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NEW CONJECTURES FOR UNION-CLOSED FAMILIES

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ABSTRACT. The Frankl conjecture, also known as the union-closed sets conjecture, states that there exists an element in at least half of the sets of any (non-empty) union-closed family. From an optimization point of view, one could instead prove that $2a$ is an upper bound to the number of sets in a union-closed family with n elements where each element is in at most a sets for all $a, n \in \mathbb{N}^+$. Similarly, one could prove that the minimum number of sets containing the most frequent element in a (non-empty) union-closed family with m sets and n elements is at least $\frac{m}{2}$ for any $m, n \in \mathbb{N}^+$. Formulating these problems as integer programs we observe that computed optimal values do not vary with n . We formalize these observations as conjectures, and show that they are not equivalent to the Frankl conjecture while still having wide-reaching implications if proven true. Finally, we partially prove the new conjectures and discuss possible approaches to solve them completely.

1. INTRODUCTION

The origins behind the Frankl conjecture, also known as the union-closed sets conjecture, are ambiguous. Although the name suggests Péter Frankl was the first to formalize it, the authors of [BBE13] claim that the conjecture was previously known. Still, it seems clear that Frankl popularized it around 1979 [Fra83].

Before stating the conjecture, we need a few definitions. Throughout this paper, we'll think of a *family of sets* $\mathcal{F} = (E(\mathcal{F}), \mathcal{S}(\mathcal{F}))$ as being a collection $\mathcal{S}(\mathcal{F})$ of distinct sets S such that every set $S \subseteq E(\mathcal{F})$ where $E(\mathcal{F})$ is the ground set of elements. In general, we'll let $E(\mathcal{F}) = \{1, \dots, n\} =: [n]$. A family of sets is said to be *union-closed* if and only if the union of two sets of the family is also a set of the family.

Conjecture 1.1 (Frankl, 1979). *In a union-closed family \mathcal{F} such that $\mathcal{S}(\mathcal{F}) \neq \{\emptyset\}$, there exists an element of $E(\mathcal{F})$ that is in at least half of the sets of $\mathcal{S}(\mathcal{F})$.*

Since 1979, a lot of progress has been made towards solving this conjecture. It has attracted the attention of both lattice theorists as well as combinatorial probabilists; more recently, computer scientists have also joined the race to solve Frankl's conjecture. To the best of our knowledge, this is the first time the problem is investigated through combinatorial optimization. After defining a few more concepts, we give an overview of the literature and present our contributions.

Let $m(\mathcal{F})$ and $n(\mathcal{F})$ be respectively the numbers of sets and elements of a family \mathcal{F} , i.e., $m(\mathcal{F}) = |\mathcal{S}(\mathcal{F})|$ and $n(\mathcal{F}) = |E(\mathcal{F})|$. Moreover, let $m_e(\mathcal{F})$ be the number of sets in \mathcal{F} containing some element $e \in E(\mathcal{F})$, let $a(\mathcal{F})$ be the maximum number of sets in \mathcal{F} containing any element of $E(\mathcal{F})$, i.e., $a(\mathcal{F}) = \max_{e \in E(\mathcal{F})} m_e(\mathcal{F})$, and let $e^*(\mathcal{F})$ be any of possibly many elements in $E(\mathcal{F})$ contained in $a(\mathcal{F})$ sets. For

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example, if \mathcal{F} is such that $E(\mathcal{F}) = [3]$ and $\mathcal{S}(\mathcal{F}) = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \emptyset\}$, then \mathcal{F} is union-closed, $m(\mathcal{F}) = 4$, $n(\mathcal{F}) = 3$, $m_1(\mathcal{F}) = 3$, $m_2(\mathcal{F}) = 2$, $m_3(\mathcal{F}) = 2$, $a(\mathcal{F}) = 3$ and $e^*(\mathcal{F}) = 1$. Since $\frac{3}{4} \geq \frac{1}{2}$, the Frankl conjecture holds for \mathcal{F} .

1.1. A bit of history. Conjecture 1.1 is known to hold for a wide range of families. Two easy results were noticed particularly early on. Their authorship is unclear.

Theorem 1.2 (Folklore). *The Frankl conjecture holds true for any family \mathcal{F} such that there exists $S \in \mathcal{S}(\mathcal{F})$ where $|S| = 1$.*

Theorem 1.3 (Folklore). *The Frankl conjecture holds true for any family \mathcal{F} such that $\frac{1}{m(\mathcal{F})} \sum_{S \in \mathcal{S}(\mathcal{F})} |S| \geq \frac{n}{2}$.*

The following result was proven many times by different people in different ways, but the first to do so were Sarvate and Renaud.

Theorem 1.4 (Sarvate and Renaud, 1989). *The Frankl conjecture holds true for any family \mathcal{F} such that there exists $S \in \mathcal{S}(\mathcal{F})$ where $|S| = 2$.*

Another early result is the following:

Theorem 1.5 (Roberts, 1992). *The inequality $m(\mathcal{F}) < 4n(\mathcal{F}) - 1$ holds for any union-closed family \mathcal{F} that is a minimum counterexample to the Frankl conjecture.*

The above results and a few others allowed the conjecture to be proven for increasing values of m and n over time. The current status is as follows:

Theorem 1.6 (Roberts and Simpson, 2010). *The Frankl conjecture is true for any family \mathcal{F} with $m(\mathcal{F}) \leq 46$.*

Theorem 1.7 (Bošnjak and Marković, 2008). *The Frankl conjecture is true for any family \mathcal{F} with $n(\mathcal{F}) \leq 11$.*

These results follow others that became incrementally better over the years ($m(\mathcal{F}) \leq 11$ in [SR89], $m(\mathcal{F}) \leq 18$ in [SR90], $m(\mathcal{F}) \leq 24$ in [Far94], $m(\mathcal{F}) \leq 27$ in [Poo92], $m(\mathcal{F}) \leq 32$ in [GY98], $m(\mathcal{F}) \leq 40$ in [Rob92], $n(\mathcal{F}) \leq 7$ in [Poo92], $n(\mathcal{F}) \leq 9$ in [Far94]).

More recently, Vučković and Živković announced the following result which is still unpublished.

Theorem 1.8 (Vučković and Živković, 2012). *The Frankl conjecture is true for any family \mathcal{F} such that $n(\mathcal{F}) \leq 12$ and $m(\mathcal{F}) \leq 50$.*

Thus the conjecture is still open for $m(\mathcal{F}) \geq 51$ and $n(\mathcal{F}) \geq 13$. We now turn our attention to three important results which are quite useful for the purposes of this paper.

Firstly, Balla, Bollobás and Eccles recently proved that the Frankl conjecture holds for families \mathcal{F} containing at least $\frac{2}{3}$ of the sets in the power set of $n(\mathcal{F})$.

Theorem 1.9 (Balla, Bollobás & Eccles, 2013). *The union-closed conjecture holds for any family \mathcal{F} where $m(\mathcal{F}) \geq \frac{2}{3}2^{n(\mathcal{F})}$.*

Secondly, instead of proving the Frankl conjecture, one could instead try to prove that any union-closed family contains an element present in at least some fraction of the sets, just as Knill did in the following theorem.

Theorem 1.10 (Knill, 1994). *In any union-closed family \mathcal{F} , there always exists an element present in at least $\frac{m(\mathcal{F})-1}{\log_2 m(\mathcal{F})}$ sets, that is, $a(\mathcal{F}) \geq \frac{m(\mathcal{F})-1}{\log_2 m(\mathcal{F})}$.*

Wójcik improved this result slightly in [Wój99], but, amazingly, still no constant fraction is known. Thirdly, Bruhn and Schaudt in [BS14] were able to find a constant close to $\frac{1}{2}$ for certain families by using theorems of [Rei03] and [Vv12].

Theorem 1.11 (Bruhn & Schaudt, 2013). *Let \mathcal{F} be any union-closed family \mathcal{F} such that $2^{n(\mathcal{F})-1} < m(\mathcal{F}) \leq 2^{n(\mathcal{F})}$. Then $a(\mathcal{F}) \geq \frac{6}{13} \cdot m(\mathcal{F})$, i.e., there exists an element in at least $\frac{6}{13}$ of the sets of the family.*

Many other results have been discovered throughout the years. For a more complete history of the problem, we refer the reader to the following excellent survey [BS14].

1.2. Our contributions. In this paper, we examine the Frankl conjecture through a different lens by viewing it as an optimization problem. Indeed, we can rewrite Conjecture 1.1 either as a maximization or minimization problem, depending on whether we fix $m(\mathcal{F})$ or $a(\mathcal{F})$.

Conjecture 1.12 (Maximization Version). *For any positive integer a , let*

$$\mathcal{F}(a) = \{\mathcal{F} \mid \mathcal{F} \text{ is a union-closed family, } \mathcal{S}(\mathcal{F}) \neq \emptyset \text{ and } a(\mathcal{F}) \leq a\}.$$

Then $\max_{\mathcal{F} \in \mathcal{F}(a)} m(\mathcal{F}) \leq 2a$ for all $a \in \mathbb{N}^+$.

Conjecture 1.13 (Minimization Version). *For any positive integer m , let*

$$\mathcal{G}(m) = \{\mathcal{F} \mid \mathcal{F} \text{ is a union-closed family, } \mathcal{S}(\mathcal{F}) \neq \emptyset \text{ and } m(\mathcal{F}) = m\}.$$

Then $\min_{\mathcal{F} \in \mathcal{G}(m)} a(\mathcal{F}) \geq \frac{m}{2}$ for all $m \in \mathbb{N}^+$.

Note that the conjectures 1.1, 1.12 and 1.13 are equivalent since there exists a counterexample to the original Frankl conjecture if and only if there exists a union-closed family \mathcal{F} such that $m(\mathcal{F}) > 0$ and $a(\mathcal{F}) < \frac{m(\mathcal{F})}{2}$.

In Section 2.1, we model the optimization versions of the Frankl conjecture as integer programs for fixed n (the number of elements in $E(\mathcal{F})$ for all families \mathcal{F} considered). In Theorem 2.1, we discuss some of the properties of the optimal values of said programs. Then in Section 2.2, we present computational results for the models. We observe that computed optimal values do not vary as n is increased, i.e.,

$$\max_{\substack{\mathcal{F} \in \mathcal{F}(a): \\ n(\mathcal{F})=n}} m(\mathcal{F}) = \max_{\substack{\mathcal{F} \in \mathcal{F}(a): \\ n(\mathcal{F})=n+1}} m(\mathcal{F})$$

and

$$\min_{\substack{\mathcal{F} \in \mathcal{G}(m): \\ n(\mathcal{F})=n}} a(\mathcal{F}) = \min_{\substack{\mathcal{F} \in \mathcal{G}(m): \\ n(\mathcal{F})=n+1}} a(\mathcal{F})$$

for any large enough n . We did not expect this and this is not necessary for the Frankl conjecture to hold. We formally present these two observations as conjectures and prove that they are equivalent. However, these new conjectures do not imply the Frankl conjecture, and conversely, the Frankl conjecture does not imply these two new conjectures. Still, in Section 2.3, we discuss some of the important

implications the new conjectures have on the Frankl conjecture. Notably, proving these conjectures would prove Conjecture 1.1 when $|\mathcal{S}(\mathcal{F})| = m$ for infinitely many values of m . Moreover, their proof would yield that there always exists an element in $\frac{6}{13}$ of the sets of a union-closed family, and would thus achieve the first known constant bound for the percentage of sets containing some element in any union-closed family. Finally, in Section 3, we prove a restricted version of the new conjectures using a construction of Falgas-Ravry in [FR11] and discuss the importance of twin sets, which we define as sets that differ only in one element.

2. FRANKL'S INTEGER PROBLEMS AND TWO NEW CONJECTURES

2.1. Modeling the Frankl optimization problems. For any positive integers a, n , let

$$\mathcal{F}(n, a) = \{\mathcal{F} \in \mathcal{F}(a) \mid n(\mathcal{F}) = n\}.$$

Then proving that $\max_{\mathcal{F} \in \mathcal{F}(n, a)} m(\mathcal{F}) \leq 2a$ for all possible n, a would prove Conjecture 1.12 (and thus, also the original conjecture). Fix n, a , and let $f(n, a) = \max_{\mathcal{F} \in \mathcal{F}(n, a)} m(\mathcal{F})$. Then we can find $f(n, a)$ by solving the following integer program

$$\begin{array}{ll} f(n, a) = \max \sum_{S \in \mathcal{S}_n} x_S & \\ \text{such that } x_U + x_T \leq 1 + x_S & \forall T \cup U = S \in \mathcal{S}_n \\ \sum_{S \in \mathcal{S}_n : e \in S} x_S \leq a & \forall e \in [n] \\ x_S \in \{0, 1\} & \forall S \in \mathcal{S}_n, \end{array}$$

where \mathcal{S}_n is the power set of $[n]$, and the variable x_S for any set $S \in \mathcal{S}_n$ is 1 if S is in the family, and 0 otherwise. Thus, we maximize the number of sets while ensuring the family is union-closed (through the first constraint) and that $a(\mathcal{F}) \leq a$ holds (through the second constraint), i.e., we calculate $f(n, a)$.

Similarly, let

$$\mathcal{G}(n, m) = \{\mathcal{F} \in \mathcal{G}(m) \mid n(\mathcal{F}) = n\}$$

for any positive integers m and large enough n . Then, it is easy to see that proving that $\min_{\mathcal{F} \in \mathcal{G}(n, m)} a(\mathcal{F}) \geq \frac{m}{2}$ for all large enough n and m would prove Conjecture 1.13. Fix n, m , and let $g(n, m) = \min_{\mathcal{F} \in \mathcal{G}(n, m)} a(\mathcal{F})$ for $m \geq 2$. Then we can find $g(n, m)$ by solving the following integer program

$$\begin{array}{ll} g(n, m) = \min \sum_{S \in \mathcal{S}_n : 1 \in S} x_S & \\ \text{such that } x_U + x_T \leq 1 + x_S & \forall T \cup U = S \in \mathcal{S}_n \\ \sum_{S \in \mathcal{S}_n : i \in S} x_S \geq \sum_{S \in \mathcal{S}_n : j \in S} x_S & \forall 1 \leq i \leq j \leq n \\ \sum_{S \in \mathcal{S}_n} x_S = m & \\ x_S \in \{0, 1\} & \forall S \in \mathcal{S}_n, \end{array}$$

where the variables x_S are as before. The second constraint says that element 1 is the element contained in the most number of sets in the family, so we are minimizing the maximum number of sets containing the most frequent element while enforcing that the family is union-closed (through the first constraint) and has m sets (through the third constraint).

Note that there are values of a, m, n for which $f(n, a)$ and $g(n, m)$ have trivial solutions that do not interest us. For example, it is clear that $f(n, a) = 2^n$ if $a \geq 2^{n-1}$. Indeed, the power set of n , \mathcal{S}_n , is union-closed, and there are 2^n sets in \mathcal{S}_n where each element is in exactly 2^{n-1} sets. It is thus a trivially optimal solution for $f(n, a)$. It is also clear that $g(n, m)$ has trivially no solution if $m > 2^n$. Indeed, even if we take all of the sets in \mathcal{S}_n , we'd have less sets than the number of sets required by the program. We first study a few properties of the functions f and g for non-trivial values of a, m, n .

Theorem 2.1. *The following properties hold.*

- (1) *The function f is non-decreasing in n , that is, $f(n, a) \leq f(n+1, a)$ for every $a, n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 a \rceil + 1$.*
- (2) *The function g is non-increasing in n , that is, $g(n, m) \geq g(n+1, m)$ for every $m, n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 m \rceil$.*
- (3) *The function f is strictly increasing in a , that is, $f(n, a) < f(n, a+1)$ for every $a, n \in \mathbb{N}^+$ such that $n > \lceil \log_2 a \rceil + 1$.*
- (4) *The function g is non-decreasing in m , that is, $g(n, m) \leq g(n, m+1)$ for every $m, n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 m \rceil$.*
- (5) *We have that $g(n, f(n, a)) = a$ for all $a, n \in \mathbb{N}^+$ such that $n > \lceil \log_2 a \rceil + 1$.*
- (6) *We have that $f(n, g(n, m)) \geq m$ for all $m, n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 m \rceil$.*

Proof. (1) Fix a, n and suppose $f(n, a) = m$. Take a family \mathcal{F} that is optimal, i.e., a family $\mathcal{F} \in \mathcal{F}(n, a)$ such that $m(\mathcal{F}) = m$. Add an element $n+1$ to the family such that this element is exactly in the same sets as some other element e of the family. Then this augmented family is still union-closed, and every element of it is still in at most a sets. Thus, $f(n+1, a) \geq m = f(n, a)$.

(2) Fix m, n and suppose $g(n, m) = a$. Take a family \mathcal{F} that is optimal, i.e., a family $\mathcal{F} \in \mathcal{G}(n, m)$ such that $a(\mathcal{F}) = a$. Add an element $n+1$ to the family such that this element is exactly in the same sets as some other element e of the family. Then this augmented family is still union-closed, and every element of it is still in at most a sets. Thus, $g(n+1, m) \leq a = g(n, m)$.

(3) Fix a, n and suppose $f(n, a) = m$. Take a family \mathcal{F} that is optimal (as before). Then add to this family one of the greatest sets that is not already present in the family, i.e., a set in $\mathcal{S}_n \setminus \mathcal{S}(\mathcal{F})$ containing as many elements as possible. This is always possible if $f(n, a) < 2^n$ (note that this is the case if $a < 2^{n-1}$ since we cannot take all of \mathcal{S}_n then). The new family we built has $m+1$ sets and is still union-closed since taking the union of any other set with the added set will give either the new set itself or a greater set (which is present in the family by construction). Moreover, every element is present in this new family at most $a+1$ sets. Thus, $f(n, a+1) \geq m+1 = f(n, a)+1$.

(4) Fix m, n and suppose $g(n, m+1) = a$. Take a family \mathcal{F} that is optimal (as before). Remove the smallest set of the family, i.e., one of the sets in $\mathcal{S}(\mathcal{F})$ with the least number of elements. The family is still union-closed since the set that was removed was not the union of any other two sets since they

would have to be smaller. Moreover, this new family has m sets and every element is still there at most a times, so $g(n, m) \leq a = g(n, m + 1)$.

- (5) Fix a, n and suppose that $f(n, a) = m$. Thus m is the maximum number of sets in any union-closed family \mathcal{F} on n elements with $a(\mathcal{F}) \leq a$. Because $m \leq 2^n$, $g(n, m)$ is feasible. Certainly, this implies that $g(n, m) = g(n, f(n, a)) \leq a$. Suppose that $g(n, f(n, a)) = a' < a$. Then $f(n, a') \geq m$. Since we have already shown that the function f strictly increases in a when $n > \lceil \log_2 a \rceil + 1$, this is a contradiction.
- (6) Suppose that $g(n, m) = a$. Then $n \geq \lceil \log_2 m \rceil$, else $g(n, m)$ would be unfeasible. Thus a is the minimum number of sets containing the most frequent element of a union-closed family \mathcal{F} on n elements with $m(\mathcal{F}) = m$. Certainly, this means that $f(n, a) = f(n, g(n, m)) \geq m$.

□

2.2. Computations and Conjectures. We computed $f(n, a)$ and $g(n, m)$ for different values of n, a, m with the mixed-integer commercial solver IBM ILOG CPLEX version 12.4. Table 1 contains some of the results we obtained.

TABLE 1. Values of $f(n, a)$ and $g(n, m)$ (respectively left and right)

$a \setminus n$	1	2	3	4	5	6	7	8	$m \setminus n$	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
2	2	4	4	4	4	4	4	4	3	-	2	2	2	2	2	2	2
3	2	4	5	5	5	5	5	5	4	-	2	2	2	2	2	2	2
4	2	4	8	8	8	8	8	8	5	-	-	3	3	3	3	3	3
5	2	4	8	9	9	9	9	9	6	-	-	4	4	4	4	4	4
6	2	4	8	10	10	10	10	10	7	-	-	4	4	4	4	4	4
7	2	4	8	12	12	12	12	12	8	-	-	4	4	4	4	4	4
8	2	4	8	16	16	16	16	16	9	-	-	-	5	5	5	5	5
9	2	4	8	16	17	17	17	17	10	-	-	-	6	6	6	6	6
10	2	4	8	16	18	18	18	18	11	-	-	-	7	7	7	7	7
11	2	4	8	16	19	19	19	19	12	-	-	-	7	7	7	7	7
12	2	4	8	16	21	21	21	21	13	-	-	-	8	8	8	8	8
13	2	4	8	16	23	23	23	23	14	-	-	-	8	8	8	8	8
14	2	4	8	16	25	25	25	25	15	-	-	-	8	8	8	8	8
15	2	4	8	16	27	27	27	27	16	-	-	-	8	8	8	8	8
16	2	4	8	16	32	32	32	32		-	-	-	8	8	8	8	8

A clear pattern emerges: For any $n \geq \lceil \log_2 a \rceil + 1$, that is, for any non-trivial value of n when a is fixed, $f(n, a)$ takes the same value as n increases. Similarly, for any $n \geq \lceil \log_2 m \rceil$, that is, for any non-trivial value of n when m is fixed, $g(n, m)$ takes the same value as n increases.

To the best of our knowledge, this has never been observed before. We formulate these observations as conjectures.

Conjecture 2.2 (*f-conjecture*). Fix $a \in \mathbb{N}^+$. Then $f(n, a) = f(n + 1, a)$ for every $n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 a \rceil + 1$.

Conjecture 2.3 (*g-conjecture*). Fix $m \in \mathbb{N}^+$. Then $g(n, m) = g(n + 1, m)$ for every $n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 m \rceil$.

We checked these conjectures computationally up to $n = 9$ for all non-trivial a and m (see Appendix). We first show that these two conjectures are equivalent.

Theorem 2.4. *We have that $f(n, a) = f(n + 1, a)$ for every $a, n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 a \rceil + 1$ if and only if $g(n', m) = g(n' + 1, m)$ for every $m, n' \in \mathbb{N}^+$, $m \geq 2$ such that $n' \geq \lceil \log_2 m \rceil$.*

Proof. Necessity. Pick an arbitrary $a \in \mathbb{N}^+$ and choose any $n \in \mathbb{N}^+$ such that $n \geq \lceil \log_2 a \rceil + 1$. By Theorem 2.1(5), we know that there exists m such that $g(n, m) = a$, namely $m := f(n, a)$. Let m^* be the greatest number for which $g(n, m^*) = a$. By definition, it follows that $f(n, a) \leq m^*$. Moreover, $f(n, a) \geq m^*$ since $f(n, g(n, m^*)) \geq m^*$ by Theorem 2.1(6). Thus, $f(n, a) = m^*$ for any $n \geq \lceil \log_2 a \rceil + 1$.

Sufficiency. For all $m' \geq 2$ such that there exists an a such that $f(n, a) = m'$, we have that $g(n, m') = g(n, f(n, a)) = a$ for all $n \geq \lceil \log_2 a \rceil + 1$ by theorem 2.1(5). For all $m'' \geq 2$ such that there exists no a' such that $f(n, a') = m''$, we have that $f(n, g(n, m'')) = m'' + b$ by theorem 2.1(6), and that $g(n, m'') = g(n, m'' + b)$. Then $g(n, m'' + b) =: a''$ for all n since $f(n, g(n, m'' + b)) = m'' + b$, and so we are back to the first case. By theorems 2.1(2) and 2.1(4), this implies that $g(n, m'') = g(n, m'' + b) = a''$ for all $n \geq \lceil \log_2 m \rceil$. \square

The two conjectures are thus equivalent: proving one would prove the other. Another way to view the f -conjecture is as follows: to construct an optimal solution for $f(n, a)$, first construct an optimal solution for $f(\lceil \log_2 a \rceil + 1, a)$, and make $n - \lceil \log_2 a \rceil - 1$ copies of some element, i.e., put these new elements in the same sets as the original element, or if you prefer, put these new elements in none of the sets. If the f -conjecture is true, such families would be optimal for $f(n, a)$. The same idea can be applied to the g -conjecture.

Note that these conjectures are different from the Frankl conjecture. For one thing, even if the f - and g -conjectures hold, one would still need to show that $f(\lceil \log_2 a \rceil + 1, a) \leq 2a$ for every a or that $g(\lceil \log_2 m \rceil, m) \geq \frac{m}{2}$ for all m to prove the Frankl conjecture; therefore the f - and g -conjectures do not imply the Frankl conjecture. Moreover, the Frankl conjecture does not imply the f - and g -conjectures. Certainly, if the Frankl conjecture is true, then

$$\max_n f(n, a) \leq 2a$$

for all $a \in \mathbb{N}^+$, else the Frankl conjecture wouldn't be true; however, how the function $f(n, a)$ behaves for different n for a fixed a is irrelevant. Certainly, one can easily observe that the Frankl conjecture implies that the function f has to stabilize at some point as n increases since, by Theorem 2.1(1), $f(n, a) \leq f(n + 1, a)$, and since $f(n, a) \in \mathbb{N}$ by definition. However, the Frankl conjecture does not imply that the function f should be stable immediately as $n \geq \lceil \log_2 a \rceil + 1$, that is, as soon as there are enough elements for there to be at least a sets containing an element e . Similarly, if the Frankl conjecture is true, then

$$\min_{n \geq \lceil \log_2 m \rceil} g(n, m) \geq \frac{m}{2}$$

for all $m \in \mathbb{N}^+$, and so again one can observe that the g function has to stabilize at some point as n increases since we show in Theorem 2.1(2) that $g(n, m) \geq g(n + 1, m)$. But again, the Frankl conjecture does not imply that the function

g has to stabilize immediately when $n \geq \lceil \log_2 m \rceil$, that is, as soon as there are enough elements for there to be at least m sets in the power set of n .

Therefore, the f - and g -conjectures do not imply the Frankl conjecture and the latter does not imply the former two either. Still, proving the f - and g -conjectures would have wide-reaching implications for the Frankl conjecture.

2.3. Consequences of the f - and g -conjectures on the Frankl conjecture.

If the f - and g -conjectures are true, then combining them with Theorem 1.9 would go a long way towards solving the Frankl conjecture.

Corollary 2.5. *If the f - and g -conjectures hold, then Conjecture 1.12 holds for all a for which there exists $n \in \mathbb{N}^+$ such that $\frac{2}{3}2^n \leq a \leq 2^n$.*

Proof. For a such that $\frac{2}{3}2^n \leq a \leq 2^n$ for some n , if $f(n, a) > 2a$, i.e., if it is a counterexample to the Frankl conjecture, then we have a union-closed family on n elements with more than $\frac{2}{3}2^{n+1}$ sets in which each element is present at most a times. However, by Theorem 1.9, this cannot be a counterexample. So, $f(\lceil \log_2 a \rceil + 1, a) \leq 2a$ for all $\frac{2}{3}2^n \leq a \leq 2^n$, and, if the f -conjecture holds, i.e., if $f(n, a) = f(n+1, a)$ for all $n \geq \lceil \log_2 a \rceil + 1$, then the Frankl conjecture holds for all a 's such that $a \geq \frac{2}{3}2^n$ for some n . \square

This would mean that Conjecture 1.12 would hold for about $\frac{2}{3}$ of all possible values of a . Recall that the Conjecture 1.12 is equivalent to the Frankl conjecture, and so Corollary 2.5 could be reformulated in such terms.

Another nice consequence of the f - and g -conjectures would be that there would finally be a known constant fraction of sets containing $e^*(\mathcal{F})$, the most frequent element in \mathcal{F} , in any union-closed family \mathcal{F} . Indeed, the f - and g -conjectures along with Theorem 1.11 would not only yield a constant fraction, $\frac{6}{13}$, but one close to the desired $\frac{1}{2}$.

Corollary 2.6. *If the f - and g -conjectures hold, then any union-closed family on m sets contains an element in at least $\frac{6}{13}m$ sets of the family.*

Proof. By Theorem 1.11, we know that $g(\lceil \log_2 m \rceil, m) \geq \frac{6}{13}m$ for any $m \in \mathbb{N}^+$, and by the g -conjecture, we know then that $g(n, m) \geq \frac{6}{13}m$ for all $n \geq \lceil \log_2 m \rceil$. Therefore, we know that any family on m sets contains an element in at least $\frac{6}{13}m$ sets of the family. \square

Thus, from our point of view, studying the f - and g -conjectures offers new ways of attacking the Frankl conjecture, in addition to being interesting in and of itself. Hence, the new conjectures warrant a closer examination.

3. TOWARDS PROVING THE NEW CONJECTURES

3.1. Twin sets and a partial proof of the new conjectures. As noted before in Theorem 2.4, the f - and g -conjectures are equivalent, so from now on, we'll focus only on the f -conjecture. We now introduce a new idea: twin sets.

Definition 3.1. We call two sets S_1, S_2 with $n > |S_1| > |S_2|$ *twin sets* if $|S_1 \Delta S_2| = 1$. We call S_1 the *big twin* and S_2 the *little twin*. Moreover, we call the element $e = S_1 \Delta S_2$ the *twin difference* of S_1 and S_2 .

Twin sets play an important role in proving the f -conjecture as made clear by the following lemma.

Lemma 3.2. *Suppose the f -conjecture is not true and that there exists values of n and a for $n \geq \lceil \log_2 a \rceil + 1$ such that $f(n, a) < f(n + 1, a)$. Then every optimal solution of $f(n + 1, a)$ is such that every element is a twin difference for at least one pair of sets.*

Proof. Suppose not. Suppose that for $f(n + 1, a) =: m$ there exists an optimal solution for which there exists an element that is not a twin difference. Then removing this element (from every set containing it) will leave all of the sets distinct, and the family will still be union-closed, but with n elements. In that case, we know that $f(n, a) \geq m = f(n + 1, a)$, and since $f(n, a) \leq f(n + 1, a)$ by Theorem 2.1(1), we obtain $f(n, a) = f(n + 1, a)$ as in the conjecture, a contradiction. \square

Therefore we only need to focus on the case where every optimal solution for some $f(n, a)$ is such that every element is a twin difference.

Example 3.3. For example, the following union-closed family on 6 elements with $a = 10$ is such that every element is a twin difference for at least one pair of sets. (We list the elements j in a set S_i vertically by putting a 1 if $j \in S_i$, and 0 otherwise.)

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}
1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
3	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
4	1	1	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0	0
5	1	1	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
6	1	0	1	0	1	0	1	0	1	1	1	0	1	0	1	0	1	0

If we remove element 3, then we're left with

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}
1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
3																		
4	1	1	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0	0
5	1	1	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
6	1	0	1	0	1	0	1	0	1	1	1	0	1	0	1	0	1	0

and we now have two sets that are the same, and so this is not a solution to the problem $f(5, 10)$. The same is true for all other elements (note that in some cases, many sets are doubled).

Using Lemma 3.2 and a construction from Falgas-Ravry [FR11], we can partially prove the f -conjecture.

Theorem 3.4. *We have that $f(n - 1, a) = f(n, a)$ for all $n > a$.*

Proof. First assume $f(n - 1, a) < f(n, a)$ for some $n > a$. By Lemma 3.2, every element in any optimal solution of $f(n, a)$ is a twin difference of at least one pair of sets. This implies that any optimal solution for n elements contains no two elements that are exactly in the same sets. Indeed, such elements would not be twin differences. We can thus apply the same construction as in [FR11]. Order the elements $[n]$ by decreasing frequency. Observe now that for all elements $1 \leq i < j \leq$

n , there exists S_{ij} such that $i \in S_{ij}$ and $j \notin S_{ij}$. For $2 \leq j \leq n$, let $S_j = \cup_{i=1}^{j-1} S_{ij}$, and let $S_{n+1} = [n]$. Note that the S_j 's are all distinct since $[1, j-1] \subseteq S_j$ and $j \notin S_j$. Since the most frequent element, element 1, is in at least these n sets of the family, we have that $a \geq n$, which is a contradiction. \square

First note that the Falgas-Ravry construction can also be applied to the function g to prove that $g(n, m) = g(n+1, m)$ for all $n \geq m-1$. Furthermore, observe that having no two elements in exactly the same sets is a much weaker constraint than having each element being a twin difference. Therefore, it might be possible to improve this result.

3.2. Models with twins and some computations. Since we are interested in union-closed families in which every element is a twin difference, we can modify the models that from section 2.1 so that they only consider such families.

Additionally, we will not count the set with every element as a possible big twin (that we'll call trivial twin) as this case also yields that $f(n+1, a) = f(n, a)$. Indeed, if an element e is the twin difference only of the trivial twin and $[n] \setminus \{e\}$, then we can remove e and replace the set $[n] \setminus \{e\}$ (now a set with all $n-1$ elements identical to the trivial twin) with the biggest set missing from the family (for example, the first set missing in the lexicographic order using any ordering of the elements). This new family is union-closed by the same argument as in theorem 2.1(3), and each element is still in at most a sets, and so here again we have that $f(n+1, a) = f(n, a)$. Thus, we would not consider the family in Example 3.3 since element 3 is the twin difference only for sets $[6]$ and $[6] \setminus \{3\}$. Thus after removing 3 from the family, we can replace one of the copies of the set $[6] \setminus \{3\}$ with $[6] \setminus \{1, 3\}$ or $[6] \setminus \{2, 3, 4\}$ or $[6] \setminus \{2, 3, 5\}$, etc., and thus obtain a union-closed family on five elements where each element is in at most ten of the eighteen sets, thus giving us a lower bound on $f(5, 10)$.

Enforcing that each element is a non-trivial twin difference for some pair of sets is easily done in our f - and g -programs by introducing the variable z_S^e for every $S \in \mathcal{S}_n$ and $e \in [n]$, which is zero if at least one of S and $S \cup e$ is absent from the family. This of course makes the program much larger. Luckily, z_S^e can be a continuous non-negative variable. Indeed, by adding the constraints $z_S^e \leq x_S$ and $z_S^e \leq x_{S \cup e}$, we ensure that the twin variable is zero if S or $S \cup e$ is missing. We also add the constraint $\sum_{\substack{S \ni e, \\ |S| \neq n-1}} z_S^e \geq 1$ for every $e \in [n]$, which ensures each element

is a non-trivial twin difference. We let $f_t(n, a)$ and $g_t(n, m)$ be the optimal values of the programs for $f(n, a)$ and $g(n, m)$ with the new variables and constraints.

We present in Table 2 computational results for $f_t(n, a)$ and $g_t(n, m)$.

From Theorem 3.4, we know that everything past the main diagonal for $f_t(n, a)$ is unfeasible. Similarly, everything past the lower diagonal for $g_t(n, m)$ is unfeasible.

The fact that $f_t(n, a)$ decreases as n increases appears counterintuitive at first, but on second thought it makes sense. If there are more elements, each forced to be a twin difference, then there will be more distinct unions of sets. Therefore the risk of violating the a -limit increases, and so the number of allowable sets decreases. If one could prove that, then the f -conjecture would be proven, and thus the g -conjecture as well. Another idea pointing in a similar direction is the following.

Lemma 3.5. *The minimum number of twin pairs for an element in an optimal union-closed family where each element is in at most a sets decreases as n increases.*

TABLE 2. Non-trivial values of $f_t(n, a)$ and $g_t(n, m)$ (respectively left and right)

$a \setminus n$	1	2	3	4	5	6	7	8	$m \setminus n$	1	2	3	4	5	6	7										
1	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-										
2		-	-	-	-	-	-	-	2	-	-	-	-	-	-	-										
3			-	-	-	-	-	-	3		-	-	-	-	-	-										
4			8	-	-	-	-	-	4		2	-	-	-	-	-										
5				8	-	-	-	-	5			4	-	-	-	-										
6					10	9	-	-	6				4	5	-	-										
7						12	11	10	-	7				4	5	6										
8							16	13	12	11	-	8			4	5	6	7								
9								15	14	13	12	9				6	6	7	8							
10									18	16	15	14	10				6	7	7	8						
11										19	19	17	16	11				7	7	8	8					
12											21	20	20	18	12				7	8	8	9				
13												23	22	21	21	13				8	8	9	9			
14													25	24	23	22	14				8	9	9	10		
15														27	25	25	24	15				8	9	10	10	
16															32	28	26	26	16				8	10	10	11

Proof. Let $t_{\mathcal{F}}$ be the minimum number of twin pairs for any element in an optimal family \mathcal{F} for $f(n, a)$. If $t_{\mathcal{F}} \geq 1$, then, by the Falgas-Ravry construction presented in Theorem 3.4, there exists sets S_j with $[1, j - 1] \subseteq S_j$ and $j \notin S_j$ for every $2 \leq j \leq n$, as well as set $S_{n+1} = [n]$. Note that, for element 1, these sets can only be big twins (since 1 is in all of these sets). Suppose u of these sets are big twins for element 1. Then there exists at least $t_{\mathcal{F}} - u$ other sets that are big twins for element 1. Thus element 1 is in at least $n + t_{\mathcal{F}} - u$ sets. Moreover, element 2 will be in at least $u - 1$ of the small twins for element 1, and it is in $n - 1$ sets among the S_j 's. Therefore, element 2 is in at least $n - 1 + u - 1$ sets. Therefore, $a \geq \max\{n + t_{\mathcal{F}} - u, n + u - 2\} \geq n - 1 + \frac{t_{\mathcal{F}}}{2}$. This implies that $t_{\mathcal{F}} \leq 2(a - n + 1)$. Thus, for a fixed a , as n increases, the potential number of twins in an optimal family decreases. \square

Some of the properties in Theorem 2.1 of the f - and g -functions also hold for f_t and g_t .

Theorem 3.6. *The following properties hold.*

- (1) We have that $f_t(n, a) < f_t(n, a + 1)$ if $a < 2^{n-1}$.
- (2) We have that $g_t(n, f_t(n, a)) = a$ for all a and n .
- (3) We have that $f_t(n, g_t(n, m)) \geq m$ for all m and n .

Proof. (1) Suppose $f_t(n, a) = m$. Take a family \mathcal{F} that is optimal. Then add to this family one of the greatest set, i.e. a set containing as many elements as possible or lexicographically greatest for some ordering of the elements, that is not already present in the family. This is always possible since $f_t(n, a) < 2^n$. This new family is still union-closed, and moreover there are still twins for each element since we didn't remove any set. Each

element is now in at most $a + 1$ sets, so this family is valid for $f_t(n, a + 1)$, so $f_t(n, a + 1) \geq f_t(n, a) + 1$.

- (2) Suppose that $f_t(n, a) = m$. This means the maximum number of sets in a union-closed family such that every element is in at most a sets and such that each element is a non-trivial twin difference is m . Certainly, this means that $g_t(n, m) = g_t(n, f_t(n, a)) \leq a$ since we're minimizing and the previous family is valid here. Suppose now that $g_t(n, f_t(n, a)) = a' < a$. Then $f_t(n, a') \geq m$. Since f_t is strictly increasing in a by (1), this is a contradiction.
- (3) Suppose that $g_t(n, m) = a$. This means that there exists a family on m sets where each element is a twin difference and is also present in at most a sets. Thus, this family is valid for $f_t(n, a)$, and so $f_t(n, a) = f_t(n, g_t(n, m)) \geq m$ since we are maximizing. \square

3.3. Number of twin pairs. Another direction worth investigating would be to prove that if $f(n, a) < f(n + 1, a)$, then every element is the difference of an increasingly large number of twins. At some point, this ceases to be possible (trivially, an element cannot be the twin difference of more than a twin pairs), and so we would reach a contradiction.

Theorem 3.7. *If $f(n, a) < f(n + 1, a)$, then every element is the difference of at least two pairs of twin sets.*

Proof. Suppose that $f(n, a) = m$ and $f(n + 1, a) = m + k$ for some $k > 0$. Then we know that any optimal solution for $f(n + 1, a)$ must be such that every element is a twin difference, otherwise we could remove that element and get an $m + k$ union-closed family spanning n elements such that none is in more than a sets, and so $f(n, a) \geq m + k$, a contradiction.

Now let

$$k' := \min_{e \in [n+1]} |\{S \in \mathcal{F} | e \notin S, e \cup S \in \mathcal{F} \text{ and } \mathcal{F} \text{ is an optimal family for } f(n+1, a)\}|,$$

i.e. k' is the minimum number of twin pairs for which an element is a twin difference in an optimal solution for $f_{n+1}(a)$.

We now show that $k \leq k'$. Suppose not. Let e' and \mathcal{F}' be an element and a family that attain minimum k' . Remove e' from \mathcal{F}' , and remove the k' sets that are now duplicated. Call this new family \mathcal{F}'' . What remains is a union-closed family of $m + k - k'$ sets on n elements. So $m = f(n, a) \geq m + k - k'$, which implies that $k \leq k'$.

Now suppose that $k = k'$. Notice then that there must exist e'' such that e'' is contained in a sets of \mathcal{F}' that must still be contained in a sets of \mathcal{F}'' . If not, each element of the new family \mathcal{F}'' would be contained in at most $a - 1$ sets of \mathcal{F}'' , and so we would have that $f(n, a - 1) \geq m + k - k' = m$, which is a contradiction on the fact that $f(n, a - 1) < f(n, a)$.

Thus if $k = k'$, then there exists e'' such that $|\{S \in \mathcal{F}' | S \ni e''\}| = a$ and such that e'' is never contained in a set of the family that does not contain e' . Indeed, if there existed $S' \in \mathcal{F}'$ such that $e' \notin S'$ and $e'' \in S'$, then for any set $S'' \in \mathcal{F}$ such that $S'' \cup e' \in \mathcal{F}$ as well, i.e. twin sets with difference e' , then one of $S' \cup S''$

and $S' \cup (S'' \cup e')$ will disappear in \mathcal{F}'' and so e'' would be present $a - 1$ times, a contradiction.

Thus $\{S \in \mathcal{F}' \mid S \ni e''\} \subseteq \{S \in \mathcal{F}' \mid S \ni e'\}$ and since $|\{S \in \mathcal{F}' \mid S \ni e''\}| = a$, then $|\{S \in \mathcal{F}' \mid S \ni e'\}| = a$ as well and $\{S \in \mathcal{F}' \mid S \ni e''\} = \{S \in \mathcal{F}' \mid S \ni e'\}$ which is a contradiction of the fact that e' and e'' are twin differences for some sets. Since they are copies of each other, we can remove either one of them without creating duplicate sets. Thus, $k < k'$, and so if $k' = 1$, then $k \leq 0$, and then $f(n, a) \geq f(n + 1, a)$. \square

Note that this also means that if we remove any element from such a solution, and remove a copy of every duplicated set created, what remains is never an optimal solution for $f(n, a)$.

4. CONCLUSION

As we have seen, a complete proof of the new conjectures has far-reaching implications: Frankl's conjecture would hold for about $\frac{2}{3}$ of all possible cases, and we could show that there always exists an element in almost half of the sets of a union-closed family. Therefore we believe our new conjectures merit additional attention. In order to encourage further progress in this direction, we conclude with a few open problems of interest.

- (1) Show that $f(n, a) = f(n + 1, a)$ for smaller values of n .
- (2) Show that if $f(n, a) < f(n + 1, a)$, each element is a twin difference for even more sets. As noted, at some point, this clearly implies that the solutions are not optimal or even feasible.
- (3) Find a constant upper bound for $f_t(n + 1, a) - f_t(n, a)$. Since we know $f(n, a)$ stops growing after $n = a$, this would provide a first constant lower bound for the number of sets containing the most frequent element in a union-closed family.

5. APPENDIX

In the following tables for $f(n, a)$ and $g(n, m)$, we remove unnecessary columns, i.e. the columns for which the values of $f(n, a)$ and $g(n, m)$ are trivial.

TABLE 3. Values of $f(n, a)$

$a \setminus n$	1	2	3	4	5	6	7	8	9	$a \setminus n$	7	8	9	$a \setminus n$	8	9	$a \setminus n$	8	9
1	2	2	2	2	2	2	2	2	2	33	65	65	65	65	129	129	97	179	179
2		4	4	4	4	4	4	4	4	34	66	66	66	66	130	130	98	180	180
3			5	5	5	5	5	5	5	35	67	67	67	67	131	131	99	182	182
4				8	8	8	8	8	8	36	68	68	68	68	132	132	100	184	184
5					9	9	9	9	9	37	69	69	69	69	133	133	101	186	186
6						10	10	10	10	38	71	71	71	71	134	134	102	188	188
7							12	12	12	39	72	72	72	72	136	136	103	189	189
8								16	16	40	74	74	74	74	137	137	104	192	192
9									17	41	75	75	75	75	139	139	105	194	194
10										42	77	77	77	77	140	140	106	196	196
11										43	79	79	79	79	142	142	107	198	198
12										44	80	80	80	80	144	144	108	200	200
13										45	82	82	82	82	145	145	109	202	202
14										46	83	83	83	83	146	146	110	204	204
15										47	85	85	85	85	147	147	111	206	206
16										48	88	88	88	88	149	149	112	209	209
17										49	89	89	89	89	150	150	113	211	211
18										50	91	91	91	91	152	152	114	214	214
19										51	93	93	93	93	154	154	115	216	216
20										52	95	95	95	95	156	156	116	220	220
21										53	98	98	98	98	157	157	117	221	221
22										54	99	99	99	99	158	158	118	224	224
23										55	101	101	101	101	160	160	119	226	226
24										56	104	104	104	104	162	162	120	229	229
25										57	105	105	105	105	164	164	121	231	231
26										58	108	108	108	108	166	166	122	233	233
27										59	110	110	110	110	168	168	123	236	236
28										60	113	113	113	113	170	170	124	240	240
29										61	115	115	115	115	171	171	125	242	242
30										62	118	118	118	118	173	173	126	245	245
31										63	121	121	121	121	175	175	127	248	248
32										64	128	128	128	128	176	176	128	256	256

TABLE 4. Values of $g(n, m)$

$m \setminus n$	1	2	3	4	5	6	7	8
2	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	2
4	2	2	2	2	2	2	2	2
5	3	3	3	3	3	3	3	3
6	4	4	4	4	4	4	4	4
7	4	4	4	4	4	4	4	4
8	4	4	4	4	4	4	4	4
9	5	5	5	5	5	5	5	5
10	6	6	6	6	6	6	6	6
11	7	7	7	7	7	7	7	7
12	7	7	7	7	7	7	7	7
13	8	8	8	8	8	8	8	8
14	8	8	8	8	8	8	8	8
15	8	8	8	8	8	8	8	8
16	8	8	8	8	8	8	8	8
17	9	9	9	9	9	9	9	9
18	10	10	10	10	10	10	10	10
19	11	11	11	11	11	11	11	11
20	12	12	12	12	12	12	12	12
21	12	12	12	12	12	12	12	12
22	13	13	13	13	13	13	13	13
23	13	13	13	13	13	13	13	13
24	14	14	14	14	14	14	14	14
25	14	14	14	14	14	14	14	14
26	15	15	15	15	15	15	15	15
27	15	15	15	15	15	15	15	15
28	16	16	16	16	16	16	16	16
29	16	16	16	16	16	16	16	16
30	16	16	16	16	16	16	16	16
31	16	16	16	16	16	16	16	16
32	16	16	16	16	16	16	16	16

$m \setminus n$	6	7	8
33	17	17	17
34	18	18	18
35	19	19	19
36	20	20	20
37	21	21	21
38	21	21	21
39	22	22	22
40	22	22	22
41	23	23	23
42	24	24	24
43	24	24	24
44	25	25	25
45	25	25	25
46	26	26	26
47	26	26	26
48	27	27	27
49	27	27	27
50	28	28	28
51	28	28	28
52	28	28	28
53	29	29	29
54	30	30	30
55	30	30	30
56	30	30	30
57	31	31	31
58	31	31	31
59	32	32	32
60	32	32	32
61	32	32	32
62	32	32	32
63	32	32	32
64	32	32	32

$m \setminus n$	7	8
65	33	33
66	34	34
67	35	35
68	36	36
69	37	37
70	38	38
71	38	38
72	39	39
73	40	40
74	40	40
75	41	41
76	42	42
77	42	42
78	43	43
79	43	43
80	44	44
81	45	45
82	45	45
83	46	46
84	47	47
85	47	47
86	48	48
87	48	48
88	48	48
89	49	49
90	50	50
91	50	50
92	51	51
93	51	51
94	52	52
95	52	52
96	53	53

$m \setminus n$	7	8
97	53	53
98	53	53
99	54	54
100	55	55
101	55	55
102	56	56
103	56	56
104	56	56
105	57	57
106	58	58
107	58	58
108	58	58
109	59	59
110	59	59
111	60	60
112	60	60
113	60	60
114	61	61
115	61	61
116	62	62
117	62	62
118	62	62
119	63	63
120	63	63
121	63	63
122	64	64
123	64	64
124	64	64
125	64	64
126	64	64
127	64	64
128	64	64

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