \max\{t_1, s_1\} &\leq v_1 < u_1 \\ \max\{s_2, u_2\} &\leq v_2 < t_2 \\ \max\{t_3, u_3\} &\leq v_3 < s_3 \end{aligned}$$

with  $u = [19, 14, 10] \in \mathcal{N}_{-1}$ ,  $t = [13, 16, 11] \in \mathcal{N}_{-2}$  and  $s = [11, 11, 14] \in \mathcal{N}_{-3}$ .

**Definition 2.3.** Let  $B(y^1, \dots, y^k) := \prod_{i=1}^k [\max_{j \in [k] \setminus \{i\}} \{y_j^i\}, y_i^i)$  be the  $k$ -ary Cartesian product of half-open intervals given by  $y^i \in \mathcal{N}_{-i}$  for  $i = 1, \dots, k$ . We will call  $B(y^1, \dots, y^k)$  a (rectangular) box.

We show next that none of the points in  $\mathcal{N}$  is contained in any of the rectangular boxes given by points in  $\mathcal{N}$ .

**Theorem 2.2.** Let  $B(y^1, \dots, y^k)$  be a rectangular box given by some  $(y^1 \in \mathcal{N}_{-1}, \dots, y^k \in \mathcal{N}_{-k})$ . Then, for all  $y \in \mathcal{N}$ , we have  $y \notin B(y^1, \dots, y^k)$ .

*Proof.* None of the  $y^1, \dots, y^k$  is contained in  $B(y^1, \dots, y^k)$  since  $B(y^1, \dots, y^k)$  is the  $k$ -ary Cartesian product of half-open intervals excluding the  $i$ -th component of  $y^i$  for  $i = 1, \dots, k$ . Now assume that there is  $y^* \in \mathcal{N} \setminus \{y^1, \dots, y^k\}$  with  $y^* \in B(y^1, \dots, y^k)$ . Then, for  $i = 1, \dots, k$ ,

$$\max_{j \in [k] \setminus \{i\}} \{y_j^i\} \leq y_i^* < y_i^i$$

holds. This implies, for  $i = 1, \dots, k$ , that

$$y_j^i \leq y_j^* \text{ for all } j \in [k] \setminus \{i\}.$$

In other words, for  $i = 1, \dots, k$ , either  $y_j^i = y_j^*$  for all  $j \in [k] \setminus \{i\}$  or  $y_j^i \leq y_j^*$  for all  $j \in [k] \setminus \{i\}$  with a strict inequality for at least one  $j \in [k] \setminus \{i\}$ . Now assume there is  $i \in [k]$  such that  $y_j^i = y_j^*$  for all  $j \in [k] \setminus \{i\}$ . Then  $y^*$  dominates  $y^i$  because we also have  $y_i^* < y_i^i$  contradicting  $y^i \in \mathcal{N}_{-i}$ . On the other hand, assuming that for  $i = 1, \dots, k$  there is  $j \in [k] \setminus \{i\}$  such that  $y_j^i < y_j^*$  in addition to  $y_j^i \leq y_j^*$ , for all  $j \in [k] \setminus \{i\}$ , implies that  $y^* \notin \mathcal{N}$ .  $\square$

## Acknowledgements

The authors would like to acknowledge that this research was funded by the German Research Foundation DFG via the Collaborative Research Center 1026 (Sonderforschungsbereich 1026).

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