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random variables**

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Zusammenfassung

Given a sequence of Cauchy-distributed random variables defined by a sequence of location parameters and a sequence of scale parameters, we consider another sequence of random variables that is obtained by perturbing the location or scale parameter sequences. Using a result of Kakutani on equivalence of infinite product measures, we provide sufficient conditions for the equivalence of laws of the two sequences.

1 Introduction

The absolute continuity of a measure with respect to a reference measure is a fundamental topic in probability theory, with important ramifications in many applications such as importance sampling and Bayesian inference. An important task in this context is the specification of sufficient conditions for absolute continuity, or equivalently for the existence of a Radon–Nikodým derivative. For example, Novikov’s and Kazamaki’s conditions for the law of a solution of stochastic differential equation (SDE) to be absolutely continuous with respect to the law of a solution of another SDE after a change of drift are standard results in the theory of continuous-time stochastic processes.

This article considers the equivalence of laws for sequences of Cauchy distributed random variables. If $\underline{U} = (U_n)_{n=1}^\infty$ is such a random sequence, specified by a sequence of location parameters $\underline{\delta}$ and a sequence $\underline{\gamma}$ of scale parameters, we consider conditions under which a perturbation of either $\underline{\delta}$ or $\underline{\gamma}$ results in a sequence \underline{V} of Cauchy random variables with law equivalent to that of \underline{U} . This work is motivated by the study of the stability of Bayesian inversion on infinite-dimensional Banach spaces with respect to perturbations of the prior, when the prior is a heavy-tailed, non-Gaussian distribution [6, 14]. Such priors may be sampled in the manner of a Karhunen–Loève expansion, i.e. via a random series expansion $\sum_{n=1}^\infty U_n \psi_n$ in a fixed basis $(\psi_n)_{n=1}^\infty$ of the Banach space (e.g. a wavelet basis of a Sobolev or Besov space), in which the coefficient sequence \underline{U} is a random sequence of the type considered here. We identify here some admissible perturbations for the corresponding sequences of location and scale parameters that yield equivalent laws in Theorems 4.1 and 4.2 respectively.

Equivalence of measures on infinite-dimensional spaces has been studied extensively in probability theory. If the reference measure is the Gaussian measure $\mathcal{N}(m_0, C_0)$ with mean m_0 and covariance operator C_0 , and the measure of interest is the Gaussian measure with mean $m_1 := m_0 + m$ and the same covariance

operator C_0 , then the Cameron–Martin theorem (see, e.g. [3, Theorem 2.4.5] and [13, Theorem 2.51]) states that the perturbed measure $\mathcal{N}(m_1, C_0)$ is equivalent to the original measure $\mathcal{N}(m_0, C_0)$ precisely when the translation m lies in the Cameron–Martin space $\text{ran } C_0^{1/2}$; otherwise, the two measures are mutually singular. For multiplicative perturbations, $\mathcal{N}(m_0, C_0)$ and $\mathcal{N}(m_0, \sigma C_0)$ are mutually equivalent precisely when $|\sigma| = 1$. The study of absolute continuity and singularity of probability measures on path spaces — including for stable processes — continues to be a topic of active research [4, 5, 7, 9, 11, 12]. See [1, Chapter 5] for changes of measure for Lévy processes. Such issues are important in applications to Bayesian inference, since fully assessing the robustness and reliability of the posterior requires perturbing not only the forward model and observed data [6], but also the prior [2, 10].

The results in this note here are similar in spirit to the Cameron–Martin theorem, but do not appear to have been treated in the literature. This may be due to the fact that we do not approach the problem from the perspective of stochastic processes, but instead treat the sequences \underline{U} and \underline{V} as random variables in their own right, and find sufficient conditions for these random variables to have mutually absolutely continuous laws. In particular, our investigation does not make use of transition probabilities or infinitesimal generators. Instead, we only consider additive perturbations of the location parameters and multiplicative perturbations of the scale parameters. Although we consider Cauchy-distributed random variables, we do not attempt to generalise the results we obtain here to the class of α -stable distributions for $\alpha \neq 1$, because the approach used here relies on the having a convenient representation of the density.

The outline of the paper is as follows. In Section 2, we formulate the problem of interest and state the main result that we shall use throughout this paper, namely a result of Kakutani on the equivalence or singularity of infinite product measures. We provide a brief preview of the main results of this paper by first considering the Gaussian case in Section 3. In Section 4.1, we consider the problem of providing sufficient conditions for equivalence when the sequence $\underline{\delta}$ of location parameters is additively perturbed; the main result is Theorem 4.1. In Section 4.2, we consider multiplicative perturbations of the sequence $\underline{\gamma}$, with the main result being Theorem 4.2.

The additive and multiplicative cases follow the same proof strategy: we define a series that depends at most on the parameter sequences $\underline{\delta}$, $\underline{\gamma}$ and the sequence of perturbations, such that convergence of the series implies absolute convergence of the series in Kakutani’s theorem; the desired equivalence follows. To find this dominating series, we rely on the Taylor series representation of logarithms and the structure of the density (4) of Cauchy-distributed random variables. Throughout, ℓ^p denotes the space of p -summable sequences for $p \geq 1$, and ℓ^∞ denotes the space of bounded sequences.

2 Problem formulation and Kakutani’s theorem

Let $\underline{U} = (U_n)_{n=1}^\infty$ and $\underline{V} = (V_n)_{n=1}^\infty$ be two sequences of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that, for each n , the law of each V_n is absolutely continuous with respect to the law of U_n , with Radon–Nikodým derivative φ_n , so that

$$\mathbb{E}[f(U_n)\varphi_n(U_n)] = \mathbb{E}[f(V_n)]$$

for all $f: \mathbb{R} \rightarrow \mathbb{R}$ that are integrable with respect to the law of V_n . It follows from the Cauchy–Schwarz inequality that, for each $n \in \mathbb{N}$,

$$\mathbb{E} \left[\sqrt{\varphi_n(U_n)} \right] \leq (\mathbb{E}[\varphi_n(U_n)]\mathbb{E}[1])^{1/2} = 1.$$

Consequently, we have

$$0 \leq -\log \mathbb{E} \left[\sqrt{\varphi_n(U_n)} \right].$$

In this note, we shall rely on the following theorem of Kakutani (see also [3, Theorem 2.12.7]):

Theorem 2.1. *[8, Theorem 1] The measures $\mathbb{P} \circ \underline{U}^{-1}$ and $\mathbb{P} \circ \underline{V}^{-1}$ are equivalent, i.e. mutually absolutely continuous, if the laws of U_n and V_n are equivalent for every $n \in \mathbb{N}$, and if the series*

$$\sum_{n=1}^{\infty} -\log \mathbb{E} \left[\sqrt{\varphi_n(U_n)} \right] \tag{1}$$

converges. If the series in (1) diverges, then the measures $\mathbb{P} \circ \underline{U}^{-1}$ and $\mathbb{P} \circ \underline{V}^{-1}$ are mutually singular.

Remark 1. By Jensen’s inequality, a sufficient condition for convergence of the series in (1) is convergence of the series

$$\sum_{n=1}^{\infty} \mathbb{E} \left[-\frac{1}{2} \log \varphi_n(U_n) \right].$$

3 The Gaussian case

As a preview of the main results, we shall first recall the classical case when U_n is a real-valued Gaussian random variable with mean $\delta_n \in \mathbb{R}$ and standard deviation $\gamma_n > 0$ for every $n \in \mathbb{N}$. The random sequence \underline{U} is then defined by the sequences $\underline{\delta}$ and $\underline{\gamma}$; we shall assume that $\underline{\delta} \in \ell^1$ and $\underline{\gamma} \in \ell^2$, since this ensures that $\underline{U} \in \ell^1$ a.s.

3.1 Additive perturbations of the means

For every $n \in \mathbb{N}$, let V_n be a \mathbb{R} -valued Gaussian random variable with mean $\delta_n + \varepsilon_n$ and the same standard deviation γ_n as U_n . The Radon–Nikodým derivative of the law of V_n with respect to the law of U_n is

$$\begin{aligned} \varphi_n(u_n) &= \exp \left(-\frac{|u_n - \delta_n - \varepsilon_n|^2 - |u_n - \delta_n|^2}{2\gamma_n^2} \right) \\ &= \exp \left(-\frac{(2(u_n - \delta_n) - \varepsilon_n)(-\varepsilon_n)}{2\gamma_n^2} \right). \end{aligned}$$

Then

$$\mathbb{E} [\log \varphi_n(U_n)] = \frac{\varepsilon_n}{2\gamma_n^2} \mathbb{E} [2(U_n - \delta_n) - \varepsilon_n] = \frac{\varepsilon_n}{2\gamma_n^2} (0 - \varepsilon_n) = -\frac{\varepsilon_n^2}{\gamma_n^2}.$$

Thus, if \underline{U} is a sequence of independent, \mathbb{R} -valued Gaussian random variables with means $\underline{\delta}$ and standard deviations $\underline{\gamma}$, and if $(\varepsilon_n/\gamma_n)_n \in \ell^2$, then by Kakutani's theorem, the random sequence \underline{V} with sequence of means $\underline{\delta} + \underline{\varepsilon}$ and sequence of standard deviations $\underline{\gamma}$ has a law that is equivalent to the law of \underline{U} . Equivalently, if the translation sequence $\underline{\varepsilon}$ belongs to the space ℓ^2 of square-summable sequences weighted by the sequence $(\gamma_n^{-2})_n$ of the inverses of the variances, then the law of \underline{V} and \underline{U} are equivalent.

For completeness, we show that the space

$$H(\mu) := \left\{ \underline{\varepsilon} : \sum_{n=1}^{\infty} \frac{\varepsilon_n^2}{\gamma_n^2} < \infty \right\}$$

is indeed the Cameron–Martin space of the product measure $\mu = \bigotimes_{n=1}^{\infty} \mu_n$, where each μ_n is a Gaussian measure on \mathbb{R} with mean $\delta_n \in \mathbb{R}$ and variance $\gamma_n^2 > 0$. Note that the infinite product measure μ is a Gaussian measure on the locally convex space $X = \prod_{n=1}^{\infty} X_n$, where $X_n = \mathbb{R}$. By [3, Example 2.3.8], it follows that the Cameron–Martin space of μ is the Hilbert direct sum of the Cameron–Martin spaces of the measures $(\mu_n)_n$, i.e.

$$H(\mu) = \left\{ \underline{\varepsilon} = (\varepsilon_n) \in X : \varepsilon_n \in H(\mu_n), \|\underline{\varepsilon}\|_{H(\mu)}^2 = \sum_{n=1}^{\infty} \|\varepsilon_n\|_{H(\mu_n)}^2 < \infty \right\}.$$

Recall that, given a Gaussian measure ν on a locally convex space Y , the Cameron–Martin space of ν is defined by

$$H(\nu) := \left\{ h \in Y : \|h\|_{H(\nu)} < \infty \right\},$$

where

$$\|h\|_{H(\nu)} = \sup \{ \lambda(h) : \lambda \in Y^*, R_\nu(\lambda)(\lambda) \leq 1 \}, \quad (2)$$

Y^* denotes the topological dual of Y , and

$$R_\nu(f)(g) = \int_Y [f(x) - a_\nu(f)][g(x) - a_\nu(g)] \nu(dx), \quad a_\nu(f) = \int_Y f(x) \nu(dx).$$

To use [3, Example 2.3.8], it suffices to show that

$$\|\varepsilon_n\|_{H(\mu_n)}^2 = \frac{\varepsilon_n^2}{\gamma_n^2}. \quad (3)$$

Note that any linear functional λ of the \mathbb{R} -valued random variable X_n is given by $\lambda(X_n) = aX_n$ for some $a \in \mathbb{R}$. This implies that

$$R_{\mu_n}(\lambda)(\lambda) = \mathbb{E}_{X_n \sim \mu_n} [a^2(X_n - \delta_n)^2] = a^2\gamma_n^2,$$

and hence constrains $a \in \{\pm\gamma_n^{-1}\}$. Therefore, the supremum in the definition (2) of $H(\mu_n)$ is (up to sign) equal to ε_n/γ_n . This yields (3).

3.2 Multiplicative perturbations of the standard deviations

For every $n \in \mathbb{N}$, let V_n be a \mathbb{R} -valued Gaussian random variable with the same mean δ_n as U_n but different standard deviation $|\sigma_n|\gamma_n$, where $\sigma_n \neq 0$. The

Radon–Nikodým derivative of the law of V_n with respect to the law of U_n is

$$\begin{aligned}\varphi_n(u_n) &= \exp\left(-\frac{|u_n - \delta_n|^2}{2\sigma_n^2\gamma_n^2}\right) \exp\left(+\frac{|u_n - \delta_n|^2}{2\gamma_n^2}\right) \\ &= \exp\left(-|u_n - \delta_n|^2 \frac{1 - \sigma_n^2}{2\sigma_n^2\gamma_n^2}\right).\end{aligned}$$

Then

$$\mathbb{E}[\log \varphi_n(U_n)] = \left(\frac{1 - \sigma_n^2}{2\sigma_n^2\gamma_n^2}\right) \mathbb{E}\left[-|U_n - \delta_n|^2\right] = \frac{1 - \sigma_n^2}{2\sigma_n^2} = -\frac{1}{2}|1 - \sigma_n^{-2}|.$$

Therefore, by Kakutani’s theorem, a necessary condition for equivalence is the summability of the sequence $(1 - \sigma_n^{-2})_n$. (In particular, if $\underline{\sigma}$ is a constant sequence, then this constant must have unit modulus; equivalence of infinite-dimensional Gaussian measures is not preserved by non-unit dilations.) In turn, summability implies that the sequence $(|\sigma_n|)_n$ converges to 1 and is bounded away from 0. We define the set

$$\mathcal{S} := \{\underline{\sigma} \in \mathbb{R}^{\mathbb{N}} : \exists c > 0, C > 1 \text{ s.t. } c < |\sigma_n| < C \text{ for all large enough } n \in \mathbb{N}\}.$$

Let $\underline{\sigma} \in \mathcal{S}$ be arbitrary, and let c and C be the corresponding constants. One can show using elementary calculus as in the proof of Lemma 4.11 below that, for large enough n ,

$$\frac{c+1}{c^2} \left||\sigma_n| - 1\right| < \left|\sigma_n^{-2} - 1\right| < \frac{C+1}{C^2} \left||\sigma_n| - 1\right|.$$

Therefore, a sufficient condition for summability of $(1 - \sigma_n^{-2})_n$ is that $(1 - |\sigma_n|)_n \in \ell^1$. Thus, if the deviations from 1 of the absolute values of the multiplicative perturbations form a summable sequence, then equivalence of the laws of \underline{U} and \underline{V} holds; compare this result with [3, Example 2.7.6].

4 The Cauchy case

We now consider the case when \underline{U} is a sequence of Cauchy random variables that is defined by a sequence $\underline{\delta} = (\delta_n)_n$ of location parameters and a sequence $\underline{\gamma}$ of strictly positive scale parameters. The \mathbb{R} -valued random variable U_n has the Cauchy distribution with location parameter δ_n and scale parameter γ_n if U_n has the density

$$f_n(x; \delta_n, \gamma_n) = \frac{1}{\pi\gamma_n} \frac{\gamma_n^2}{(x - \delta_n)^2 + \gamma_n^2}. \quad (4)$$

As in Section 3 above, we consider perturbations of the location parameters and scale parameters separately.

As a preliminary remark, we assume that the sequence $\underline{\delta}$ of location parameters lies in ℓ^1 , and that the sequence $\underline{\gamma}$ of scale parameters has all terms strictly positive and lies in $\ell^1 \cap \ell \log \ell$, where

$$\ell \log \ell := \left\{ r \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |r_n \log |r_n|| < \infty \right\}, \quad (5)$$

with the convention that $0 \log 0 = 0$. Note that $\ell^1 \cap \ell \log \ell$ is a non-empty proper subset of both ℓ^1 and $\ell \log \ell$, since, for example,

$$(n^{-1}(\log n)^{-2})_n \in \ell^1 \cap (\ell \log \ell)^{\mathbb{G}} \quad (6a)$$

$$(\exp(-a^n))_n \in (\ell^1)^{\mathbb{G}} \cap \ell \log \ell \quad \text{for } 0 < a < 1, \quad (6b)$$

$$(n^{-p})_n \in \ell^1 \cap \ell \log \ell \quad \text{for } p > 1. \quad (6c)$$

The spaces ℓ^1 and $\ell^1 \cap \ell \log \ell$ play a key role for sequences of Cauchy random variables: by [14, Theorem 3.3 and Example 3.5], $\underline{\delta} \in \ell^1$ and $\underline{\gamma} \in \ell^1 \cap \ell \log \ell$ together imply that $\underline{U} \in \ell^1$ a.s., so these assumptions on $\underline{\delta}$ and $\underline{\gamma}$ will be assumed throughout this section.

4.1 Additive perturbation of location parameters

For every $n \in \mathbb{N}$, let V_n be a \mathbb{R} -valued Cauchy random variable with location parameter $\delta_n + \varepsilon_n$ and the same scale parameter γ_n as U_n . Then V_n has the density $f_n(x - \varepsilon_n; \delta_n, \gamma_n) = f_n(x; \delta_n + \varepsilon_n; \gamma_n)$, where the function f is defined in (4). Since $\underline{\delta} \in \ell^1$, we make the following minimal assumption on the sequence $\underline{\varepsilon}$ of additive perturbations of the means:

Assumption 4.1. *The sequence $\underline{\varepsilon} = (\varepsilon_n)_{n \in \mathbb{N}}$ belongs to ℓ^1 .*

The Radon–Nikodým derivative of the law of V_n with respect to U_n is given by

$$\varphi_n(x) := \frac{f_n(x - \varepsilon_n)}{f_n(x)} = \frac{1}{\pi \gamma_n} \frac{\gamma_n^2}{(x - \delta_n - \varepsilon_n)^2 + \gamma_n^2} \left(\frac{1}{\pi \gamma_n} \frac{\gamma_n^2}{(x - \delta_n)^2 + \gamma_n^2} \right)^{-1}. \quad (7)$$

Note that if $\varepsilon_n = 0$, then $\varphi_n(x) = 1$ for all $x \in \mathbb{R}$. The result below follows immediately from this observation.

Corollary 4.2. *If the cardinality of the set $\{\varepsilon_n \mid \varepsilon_n \neq 0\}$ is finite, then the series in (1) converges, and the measures $\mathbb{P} \circ \underline{U}^{-1}$ and $\mathbb{P} \circ \underline{V}^{-1}$ are equivalent.*

In view of Corollary 4.2, we simplify the analysis that follows by assuming that the translation $\underline{\varepsilon}$ is a sequence with all terms nonzero:

Assumption 4.3. *For all $n \in \mathbb{N}$, $\varepsilon_n \neq 0$.*

Define

$$y_n := x_n - \delta_n \quad (8a)$$

$$\rho_n(y_n; \varepsilon_n) := \frac{y_n^2 + \gamma_n^2}{(y_n - \varepsilon_n)^2 + \gamma_n^2}. \quad (8b)$$

The next result follows from (8b), and the fact that $\underline{\varepsilon} \in \ell^1$ implies that $\varepsilon_n \rightarrow 0$.

Lemma 4.4. *The ratio $\rho_n(y_n; \varepsilon_n)$ converges to 1 as $n \rightarrow +\infty$, for all $\underline{y} := (y_n)_n \in \mathbb{R}^{\mathbb{N}}$.*

We now return to the question of the equivalence of the laws of \underline{U} and \underline{V} . Recall that the convergence of the series (1) provided a sufficient condition for

equivalence, according to Kakutani's theorem. It follows from (4), (7), (8a), and (8b) that

$$\log \varphi_n(x_n) = \log \rho_n(y_n; \varepsilon_n).$$

Thus, to show that the series in (1) converges, we must show that

$$\sum_{n=1}^{\infty} \mathbb{E} [\log \rho_n(U_n - \delta_n; \varepsilon_n)] \quad (9)$$

converges. To show that the series in (9) converges, define

$$z_n(y_n; \varepsilon_n) := \frac{\varepsilon_n^2 - 2y_n\varepsilon_n}{(y_n - \varepsilon_n)^2 + \gamma_n^2}, \quad (10)$$

so that

$$1 = \rho_n(y_n; \varepsilon_n) + z_n(y_n; \varepsilon_n). \quad (11)$$

Fix an arbitrary $\underline{y} = (y_n)_n$. By Lemma 4.4, there are at most finitely many $z_n(y_n; \varepsilon_n)$ that are greater than 1 in absolute value. Therefore, without loss of generality, we may assume that the sequence $(z_n(y_n; \varepsilon_n))_n$ satisfies the bound $|z_n(y_n; \varepsilon_n)| < 1$ for all $n \in \mathbb{N}$ (note that one can specify conditions on $\underline{\varepsilon}$ and $\underline{\gamma}$ such that this holds for all \underline{y} ; see Remark 2 below). Thus, the value of $\log \rho_n(y_n; \varepsilon_n)$ agrees with its Taylor series, and we have

$$\sum_{n=1}^{\infty} \log \rho_n(y_n; \varepsilon_n) = \sum_{n=1}^{\infty} \log(1 - z_n(y_n; \varepsilon_n)) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (-z_n(y_n; \varepsilon_n))^m. \quad (12)$$

The next result makes use of elementary calculus to establish an upper bound on the value of $|z_n|$ that depends on ε_n and γ_n , and is uniform in y_n . The result is essential for the main result of this section (Theorem 4.1).

Lemma 4.5. *For each $n \in \mathbb{N}$, there exists a constant $C_n := C_n(\varepsilon_n/\gamma_n)$ such that*

$$|z_n(y_n; \varepsilon_n)| \leq C_n \frac{|\varepsilon_n|}{\gamma_n} \quad \forall y_n \in \mathbb{R}, \quad (13)$$

where

$$1 + \frac{|\varepsilon_n|}{2\gamma_n} < C_n < 1 + \frac{|\varepsilon_n|}{\gamma_n}. \quad (14)$$

Corollary 4.6. *If $|\varepsilon_n|/\gamma_n \searrow 0$, then $C_n \searrow 1$, and $|z_n(y_n; \varepsilon_n)| \searrow 0$ uniformly in y_n .*

Proof of Lemma 4.5. Fix $n \in \mathbb{N}$. Define

$$w_n(y_n; \varepsilon_n) := \frac{y_n - \varepsilon_n}{\gamma_n}, \quad \zeta_n(\varepsilon_n, \gamma_n) := \frac{|\varepsilon_n|}{\gamma_n}.$$

Henceforth, we omit the arguments of z_n , w_n , and ζ_n . The definition (10) of z_n yields

$$\begin{aligned} |z_n| &= |\varepsilon_n| \frac{|2(y_n - \varepsilon_n) + \varepsilon_n|}{(y_n - \varepsilon_n)^2 + \gamma_n^2} = \frac{|\varepsilon_n|}{\gamma_n} \frac{|2(y_n - \varepsilon_n) + \varepsilon_n|}{\gamma_n} \frac{1}{(y_n - \varepsilon_n)^2/\gamma_n^2 + 1} \\ &= \zeta_n \frac{|2w_n + \zeta_n|}{w_n^2 + 1}. \end{aligned}$$

To show (13), it suffices to show that the ratio $|2w_n + \zeta_n|/(w_n^2 + 1)$ is bounded by the constant C_n . Note that

$$|2w_n + \zeta_n| = \begin{cases} -(2w_n + \zeta_n) & \text{if } w_n < -\frac{\zeta_n}{2}, \\ 0 & \text{if } w_n = -\frac{\zeta_n}{2}, \\ 2w_n + \zeta_n & \text{if } w_n > -\frac{\zeta_n}{2}. \end{cases}$$

Taking the derivative of the ratio $|2w_n + \zeta_n|/(w_n^2 + 1)$ with respect to w_n , we observe that

$$\frac{d}{dw_n} \frac{|2w_n + \zeta_n|}{w_n^2 + 1} = 0 \iff 2(w_n^2 + 1) - (2w_n + \zeta_n)2w_n = 0, \quad w_n \neq -\frac{\zeta_n}{2}.$$

Simplifying the quadratic equation in the statement on the right-hand side, it follows that the derivative vanishes at $w_{n,\pm} := \frac{1}{2}(-\zeta_n \pm \sqrt{\zeta_n^2 + 4})$. The corresponding values of the ratio $|2w_n + \zeta_n|/(w_n^2 + 1)$ are given by

$$\frac{|2w_{n,\pm} + \zeta_n|}{w_{n,\pm}^2 + 1} = \frac{4|\sqrt{\zeta_n^2 + 4}|}{(\zeta_n^2 + 4) + \zeta_n^2 \mp 2\zeta_n\sqrt{\zeta_n^2 + 4} + 4} = \frac{2}{\sqrt{\zeta_n^2 + 4} \mp \zeta_n},$$

and therefore the maximum value of the ratio is attained at $w_{n,+}$. It remains to identify the smallest constant $C_n = C_n(\varepsilon_n/\gamma_n)$ that bounds the maximum value of the ratio, i.e. to find the smallest C_n such that

$$2 \leq C_n \left(\sqrt{\zeta_n^2 + 4} - \zeta_n \right). \quad (15)$$

The above is equivalent to

$$4 + (C_n\zeta_n)^2 + 4C_n\zeta_n \leq C_n^2(\zeta_n^2 + 4).$$

Setting the inequality above to equality yields the quadratic polynomial $C_n^2 - C_n\zeta_n - 1 = 0$, the unique positive root of which is

$$C_n := \frac{\zeta_n + \sqrt{\zeta_n^2 + 4}}{2}. \quad (16)$$

The upper and lower bounds in (14) follow from the strict positivity of ζ_n respectively, since

$$1 + \frac{\zeta_n}{2} = \frac{\zeta_n + \sqrt{4}}{2} < C_n < \frac{\zeta_n + \sqrt{\zeta_n^2 + 4\zeta_n + 4}}{2} = 1 + \zeta_n.$$

This completes the proof. That (16) is indeed optimal can be verified directly by substitution in (15). \square

The next result will be useful for proving the main result of this section.

Lemma 4.7. *Let $0 < |\varepsilon_n|/\gamma_n \leq \sqrt{2} - 1$ and $C_n = C_n(\varepsilon_n/\gamma_n)$ be defined as in (16). Then*

$$\sum_{m=1}^{\infty} \frac{C_n^m}{m} \left(\frac{|\varepsilon_n|}{\gamma_n} \right)^m < \left| \log \frac{|\varepsilon_n|}{\gamma_n} \right|. \quad (17)$$

Proof of Lemma 4.7. Since $0 < \sqrt{2} - 1 < 1$, it follows from the assumption on $|\varepsilon_n|/\gamma_n$ that the logarithm of $|\varepsilon_n|/\gamma_n$ agrees with its Taylor series, and that $\text{sign}(|\varepsilon_n|/\gamma_n - 1) = -1$. Thus we obtain

$$\begin{aligned} \left| \log \frac{|\varepsilon_n|}{\gamma_n} \right| &= \left| \sum_{m=1}^{\infty} \left(\frac{|\varepsilon_n|}{\gamma_n} - 1 \right)^m \frac{(-1)^{m-1}}{m} \right| = \left| \sum_{m=1}^{\infty} \left| \frac{|\varepsilon_n|}{\gamma_n} - 1 \right|^m \frac{(-1)^{2m-1}}{m} \right| \\ &= \left| - \sum_{m=1}^{\infty} \left| \frac{|\varepsilon_n|}{\gamma_n} - 1 \right|^m \frac{1}{m} \right| = \sum_{m=1}^{\infty} \left| \frac{|\varepsilon_n|}{\gamma_n} - 1 \right|^m \frac{1}{m}. \end{aligned}$$

By the upper bound in (14), it suffices to verify that

$$\left(1 + \frac{|\varepsilon_n|}{\gamma_n} \right) \left(\frac{|\varepsilon_n|}{\gamma_n} \right) \leq 1 - \frac{|\varepsilon_n|}{\gamma_n} \iff \left(\frac{|\varepsilon_n|}{\gamma_n} \right)^2 + 2 \left(\frac{|\varepsilon_n|}{\gamma_n} \right) - 1 \leq 0. \quad (18)$$

The unique positive root of the quadratic polynomial on the right-hand side is given by the upper bound $\sqrt{2} - 1$. The result now follows from the hypothesis on $|\varepsilon_n|/\gamma_n$. \square

Remark 2. If $|\varepsilon_n|/\gamma_n < \sqrt{2} - 1$, then it follows that, for arbitrary $\underline{y} \in \mathbb{R}^{\mathbb{N}}$, $|z_n(\underline{y}_n; \varepsilon_n)| < 1$ for all $n \in \mathbb{N}$; this bound on z_n justifies replacing the logarithm with its Taylor series. Note that combining (13) and (14) yields

$$|z_n(\underline{y}_n; \varepsilon_n)| < \frac{|\varepsilon_n|}{\gamma_n} \left(1 + \frac{|\varepsilon_n|}{\gamma_n} \right).$$

By solving a quadratic polynomial in $|\varepsilon_n|/\gamma_n$, one can show that the right-hand-side is less than or equal to 1 if and only if

$$\frac{|\varepsilon_n|}{\gamma_n} \leq \frac{\sqrt{5} - 1}{2}.$$

Since the above holds when $|\varepsilon_n|/\gamma_n < \sqrt{2} - 1$, then we have the desired bound on $|z_n|$.

Our final preparation for the main result of this section is the following result, which is similar to Lemma 4.7.

Lemma 4.8. *Let ζ be a strictly positive sequence that satisfies $0 < \zeta_n \leq \sqrt{2} - 1$ for all $n \in \mathbb{N}$, and let $C_n = C_n(\zeta_n)$ be defined as in (16). Then*

$$\sqrt{2} \left(\sum_{n=1}^{\infty} \zeta_n + \sum_{n=1}^{\infty} \zeta_n |\log \zeta_n| \right) > \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{C_n^m}{m} \zeta_n^m.$$

Proof of Lemma 4.8. It follows from (14) in Lemma 4.5 that $C_n \leq \sqrt{2}$ for all $n \in \mathbb{N}$. Therefore, it follows from Lemma 4.7 that

$$\sqrt{2} \sum_{n=1}^{\infty} \zeta_n |\log |\zeta_n|| \geq \sum_{n=1}^{\infty} C_n \zeta_n \sum_{m=1}^{\infty} \frac{C_n^m}{m} \zeta_n^m = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{C_n^{m+1}}{m} \zeta_n^{m+1}.$$

Setting $\ell = m + 1$, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{C_n^{m+1}}{m} \zeta_n^{m+1} = \sum_{n=1}^{\infty} \sum_{\ell=2}^{\infty} \frac{C_n^{\ell}}{\ell-1} \zeta_n^{\ell} > \sum_{n=1}^{\infty} \sum_{\ell=2}^{\infty} \frac{C_n^{\ell}}{\ell} \zeta_n^{\ell}.$$

Therefore, as desired,

$$\sqrt{2} \left(\sum_{n=1}^{\infty} \zeta_n + \sum_{n=1}^{\infty} \zeta_n |\log \zeta_n| \right) > \sum_{n=1}^{\infty} C_n \zeta_n + \sum_{n=1}^{\infty} \sum_{\ell=2}^{\infty} \frac{C_n^\ell}{\ell} \zeta_n^\ell = \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{C_n^\ell}{\ell} \zeta_n^\ell. \quad \square$$

We now turn to the main result of this section, in which we use the preceding results to specify sufficient conditions on the sequence $\underline{\varepsilon}$ of additive perturbations of the sequence $\underline{\delta}$ of location parameters of the Cauchy density in (4), such that the law of the perturbed random sequence \underline{V} is equivalent to the law of the reference random sequence \underline{U} .

Theorem 4.1. *If $(|\varepsilon_n|/\gamma_n)_{n \in \mathbb{N}} \in \ell^1 \cap \ell \log \ell$, then the laws of \underline{U} and \underline{V} are equivalent.*

Proof of 4.1. Recall that we want to show the equivalence of the infinite product measures using Kakutani's theorem (Theorem 2.1). To apply Kakutani's theorem, we need to show that the series in (9) converges absolutely. Since $(|\varepsilon_n|/\gamma_n)_{n \in \mathbb{N}}$ belongs to the space ℓ^1 of summable sequences, it must converge to zero, and thus only finitely many terms in the sequence are greater than or equal to $\sqrt{2} - 1$. Therefore, without loss of generality, we may assume that $(|\varepsilon_n|/\gamma_n)_{n \in \mathbb{N}}$ belongs to the space $\ell_{< \sqrt{2}-1}^\infty$. By the inequality on the left-hand side of (18), it follows that for every $n \in \mathbb{N}$, $|z_n(y_n; \varepsilon_n)|$ is strictly less than 1 for all $y_n \in \mathbb{R}$. By (12), Lemma 4.5, and Lemma 4.7, we have

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \mathbb{E} [\log \rho_n(U_n; \varepsilon_n)] \right| &= \left| \sum_{n=1}^{\infty} \mathbb{E} \left[\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (-z_n(U_n; \varepsilon_n))^m \right] \right| \\ &\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{C_n^m}{m} \left(\frac{|\varepsilon_n|}{\gamma_n} \right)^m. \end{aligned}$$

The result then follows by Lemma 4.8. \square

4.2 Multiplicative perturbations of the scale parameters

Recall that the random sequence $(U_n)_n$ is defined by the sequence $\underline{\delta}$ of location parameters and the sequence $\underline{\gamma} \in \mathbb{R}_{>0}^{\mathbb{N}}$ of scale parameters, and by the density (4). In this section, we define the perturbed sequence $(V_n)_n$ by defining (for every $n \in \mathbb{N}$) V_n to be a Cauchy random variable with the same location parameter δ_n as U_n but a different scale parameter $\sigma_n \gamma_n$, where $\sigma_n \neq 0$. Therefore, V_n has density $f_n(x; \delta_n, |\sigma_n| \gamma_n)$. Using (4), it follows that the Radon–Nikodým derivative of the law of V_n with respect to the law of U_n is given by

$$\varphi_n(x_n) := \frac{f_n(x_n; \delta_n, |\sigma_n| \gamma_n)}{f_n(x_n; \delta_n, \gamma_n)} = |\sigma_n| \frac{(x_n - \delta_n)^2 + \gamma_n^2}{(x_n - \delta_n)^2 + \sigma_n^2 \gamma_n^2}. \quad (19)$$

Observe that $\varphi_n(x_n) = 1$ for all $x_n \in \mathbb{R}$ if and only if $|\sigma_n| = 1$. We make the following simplifying assumption:

Assumption 4.9. *For all $n \in \mathbb{N}$, $|\sigma_n| \notin \{0, 1\}$.*

As in (8a), we let $y_n := x_n - \delta_n$. Then using (19), we obtain

$$\varphi_n(y_n + \delta_n) + |\sigma_n| \frac{\gamma_n^2 (\sigma_n^2 - 1)}{y_n^2 + \sigma_n^2 \gamma_n^2} = |\sigma_n|,$$

so that

$$\log \varphi_n(y_n + \delta_n) = \log |\sigma_n| + \log \left(1 - \frac{\gamma_n^2(\sigma_n^2 - 1)}{y_n^2 + \sigma_n^2 \gamma_n^2} \right).$$

Thus, the following statements together constitute equivalent conditions for the convergence of the series (1), by Kakutani's theorem (Theorem 2.1):

$$\sum_{n=1}^{\infty} -\log |\sigma_n| < \infty, \quad (20a)$$

$$\sum_{n=1}^{\infty} -\mathbb{E} \left[\log \left(1 - \frac{\gamma_n^2(\sigma_n^2 - 1)}{U_n^2 + \sigma_n^2 \gamma_n^2} \right) \right] < \infty. \quad (20b)$$

From (20a), we obtain a necessary condition for equivalence of the laws of the random sequences:

Proposition 4.10. *Let \underline{U} and \underline{V} be random sequences of Cauchy random variables, where the Radon–Nikodým density of the law of V_n with respect to the law of U_n is given by (19). If $\mathbb{P} \circ \underline{U}^{-1}$ is equivalent to $\mathbb{P} \circ \underline{V}^{-1}$, then $|\sigma_n| \rightarrow 1$.*

By Proposition 4.10, we may without loss of generality assume that $|\sigma_n| < 2$ for all $n \in \mathbb{N}$. Thus, we may replace the logarithm of $|\sigma_n|$ with its Taylor series, which yields

$$\sum_{n=1}^{\infty} |\log |\sigma_n|| = \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{(|\sigma_n| - 1)^m}{m} (-1)^{m-1} \right| \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \|\sigma_n - 1\|^m. \quad (21)$$

Note that demanding that the right-hand side of (21) be finite implies that (20a) holds. We now consider a sufficient condition for (20b). Observe that

$$\left| \frac{\gamma_n^2(\sigma_n^2 - 1)}{y_n^2 + \sigma_n^2 \gamma_n^2} \right| \leq \frac{\gamma_n^2}{\sigma_n^2 \gamma_n^2} |\sigma_n^2 - 1| \leq |1 - \sigma_n^{-2}| \quad \forall y_n \in \mathbb{R}.$$

Observe also that

$$2^{-1/2} < |\sigma_n| \iff -1 < \sigma_n^{-2} - 1 < 1. \quad (22)$$

Thus, if we also demand that $|\sigma_n| > 2^{-1/2}$, we may replace the logarithm in (20b) with its Taylor series, and obtain that for any $y \in \mathbb{R}^{\mathbb{N}}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \log \left(1 - \frac{\gamma_n^2(\sigma_n^2 - 1)}{y_n^2 + \sigma_n^2 \gamma_n^2} \right) \right| &= \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\frac{\gamma_n^2(1 - \sigma_n^2)}{y_n^2 + \sigma_n^2 \gamma_n^2} \right)^m \right| \\ &\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} |1 - \sigma_n^{-2}|^m. \end{aligned} \quad (23)$$

We define the set of $\underline{\sigma}$ for which both (23) and (21) hold as

$$\mathcal{S} := \left\{ \underline{\sigma} \in \mathbb{R}^{\mathbb{N}} : 2^{-1/2} < |\sigma_n| < 2, |\sigma_n| \neq 1 \text{ for all large enough } n \in \mathbb{N} \right\}. \quad (24)$$

For simplicity and without loss of generality, we take “large enough” to mean $n \geq 1$. In our first step to finding one sufficient condition for the convergence of the right-hand sides of (23) and (21), we have

Lemma 4.11. *If $\underline{\sigma} \in \mathcal{S}$, then*

$$\frac{3}{4} \|\sigma_n\| - 1 < |\sigma_n^{-2} - 1| < 4 \|\sigma_n\| - 1. \quad (25)$$

Proof of 4.11. Observe that

$$|\sigma_n^{-2} - 1| = \frac{|\sigma_n| + 1}{\sigma_n^2} \|\sigma_n\| - 1. \quad (26)$$

Given the function $f(x) := (x + 1)/x^2$, its derivative $f'(x) = -x^{-2} - 2x^{-3}$ is negative over the interval $(2^{-1/2}, 2)$, so f attains its maximum (resp. minimum) at the left (resp. right) endpoint of the interval. Since $c := 3/4 = f(2)$ and $C := 4 > f(2^{-1/2})$, the result follows. \square

Corollary 4.12. *If $\underline{\sigma} \in \mathcal{S}$, then*

$$\left| \sum_{n=1}^{\infty} -\mathbb{E} \left[\log \left(1 - \frac{\gamma_n^2 (\sigma_n^2 - 1)}{U_n^2 + \sigma_n^2 \gamma_n^2} \right) \right] \right| \leq 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \|\sigma_n\| - 1^m.$$

Proof of 4.12. The statement follows from (23) and Lemma 4.11. \square

Recall the definition (24) of \mathcal{S} . The following analogue of Theorem 4.1 is the main result of this section:

Theorem 4.2. *Let \underline{U} and \underline{V} be random sequences of Cauchy random variables, where the Radon–Nikodým density of the law of V_n with respect to the law of U_n is given by (19). If $\underline{\sigma} \in \mathcal{S}$ satisfies*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \|\sigma_n\| - 1^m < \infty, \quad (27)$$

then $\mathbb{P} \circ \underline{V}^{-1}$ is equivalent to $\mathbb{P} \circ \underline{U}^{-1}$. In particular, if $(\|\sigma_n\| - 1)_n \in \ell^1 \cap \ell \log \ell$, then $\mathbb{P} \circ \underline{V}^{-1}$ is equivalent to $\mathbb{P} \circ \underline{U}^{-1}$.

Proof of 4.2. The sufficiency of the first hypothesis follows directly from (21) and Corollary 4.12, since these imply that the equations (20) hold, and therefore that the sufficient condition for Kakutani’s theorem is valid. The sufficiency of the second hypothesis follows from the fact that $C_n > 1$, as was shown in (14). \square

Remark 3. If there exists a $s > 0$ such that

$$\sum_{n=1}^{\infty} \|\sigma_n\| - 1^m \leq C m^{-s},$$

where C does not depend on m or $\underline{\sigma}$, then (27) holds, since the double series is in fact a p -series with $p > 1$.

Example 4.1. Recall that the sequence $(1 - \sigma_n^{-2})_n$ plays a role in the validity of (20b) by the inequality (23), as well as in the validity of (20a) by Lemma 4.11. By (6c), an example of a non-trivial sequence $\underline{\sigma} \in \mathcal{S}$ that satisfies $(1 - \sigma_n^{-2})_n \in \ell^1 \cap \ell \log \ell$ is, for any choice of $p > 1$,

$$\sigma_n := (1 - (n + 1)^{-p})^{-1/2}.$$

5 Conclusions and Discussion

This note shows how the classical Cameron–Martin theorem for translations and dilations of an infinite product Gaussian measure on sequence space may be extended to an infinite product Cauchy measure. Equivalence of the original and translated measures in the Gaussian case requires that a weighted ℓ^2 norm of the translation vector be finite; in the Cauchy case, by Theorem 4.1, equivalence requires that a weighted version of the translation vector lie in $\ell^1 \cap \ell \log \ell$. Together with [14, Theorem 3.3 and Remark 3.5], this result highlights the central role of the sequence space $\ell^1 \cap \ell \log \ell$ when working with sequences of Cauchy random variables, and hence Cauchy-distributed random fields.

Thus, the Gaussian and Cauchy cases inspire the following conjecture:

Conjecture 5.1. *Let $\alpha \in (0, 1) \cup (1, 2)$. If \underline{U} is a sequence with α -stable terms with location parameters $\underline{\delta} \in \ell^1$ and scale parameters $\underline{\gamma} \in \ell^\alpha$, $\underline{V} := \underline{\varepsilon} + \underline{U}$, and $(|\varepsilon_n|/\gamma_n)_n \in \ell^\alpha$, then the laws of \underline{U} and \underline{V} are equivalent.*

However, in the absence of explicit formulae for the densities of α -stable distributions, the proof of Conjecture 5.1 would require different techniques to those used here.

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