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This work has been supported by the Research Campus MODAL *Mathematical Optimization and Data Analysis Laboratories* funded by the Federal Ministry of Education and Research (BMBF Grant 05M14ZAM). All responsibilty for the content of this publication is assumed by the authors.

ZIB Report 16-58 (November 2016)

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URL: http://www.zib.de

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

Verifying Integer Programming Results

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November 20, 2016

Abstract

Software for mixed-integer linear programming can return incorrect results for a number of reasons, one being the use of inexact floating-point arithmetic. Even solvers that employ exact arithmetic may suffer from programming or algorithmic errors, motivating the desire for a way to produce independently verifiable certificates of claimed results. Due to the complex nature of state-of-the-art MILP solution algorithms, the ideal form of such a certificate is not entirely clear. This paper proposes such a certificate format, illustrating its capabilities and structure through examples. The certificate format is designed with simplicity in mind and is composed of a list of statements that can be sequentially verified using a limited number of simple yet powerful inference rules. We present a supplementary verification tool for compressing and checking these certificates independently of how they were created. We report computational results on a selection of mixed-integer linear programming instances from the literature. To this end, we have extended the exact rational version of the MIP solver SCIP to produce such certificates.

Keywords: correctness, verification, proof, certificate, optimality, infeasibility, mixed-integer linear programming

1 Introduction

Let (MIP) denote the following mixed-integer linear programming (MILP) problem:

$$\begin{array}{ll} \min & c^{\mathsf{T}} x \\ \text{s.t.} & \mathcal{C}(x) \\ & x \in \mathbb{R}^n \\ & x_i \in \mathbb{Z} \ \forall \ i \in I \end{array}$$

where n is a positive integer, $c \in \mathbb{Q}^n$, $\mathcal{C}(x)$ is a finite set of linear constraints on x with rational coefficients, and $I \subseteq \{1, \ldots, n\}$.

Methods and techniques for solving instances of (MIP) have been developed over the years. Solvers can now routinely handle large-scale instances. As the size of the instances that can be solved and the complexity of the solvers increase, a question emerges: How does one know if the computational results are correct?

Though rare, MILP solvers do occasionally return incorrect or dubious results (see [11]). Despite such errors, maintaining a skeptical attitude that borders on paranoia is arguably neither healthy nor practical. After all, machines do outperform humans on calculations by many orders of magnitude and many tasks in life are now entrusted to automation.

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Hence, the motivation for asking how to verify correctness of computational results is not necessarily because of an inherent distrust of solvers. Rather, it is the desire to seek ways to identify and reduce errors and to improve confidence in the computed results. After all, if one believes in Murphy's Law, errors tend to arise when one can least afford to have errors. Therefore, there is a practical need to improve the robustness and correctness of solvers. Previous research on computing accurate solutions to MIPs has utilized various techniques including interval arithmetic [33], exact rational arithmetic [4, 11, 15], and safely derived cuts [10]. Nevertheless, correctness cannot be guaranteed as attempts along this line still fall short of provably-correct code using formal methods that are found in mission-critical applications such as software for medical applications and avionics. Incidentally, a footnote in [11] states: "even with a very careful implementation and extensive testing, a certain risk of an implementation error remains".

One way to satisfy skeptics is to build solvers that output extra information that facilitates independent checking. We shall use the word *certificate* to refer to such extra information for a given problem that has been solved. Ideally, the certificate should allow for checking the results using fewer resources than what are needed to solve the problem from scratch. Such a certificate could in principle be used in formal verification using a proof checker as done in the Flyspeck Project [17, 35, 37] for a formal proof of Kepler's Conjecture, or informal verification as done by Applegate *et al.* [3] for the Traveling Salesman Problem and by Carr *et al.* [9] in their unpublished work for MILP in general. Naturally, certificates should be as simple to verify as possible if they are to be convincing.

We highlight two specific applications where solution verification is desirable. First, Achterberg [1] presented MILP formulations for circuit design verification problems. For design verification, obtaining correct results is of critical importance; yet floating-point based solvers have been shown to return incorrect results on some of these instances [11]. Second, Pulaj [36] has recently used integer programming models to settle some open questions related to Frankl's conjecture (the union-closed sets conjecture). This, and other applications in pure mathematics, are also cases where correctness of the results is of the utmost importance. Software developed in connection with our paper has been successfully used to generate and check certificates for MILPs coming from both of these applications.

For linear programming (LP), duality theory tells us that an optimal primal solution and an optimal dual solution are sufficient to facilitate effective verification of optimality. In the case of checking infeasibility, a Farkas certificate will do. Therefore, verifying LP results, at least in the case when exact rational arithmetic is used, is rather straightforward. However, the situation with MILP is drastically different. From a theoretical perspective, even though some notions of duality for MILP have been formulated (see the survey [21]), small (i.e. polynomial size) certificates for infeasibility or optimality of MILPs may not even exist. As a result, there are many forms that certificates could take: a branch-and-bound tree, a list of derived cutting planes, a superadditive dual function, or other possibilities for problems with special structures such as pure integer linear programming and binary programming (see, for example, [8, 14, 28, 30]). Which format would be preferred for certificate verification is not entirely clear, and in this paper we provide reasoning behind our choice.

From a software perspective, MILP result certification is also considerably more complicated than LP certification. Even though most solvers adopt the branch-and-cut paradigm, they typically do not make the computed branch-and-bound tree or generated cuts readily available, and they may also utilize many other techniques including constraint propagation, conflict analysis, or reduced cost fixing. Thus, even if a solver did print out all information used to derive its solution, a verifier capable of interpreting such information would itself be highly complex, contradicting our desire for a simple verifier. As a result, other than accepting the results of an exact solver such as [11], the best that many people can do today to "verify" the results of a solver on a MILP instance is to solve the instance by several different solvers and check if the results match or minimally check that an opti-

mal solution is indeed feasible and has the correct objective function value as done by the solution checker in [29].

The main contribution of this paper is the development of a certificate format for the verification of mixed-integer linear programs. Compared to the previous work of Applegate *et al.* [3] for the Traveling Salesman Problem and the unpublished work of Carr *et al.* [9] for general MILP, our certificate format has a significantly simpler structure. It consists of a sequence of statements that can be verified one by one using simple inference rules, facilitating verification in a manner akin to natural deduction. The approach is similar to that for verification of unsatisfiability proofs for SAT formulas. (See for example [24, 38].) This simple certificate structure makes it easier for researchers to develop their own independent certificate verification programs, or check the code of existing verifiers, even without any expert knowledge of MILP solution algorithms.

To demonstrate the utility of the proposed certificate format, we have developed a reference checker in C++ and added the capability to produce such certificates to the exact MIP solver [11] of SCIP [18]. We used these tools to verify results reported in [11]. To the best of our knowledge, this work also represents the first software for general MILP certificate verification that has been made available to the mathematical optimization community.

Organization of the paper. Even though the proposed format for the certificate is rather straightforward, some of the details are nevertheless technical. Therefore, in this paper, we discuss the certificate format at a conceptual level. The full technical specification is found in the accompanying computer files. We begin with the necessarily ingredients starting with the simple case of LP in Section 2. In Section 3, the ideas for dealing with LP are extended to pure integer linear programming. The full conceptual description of the format of the certificate is then given in Section 4. Computational experiments are reported in Section 5, and concluding remarks are given in Section 6.

Accompanying computer files. Software, technical documentation and certificate files are available at: `https://github.com/ambros-gleixner/VIPR`

2 Certificates for linear programming

Throughout this paper, we assume that problems are specified and solved with exact rational arithmetic.

A typical certificate of optimality for an LP is a dual-feasible solution whose objective function value matches the optimal value. However, there is no need to specify the dual when one views the task of certification as an inference procedure. (For instance, see [25].) To facilitate discussion, we call this inference procedure *linear inequality inference* and the details are as follows.

Let (S) denote the system of linear constraints:

$$\begin{aligned} Ax &\geq b \\ A'x &\leq b' \\ A''x &= b''. \end{aligned}$$

Here, x is a vector of variables, $A \in \mathbb{R}^{m \times n}$, $A' \in \mathbb{R}^{m' \times n}$, $A'' \in \mathbb{R}^{m'' \times n}$, $b \in \mathbb{R}^m$, $b' \in \mathbb{R}^{m'}$, and $b'' \in \mathbb{R}^{m''}$ for some nonnegative integers n, m, m', and m''.

We say that $c^\mathsf{T} x \geq v$ is obtained by taking a *suitable linear combination* of the constraints in (S) if

$$c^\mathsf{T} = d^\mathsf{T} A + d'^\mathsf{T} A' + d''^\mathsf{T} A'', \;\; v = d^\mathsf{T} b + d'^\mathsf{T} b' + d''^\mathsf{T} b''$$

for some $d \in \mathbb{R}^m$, $d' \in \mathbb{R}^{m'}$, and $d'' \in \mathbb{R}^{m''}$ with $d \geq 0$ and $d' \leq 0$. Clearly, if x satisfies (S), then it necessarily satisfies $c^\mathsf{T} x \geq v$. We say that the inequality $c^\mathsf{T} x \geq v$ is *inferred* from (S).

Remarks.

1. Together, d, d', d'' is simply a feasible solution to the linear programming dual of the linear program (LP) given by:

$$\begin{array}{ll} \min & c^{\mathsf{T}}x \\ \text{s.t.} & Ax \geq b \\ & A'x \leq b' \\ & A''x = b''. \end{array}$$

2. Hooker [25] calls $c^{\mathsf{T}}x \geq v$ a *surrogate* of (S).

Suppose that an optimal solution to (LP) exists and the optimal value is v. Linear programming duality theory guarantees that $c^{\mathsf{T}}x \geq v$ can be inferred from (S). Therefore, linear inequality inference is sufficient to certify optimality for linear programming. Conceptually, the certificate that we propose is a listing of the constraints in (S) followed by the inequality $c^{\mathsf{T}}x \geq v$ with the associated multipliers used in the inference as illustrated in the following example.

Example 1. The following shows an LP problem and its associated certificate.

$$\begin{array}{ll} \min & 2x + y \\ \text{s.t.} & \\ C1: & 5x - y \geq 2 \\ C2: & 3x - 2y \leq 1. \end{array}$$

Given		
$C1:$	$5x - y \geq 2$	
$C2:$	$3x - 2y \leq 1$	
Derived		**Reason**
`obj`$:$	$2x + y \geq 1$	$\{1 \times C1 + (-1) \times C2\}$

Here, $C1$ and $C2$ are constraint labels. Taking the suitable linear combination $1 \times C1 + (-1) \times C2$ gives $2x + y \geq 1$, thus establishing that 1 is a lower bound for the optimal value.

We remark that this type of linear inference could also be used to derive $\leq$-inequalities or equality constraints. Assuming that all problem data is rational, rational multipliers are sufficient to certify infeasibility or optimality.

3 Handling Chvátal-Gomory cutting planes

Gomory [20] showed in theory that, for pure integer linear programming (ILP), optimality or infeasibility can be established by a pure cutting-plane approach. Such an approach can also work in practice (see [6, 39]). In addition to linear inequality inference, a rounding operation is needed.

Suppose that $c^{\mathsf{T}}x \geq v$ can be inferred from (S) by taking a suitable linear combination of the constraints. If $c_i \in \mathbb{Z}$ for $i \in I$ for some $I \subseteq \{1, \ldots, n\}$ and $c_i = 0$ for $i \notin I$, then any $x \in \mathbb{R}^n$ satisfying (S) with $x_i \in \mathbb{Z}$ for $i \in I$ must also satisfy $c^{\mathsf{T}}x \geq \lceil v \rceil$. We say that $c^{\mathsf{T}}x \geq \lceil v \rceil$ is obtained from $c^{\mathsf{T}}x \geq v$ by *rounding*.

When $I = \{1, \ldots, n\}$, the inequality $c^{\mathsf{T}}x \geq \lceil v \rceil$ is known as a *Chvátal-Gomory cut* (CG-cut in short). It can then be added to the system and the process of obtaining another CG-cut can be repeated.

Conceptually, a certificate for an ILP instance solved using only CG-cuts can be given as a list of the original constraints followed by the derived constraints.

Example 2. The following shows an ILP problem and its associated certificate.

$$\begin{array}{ll} \min & x+y \\ \text{s.t.} & \\ C1: & 4x+y \geq 1 \\ C2: & 4x-y \leq 2 \\ & x,y \in \mathbb{Z} \end{array}$$

Given		
	$x, y \in \mathbb{Z}$	
$C1:$	$4x+y \geq 1$	
$C2:$	$4x-y \leq 2$	
Derived		**Reason**
$C3:$	$y \geq -\frac{1}{2}$	$\left\{\frac{1}{2} \times C1 + \left(-\frac{1}{2}\right) \times C2\right\}$
$C4:$	$y \geq 0$	{round up $C3$}
$C5:$	$x+y \geq \frac{1}{4}$	$\left\{\frac{1}{4} \times C1 + \frac{3}{4} \times C4\right\}$
$C6:$	$x+y \geq 1$	{round up $C5$}

Note that the derived constraints in the certificate can be processed in a sequential manner. In the next section, we see how to deal with branching without sacrificing sequential processing.

4 Branch-and-cut certificates

In practice, most instances of (MIP) are not solved by cutting planes alone. Thus, certificates as described in the previous section are of limited utility.

We now propose a type of certificate for optimality or infeasibility established by a branch-and-cut procedure in which the generated cuts at any node can be derived as split cuts and branching is performed on a disjunction of the form $a^\mathsf{T}x \leq \delta \vee a^\mathsf{T}x \geq \delta + 1$ where $\delta \in \mathbb{Z}$ and $a^\mathsf{T}x$ is integral for all feasible x.

The use of split disjunctions allows us to consider branching and cutting under one umbrella. Many of the well-known cuts generated by MILP solvers can be derived as split cuts (see [12]) and they are effective in closing the integrality gap in practice (see [16]). Branching typically uses only simple split disjunctions (where the a above is a unit vector), although some studies have considered the computational performance of branching on general disjunctions [13, 27, 34].

Recall that each branching performed splits the solution space into two subcases. At the end of a branch-and-bound (or branch-and-cut) procedure, each leaf of the branch-and-bound tree corresponds to one of the cases and the leaves together cover all the cases needed to be considered. Hence, if the branch-and-bound tree is valid, all one needs to look at are the LP results at the leaves.

Our proposal is to "flatten" the branch-and-bound tree into a list of statements that can be verified sequentially. Thus, our approach departs from the approach in [3] and [9] which requires explicit handling of the branch-and-bound tree. The price we pay is that we can no longer just examine the leaves of the tree. Instead, we process the nodes in a bottom-up fashion and discharge assumptions as we move up towards the root. We illustrate the ideas with an example.

Example 3. It is known that the following has no solution.

$$\begin{array}{rr} C1: & 2x_1 + 3x_2 \geq 1 \\ C2: & 3x_1 - 4x_2 \leq 2 \\ C3: & -x_1 + 6x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Note that $(x_1, x_2) = (\frac{10}{17}, -\frac{1}{17})$ is an extreme point of the region defined by $C1$, $C2$, and $C3$. Branching on the integer variable x_1 leads to two cases:

1. $A1: \ x_1 \leq 0$;
2. $A2: \ x_1 \geq 1$.

We consider each case in turn.

Case 1. $A1:\ x_1 \leq 0$

Note that $(x_1, x_2) = (0, \frac{1}{3})$ satisfies $C1, C2, C3, A1$. We branch on x_2:

Case 1a. $A3:\ x_2 \leq 0$

Taking $C1 + (-2) \times A1 + (-3) \times A3$ gives the absurdity $C4:\ 0 \geq 1$.

Case 1b. $A4:\ x_2 \geq 1$

Taking $\left(-\frac{1}{3}\right) \times C3 + \left(-\frac{1}{3}\right) \times A1 + 2 \times A4$ gives the absurdity $C5:\ 0 \geq 1$.

Case 2. $A2:\ x_1 \geq 1$

Taking $\left(-\frac{1}{4}\right) \times C2 + \left(\frac{3}{4}\right) \times A2$ gives $C6:\ x_2 \geq \frac{1}{4}$. Rounding gives $C7:\ x_2 \geq 1$.
Taking $\left(-\frac{1}{3}\right) \times C2 + (-1) \times C3 + \frac{14}{3} \times C7$ gives the absurdity $C8:\ 0 \geq 1$.

As all cases lead to $0 \geq 1$, we conclude that there is no solution. To issue a certificate as a list of derived constraints, we need a way to specify the different cases. To this end, we allow the introduction of constraints as assumptions.

Figure 1 shows conceptual certificate for the problem. Notice how the constraints $A1$, $A2$, $A3$, and $A4$ are introduced to the certificate as assumptions. Since we want to end with $0 \geq 1$ without additional assumptions attached, we get there by gradually undoing the case-splitting operations. We call the undoing operation *unsplitting*. For example, $C4$ and $C5$ are both the absurdity $0 \geq 1$ with a common assumption $A1$. Since $A3 \vee A4$ is true for all feasible x, we can infer the absurdity $C9:\ 0 \geq 1$ assuming only $A1$ in addition to the original constraints. We say that $C9$ is obtained by *unsplitting* $C4, C5$ on $A3, A4$. Similarly, both $C8$ and $C9$ are the absurdity $0 \geq 1$ and $A2 \vee A1$ is true for all feasible x, we can therefore unsplit on $C8, C9$ on $A2, A1$ to obtain $C10:\ 0 \geq 1$ without any assumption in addition to the original constraints.

Given			
	$x, y \in \mathbb{Z}$		
$C1:$	$2x_1 + 3x_2 \geq 1$		
$C2:$	$3x_1 - 4x_2 \leq 2$		
$C3:$	$-x_1 + 6x_2 \leq 3$		
Derived		**Reason**	**Assumptions**
$A1:$	$x_1 \leq 0$	{assume}	
$A2:$	$x_1 \geq 1$	{assume}	
$A3:$	$x_2 \leq 0$	{assume}	
$C4:$	$0 \geq 1$	$\{C1 + (-2) \times A1 + (-3) \times A3\}$	$A1, A3$
$A4:$	$x_2 \geq 1$	{assume}	
$C5:$	$0 \geq 1$	$\left\{\left(-\frac{1}{3}\right) \times C3 + \left(-\frac{1}{3}\right) \times A1 + 2 \times A4\right\}$	$A1, A4$
$C6:$	$x_2 \geq \frac{1}{4}$	$\left\{\left(-\frac{1}{4}\right) \times C2 + \left(\frac{3}{4}\right) \times A2\right\}$	$A2$
$C7:$	$x_2 \geq 1$	{round up $C6$}	$A2$
$C8:$	$0 \geq 1$	$\left\{\left(-\frac{1}{3}\right) \times C2 + (-1) \times C3 + \frac{14}{3} \times C7\right\}$	$A2$
$C9:$	$0 \geq 1$	{unsplit $C4, C5$ on $A3, A4$}	$A1$
$C10:$	$0 \geq 1$	{unsplit $C8, C9$ on $A2, A1$}	

Figure 1: Certificate for Example 3

In practice, the list of assumptions associated with each derived constraint needs not be pre-specified as it can be deduced on the fly by a checker. For example, when processing $C4$, we see that it uses $A1$ and $A3$, both of which are assumptions. Hence, we associate $C4$ with the list of assumptions $A1, A3$.

As any linear inequality can be introduced as an assumption, branching can in fact be on general disjunctions.

We now show conceptually how our proposed certificate format accommodates split cuts for (MIP). The discussion that follows focuses on split cuts at the root node but the ideas readily extend to split cuts at other nodes.

We say that $a^\mathsf{T}x \geq \beta$ *dominates* ${a'}^\mathsf{T}x \geq \beta'$ only if $a = a'$ and $\beta \geq \beta'$, or $a = 0$ and $\beta > 0$.

Let $a \in \mathbb{Z}^n$ be such that $a_i = 0$ for $i \notin I$. Let $\delta \in \mathbb{Z}$. Recall that the inequality $d^\mathsf{T}x \geq \beta$ is a split cut if it is valid for the convex hull of $\{x \in \mathbb{R}^n : a^\mathsf{T}x \leq \delta \vee a^\mathsf{T}x \geq \delta + 1,\ \mathcal{C}(x),\ x_i \in \mathbb{Z}\ \forall\ i \in I\}$. The split cut can then be specified in the certificate as follows: Introduce $A:\ a^\mathsf{T}x \leq \delta$ as an assumption and list $S:\ d^\mathsf{T}x \geq \beta$ with the associated reason being dominated by a suitable linear combination of A and $\mathcal{C}(x)$ Then introduce $A':\ a^\mathsf{T}x \geq \delta + 1$ as an assumption and list $S':\ d^\mathsf{T}x \geq \beta$ with the reason being dominated by a suitable linear combination of A' and $\mathcal{C}(x)$. Then list $C:\ d^\mathsf{T}x \geq \beta$ with $\{$unsplit S, S' on $A, A'\}$ as the reason. To verify C, the following conditions must be checked:

1. A and A' must be $a^\mathsf{T}x \leq \delta$ and $a^\mathsf{T}x \geq \delta + 1$, though not necessarily in that order, with $\delta \in \mathbb{Z}$ and $a_i \in \mathbb{Z}$ for all $i \in I$ and $a_i = 0$ for all $i \notin I$.
2. D dominates C and D' dominates C

The assumption list of C contains all the assumptions of D and D' except A and A'.

5 Computational experiments

In this section, we describe software developed to produce and check certificates for MILP results using the certificate format developed in this paper. Software and resources are provided at: `https://github.com/ambros-gleixner/VIPR`
We again emphasize that the format was designed with simplicity in mind; the certificate verification program we have provided is merely a reference and others should be able to write their own verifiers without much difficulty, even without knowledge of how MIP solvers operate.

The supporting documentation provides a more detailed technical description of the file format along with examples. A feature of the file format is that along with each derived constraint, the largest index of any derived constraint that references it is specified, thus allowing constraints to be freed from memory when they will no longer be needed as a certificate is being read and verified, leading to reduced memory consumption. The following C++ programs are provided:

- `viprchk`. A program that verifies MILP results provided in our specified file format. All computations are performed in exact rational arithmetic using the GMP library [19].
- `viprttn`. A program that performs simple modifications to "tighten" certificates. For each derived constraint, it computes the largest index over constraints referencing the target constraint to reduce memory required by a checker. It also has an option to remove unnecessary derived constraints.
- `vipr2html`. A program that converts a certificate file to a "human-readable" HTML file.

We also created a modified version of the exact rational MIP solver described in [11] and used it to compute certificates for several MIP instances from the literature. The exact rational MIP solver is based on SCIP [18] and uses a hybrid of floating-point and exact rational arithmetic to efficiently compute exact solutions using a pure branch-and-bound

Table 1: Aggregated computational results over 106 instances from [11].

		SCIP		SCIP+C		VIPR				
Test set	N	N_{sol}	t_{MIP}	N_{sol}	t_{MIP}	t_{ttn}	t_{chk}	size_{raw}	size_{ttn}	size_{gz}
easy-all	56	53	62.0	39	180.8	25.8	28.9	214	72	22
-solved	39	39	23.2	39	48.0	9.6	13.4	77	34	10
-memout	5	4	600.6	0	1769.4	377.5	169.8	10286	513	159
-timeout	12	10	357.4	0	3600.0	83.7	97.5	1151	368	108
hard-all	50	23	725.1	14	976.6	31.2	15.1	372	38	11
-solved	13	13	22.9	13	40.8	7.1	6.3	49	15	5
-memout	10	0	3600.0	0	1833.9	275.7	53.8	10269	146	39
-timeout	27	10	1811.1	1	3255.1	20.7	11.9	286	35	9

algorithm. In our experiments, the rational MIP solver uses CPLEX 12.6.0.0 [26] as its underlying floating-point LP solver and a modified version of QSOPT_EX 2.5.10 [5] as its underlying exact LP solver. The exact MIP solver supports several methods for computing valid dual bounds and our certificate printing functionality is currently supported by the *Project-and-shift* method (for dual solutions only) and the *Exact LP* method (for both dual solutions and Farkas proofs), for details on these methods see [11]. Future plans are to include certificate printing functionality in all dual bounding methods and release this in subsequent versions of exact SCIP; our developmental version is currently available from the authors by request.

In the following, we report some computational results on the time and memory required to produce and verify certificates. We considered the *easy* and *numerically difficult* (referred to here as '*hard*') test sets from [11]; these test sets consist of instances from well known libraries including [2, 7, 29, 31, 32].

Experiments were conducted on a cluster of Intel(R) Xeon(R) CPU E5-2660 v3 at 2.60GHz; jobs were run exclusively to ensure accurate time measurement. Table 1 reports a number of aggregate statistics on these experiments. The columns under the heading SCIP report information on tests using the exact version of SCIP, using its default dual bounding strategy. The columns under SCIP+C report on tests involving the version of exact SCIP that generates certificates as it solves instances; since certificate printing is not supported for all dual bounding methods it uses only a subset of the dual bounding methods, contributing to its slower speed. Columns under the heading VIPR report time and memory usage for certificate checking. For each of the easy and hard test sets, we report information aggregated into four categories: 'all' reports statistics over all instances; 'solved' reports over instances solved by both SCIP and SCIP+C within a 1 hour time limit and a 10GB limit on certificate file size, 'memout' and 'timeout' report on instances where one of the solvers experienced a filesize limit or timeout. All averages are reported as shifted geometric means with a shift of 10 sec. for time and 1MB for memory. The column N represents the number of instances in each category; N_{sol} represents the number in each category that were solved to optimality (or infeasibility) by a given solver; t_{MIP} represents the time (sec.) used to solve the instance and, when applicable, output a certificate; t_{ttn} is the time (sec.) required by the `viprttn` routine to tighten the certificate file; t_{chk} is the time (sec.) required to for `viprchk` to check the certificate file – on instances in the memout and timeout rows this represents the time to verify the primal and dual bounds present in the intermediate certificate printed before the solver was halted. The final three columns list the size of the certificate (in MB), before tightening, after tightening and then after being compressed to a gzipped file. The instance `nw04` is excluded from the easy test set due to a memout by the `viprttn` routine. Timings and memory usage for individual instances are available in a document hosted together with the accompanying software.

From this table, we can make a number of observations. First, there is a noticeable,

but not prohibitive, cost to generate the certificates. The differences in t_{MIP} between SCIP and SCIP+C are due to both the difference in dual bounding strategies, and the overhead for writing the certificate files. In some additional experiments, we observed that on the 39 instances in the easy-solved category, the file I/O amounted to roughly 7% of the solution time, based on this we believe that future modifications to the code will allow us to solve and print certificates in times much closer to those in the SCIP column. Perhaps most importantly, we observe that the time to check the certificates is significantly less than the time to solve the instances. Moreover, the certificate tightening program `viprttn` is able to make significant reductions in the certificate size, and the resulting certificate sizes are often surprisingly manageable.

6 Conclusion

This paper presented a certificate format for verifying integer programming results. We have demonstrated the practical feasibility of generating and checking such certificates on well-known MIP instances. We see this as the first step of many in verifying the results of integer programming solvers. We now discuss some future directions made possible by this work.

Even in the context of floating-point arithmetic, our certificate format could serve a number of purposes. Using methods described by [10, 33], directed rounding and interval arithmetic may allow us to compute and represent valid certificates exclusively using floating-point data, allowing for faster computation and smaller certificate size. Additionally, generating approximate certificates with inexact data could be used for debugging solvers, or measuring the maximum or average numerical violation over all derivations. In a more rigorous direction, one could also convert our certificates to a form that could be formally verified by a proof assistant such as HOL Light [22].

Finally, we note that there are many potential ways to further simplify or optimize the certificates, beyond what is done by our `viprttn` routine. One natural adjustment is to reorder the deduction steps to minimize memory requirement by a checker, but much more is possible in this direction.

Acknowledgements. We want to thank Kati Wolter for the exact version of SCIP [11] and Daniel Espinoza for QSopt_ex [5], which provided the basis for our computational experiments. We thank Gregor Hendel for his Ipet package [23], which was a big help in analyzing the experimental results. This work has been supported by the Research Campus MODAL *Mathematical Optimization and Data Analysis Laboratories* funded by the Federal Ministry of Education and Research (BMBF Grant 05M14ZAM). All responsibilty for the content of this publication is assumed by the authors.

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