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**Perfect f -Matchings and f -Factors in
Hypergraphs - A Combinatorial
Approach**

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Perfect f -Matchings and f -Factors in Hypergraphs - A Combinatorial Approach

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Abstract

We prove characterizations of the existence of perfect f -matchings in uniform mengerian and perfect hypergraphs. Moreover, we investigate the f -factor problem in balanced hypergraphs. For uniform balanced hypergraphs we prove two existence theorems with purely combinatorial arguments, whereas for non-uniform balanced hypergraphs we show that the f -factor problem is \mathcal{NP} -hard.

Keywords: perfect f -matchings in hypergraphs, f -factors in hypergraphs, mengerian hypergraph, balanced hypergraph, perfect hypergraph, Hall's Theorem

1 Introduction

We investigate conditions for the existence of perfect f -matchings and f -factors in hypergraphs generalizing bipartite graphs, namely balanced, mengerian, and perfect hypergraphs. For the subclass of unimodular hypergraphs, which are hypergraphs with a totally unimodular incidence matrix M , Hoffman characterized in [Hof60] when the system

$$f \leq Mx \leq g, l \leq x \leq u \tag{1}$$

has a solution. He called his result (which was not proven in [Hof60]) “*the most general theorem of the Hall type*” as many Hall type theorems (existence of a system of distinct representatives, perfect matchings in bipartite graphs, flows in networks, etc.) can be reduced to it. Sixteen years later Hoffman published a proof of a slight generalization of this theorem (see [Hof76]).

Hoffman's result includes a characterization of the existence of perfect f -matchings and f -factors in unimodular hypergraphs generalizing the known ones for bipartite graphs (see for example [LP86] or [Hof60], [Hof76] as the incidence matrix of a bipartite graph is totally unimodular):

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Namely, a bipartite graph $G = (V, E)$ with vertex partition $V = A \cup B$ has a perfect f -matching if and only if

$$\begin{aligned} f(A) &= f(B) \text{ and} \\ f(S) &\leq f(\Gamma(S)) \text{ for all } S \subseteq A. \end{aligned} \quad (2)$$

Here, $\Gamma(S) := \{v \in V \setminus S \mid v \text{ is adjacent to } s \in S\}$ denotes the neighborhood of a set of vertices, and we define $f(U) := \sum_{v \in U} f(v)$ for a set of vertices $U \subseteq V$.

Furthermore, G has an f -factor if and only if

$$\begin{aligned} f(A) &= f(B) \text{ and} \\ f(S) - f(T) &\leq |\{e \in E \mid e \cap S \neq \emptyset, e \cap T = \emptyset\}| \text{ for all } S \subseteq A, T \subseteq B. \end{aligned} \quad (3)$$

Balanced, perfect, and mengerian hypergraphs do not have totally unimodular incidence matrices in general. Nevertheless, these classes are considered as generalizations of bipartite graphs to hypergraphs. In the following we will show that conditions similar to (2) and (3) hold for some of these classes as well.

In Section 2 we proof three theorems, namely Theorem 2.2, Theorem 2.4, and Theorem 2.6, which can be interpreted as a generalization of Ore's deficient variant of Hall's Theorem, condition (2), and Hall's Theorem for balanced hypergraphs (see [CCKV96]), respectively. Therefore, we introduce a new concept of neighborhood in hypergraphs.

Section 3 deals with the f -factor problem in uniform balanced hypergraphs. We state two existence characterizations, a one-set and a two-set condition. The first one is proven using a min-max theorem given in [ST15]. Whereas the second one can be derived from Theorem 2.4 using a construction which reduces the f -factor problem in uniform balanced hypergraphs to the perfect f -matching problem.

Finally, in Section 4 we show that the f -matching and the f -factor problem are \mathcal{NP} -hard for non-uniform balanced hypergraphs. Moreover, it turns out that for non-uniform mengerian hypergraphs even the perfect matching problem is \mathcal{NP} -hard. This justifies that we mainly investigate uniform hypergraphs in the other sections.

As a last step, we comment on our work and give some directions for future work. We would like to mention that it is possible to obtain some of the results by linear programming methods. However, our main contribution are the purely combinatorial arguments given in this paper.

2 Perfect f -Matchings

In this section we investigate the perfect f -matching problem in two classes of hypergraphs generalizing bipartite graphs, namely mengerian and perfect hypergraphs. Formally, a perfect f -matching is defined as follows.

Definition 2.1. Let $H = (V, E)$ and $f : V \rightarrow \mathbb{Z}_{\geq 0}$ be given. An f -matching is a function $x : E \rightarrow \mathbb{Z}_{\geq 0}$ such that

$$\sum_{e \in E: v \in e} x(e) \leq f(v)$$

for every vertex $v \in V$. An f -matching is called *perfect* if equality holds for every $v \in V$.

2.1 Perfect f -Matchings in Mengerian Hypergraphs

In this subsection we investigate the perfect f -matching problem in mengerian hypergraphs and give some existence characterizations. First, we need the notion of vertex-expansion in order to define a mengerian hypergraph.

Definition 2.2. Let $H = (V, E)$ be a hypergraph, $\lambda \geq 0$ an integer, and $v \in V$. *Expanding* a vertex v by λ means replacing v by λ new vertices v^1, \dots, v^λ , and each hyperedge e which contains v by λ new hyperedges $e^1 = e \setminus \{v\} \cup \{v^1\}, \dots, e^\lambda = e \setminus \{v\} \cup \{v^\lambda\}$. If $\lambda = 0$, we delete v and all hyperedges e containing v . Given $f : V \rightarrow \mathbb{Z}_{\geq 0}$ the *expansion* of H by f is the hypergraph H^f obtained from H by expanding each vertex v by $f(v)$.

Furthermore, we denote by $\nu(H)$ the maximum size of a matching (set of disjoint hyperedges) in H , and by $\tau(H)$ the minimum size of a vertex cover (a set of vertices intersecting each hyperedge at least once). We say that a matching M is perfect, if its hyperedges cover every vertex of the hypergraph. With these notions we can define a mengerian hypergraph.

Definition 2.3. A hypergraph H is called *mengerian* if $\nu(H^f) = \tau(H^f)$ for every expansion H^f of H .

By König's Theorem (see e.g. [LP86]) a bipartite graph G satisfies $\nu(G) = \tau(G)$. As the expansion of a bipartite graph is bipartite as well, every bipartite graph is mengerian. Thus, mengerian hypergraphs can be interpreted as a generalization of bipartite graphs.

We say that a hypergraph is *r -uniform* if all hyperedges have the same size r . If furthermore the vertex set of an r -uniform hypergraph can be partitioned into r subsets such that each hyperedge intersects each of the r subsets exactly once, the hypergraph is called *r -partite*. In general, an r -partite hypergraph is r -uniform but not the other way around. However, for ideal hypergraphs, which are hypergraphs for which the vertex cover polytope

$$P_C := \left\{ x \in \mathbb{R}^V \mid \sum_{v \in e} x_v \geq 1 \text{ for all } e \in E, 0 \leq x \leq 1 \right\}$$

is integral, an r -uniform hypergraph is also r -partite.

Theorem 2.1. *Let $H = (V, E)$ be an ideal hypergraph. If H is r -uniform, then H is r -partite.*

Proof. We prove the theorem by induction on r . For $r = 1$ the claim is trivial.

Now, suppose $r \geq 2$. We show that there is a vertex cover $C \subseteq V$ covering every hyperedge exactly once. The vector $z \in \mathbb{R}^V$ with $z_v := \frac{1}{r}$ is an element of P_C , and can therefore be written as a convex combination of the vertices of P_C . The vertices of P_C are exactly the incidence vectors of vertex covers of H . Thus, there exist k vertex covers C_i with incidence vectors χ^{C_i} for $i = 1, \dots, k$ and $\lambda_1, \dots, \lambda_k \geq 0$ such that $\sum_{i=1}^k \lambda_i = 1$ and $z = \sum_{i=1}^k \lambda_i \chi^{C_i}$. For every hyperedge e we have

$$1 = \sum_{v \in e} z_v = \sum_{v \in e} \sum_{i=1}^k \lambda_i \chi_v^{C_i} = \sum_{i=1}^k \lambda_i |e \cap C_i| \geq \sum_{i=1}^k \lambda_i = 1. \quad (4)$$

Inequality (4) implies that $|e \cap C_i| = 1$ for all $i = 1, \dots, k$. We choose any of the k vertex covers to be C , and consider the hypergraph $H/C := (V \setminus C, \{e \setminus C \mid e \in E\})$. As $|e \cap C| = 1$ for all $e \in E$, the hypergraph H/C is $(r-1)$ -uniform, and, by Theorem 78.2 in [Sch03], H/C is still ideal. By induction, there exists a partition V_1, V_2, \dots, V_{r-1} of $V \setminus C$ such that $|(e \setminus C) \cap V_i| = 1$ for all $e \in E$, $i = 1, \dots, r-1$. We obtain a partition $V_1, V_2, \dots, V_r := C$ of V such that every hyperedge e contains exactly one vertex from each V_i , $i = 1, \dots, r$. \square

As mengerian hypergraphs are also ideal, an r -uniform mengerian hypergraph is always r -partite. In the remainder we will write $H = (V_1, V_2, \dots, V_r, E)$ if H is an r -partite hypergraph with vertex classes V_1, \dots, V_r , i.e., $|e \cap V_i| = 1$ for $i = 1, \dots, r$, and all $e \in E$.

The following definition will be useful to characterize the existence of a perfect f -matching in an r -partite hypergraph by reducing it to a problem in an $(r-1)$ -partite hypergraph.

Definition 2.4. Let $H = (V_1, V_2, \dots, V_r, E)$ be a mengerian r -partite hypergraph, and let $i_1, \dots, i_t \in \{1, \dots, r\}$ with $i_1 < i_2 < \dots < i_t$ for a natural number $t < r$. Moreover, let sets $S_1 \subseteq V_{i_1}, S_2 \subseteq V_{i_2}, \dots, S_t \subseteq V_{i_t}$ for $t < r$ be given. H_{S_1, \dots, S_t} is the $(r-t)$ -partite hypergraph with vertex set $V \setminus (\bigcup_{k=1}^t V_{i_k})$ and edge set $\{e \setminus (S_1 \cup S_2 \cup \dots \cup S_t) \mid e \in E \text{ with } e \cap S_i \neq \emptyset \text{ for } i = 1, \dots, t\}$.

The next remark shows that this construction preserves the mengerian property.

Remark. Observe that the construction H_{S_1, \dots, S_t} can be achieved by first deleting the vertices $V_{i_1} \setminus S_1 \cup \dots \cup V_{i_t} \setminus S_t$, and then contracting S_1, \dots, S_t . By Theorem 79.1 in [Sch03] both operations preserve the mengerian property.

In the following, we slightly abuse notation, i.e., with H_S^f respectively $H_{S_1, \dots, S_{r-1}}^f$ we will denote $(H_S)^f$ respectively $(H_{S_1, \dots, S_{r-1}})^f$. Thereby, we restrict the expansion function f implicitly to the vertex set of H_S respectively $H_{S_1, \dots, S_{r-1}}$.

Now, we can prove a generalization of the defect version of Hall's Theorem for the f -matching problem in r -partite, mengerian hypergraphs.

Theorem 2.2. *Let $H = (V_1, V_2, \dots, V_r, E)$ be an r -partite mengerian hypergraph, and let $f : V \rightarrow \mathbb{Z}_{\geq 0}$ a function. It holds that*

$$\nu(H^f) = f(V_i) - \max_{S \subseteq V_i} (f(S) - \nu(H_S^f)).$$

Proof. Let $x : E \rightarrow \mathbb{Z}_{\geq 0}$ be a maximum size f -matching and $S \subseteq V_i$ attaining the maximum of $f(S) - \nu(H_S^f)$. As mentioned above, we restrict f and also x to H_S by setting $f_S : V \setminus V_i \rightarrow \mathbb{Z}_{\geq 0}$, $f_S(v) = f(v)$ for all $v \in V \setminus V_i$ and $x_S : E(H_S) \rightarrow \mathbb{Z}_{\geq 0}$, $x_S(e \setminus S) = x(e)$ for all $e \in E$ with $e \cap S \neq \emptyset$. Then, x_S is an f_S -matching of H_S , thus $\nu(H_S^f) \geq \sum_{e \in E: e \cap S \neq \emptyset} x_S(e \setminus S) = \sum_{e \in E: e \cap S \neq \emptyset} x(e)$.

We get

$$\begin{aligned}
f(S) - \nu(H_S^f) &\leq f(S) - \sum_{e \in E: e \cap S \neq \emptyset} x(e) \\
&= \sum_{v \in S} (f(v) - \sum_{e \in E: v \in e} x(e)) \\
&\leq \sum_{v \in V_i} (f(v) - \sum_{e \in E: v \in e} x(e)) \\
&= f(V_i) - \sum_{e \in E} x(e) = f(V_i) - \nu(H^f).
\end{aligned}$$

In particular, $\nu(H^f) \leq f(V_i) - \max_{S \subseteq V_i} (f(S) - \nu(H_S^f))$.

For the other direction, let C be a minimum f -size vertex cover of H . Then $f(C) = \tau(H^f) = \nu(H^f)$ holds because H is mengerian. We claim that $C \setminus V_i$ is a vertex cover of $H_{V_i \setminus C}$. All hyperedges of $H_{V_i \setminus C}$ are of the form $e \setminus \{v\}$ for $e \in E(H)$ and some $v \in V_i \setminus C$ with $v \in e$. This implies that $\emptyset \neq e \cap C = (e \setminus \{v\}) \cap C = (e \setminus \{v\}) \cap (C \setminus V_i)$, thus $C \setminus V_i$ is a vertex cover of $H_{V_i \setminus C}$. It follows that

$$\begin{aligned}
f(V_i) - \max_{S \subseteq V_i} (f(S) - \nu(H_S^f)) &\leq f(V_i) - f(V_i \setminus C) + \nu(H_{V_i \setminus C}^f) \\
&= f(V_i \cap C) + \tau(H_{V_i \setminus C}^f) \\
&\leq f(V_i \cap C) + f(C \setminus V_i) \\
&= f(C) = \nu(H^f).
\end{aligned}$$

□

In the case of bipartite graphs and $f \equiv 1$, Theorem 2.2 boils down to Ore's deficient variant of Hall's Theorem.

Corollary 2.3. *Let $H = (V_1, V_2, \dots, V_r, E)$ be an r -partite mengerian hypergraph, with a given function $f : V \rightarrow \mathbb{Z}_{\geq 0}$ such that $f(V_i) = f^*$ for all $i = 1, \dots, r$. H has no perfect f -matching if and only if there exist $S_i \subseteq V_i$ for $i = 1, \dots, r-1$ such that*

$$\nu(H_{S_1, \dots, S_{r-1}}^f) < f(S_1) + \dots + f(S_{r-1}) - (r-2)f^*.$$

Proof. By Theorem 2.2 and the fact that $H_{S_1, \dots, S_k, S_{k+1}}^f = ((H_{S_1, \dots, S_k})_{S_{k+1}})^f$ there exists $S_{k+1} \subseteq V_{k+1}$ such that

$$\nu(H_{S_1, \dots, S_k}^f) = f(V_{k+1}) - f(S_{k+1}) + \nu(H_{S_1, \dots, S_k, S_{k+1}}^f)$$

for every $1 \leq k \leq r-1$. Applied iteratively, we get

$$\begin{aligned}
\nu(H_{S_1}^f) &= f(V_2) - f(S_2) + \nu(H_{S_1, S_2}^f) \\
&= \dots = \sum_{i=2}^{r-1} (f(V_i) - f(S_i)) + \nu(H_{S_1, \dots, S_{r-1}}^f).
\end{aligned}$$

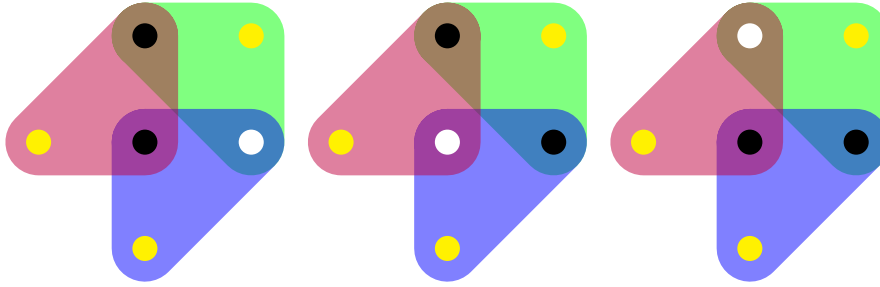


Figure 1: An example of a hypergraph $H = (V, E)$, a set $S \subseteq V$ (yellow), and the three sets in $\mathcal{N}(S)$ (black).

Furthermore, H has no perfect f -matching if and only if $\nu(H^f) < f^*$. This implies, that there exists $S_1 \subseteq V_1$ with $\nu(H_{S_1}^f) < f(S_1)$. As a result the desired inequality

$$\begin{aligned} \nu(H_{S_1, \dots, S_{r-1}}^f) &= \nu(H_{S_1}^f) + f(S_2) + \dots + f(S_{r-1}) - (r-2)f^* \\ &< f(S_1) + \dots + f(S_{r-1}) - (r-2)f^* \end{aligned}$$

follows. □

In order to generalize the results of the existence of f -matchings in bipartite graphs to r -uniform mengerian hypergraphs we introduce a new concept for the neighborhood in hypergraphs. Compared to the graph case it will be defined slightly different, i.e., the neighborhood in a hypergraph will be a set of subsets of V and not just one subset of V .

Definition 2.5. Let $H = (V, E)$ be a hypergraph. The *neighborhood* $\mathcal{N}(A)$ of $A \subseteq V$ is defined by

$$\mathcal{N}(A) := \{B \subseteq V \setminus A \mid B \text{ is (inclusionwise-)minimal such that} \\ \text{if } e \cap A \neq \emptyset, \text{ then } e \cap B \neq \emptyset \forall e \in E\}.$$

If $G = (V, E)$ is a graph and $S \subseteq V$ a stable set then $\mathcal{N}(S)$ contains only the neighborhood $\Gamma(S)$ of S . In general hypergraphs the neighborhood can contain more than one minimal set (compare Figure 1), and it can be empty.

Now, we prove a characterization for the existence of perfect f -matchings in uniform mengerian hypergraphs as an application of Corollary 2.3. It is a direct generalization of characterization (2) stated in Section 1.

Theorem 2.4. Let $H = (V_1, \dots, V_r, E)$ be an r -uniform, mengerian hypergraph, and let $f : V \rightarrow \mathbb{Z}_{\geq 0}$ be a function. H has a perfect f -matching if and only if

- (a) $f(V_1) = f(V_2) = \dots = f(V_r)$ and
- (b) $f(X) \leq f(Y)$ for all $X \subseteq V_1, Y \subseteq V \setminus V_1, Y \in \mathcal{N}(X)$.

Proof. Suppose H has a perfect f -matching $x : E \rightarrow \mathbb{Z}_{\geq 0}$. Then (a) holds because for every $i = 1, \dots, r$ we have

$$f(V_i) = \sum_{v \in V_i} f(v) = \sum_{v \in V_i} \sum_{e \in E: v \in e} x(e) = \sum_{e \in E} |e \cap V_i| x(e) = \sum_{e \in E} x(e).$$

For (b), let $X \subseteq V_1$, $Y \subseteq V \setminus V_1$, $Y \in \mathcal{N}(X)$. As Y is in the neighborhood of X we have $e \cap Y \neq \emptyset$ for all $e \in E$ with $e \cap X \neq \emptyset$. It follows that $|e \cap Y| \geq |e \cap X|$ for all $e \in E$ because $|e \cap X| \leq 1$, and if $|e \cap X| = 1$, then $|e \cap Y| \geq 1$ as $Y \in \mathcal{N}(X)$. This yields

$$f(X) = \sum_{v \in X} \sum_{e \in E: v \in e} x(e) = \sum_{e \in E} |e \cap X| x(e) \leq \sum_{e \in E} |e \cap Y| x(e) = f(Y).$$

For the other direction, suppose H has no perfect f -matching and $f(V_i) = f^*$ for $i = 1, \dots, r$. By Corollary 2.3 there exist sets $S_i \subseteq V_i$ for $i = 1, \dots, r-1$ with $\nu(H_{S_1, \dots, S_{r-1}}^f) < f(S_1) + \dots + f(S_{r-1}) - (r-2)f^*$.

The hypergraph $H_{S_1, \dots, S_{r-1}}$ is 1-uniform and consists of all singleton hyperedges $\{v\}$ such that there exist vertices $s_i \in S_i$, $i = 1, \dots, r-1$ with $\{s_1, \dots, s_{r-1}, v\} \in E$. Let $S_r := \{v \in S_r \mid \{v\} \in E(H_{S_1, \dots, S_{r-1}})\}$ be the set of these vertices. A maximum size f -matching in $H_{S_1, \dots, S_{r-1}}$ consists just of $f(v)$ copies of $\{v\}$ for each $v \in S_r$, thus $\nu(H_{S_1, \dots, S_{r-1}}^f) = f(S_r)$.

Now, we define the sets $X \subseteq V_1$ and $\tilde{Y} \subseteq V \setminus V_1$ as $X := S_1$ and $\tilde{Y} := (V_2 \setminus S_2) \cup (V_3 \setminus S_3) \cup \dots \cup (V_{r-1} \setminus S_{r-1}) \cup S_r$. With this definition we have

$$\begin{aligned} f(\tilde{Y}) &= f(S_r) + \sum_{i=2}^{r-1} (f(V_i) - f(S_i)) \\ &= \nu(H_{S_1, \dots, S_{r-1}}^f) - \sum_{i=2}^{r-1} f(S_i) + (r-2)f^* < f(S_1) = f(X). \end{aligned}$$

Let e be a hyperedge with $e \cap X \neq \emptyset$. If $e \cap S_i = \emptyset$ for some $i = 2, \dots, r-1$, then $e \cap \tilde{Y} \supseteq e \cap (V_i \setminus S_i) \neq \emptyset$. Otherwise, $e \cap S_i \neq \emptyset$ for $i = 1, 2, \dots, r-1$, and thus $e \cap \tilde{Y} = e \cap S_r \neq \emptyset$. Therefore, \tilde{Y} is a superset of some $Y \in \mathcal{N}(X)$ and, since $f(\tilde{Y}) < f(X)$, also $f(Y) < f(X)$ holds. \square

In the case of bipartite graphs, the condition of the foregoing theorem is equivalent to Condition (2) stated in Section 1.

All the results stated so far hold just for uniform mengerian hypergraphs. For non-uniform mengerian hypergraphs we can state the following two conditions.

Lemma 2.5. *Let $H = (V, E)$ be a mengerian hypergraph with $s := \min\{|e| \mid e \in E\}$, $r := \max\{|e| \mid e \in E\}$, and $f : V \rightarrow \mathbb{Z}_{\geq 0}$ a function on the vertices of H .*

- (a) *If H has a perfect f -matching then $f(A) \leq (r-1)f(B)$ for all $A \subseteq V$ and $B \in \mathcal{N}(A)$.*
- (b) *If $f(A) \leq (s-1)f(B)$ for all $A \subseteq V$ and $B \in \mathcal{N}(A)$ then H has a perfect f -matching.*

Proof. (a) Can be shown by a straightforward calculation.

- (b) Suppose H has no perfect f -matching, then, from the equation $\nu(H^f) = \tau(H^f)$, we obtain a vertex cover C of H such that $f(C) < f(V)/s$. With $A := V \setminus C$ we get $f(A) = f(V) - f(C) > (s-1)f(C)$. Because of $e \cap C \neq \emptyset$ for all $e \in E$ we can choose a minimal set $B \subseteq C$ with $B \in \mathcal{N}(A)$. Clearly, also the inequality $f(A) > (s-1)f(B)$ holds. \square

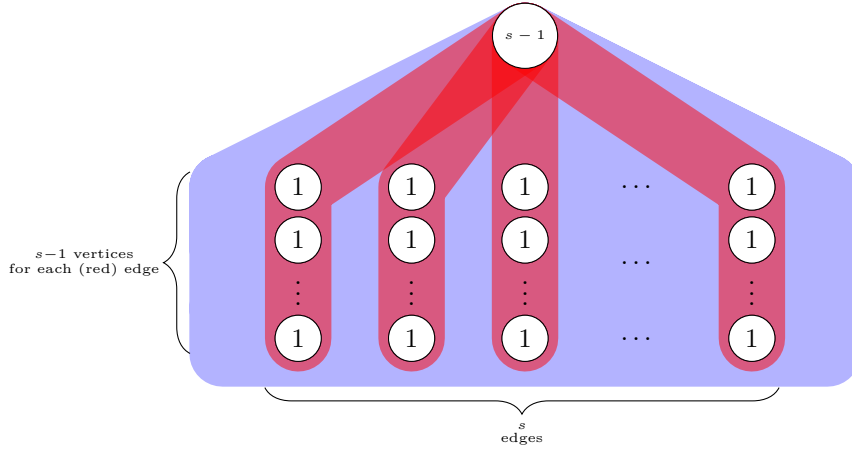


Figure 2: Sharpness Example for Lemma 2.5 b).

Remark. For non-uniform hypergraphs the conditions in Lemma 2.5 are only sufficient or necessary but in general not both. Moreover, the constants $r - 1$, $s - 1$ in (a) and (b) are best possible as the following examples show:

- (a) Let $H = (V_1, V_2, \dots, V_r, E)$ be the complete r -partite hypergraph with $|V_1| = |V_2| = \dots = |V_r| = r$, i.e., E consists of all r^r hyperedges $\{v_1, v_2, \dots, v_r\}$ with $v_1 \in V_1, v_2 \in V_2, \dots, v_r \in V_r$. It is easy to show that H is mengerian and has a perfect matching. With $f \equiv 1$ and choosing $A := V \setminus V_1$ and $B := V_1 \in \mathcal{N}(A)$ we obtain $f(A) = |A| = (r-1)|B| = (r-1)f(B)$. Thus $(r-1)$ cannot be replaced by some smaller natural number.
- (b) For every $s \geq 3$ we construct a hypergraph on

$$V_s := \{v^*\} \cup \{v_j^i \mid 1 \leq i \leq s, 1 \leq j \leq s-1\}$$

with hyperedge set

$$E_s := \{V_s\} \cup \{\{v^*, v_1^i, v_2^i, \dots, v_{s-1}^i\} \mid 1 \leq i \leq s\},$$

and let $f : V_s \rightarrow \mathbb{Z}_{\geq 0}$ be defined by $f(v^*) := s - 1$ and $f(v) := 1$ for all $v \in V_s \setminus \{v^*\}$ (compare Figure 2). With these definitions H_s is mengerian, the minimum size of a hyperedge in E_s is exactly s , and $H_s := (V_s, E_s)$ has no perfect f -matching. However, we claim that $f(A) \leq sf(B)$ for all $A \subseteq V_s, B \in \mathcal{N}(A)$. We distinguish whether $v^* \in A$ or $v^* \notin A$. In the first case, $B \in \mathcal{N}(A)$ has to contain at least one vertex v_j^i for every $i = 1, \dots, s$, and by minimality it contains exactly one, e.g. $B := \{v_1^{s-1}, \dots, v_s^{s-1}\}$. Then $f(A)$ is maximum if $A := \{v^*\} \cup \{v_j^i \mid 1 \leq i \leq s, 1 \leq j \leq s-2\}$. For that choice of $A \subseteq V_s$ and $B \in \mathcal{N}(A)$ we have

$$f(A) = s - 1 + s(s - 2) = s(s - 1) - 1 < s(s - 1) = sf(B).$$

If $v^* \notin A$, then $f(A)$ is maximum and $f(B)$ minimum for $A := V_s \setminus \{v^*\}$ and $B := \{v^*\}$. In this case, we still have $f(A) = s(s - 1) = sf(B)$.

Altogether, this example shows that $(s - 1)$ in Lemma 2.5 (b) cannot be replaced by a larger natural number.

2.2 Perfect f -Matchings in Perfect Hypergraphs

In this subsection we look at the perfect f -matching problem in another class of hypergraphs generalizing bipartite graphs, namely so-called perfect hypergraphs. By a result of Lovász in [Lov72] a perfect hypergraph can be defined in the following way (compare Chapter 82 in [Sch03]).

Definition 2.6. A hypergraph $H = (V, E)$ is called *perfect* if a perfect graph G on V exists such that the maximal hyperedges of H correspond to the maximal cliques of G .

This graph theoretical characterization allows us to show that Hall's condition for balanced hypergraph of [CCKV96] also holds for uniform perfect hypergraphs. We can even show that it gives rise to a characterization for the existence of perfect f -matchings.

Theorem 2.6. *Let $H = (V, E)$ be a perfect r -uniform hypergraph, $f : V \rightarrow \mathbb{Z}_{\geq 0}$ be a given function. H has no perfect f -matching if and only if there exists a pair $R, B \subseteq V$ of disjoint node sets such that $|e \cap R| \geq |e \cap B|$ for all hyperedges $e \in E$ but $f(R) < f(B)$.*

Proof. By an easy counting argument, H cannot have a perfect f -matching if such a pair R, B exists.

Now, suppose H has no perfect f -matching. Let G be a perfect graph on V such that the hyperedges of H correspond to the maximal cliques of G . On the one hand, as G is perfect, there exists an r -coloring of the vertices V . Let S be the color class of this coloring with the smallest f -value. Then, S is a stable set with $f(S) \leq f(V)/r$. On the other hand, G has a stable set \tilde{S} of f -value greater than $f(V)/r$. This follows from the fact that every minimum size set of cliques covering each vertex v at least $f(v)$ -times must have size greater than $f(V)/r$, otherwise it would correspond to a perfect f -matching of H .

Now, we set $R := S \setminus \tilde{S}$, $B := \tilde{S} \setminus S$. It follows that

$$f(R) = f(S) - f(S \cap \tilde{S}) < f(\tilde{S}) - f(S \cap \tilde{S}) = f(B).$$

Furthermore, $|e \cap B| \leq 1$ holds for all $e \in E$ as \tilde{S} is a stable set of G and every hyperedge $e \in E$ corresponds to a maximal clique in G . If $|e \cap B| = 1$, then $1 = |e \cap \tilde{S}| - |e \cap (\tilde{S} \cap S)|$, in particular it follows that $|e \cap (\tilde{S} \cap S)| = 0$. This and the fact that S intersects every maximal clique of G , and thus every $e \in E$, exactly once implies $|e \cap R| = |e \cap S| = 1$. Altogether, we have $|e \cap R| \geq |e \cap B|$ for all $e \in E$. \square

The theorem above implies that every uniform perfect hypergraph satisfies Hall's condition. This is not true for non-uniform perfect hypergraphs, as the following example shows:

Example. Consider the hypergraph H on the vertex set $\{1, 2, 3, 4\}$ with hyperedges $\{1, 2, 4\}, \{2, 3, 4\}$ of size three, and edges $\{2, 3\}, \{3, 4\}, \{2, 4\}$ depicted in Figure 3. This hypergraph is perfect because the maximal hyperedges $\{1, 2, 4\}, \{2, 3, 4\}$ correspond to the maximal cliques in the perfect graph

$$G = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}).$$

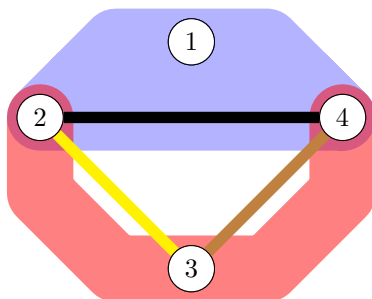


Figure 3: Example showing that Theorem 2.6 does not hold for non-uniform perfect hypergraphs.

H has no perfect matching, however, one can check that $|R| \geq |B|$ for all disjoint vertex sets $R, B \subseteq V$ with $|e \cap R| \geq |e \cap B|$.

3 f -Factors in Balanced Hypergraphs

In this section we consider the f -factor problem in balanced hypergraphs which were defined by Berge as follows:

Definition 3.1 ([Ber89]). Let $H = (V, E)$ be a hypergraph. A *cycle* of length k in H is an alternating sequence $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ such that $v_{k+1} = v_1$, the vertices v_1, v_2, \dots, v_k are distinct, the hyperedges e_1, e_2, \dots, e_k are distinct, and $v_i, v_{i+1} \in e_i$ for $i = 1, \dots, k$.

A cycle $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ is called *strong* if $e_i \cap \{v_1, \dots, v_k\} = \{v_i, v_{i+1}\}$ for all $i = 1, \dots, k$. H is *balanced* if H does not contain any strong cycle of odd length.

Balanced hypergraphs form a subclass of mengerian hypergraphs. This was shown by Fulkerson, Hoffman, and Oppenheim using linear programming arguments ([FHO74]).

Remark. [BSS16] With combinatorial methods this result can be proven using a similar idea as in [ST15]. Instead of considering a relaxation of matchings use relaxed vertex covers. Therefore, introduce penalty costs for edges which are not covered and proceed along the lines of the proof of Theorem 1 in [ST15].

Now, we define formally what an f -factor in a hypergraph is.

Definition 3.2. Let $H = (V, E)$ and $f : V \rightarrow \mathbb{Z}_{\geq 0}$ be given. An f -factor of H is a subset $F \subseteq E$ such that $|\{e \in F \mid v \in e\}| = f(v)$ for all $v \in V$.

Our next theorem is a characterization of the existence of f -factors in uniform balanced hypergraphs.

Theorem 3.1. Let $H = (V, E)$ be a balanced, r -uniform hypergraph with vertex partition $V = V_1 \cup V_2 \cup \dots \cup V_r$. H has an f -factor if and only if

$$(a) \quad f(V_1) = f(V_2) = \dots = f(V_r) \text{ and}$$

(b) $f(S) - f(V_1) \leq \sum_{e \in E: |e \cap S| \geq 1} (|e \cap S| - 1)$ for all $S \subseteq V$.

Proof. First, suppose H has an f -factor $F \subseteq E$, then for every $S \subseteq V$ the following holds:

$$\begin{aligned} f(S) - f(V_1) &= \sum_{v \in S} \deg_F(v) - \sum_{v \in V_1} \deg_F(v) = \sum_{e \in F} |e \cap S| - \sum_{e \in F} |e \cap V_1| \\ &\leq \sum_{e \in F: |e \cap S| \geq |e \cap V_1|} (|e \cap S| - |e \cap V_1|) \\ &\leq \sum_{e \in E: |e \cap S| \geq 1} (|e \cap S| - 1). \end{aligned}$$

Choosing $S = V_i$ yields $f(V_i) - f(V_1) \leq 0$ and by the same argument $f(V_1) - f(V_i) \leq 0$.

For the other direction, suppose H has no f -factor and (a) holds. Then the optimal value of

$$\begin{aligned} &\min \sum_{e \in E} x(e) \\ &\sum_{e \ni v} x(e) \geq f(v) \text{ for all } v \in V \\ &0 \leq x(e) \leq 1 \text{ for all } e \in E \end{aligned}$$

is larger than $f(V)/r$ as H is r -uniform and balanced. Now, we use Theorem 1 of [ST15] but apply it to the dual hypergraph H^* with $d = f$ and penalty costs $b(e) = 1$ for all $e \in E$. This shows that there exists $S \subseteq V$ with $f(S) - \sum_{e: |e \cap S| \geq 1} (|e \cap S| - 1) > f(V)/r$. Rearranging this inequality and using (a) yields the desired contradiction to (b). \square

Observe that the foregoing theorem gives a one-set condition. For bipartite graphs it is equivalent to a known condition by Heinrich et al. [HHKL90].

The next theorem establishes an alternative characterization which generalizes Condition (3) mentioned in Section 1 for f -factors in bipartite graphs to uniform, balanced hypergraphs. We just state the idea of a combinatorial proof and omit details. Note, that it can also be proven by linear programming arguments.

Theorem 3.2. *Let $H = (V, E)$ be a balanced r -uniform hypergraph with vertex partition $V = V_1 \cup V_2 \cup \dots \cup V_r$. H has an f -factor if and only if*

- (a) $f(V_1) = f(V_2) = \dots = f(V_r)$ and
- (b) $f(X) - f(Y) \leq |\{e \in E \mid e \cap X \neq \emptyset, e \cap Y = \emptyset\}|$ for all $X \subseteq V_1$ and $Y \subseteq V \setminus V_1$.

In order to give a chain of combinatorial arguments we reduce Theorem 3.2 to Theorem 2.4 using the following new construction that generalizes the one described by Lovász and Plummer in Chapter 10 of [LP86] for graphs.

Let $H = (V_1, V_2, \dots, V_r, E)$ be an r -partite hypergraph. We will define an auxiliary hypergraph $\tilde{H} = (\tilde{V}, \tilde{E})$. In \tilde{H} each hyperedge e of H is replaced

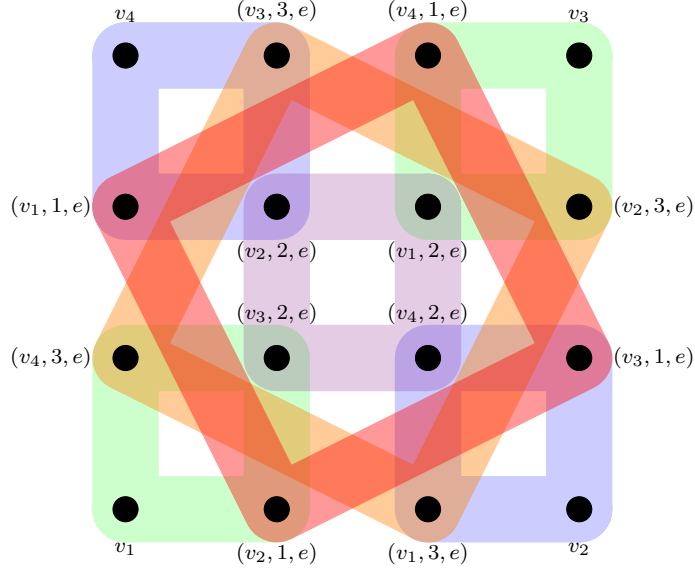


Figure 4: Construction of \tilde{H} .

by $2r - 1$ new hyperedges in such a way that the balancedness is preserved. Formally, the construction of \tilde{H} works as follows:

For every hyperedge $e = \{v_1, v_2, \dots, v_r\}$ of the original hypergraph H we introduce $r(r - 1)$ new vertices (v_i, j, e) where $1 \leq i \leq r, 1 \leq j \leq r - 1$, and for every e we define $2r - 1$ new hyperedges by

$$\begin{aligned}
 e_1 &:= \{v_1, (v_2, 1, e), (v_3, 2, e), \dots, (v_r, r - 1, e)\}, \\
 e_2 &:= \{(v_1, 1, e), (v_2, 2, e), \dots, (v_{r-1}, r - 1, e), v_r\}, \\
 &\vdots \\
 e_i &:= \{(v_1, i - 1, e), (v_2, i, e), \dots, (v_{r-i+1}, r - 1, e), v_{r-i+2}, \\
 &\quad (v_{r-i+3}, 1, e), (v_{r-i+4}, 2, e), \dots, (v_r, i - 2, e)\}, \\
 &\vdots \\
 e_r &:= \{(v_1, r - 1, e), v_2, (v_3, 1, e), (v_4, 2, e), \dots, (v_r, r - 2, e)\}, \\
 e_{r+j} &:= \{(v_1, j, e), (v_2, j, e), \dots, (v_r, j, e)\} \text{ for all } 1 \leq j \leq r - 1.
 \end{aligned}$$

We set $\tilde{V} := V(H) \cup \{(v_i, j, e) \mid \forall e \in E, v_i \in e, 1 \leq j \leq r - 1\}$ and $\tilde{E} := \{e_1, \dots, e_{2r-1} \mid e \in E\}$. An example of this construction for $r = 4$ and an edge $e = \{v_1, v_2, v_3, v_4\}$ is depicted in Figure 4. One can prove that the resulting hypergraph \tilde{H} has a perfect f -matching if and only if H has an f -factor. Moreover, the construction preserves balancedness. Thus, \tilde{H} is mengerian. Translating the condition of Theorem 2.4 for \tilde{H} to H yields Theorem 3.2.

4 Complexity Results

In the foregoing sections we dealt mainly with uniform hypergraphs which is justified by the fact that most non-uniform existence questions turn out to be hard. Indeed, the f -factor problem is \mathcal{NP} -complete for non-uniform balanced hypergraphs.

Theorem 4.1. *Let $H = (V, E)$ be a balanced hypergraph and $f : V \rightarrow \mathbb{Z}_{\geq 0}$ be a function on the vertices of H . Deciding whether H has an f -factor, respectively, a perfect f -matching is \mathcal{NP} -complete.*

Proof. The two problems are clearly in \mathcal{NP} . In the following, we show that 3-dimensional matching, which is one of Karp's 21 \mathcal{NP} -complete problems [Kar72], is reducible to the f -factor problem in balanced hypergraphs. The same reduction works for the perfect f -matching problem as well.

An instance of the 3-dimensional matching problem consists of an arbitrary three-partite hypergraph $H = (V, E)$ with vertex set $V = V_1 \cup V_2 \cup V_3$ and $|V_1| = |V_2| = |V_3|$. One has to decide whether H has a perfect matching. Given such an instance, we define an auxiliary balanced hypergraph $\tilde{H} := (\tilde{V}, \tilde{E})$ as follows (compare Figure 5).

- The vertex set \tilde{V} is the union of V_1, V_2, V_3, E and four new elements $h_{e,1}, h_{e,2}, h_{e,3}, h_{e,4}$ for every $e \in E$.
- The hyperedge set \tilde{E} consists of all edges $\{v, e\}$ for all $e \in E$ and $v \in e$, together with all edges of the form $\{h_{e,i}, e\}$ for $e \in E, i = 1, 2, 3, 4$, and all hyperedges $\{h_{e,1}, h_{e,2}, h_{e,3}, h_{e,4}, e\}$ for $e \in E$.

We define a function $f : \tilde{V} \rightarrow \mathbb{Z}_{\geq 0}$ by $f(v) = 1 = f(h_{e,i})$ for all $v \in V, e \in E, i = 1, 2, 3, 4$, and $f(e) := 4$ for all $e \in E$.

The hypergraph \tilde{H} defined in this way is balanced because a strong odd cycle cannot contain any hyperedge of size 5, and the remaining edges form a bipartite graph. It remains to show that H has an f -factor if and only if the 3-dimensional matching instance is a ‘Yes’ instance.

Suppose there exists a perfect matching $M \subseteq E$ of H . We define an f -factor $F \subseteq \tilde{E}$ of \tilde{H} by

$$F := \{\{v, e\} \mid e \in M, v \in e\} \cup \{\{h_{e,1}, h_{e,2}, h_{e,3}, h_{e,4}, e\} \mid e \in M\} \\ \cup \{\{h_{e,i}, e\} \mid e \notin M, i = 1, 2, 3, 4\}.$$

I.e., for $e \in M$ take the blue hyperedges and for $e \notin M$ the red ones in Figure 5. Then the degree of every vertex $e \in E$ in (\tilde{V}, F) is four and the degree of the other vertices is one, therefore F is an f -factor of \tilde{H} .

On the other hand, let $F \subseteq \tilde{E}$ be an f -factor of \tilde{H} . By the construction of \tilde{H} and f , for every $e \in E$ we have either $\{h_{e,1}, h_{e,2}, h_{e,3}, h_{e,4}, e\} \in F$ or $\{h_{e,1}, e\}, \{h_{e,2}, e\}, \{h_{e,3}, e\}, \{h_{e,4}, e\} \in F$ but not both. We define a perfect matching $M \subseteq E$ of H by

$$M := \{e \in E \mid \{h_{e,1}, h_{e,2}, h_{e,3}, h_{e,4}, e\} \in F\}.$$

For every $v \in V$ there exists $e \in E$ such that $\{v, e\} \in F$ which implies that the hyperedge $\{h_{e,1}, h_{e,2}, h_{e,3}, h_{e,4}, e\}$ lies in F . In particular, every $v \in V$ is covered

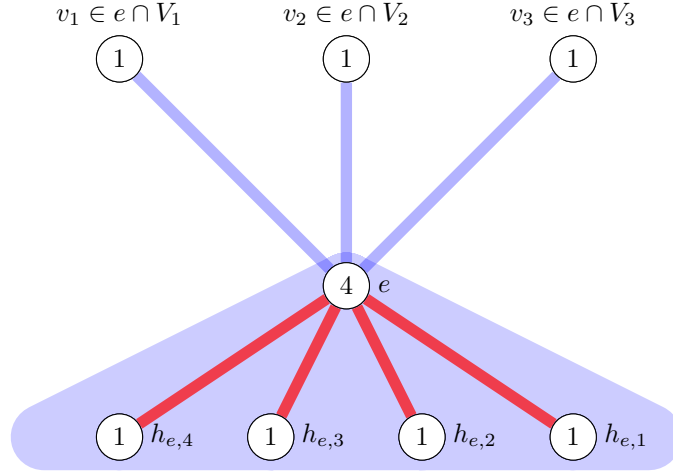


Figure 5: The Gadget of the Complexity Reduction.

by some $e \in M$. Suppose there exist two distinct $e, e' \in M$ covering the same $v \in V$. $e, e' \in M$ means $\{h_{e,1}, h_{e,2}, h_{e,3}, h_{e,4}, e\}, \{h_{e',1}, h_{e',2}, h_{e',3}, h_{e',4}, e'\} \in F$ which implies $\{v, e\}, \{v, e'\} \in F$. However, the degree of v in (\bar{V}, F) is exactly one. Thus, M is a perfect matching of H . \square

Remark. The proof shows that the f -factor and the perfect f -matching problem are \mathcal{NP} -complete for non-uniform, balanced hypergraphs of rank at least five. A balanced hypergraph of rank at most three has a totally unimodular incidence matrix, and therefore the f -factor and the perfect f -matching problem can be solved in polynomial time by linear programming. So the only open case is rank four. As there is a balanced hypergraph of rank four such that the system $Ax = f, x \geq 0$ has a fractional but no integral solution, it is likely that both problems remain \mathcal{NP} -complete for non-uniform, balanced hypergraphs of rank four.

The perfect f -matching problem can be solved in polynomial time in uniform mengerian hypergraphs because an r -uniform mengerian hypergraph has a perfect matching if and only if $\nu(H^f) = f(V)/r$. In this case a perfect matching can be calculated by solving a linear program. In the non-uniform case even the perfect matching problem ($f \equiv 1$) is \mathcal{NP} -hard.

Corollary 4.2. *Deciding whether a mengerian hypergraph has a perfect matching is \mathcal{NP} -complete.*

Proof. Let $H = (V, E)$ be a balanced hypergraph, and $f : V \rightarrow \mathbb{Z}_{\geq 0}$ a function. H has a perfect f -matching if and only if H^f has a perfect matching. Furthermore, H^f is mengerian because for every function $g : V(H^f) \rightarrow \mathbb{Z}_{\geq 0}$ it holds that $(H^f)^g = H^h$ where $h : V \rightarrow \mathbb{Z}_{\geq 0}$ with $h(v) = \sum_{i=1, \dots, f(v)} g(v^i)$ for all $v \in V$. Thus, $\nu((H^f)^g) = \tau((H^f)^g)$ for all $g : V(H^f) \rightarrow \mathbb{Z}_{\geq 0}$. Of course, the number of vertices and hyperedges of H^f depend on the values of f but in the reduction of the 3-dimensional matching problem to the perfect f -matching

problem in balanced hypergraphs we had $f(v) \leq 4$ for all $v \in V$. This shows, that there is also a polynomial time reduction of the 3-dimensional matching problem to the perfect matching problem in mengerian hypergraphs. \square

Conclusion

We proved several existence conditions for perfect f -matchings and f -factors in classes of “bipartite hypergraphs” via combinatorial arguments. There are still some classes left which generalize bipartite graphs and were not mentioned so far, for example normal hypergraphs or arboreal hypergraphs. It turns out that the f -matching problem is already \mathcal{NP} -hard for uniform arboreal hypergraphs of rank at least four (Reduce a 3-dimensional matching instance H to an f -matching problem by putting an auxiliary vertex into all edges of H). Therefore, uniform and normal hypergraphs of rank three and balanced hypergraphs of rank four are the only interesting cases left. Additionally, further necessary and sufficient conditions for the existence of factors and perfect f -matchings in “bipartite hypergraphs” can be investigated.

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