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ZIB Report 16-16 (March 2016)

Herausgegeben vom
Konrad-Zuse-Zentrum für Informationstechnik Berlin
Takustraße 7
D-14195 Berlin-Dahlem

Telefon: 030-84185-0
Telefax: 030-84185-125

e-mail: bibliothek@zib.de
URL: http://www.zib.de

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

Robust Allocation of Operating Rooms with Lognormal case Durations

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Keywords: Robust optimization, Lognormal duration, Hilbert's projective metric.

1 Introduction

The operating theater is one of the most expensive hospital resources. Therefore the allocation of surgeries to operating rooms is a critical planning step for large hospitals, involving both combinatorial optimization problems and uncertainty handling, see e.g. Guerriero and Guido (2011). In this paper, we consider a robust optimization problem introduced by Denton *et al.* (2010) for assigning operating rooms (OR) to a list of *patient blocks*, that is, groups of elective patients to be operated one after another by the same surgeon. This paper presented a mixed integer programming (MIP) model to find an optimal allocation of the ORs, robust against all duration scenarios $\boldsymbol{d}$ for the patient blocks in the set

$$\mathcal{D} = \{\boldsymbol{d} \in \mathbb{R}^n : \ \forall i, \ell_i \leq d_i \leq u_i; \ \sum_i \frac{d_i - \ell_i}{u_i - \ell_i} \leq \tau\}. \tag{1}$$

However, it is well known that surgical durations closely fit a lognormal distribution, see e.g. Kayış *et al.* (2014) and the references therein. Therefore, we expect departures from the nominal scenario to be highly nonlinear, a fact poorly captured by the set $\mathcal{D}$. We present a cutting-plane approach to solve a robust optimization problem protecting over confidence regions of a lognormal distribution. In particular, we show that fixed point iterations can be used to solve a nonconvex optimization problem to generate cut inequalities.

Throughout this article, we adopt the terminology of the job shop scheduling literature because we believe that the problem studied here could have other fields of application. Hence, patient blocks are called *jobs* and operating rooms are called *machines*.

2 Problem Formulation

We denote by $\mathcal{J}$ and $\mathcal{M}$ the sets of *jobs* and *machines*, of respective cardinality n and p. The binary variable z_m indicates whether machine $m \in \mathcal{M}$ is activated, and the binary variable x_{jm} tells whether job j is allocated to machine m. Each job must be allocated to one activated machine, so the set of all feasible solutions reads

$$\mathcal{X} := \left\{ (\boldsymbol{x}, \boldsymbol{z}) \in \{0,1\}^{n \times p} \times \{0,1\}^p : \ \forall j \in \mathcal{J}, \sum_{m \in \mathcal{M}} x_{jm} = 1, \quad \forall j, m \in \mathcal{J} \times \mathcal{M}, \ x_{jm} \leq z_m \right\}.$$

Denote by T_m the time available on machine m (if it is activated), c_f^m the fixed cost for activating machine m and c_o^m the cost of overtime per unit of time on machine m. If the duration of job j is $d_j > 0$, the total cost of an allocation $(\boldsymbol{x}, \boldsymbol{z}) \in \mathcal{X}$ can be measured as

$$F(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{d}) := \sum_{m \in \mathcal{M}} c_f^m z_m + c_o^m \left(\sum_{j \in \mathcal{J}} x_{jm} d_j - T_m \right)^+,$$

where $(u)^+ := \max(u, 0)$ denotes the nonnegative part of $u \in \mathbb{R}$.

In this paper, we consider the problem of finding the allocation $(\boldsymbol{x}, \boldsymbol{z}) \in \mathcal{X}$ minimizing the overtime, while protecting ourselves against a set of likely scenarios $\mathcal{D}$. This leads to the following robust optimization problem:

$$\min_{(\boldsymbol{x},\boldsymbol{z})\in\mathcal{X}} \max_{\boldsymbol{d}\in\mathcal{D}} F(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{d}). \tag{2}$$

We propose to use a cutting plane approach to solve Problem (2). Given a finite set of scenarios $\hat{\mathcal{D}} = \{\boldsymbol{d}^{(1)}, \ldots, \boldsymbol{d}^{(s)}\} \subseteq \mathcal{D}$, we first observe that the *restricted master problem* $\min_{(\boldsymbol{x},\boldsymbol{z})\in\mathcal{X}} \max_{\boldsymbol{d}\in\hat{\mathcal{D}}} F(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{d})$ can be formulated as a mixed integer linear program:

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{z},\Delta,\boldsymbol{\delta}} \quad & \sum_{m\in\mathcal{M}} c_f^m z_m + \Delta && && \text{(3a)}\\
\text{s.t.} \quad & \delta_{im} \geq \sum_{j\in\mathcal{J}} x_{jm} d_j^{(i)} - z_m T_m, && \forall i \in \{1,\ldots,s\}, \forall m \in \mathcal{M}, && \text{(3b)}\\
& \delta_{im} \geq 0, && \forall i \in \{1,\ldots,s\}, \forall m \in \mathcal{M}, && \text{(3c)}\\
& \Delta \geq \sum_{m\in\mathcal{M}} c_o^m \delta_{im}, && \forall i \in \{1,\ldots,s\}, && \text{(3d)}\\
& (\boldsymbol{x}, \boldsymbol{z}) \in \mathcal{X} && && \text{(3e)}
\end{aligned}$$

The objective function (3a) minimizes the fixed cost $\sum_m c_f^m z_m$ and the robust overtime cost Δ, equations (3b) and (3c) define the overtime δ_{im} for machine m and scenario $\boldsymbol{d}^{(i)}$, and (3d) makes sure that Δ is the worst-case overtime cost over all scenarios in $\hat{\mathcal{D}}$. Finally, (3e) ensures that $(\boldsymbol{x}, \boldsymbol{z})$ is a valid allocation.

Next, we introduce the adversarial problem, which, given a solution $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ of the restricted master problem (3), finds the worst scenario within the uncertainty set $\mathcal{D}$,

$$\max_{\boldsymbol{d}\in\mathcal{D}} F(\boldsymbol{x}^*, \boldsymbol{z}^*; \boldsymbol{d}). \tag{4}$$

The cutting plane algorithm to solve Problem (2) can be described as follows. Start with $\mathcal{D}^{(1)} = \{\bar{\boldsymbol{d}}\}$, where $\bar{d}_i$ is the expected value of d_i. At iteration $k \in \mathbb{N}$, solve Problem (3) for $\hat{\mathcal{D}} = \mathcal{D}^{(k)}$ and set $(\boldsymbol{x}^{(k)}, \boldsymbol{z}^{(k)})$ to the optimal solution. Then, solve Problem (4) with $(\boldsymbol{x}^*, \boldsymbol{z}^*) = (\boldsymbol{x}^{(k)}, \boldsymbol{z}^{(k)})$, insert the worst case scenario $\boldsymbol{d}^{(k)}$ in the restricted uncertainty set, $\mathcal{D}^{(k+1)} = \mathcal{D}^{(k)} \cup \{\boldsymbol{d}^{(k)}\}$, and iterate.

It is straightforward that at each iteration, the optimal value of Problem (4) is an upper bound for the value of (2), while the optimal value of (3) provides a lower bound. This process can also be refined by generating worst-case scenarios directly at nodes of the branch-and-bound tree of the MIP (3), see Bertsimas *et al.* (2014) for more details.

As mentioned in the introduction, this work is motivated by an application to surgery scheduling, where each job typically follows a log-normal distribution. In the next section, we show how to solve Problem (4) efficiently for adequate uncertainty sets.

3 Solving the adversarial problem

If we assume that $\log d_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$, it is natural to consider an uncertainty set of the form $\mathcal{D} := \{\boldsymbol{d} \in \mathbb{R}_+^n : \log(\boldsymbol{d}) \in \mathcal{E}\}$, where $\mathcal{E} := \{\boldsymbol{y} \in \mathbb{R}^n : \sum_{j=1}^n \sigma_j^{-2}(y_j - \mu_j)^2 \leq r^2\}$ for some $r > 0$. Note that the set $\mathcal{D}$ defined above is simply a log-transformation of some confidence ellipsoid of the multivariate normal law $\mathcal{N}(\boldsymbol{\mu}, \mathrm{Diag}(\boldsymbol{\sigma}))$.

Problem (4) may be reformulated as

$$\max_{\boldsymbol{\epsilon}\in\{0,1\}^p} \max_{\boldsymbol{d}\in\mathcal{D}} \sum_{m\in\mathcal{M}} c_f^m z_m^* + \epsilon_m c_o^m (\sum_{j\in\mathcal{J}} x_{jm}^* d_j - T_m), \tag{5}$$

which reduces to solving the inner maximization problem for the 2^p values of the vector $\boldsymbol{\epsilon} \in \{0,1\}^p$. Now, we make the change of variable $y_j = \log d_j$. For a fixed $\boldsymbol{\epsilon}$, the value of the inner maximization problem equals

$$\sum_{m\in\mathcal{M}} c_f^m z_m^* - \epsilon_m c_o^m T_m + \max_{\boldsymbol{y}\in\mathcal{E}} \sum_{j\in\mathcal{J}} u_j e^{y_j}, \tag{6}$$

where we have set $u_j := \sum_{m\in\mathcal{M}} \epsilon_m c_o^m x_{jm}^* \geq 0$. If we put aside the trivial case $\boldsymbol{u} = \boldsymbol{0}$, the necessary Karush-Kuhn-Tucker (KKT) conditions for the maximization problem in (6) can be written as follows:

$$\exists\lambda > 0 : \begin{cases} \forall j \in \mathcal{J}, \lambda(y_j - \mu_j)\sigma_j^{-1} = \sigma_j u_j e^{y_j} \\ \sum_{j\in\mathcal{J}} (y_j - \mu_j)^2 \sigma_j^{-2} = r^2. \end{cases} \tag{7}$$

We can find the value of λ by substituting $(y_j - \mu_j)\sigma_j^{-1} = \lambda^{-1}\sigma_j u_j e^{y_j}$ in the second equation: $\lambda = r^{-1}(\sum_j \sigma_j^2 u_j^2 e^{2y_j})^{1/2}$. Substituting back in the first equation, we find that for all $j \in \mathcal{J}$, $(y_j - \mu_j)(r\sigma_j)^{-1} = \sigma_j u_j e^{y_j} (\sum_j \sigma_j^2 u_j^2 e^{2y_j})^{-1/2}$. In other words, the vector $\boldsymbol{w} := \mathrm{Diag}(r\boldsymbol{\sigma})^{-1}(\boldsymbol{y} - \boldsymbol{\mu})$ is a fixed point of the map $g : \boldsymbol{w} \mapsto f(\boldsymbol{w})/\|f(\boldsymbol{w})\|$ which maps the unit sphere $\mathcal{S}_{n-1}$ of $\mathbb{R}^n$ onto itself, where $f(\boldsymbol{w}) := \boldsymbol{\sigma} \circ \boldsymbol{u} \circ \exp(\boldsymbol{\mu} + r\boldsymbol{\sigma} \circ \boldsymbol{w})$, the exponential is elementwise, and $\circ$ denotes the Hadamard (elementwise) product: $(\boldsymbol{a} \circ \boldsymbol{b})_i = a_i b_i$.

The next results give a condition –almost always verified in practice, cf. discussion at the end of the current section– which guarantees that fixed point iterations of g converge, and we can use the fixed point to find a global optimum of (6). To do this, we prove the following result, which relies on *Hilbert's projective metric* on the cone $K := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \boldsymbol{0}\}$. It is defined by $\forall \boldsymbol{x}, \boldsymbol{y} \in K,\ d_H(\boldsymbol{x}, \boldsymbol{y}) := \log\max_i \frac{x_i}{y_i} + \log\max_j \frac{y_j}{x_j}$, see Nussbaum (1994). Note that $d_H(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ implies $\boldsymbol{x} = \alpha\boldsymbol{y}$ for some $\alpha > 0$, and d_H defines a metric over $K \cap \mathcal{S}_{n-1}$.

Proposition 1. *The function $h : \boldsymbol{x} \mapsto \exp(\boldsymbol{x})$ is contractant for the Hilbert's metric over the unit sphere $\mathcal{S}_{n-1} \subset \mathbb{R}^n$, with a global Lipschitz constant equal to $\frac{1}{\sqrt{2}}$:*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_{n-1} \cap K,\ d_H\big(h(\boldsymbol{x}), h(\boldsymbol{y})\big) \leq \frac{1}{\sqrt{2}}\, d_H(\boldsymbol{x}, \boldsymbol{y}).$$

The proof of this result is omitted. It will be included in a full version of this article, and relies on (Nussbaum 1994, Theorem 2.4), which can be used to obtain a formula for the local Lipschitz constant at $\boldsymbol{x} \in \mathcal{S}_{n-1} \cap K$. We are now ready to prove the following proposition, which gives a simple condition ensuring convergence of the fixed point iterations.

Proposition 2. *Assume that for all $j \in \mathcal{J}$, $r\sigma_j < \sqrt{2}$. Then, there exists a point $\boldsymbol{w}^* \in \mathcal{S}_{n-1} \cap K$ such that the fixed point iterations $g(g(\cdots g(\boldsymbol{w}_0)))$ converge to $\boldsymbol{w}^*$ for all $\boldsymbol{w}_0 \in \mathbb{R}^n$. Moreover, $\boldsymbol{y}^* := \boldsymbol{\mu} + r\,\mathrm{Diag}(\boldsymbol{\sigma})\boldsymbol{w}^*$ is a global optimum of problem* (6).

Proof. First note that the existence of a fixed point of g is guaranteed by Brouwer's theorem, and any fixed point must lie in $\mathcal{S}_{n-1} \cap K$. Elementary calculus shows that $\forall \boldsymbol{x}, \boldsymbol{y} \in K$, $d_H\big(g(\boldsymbol{x}), g(\boldsymbol{y})\big) = d_H\big(f(\boldsymbol{x}), f(\boldsymbol{y})\big) = d_H\big(\exp(r\boldsymbol{\sigma} \circ \boldsymbol{x}), \exp(r\boldsymbol{\sigma} \circ \boldsymbol{y})\big) \leq r\|\sigma\|_\infty d_H(e^{\boldsymbol{x}}, e^{\boldsymbol{y}})$. Therefore, Proposition 1 implies that g is contractant for the Hilbert's metric over $\mathcal{S}_{n-1} \cap K$ if $r\|\boldsymbol{\sigma}\|_\infty < \sqrt{2}$. It is well known that $(\mathcal{S}_{n-1} \cap K, d_H)$ is a complete metric space, see Nussbaum (1994), so Banach fixed point theorem ensures the unicity of a fixed point $\boldsymbol{w}^*$ and the convergence of fixed point iterations when $r\|\boldsymbol{\sigma}\|_\infty < \sqrt{2}$. In this case, $\boldsymbol{y}^* := \boldsymbol{\mu} + r\boldsymbol{\sigma} \circ \boldsymbol{w}^*$ is the unique solution of the necessary conditions (7), so $\boldsymbol{y^*}$ maximizes $\sum_j u_j e^{y_j}$ over $\mathcal{E}$.

Choice of r: Care must be taken while setting the value of r defining $\mathcal{E}$, to avoid overconservatism. Indeed, the optimal solution of Problem (2) does not only protect against scenarios in $\mathcal{D}$, but also against all duration scenarios in $\bar{\mathcal{D}} = \{\boldsymbol{d} - \boldsymbol{u} : \boldsymbol{d} \in \mathcal{D}, \boldsymbol{u} \geq \boldsymbol{0}\}$. For the lognormal model $\log d_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$, we can see using the inclusion-exclusion principle that a scenario lies in $\bar{\mathcal{D}}$ with probability $P_n(r) := \Phi(r)^n - (\Phi(r) - \frac{1}{2})^n + \frac{1}{2^n}\sqrt{\chi_n^2(r)}$, where Φ is the standard normal cumulative distribution function (CDF), and χ_n^2 is the CDF of the χ^2-distribution with n degrees of freedom. For a confidence level α, we can hence choose

Instance	n	p	Expected value LRS	Expected value RIP	90th percentile LRS	90th percentile RIP
I1	11	5	0.97	1.01	0.98	1.00
I2	12	5	1.04	1.13	0.99	1.06
I3	11	5	0.98	1.05	0.99	1.02
I4	9	5	1.03	1.10	0.98	1.03
I5	12	5	0.77	0.74	0.66	0.63
I6	11	5	0.99	1.01	0.99	1.01
I7	12	5	0.98	1.09	0.95	1.02
I8	12	5	1.02	1.01	1.01	1.01
I9	12	5	0.97	1.07	0.97	1.02
I10	11	5	1.00	1.00	0.96	0.99

Table 1. Comparison of the expected value (*resp.* 90th percentile) of $F(\boldsymbol{x}^*, \boldsymbol{z}^*; \boldsymbol{d})$, for the solution $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ found by the lognormal robust schedule (LRS) approach of the present paper and the robust IP (RIP) of Denton *et al.* (2010), measured as a ratio to the expected value (*resp.* 90th percentile) of $F(\boldsymbol{x}_0, \boldsymbol{z}_0; \boldsymbol{d})$ for the reference solution $(\boldsymbol{x}_0, \boldsymbol{z}_0)$.

r by solving the equation $P_n(r) = 1 - \alpha$. Then, Problem (2) minimizes an upper bound of the $(1-\alpha)-$quantile of $F(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{d})$.

Discussion on the assumptions of Proposition 2. Estimates of μ_j and σ_j usually come from an analysis of historical data. It seems reasonable to assume that one can obtain estimates $\sigma_j \leq 0.5$, because $\sigma_j = 0.5$ already allows huge deviations from the nominal scenario: 95%-confidence interval is $[0.37m_j, 2.67m_j]$, where $m_j := e^{\mu_j}$ is the median of d_j. In this situation, if we choose r by solving $P_n(r) = 1 - \alpha$, the condition $r\|\boldsymbol{\sigma}\|_\infty < \sqrt{2}$ is satisfied for $n \leq 21$ jobs at the confidence level $\alpha = 0.05$, and for $n \leq 45$ at $\alpha = 0.1$.

4 Application to Allocation of Operating Rooms

We present brief results for instances based on real data from the department of general surgery of the Charité university hospital in Berlin. For each instance, we used as a reference the solution provided by the longest processing time (LPT) heuristic, which is known to give excellent results when the goal is to minimize the expected value of $F(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{d})$, and has an approximation guarantee of $\frac{13}{12}$ in the deterministic case, cf. Denton *et al.* (2010). We solved Problem (2) for lognormal activity durations with parameters estimated from historical data, and a value of r corresponding $P_n(r) = 0.90$. Table 1 compares this solution to the solution of the robust IP called *MRORA* in (Denton *et al.* 2010), based on Monte-Carlo simulations with $N = 10^6$ runs. The uncertainty set $\mathcal{D}$ for MRORA is defined as in (1) with $[\ell_i, u_i]$ set to a 90% confidence interval of d_i, and τ computed with the *newsvendor rule* described in the aforementioned paper. The table evidences that our solution is more robust indeed (better 90th percentile), while remaining very good in terms of expected value (it even beats the LPT reference solution on many instances). On average, our approach reduces the 90th percentile of the overtime of 21.5 minutes per day (95% confidence interval ± 6.1 min.) on all instances of 2013 (compared to MRORA). In conclusion, our approach takes advantage from the knowledge of the distribution of d_i to handle extreme scenarios, an essential feature for the stability of schedules in the operating theater.

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