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A Hall Condition for Normal Hypergraphs

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Conforti, Cornuéjols, Kapoor, and Vušković gave a Hall-type condition for the non-existence of a perfect matching in a balanced hypergraph. We generalize this result to the class of normal hypergraphs.

1 Introduction

Hall's theorem gives a necessary and sufficient condition for the existence of a system of distinct representatives of a family of finite sets. It is equivalent to the following result on matchings in bipartite graphs.

Theorem 1.1. *[Hal35] A bipartite graph has a perfect matching if and only if for all stable sets S the set $N(S)$ of its neighbors is as least as big as S.*

This result has been generalized to balanced hypergraphs by Conforti, Cornuéjols, Kapoor, and Vušković [CCKV96] using a linear programming argument. Later, Huck and Triesch [HT02] gave the first combinatorial proof, Schrijver provided the probably shortest proof (Corollary 83.1d in [Sch03]), and recently Scheidweiler [Sch11] gave an alternative one based on an elegant Gallai-Edmonds decomposition.

Theorem 1.2. *[CCKV96] A balanced hypergraph has no perfect matching if and only if there exists a pair (R, B) of disjoint node sets such that $|e \cap R| \geq |e \cap B|$ for all hyperedges e but $|R| < |B|$.*
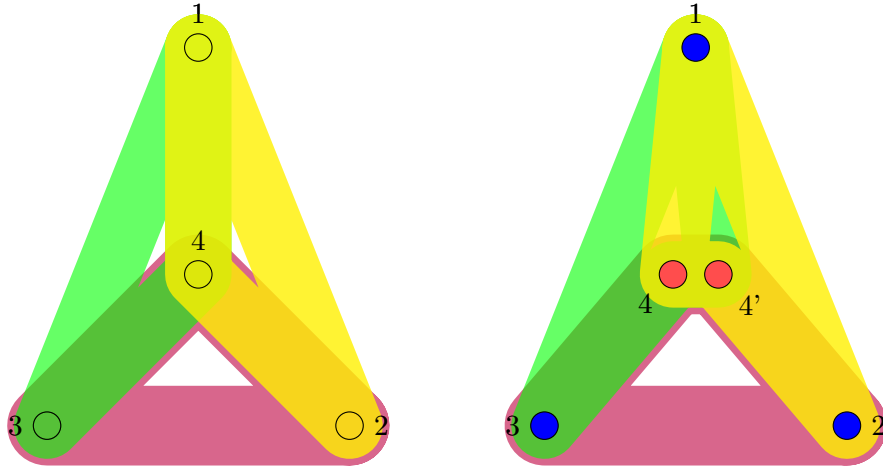
The optimization version of Hall's theorem is Kőnig's theorem which holds for bipartite graphs, balanced hypergraphs, and for the larger class of normal hypergraphs. This brings up the question whether Theorem 1.2 can be generalized to normal hypergraphs. However, the theorem's condition is only necessary but not sufficient, as the following example shows.

(a) A normal hypergraph without perfect matching which has no pair (R, B) violating Hall's condition.

(b) The hypergraph depicted in (a) after duplicating vertex 4 has a pair (R, B) violating Hall's condition.

Figure 1: Does Hall's condition hold for normal hypergraphs?

Example 1.1. Take $H = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}\}$, see Figure 1a. This is the smallest normal hypergraph that is not balanced. H has no perfect matching and $|R| \geq |B|$ for every pair of disjoint node sets R, B with $|e \cap R| \geq |e \cap B|$ for all $e \in E$.

We can fix Hall's condition in Example 1.1 by considering the hypergraph $H' = (V', E')$ obtained from H by multiplying vertex 4 by 2, that is, by setting $V' = \{1, 2, 3, 4, 4'\}$ and $E' = \{\{1, 2, 4, 4'\}, \{2, 3, 4, 4'\}, \{1, 3, 4, 4'\}\}$. Now, $B := \{1, 2, 3\}$ and $R := \{4, 4'\}$ is a pair violating Hall's condition, see Figure 1b.

In Section 2 we show how this idea can be generalized in order to derive a Hall condition for normal hypergraphs, including an analysis of bounds for the multiplicity of a node. We also investigate a deficiency version. Section 3 discusses relations and differences between Kőnig's and Hall's theorem in the graph and hypergraph case.

2 A Hall Condition for Normal Hypergraphs

We say that a hypergraph satisfies *Hall's condition* if $|R| \geq |B|$ for all disjoint node sets R, B with $|e \cap R| \geq |e \cap B|$ for all hyperedges e. Otherwise, H violates Hall's condition for hypergraphs.

Analogous to graphs [LP86], we define the *deficiency* of a hypergraph

$$\operatorname{def}(H) := \min\{|V| - |V(M)| : \text{ M is a matching of } H\}$$

Figure 2: A balanced hypergraph with $\text{def}(H) = 3 > 1 = d(H)$

to be the minimum number of vertices that cannot be covered by a matching. In the same vein, we call the maximum violation of Hall's condition in a hypergraph its *critical difference*

$$d(H) := \max\{|B|-|R| : R, B \subseteq V(H), R\cap B = \emptyset, |e\cap R| \geq |e\cap B| \forall e \in E(H)\},$$

and a pair R, B of nodes attaining the maximum at the right hand side of this definition a *critical pair*; this generalizes the definition of the critical difference in graphs [LM12].

A hypergraph has a perfect matching if and only if its deficiency is zero, and it satisfies Hall's condition if and only if its critical difference is zero. Thus, the result of Conforti, Cornuéjols, Kapoor, and Vušković [CCKV96] can be restated as $\text{def}(H) = 0 \Leftrightarrow d(H) = 0$ for all balanced hypergraphs H.

For bipartite graphs, and more generally Kőnig-Egerváry graphs (for a characterization of Kőnig-Egerváry graphs see [Ste79], [Dem79], or [KNP06]), it is known that the deficiency is equal to the critical difference, see for example [LM12].

In hypergraphs, the critical difference gives a lower bound on the deficiency. Indeed, if M is a matching covering as many vertices as possible in a hypergraph H, and R, B is a critical pair of H, then

$$d(H) = |B| - |R| = \left(\sum_{e\in M} |e \cap B| + |B \setminus V(M)|\right) - \left(\sum_{e\in M} |e \cap R| + |R \setminus V(M)|\right)$$

$$\leq |B \setminus V(M)| - |R \setminus V(M)| \leq |B \setminus V(M)| \leq |V \setminus V(M)| = \text{def}(H).$$

Huck and Triesch [HT02] observed that the gap between the critical difference and the deficiency can be arbitrarily large even for very simple balanced hypergraphs.

Example 2.1. [HT02] Let H by the hypergraph with vertices $1, 2, \ldots, 2n+1$ and hyperedges $e_1 = \{1, \ldots, n+1\}, e_2 = \{n+1, \ldots, 2n+1\}$, see Figure 2 for an example with $n = 3$.

As H has only two hyperedges it is obviously balanced. Furthermore, every non-empty matching misses n vertices, so $\text{def}(H) = n$. However, $d(H) = 1$ as there is no pair $R, B \subseteq V$ with $|e_i \cap B| \leq |e_i \cap R|$ for $i = 1, 2$ and $|B|-|R| > 1$, and thus $R = \{n+1\}, B = \{n, n+2\}$ is a critical pair.

If we could take n copies of the vertex $n+1$ into the set R, and all other vertices into B, then we would get a pair R, B with $|e_i \cap R| = n = |e_i \cap B|$ $(i = 1, 2)$ and $|B| - |R| = 2n - n = n$. This means that the deficiency of

H equals the critical difference of the hypergraph in which node $n + 1$ is "multiplied" n times.

The multiplication trick of Example 2.1 can be formalized and generalized to derive a Hall condition for normal hypergraphs. To this purpose, we use the vertex multiplication definition according to Berge.

Definition 2.1. [Ber89] Let $H = (V, E)$ be a hypergraph, $v \in V$ a fixed vertex, and $\lambda \in \mathbb{N}$. The hypergraph obtained by *multiplying v by λ* is the hypergraph that arises from H by replacing the vertex v by λ new vertices $(v, 1), \ldots, (v, \lambda)$ and every hyperedge e containing v by the new hyperedge $e \setminus \{v\} \cup \{(v, 1), \ldots, (v, \lambda)\}$.
For $c \in \mathbb{N}^V$, $H^{(c)}$ is the hypergraph obtained from H by multiplying each vertex v by c_v. We denote by $V^{(c)}$ the set of vertices of $H^{(c)}$, by $E^{(c)}$ the set of hyperedges of $H^{(c)}$, and for every $e \in E$ we denote by $e^{(c)}$ the corresponding hyperedge in $E^{(c)}$. If all entries of c are equal to some constant $k \in \mathbb{N}$, we also write $H^{(k)}, V^{(k)}, E^{(k)}$, and $e^{(k)}$.

Berge observed that $H^{(c)}$ is balanced if H is balanced. Similarly, if H is normal, then $H^{(c)}$ is also normal. So multiplying the vertices of a hypergraph does not destroy normality.
One problem arises when looking at the critical difference of the multiplied hypergraph $H^{(k)}$. Namely, if $R, B \subseteq V^{(k)}$ is a critical pair of $H^{(k)}$ with $d(H^{(k)}) = |B| - |R| > 0$, then we can define a pair of disjoint vertex sets $R', B' \subseteq V^{(lk)}$ by taking l times the number of copies of $v \in V$ that R or B contains. It holds that $|e \cap R'| = l \cdot |e \cap R| \geq l \cdot |e \cap B| = |e \cap B'|$ for all $e \in E^{(lk)}$ and $|B'| - |R'| = l \cdot (|B| - |R|)$, and thus $d(H^{(lk)}) \geq l \cdot d(H^{(k)})$. This means, that $d(H^{(k)})$ is unbounded for $k \to \infty$. Which is a problem as we do not know a priori how often we have to multiply the vertices. We can overcome this problem by considering a restricted *multiplied critical difference* defined as

$$d^*(H^{(k)}) := \max\{|B| - |R| : R, B \subseteq V^{(k)}, \ R \cap B = \emptyset,$$
$$|e \cap R| \geq |e \cap B| \ \forall e \in E^{(k)},$$
$$|B \cap \{(v, 1), (v, 2), \ldots, (v, k)\}| \leq 1 \ \forall v \in V\},$$

where the set R is arbitrary but B is only allowed to contain at most one copy of each multiplied node.
In particular, we have that $d^*(H^k)$ is bounded for $k \to \infty$. This, together width $d^*(H^k) \geq d^*(H^{k'})$ for $k \geq k'$, implies that $d^*(H^{(k)})$ becomes constant, i.e., there exists an $N \in \mathbb{N}$ with $d^*(H^{(k)}) = d^*(H^{(N)})$ for all $k \geq N$. Our main theorem shows that $N \leq r - 1$ and the limit $d^*(H^{(N)})$ is equal to the deficiency for a normal hypergraph H. In short, we obtain a deficiency version of Hall's condition for normal hypergraphs.

Theorem 2.1. $\mathrm{def}(H) = d^*(H^{(r-1)})$ *holds for every normal hypergraph H of rank r.*

Proof. First, we show that $\mathrm{def}(H) \geq d^*(H^{(k)})$ for all $k \in \mathbb{N}$, in particular this implies $\mathrm{def}(H) \geq d^*(H^{(r-1)})$. Let M be a matching covering as many vertices as possible, and $R, B \subseteq V^{(k)}$ a critical pair in which B contains at most one copy of every multiplied node. We set $M^{(k)} := \{e^{(k)} | e \in M\} \subseteq E^{(k)}$ and $C := V(M^{(k)}) \subseteq V^{(k)}$. Then $M^{(k)}$ is a matching in $H^{(k)}$ corresponding to M, and C is the set of vertices covered by $M^{(k)}$. It follows that

$$
\begin{aligned}
d^*(H^{(k)}) &= |B| - |R| \\
&= \sum_{e \in M^{(k)}} |e \cap B| + |B \setminus C| - \sum_{e \in M^{(k)}} |e \cap R| - |R \setminus C| \\
&\leq |B \setminus C| - |R \setminus C| \leq |B \setminus C| \\
&\leq |V \setminus V(M)| = \mathrm{def}(H).
\end{aligned}
$$

The first equality holds by the choice of the pair (R, B), the second equality follows from counting B and R hyperedge wise, the first inequality holds because of $|e \cap R| \geq |e \cap B|$ for all $e \in E^{(k)}$, the second inequality is trivial, and the last inequality holds because B contains at most one copy of every multiplied node.

Now, we prove $\mathrm{def}(H) \leq d^*(H^{(r-1)})$. The linear program

$$
\begin{aligned}
\max \sum_{e \in E} |e| x_e && (1) \\
s.t. \sum_{e \in \delta(v)} x_e \leq 1 && \forall v \in V \\
x_e \geq 0 && \forall e \in E
\end{aligned}
$$

and its dual

$$
\begin{aligned}
\min \sum_{v \in V} y_v && (2) \\
s.t. \sum_{v \in e} y_v \geq |e| && \forall e \in E \\
y_v \geq 0 && \forall v \in V
\end{aligned}
$$

have integral optimal solutions, see [Lov72].

Let $x^* \in \mathbb{Z}^E$ be an optimal solution to (1) and $y^* \in \mathbb{Z}^V$ be an optimal solution to (2). x^* corresponds to a matching M covering as many vertices as possible and y^* corresponds to a multiset S of vertices such that $|S \cap e| \geq |e|$ for all hyperedges e (counted with multiplicity). Using duality, we have $|S| = |V(M)|$.

It is clear that $y_v^* \le \max_{e \in E} |e| = r$ for all $v \in V$, so we can use y^* to define a pair (R, B) of disjoint vertex sets in $V^{(r-1)}$ as follows:

$$
\begin{aligned}
B &:= \{(v,1)|y_v^* = 0\}, & (3) \\
R &:= \{(v,i)|y_v^* \ge 2, \ 1 \le i \le y_v^* - 1\}. & (4)
\end{aligned}
$$

Note that B contains at most one copy of each multiplied node, but R can contain several copies. Furthermore,

$$
|e \cap R| - |e \cap B| = \sum_{v \in e}(y_v^* - 1) = |S \cap e| - |e| \ge 0,
$$

and

$$
|B| - |R| = \sum_{v \in V}(1 - y_v^*) = |V| - |S| = |V| - |V(M)| = \mathrm{def}(H).
$$

It follows that $\mathrm{def}(H) = |B| - |R| \le d^*(H^{(r-1)})$. ∎

Theorem 2.1 gives the following combinatorial characterization for the non-existence of a perfect matching in a normal hypergraph.

Corollary 2.2 (A Hall Condition for Normal Hypergraphs)**.** *A normal hypergraph H of rank r has no perfect matching if and only if $H^{(r-1)}$ violates Hall's condition, i.e., $\mathrm{def}(H) > 0$ if and only if $d(H^{(r-1)}) > 0$.*

Proof. We have already seen that $d(H^{(r-1)}) > 0$ implies $\mathrm{def}(H^{(r-1)}) > 0$ and this shows that $\mathrm{def}(H) > 0$. On the other hand, if H has no perfect matching, then Theorem 2.1 implies $0 < \mathrm{def}(H) = d^*(H^{(r-1)}) \le d(H^{(r-1)})$. ∎

The rank bound in Theorem 2.1 and Corollary 2.2 on the node multiplication factor is best possible as the following example shows.

Example 2.2. For every natural number $n \ge 3$, let $H_n = (V_n, E_n)$ be the hypergraph with vertex set $\{1, \ldots, n, n+1\}$ and hyperedges $S \cup \{n+1\}$ for every subset S of $\{1, \ldots, n\}$ of size $n-1$, see Figure 1a for an illustration of H_3. Every two hyperedges of H_n intersect in the $(n+1)$-th vertex, so H_n is a normal hypergraph without a perfect matching.

We claim that $N = n - 1$ is the smallest natural number such that $H_n^{(N)}$ violates Hall's condition. Indeed, let $N \in \mathbb{N}$ such that $H_n^{(N)}$ violates Hall's condition. So there exists a pair $R, B \subseteq V_n^{(N)}$ with $|e \cap R| \ge |e \cap B|$ for all $e \in E_n^{(N)}$ but $|R| < |B|$.

For every $i \in V_n = \{1, \ldots, n+1\}$ we define

$$
y_i := |\{\text{copies of } i \text{ in } R\}| - |\{\text{copies of } i \text{ in } B\}|.
$$

Every hyperedge of H_n is of the form $e = \{1, \ldots, n+1\} \setminus \{i\}$ for some $i \in \{1, \ldots, n\}$. As $|e^{(N)} \cap R| \geq |e^{(N)} \cap B|$, we get

$$y_1 + y_2 + \ldots + y_{n+1} - y_i \geq 0 \qquad (5)$$

for all $i = 1, \ldots, n$. On the other hand $\sum_{i=1}^{n+1} y_i = |R| - |B| < 0$, thus

$$y_i \leq \sum_{i=1}^{n+1} y_i < 0$$

holds for all $i = 1, \ldots, n$. The integrality of y_i implies $y_i \leq -1$ for $i = 1, \ldots, n$. This together with inequality (5) for $i = n$ gives

$$y_{n+1} \geq -y_1 - y_2 - \ldots - y_{n-1} \geq n - 1.$$

It follows that $N \geq n - 1$.

3 Relations to other Hypergraph and Graph Properties

In this section we look at the relationship between Kőnig's Theorem, Hall's Theorem, and the deficiency version of Hall's Theorem in several graph and hypergraph classes, including bipartite and Kőnig-Egerváry graphs, as well as balanced and normal hypergraphs. In particular, we show that the hypergraphs that satisfy the multiplied Hall condition and those for which Kőnig's Theorem hold do not coincide.

We denote by $\nu(H)$ the maximum size of a matching in a hypergraph H, by $\nu^*(H)$ the maximum size of a fractional matching, by $\tau(H)$ the minimum size of a vertex cover, and by $\tau^*(H)$ the minimum size of a fractional vertex cover. Clearly, $\nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H)$ holds for all hypergraphs H, see for example Chapter 3 in Berge's book [Ber89].

A graph G is said to be *stable* iff $\nu(G) = \nu^*(G)$. Stable graphs are exactly those graphs for which the set of vertices that are not covered by some maximum matching form a stable set. In particular, we can test whether a graph is stable in time $O(|V(G)|^3)$ using the Gallai-Edmonds-Decomposition of G. Furthermore, a graph G is *semi-bipartite* if every two node disjoint odd cycles are connected by an edge.

We say that a hypergraph H of rank r has the *Kőnig property*, the *Hall property*, the *multiplied Hall property*, or the *strong multiplied Hall property* if $\nu(H) = \tau(H)$ (Kőnig's Theorem), $\text{def}(H) \neq 0 \Leftrightarrow d(H) \neq 0$ (Theorem 1.2), $\text{def}(H) \neq 0 \Leftrightarrow d(H^{(r-1)}) \neq 0$ (Corollary 2.2), or $\text{def}(H) = d^*(H^{(r-1)})$ (Theorem 2.1), respectively. In the graph case the multiplication factor is one, so the Hall and the multiplied Hall property are the same, and we call the strong multiplied Hall property just the *strong Hall property*.

First, we analyze the relations between the classes of graphs defined by the properties above, and later between the respective classes of hypergraphs.

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3.1 Relations between Graph Properties

A summary of our results on graphs is given in Figure 3 where the rectangles indicate graph properties and an arrow from one rectangle to another indicates an implication.
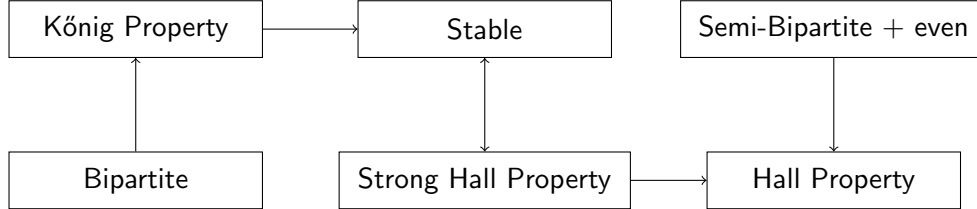

Figure 3: Summary of relations between graph properties

In the following, we argue that all solid arrows correspond to valid implications, and every valid implication is contained in the transitive closure of the diagram illustrated in Figure 3.

Bipartite → Kőnig Property

This is the result of Kőnig's Matching Theorem, see for example Theorem 16.2 in [Sch03].

Kőnig Property ↛ Bipartite

[Ste79], [Dem79], and [KNP06] characterized graphs with the Kőnig property. From their results it follows that there are non-bipartite graphs with the Kőnig property.

Kőnig Property → Stable

If G has the Kőnig property, then G is stable because $\nu(G) \leq \nu^*(G) \leq \tau(G) = \nu(G)$ implies $\nu(G) = \nu^*(G)$.

Stable ↛ Bipartite, Kőnig Property

K_4 is stable but it is not bipartite and $\nu(K_4) = 2 < 3 = \tau(K_4)$.

Stable ↔ Strong Hall Property

By introducing variables $y, z \in \{0,1\}^V$ we can calculate the critical difference $d(G)$ of G via the following integer program:

$$\max \sum_{v \in V}(y_v - z_v) \tag{6}$$

$$s.t. \sum_{v \in e}(z_v - y_v) \geq 0 \quad \forall e \in E \tag{7}$$

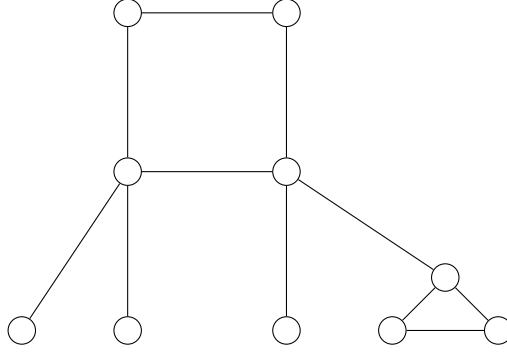$$y_v, z_v \in \{0,1\} \qquad \forall v \in V \tag{8}$$

Figure 4: A semi-bipartite graph which is not stable.

Setting $w_v := \frac{z_v - y_v + 1}{2}$ transforms (6-8) into the half-integer program

$$\max \left(|V| - 2 \cdot \sum_{v \in V} w_v \right) \tag{9}$$
$$s.t. \sum_{v \in e} w_v \geq 1 \qquad \forall e \in E \tag{10}$$
$$w_v \in \{0, \tfrac{1}{2}, 1\} \qquad \forall v \in V. \tag{11}$$

(9-11) has the same optimal value as the linear program

$$|V| - \quad 2 \cdot \min \sum_{v \in V} w_v \tag{12}$$
$$s.t. \sum_{v \in e} w_v \geq 1 \quad \forall e \in E \tag{13}$$
$$0 \leq w_v \leq 1 \qquad \forall v \in V \tag{14}$$

because (12-14) has always an optimal solution with entries in $\{0, \tfrac{1}{2}, 1\}$.
On the other hand, the deficiency of G is equal to the optimal value of the following integer program:

$$|V| - \quad 2 \cdot \max \sum_{e \in E} x_e \tag{15}$$
$$\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \tag{16}$$
$$x_e \in \{0, 1\} \qquad \forall e \in E. \tag{17}$$

As the LP-relaxation of (15-17) is the dual of (12-14), $d(G) = \mathrm{def}(G)$ holds if and only if $\nu(G) = \nu^*(G)$. In other words, G has the strong Hall property if and only if G is stable.

Strong Hall Property $\rightarrow$ Hall Property

Follows directly from the definition of the strong Hall property and the Hall property.

Semi-Bipartite + even $\rightarrow$ Hall Property

See for example Theorem 8 in Chapter 7 of [Ber73].

Bipartite, Stable, Kőnig, (Strong) Hall Property ↛ Semi-Bipartite + even

For all graph properties on the left hand side there are graphs with this property and an odd number of vertices.

Semi-Bipartite + even, Hall Property ↛ Bipartite, Stable, Kőnig, Strong Hall Property

Figure 4 shows a semi-bipartite graph (with the Hall property) that is neither bipartite nor stable, and has neither the Kőnig nor the strong Hall property.

Remaining implications

"**Bipartite → Stable, (Strong) Hall Property**", "**Kőnig Property → (Strong) Hall Property**", and "**Stable → Hall Property**" follow by transitivity.

3.2 Relations between Hypergraph Properties

The hypergraph case deals with more classes, and not all results of the graph case carry over.

Figure 5 shows a diagram summarizing the relationships of the investigated hypergraph properties; solid lines indicate hypergraph results and dashed lines an "overlay" of Figure 3.
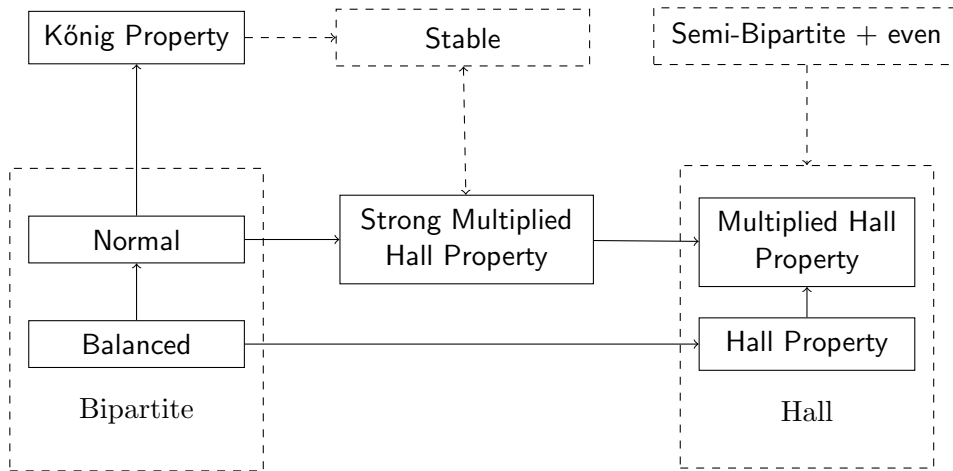

Figure 5: Summary of relations between hypergraph properties

In the remainder, we argue that the implications indicated by the transitive closure of the solid arrows, and only these implications, are valid.
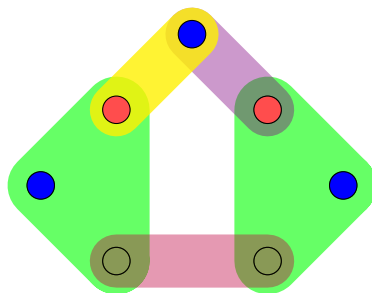
Figure 6: A hypergraph with the strong multiplied Hall-Property and $\nu(H) = 2 < 3 = \tau(H)$.

Balanced → Normal

See Theorem 4 in [Lov72].

Normal ↛ Balanced

See Figure 1a for a normal, non-balanced hypergraph.

Normal → Kőnig Property

See Theorem 3 in [Lov72].

Kőnig Property ↛ Balanced, Normal

Choose some Kőnig-Egerváry graph which is not bipartite. This is also a hypergraph with the Kőnig property which is neither balanced nor normal.

Balanced → Hall Property

This is the main result of [CCKV96].

Hall Property ↛ Balanced, Normal

The hypergraphs depicted in Figure 6 and Figure 7b are not balanced and not normal but have the Hall property.

Normal → Strong Multiplied Hall Property

See Theorem 2.1.

Strong Multiplied Hall → Multiplied Hall-Property

See the proof of Corollary 2.2.

Multiplied Hall Property $\not\rightarrow$ Strong Multiplied Hall Property

The hypergraph shown in Figure 7b has the multiplied Hall property but it does not have the strong multiplied Hall property as $\mathrm{def}(H) = 3 > 1 = d^*(H^{(2)})$.

Balanced $\rightarrow$ Kőnig, (Strong) Multiplied Hall Property & Normal $\rightarrow$ Multiplied Hall Property

By transitivity.

Normal, Kőnig, (Strong) Multiplied Hall Property $\not\rightarrow$ Hall Property

Figure 1a shows a hypergraph without the Hall property which is normal, and thus has the Kőnig property. Furthermore, as the depicted hypergraph is normal it has the multiplied and the strong multiplied Hall property.

(Strong) Multiplied Hall Property $\not\rightarrow$ Balanced, Normal, Kőnig Property

Figure 6 shows a hypergraph H with $\nu(H) = 2$ and $\tau(H) = 3$, so it does not have the Kőnig property, is not balanced, and not normal. However, as indicated by the red and blue nodes, the critical difference of the hypergraph is one which is the same as its deficiency. In particular, the depicted hypergraph has the strong multiplied Hall property.
In addition, in Example 3.1 we give a family of hypergraphs with the multiplied Hall property which are not necessarily balanced, normal, or have the Kőnig property.

Example 3.1. We define for a graph G the hypergraph $K(G)$ with the same vertex set as G and whose hyperedges are all maximal cliques of G as it was done in [FLV84]. If G is perfect, then the dual hypergraph of $K(G)$ is normal (see [Lov72]), however, $K(G)$ itself must not be normal.

Claim: If G is a perfect graph in which every maximal clique has size k, then $\mathrm{def}(K(G)) \neq 0$ if and only if $d(K(G)^{(k-1)}) \neq 0$.

Proof. If $d(H^{(k-1)}) \neq 0$, then H has no perfect matching and thus H has positive deficiency.
It remains to show that whenever H has no perfect matching the hypergraph $H^{(k-1)}$ contains a pair $B, R \subseteq V^{(k-1)}$ such that $|B| > |R|$ and $|e \cap R| \geq |e \cap B|$ $\forall e \in E^{(k-1)}$. As H has no perfect matching, the size of a minimum hyperedge cover of H is greater than $\frac{|V|}{k}$. By the construction of H, every hyperedge cover of H corresponds to a clique cover of G, and vice versa. This implies that the minimum size of a clique cover of G is greater than $\frac{|V|}{k}$. As G is perfect the minimum size of a clique cover in G is equal to the maximum

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(a) A perfect graph G in which every maximal clique has size three.
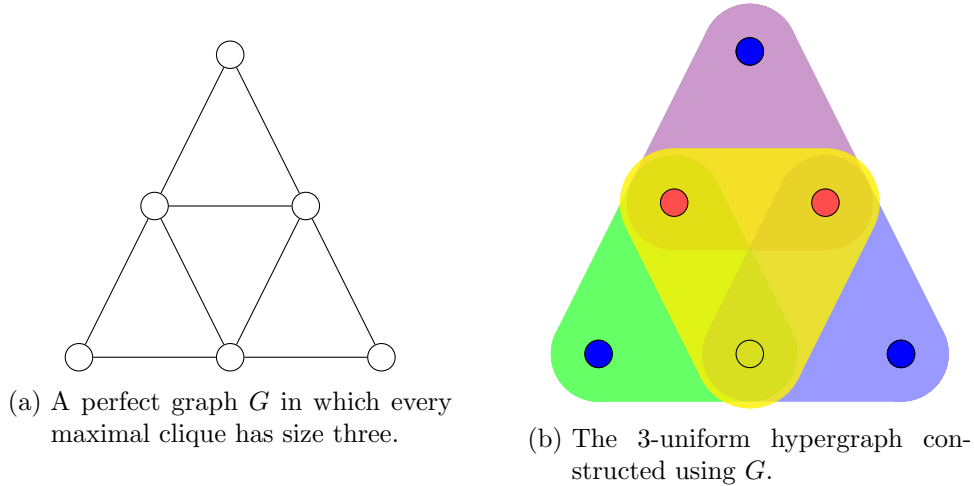
(b) The 3-uniform hypergraph constructed using G.

Figure 7: Illustration of the construction in Example 3.1.

size of a stable set of G. Let $S \subseteq V$ be a maximum stable set. We use S to define $B, R \subseteq V^{(k-1)}$ in the following way:

$$B := \{(v,1),(v,2),\ldots,(v,k-1)|v \in S\},$$
$$R := \{(v,1)|v \in V \setminus S\}.$$

We have that

$$|B| > \frac{k-1}{k}|V| = |V| - \frac{|V|}{k} > |V| - |S| = |R|.$$

As S is a stable set and B contains just $k-1$ copies of every vertex of S, we have that $|e \cap B| \le k-1$ for every $e \in E^{(k-1)}$. On the other hand, $|e \cap R| \ge k-1$ for all $e \in E^{(k-1)}$ holds. Thus, $|e \cap R| \ge |e \cap B| \, \forall e \in E^{(k-1)}$. ∎

All $(k-1)$-trees are perfect graphs in which every maximal clique has size k, see Figure 7 for the construction of $K(G)$ where G is a 2-tree.

Kőnig Property ↛ (Strong) Multiplied Hall Property

The hypergraph H with vertices v_1, v_2, v_3, v_4, v_5 and hyperedges $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_5\}$, $\{v_3, v_4\}$ (see Figure 8) has matching number $\nu(H) = 2$ and vertex covering number $\tau(H) = 2$, so it has the Kőnig property. In the following we show that H does not have the multiplied Hall property.
Suppose to the contrary that H has the multiplied Hall property. As H has no perfect matching, there must exist a pair $R, B \subseteq V^{(3)}$ of disjoint node sets such that $|e \cap R| \ge |e \cap B|$ for all $e \in E^{(3)}$ and $|R| < |B|$.
For $i = 1, \ldots, 5$ we define

$$y_i := |\{\text{copies of } v_i \text{ in } R\}| - |\{\text{copies of } v_i \text{ in } B\}|.$$
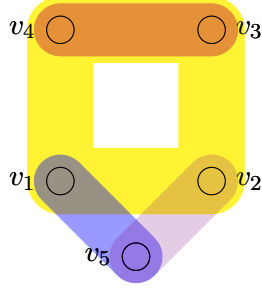
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Figure 8: A hypergraph with the Kőnig property and without the multiplied Hall property.

Then $|R| < |B|$ and $|e \cap R| \geq |e \cap B| \; \forall e \in E^{(3)}$ imply the following inequalities:

$$y_1 + y_2 + y_3 + y_4 + y_5 \leq -1 \tag{18}$$
$$y_1 + y_2 + y_3 + y_4 \geq 0 \tag{19}$$
$$y_1 + y_5 \geq 0 \tag{20}$$
$$y_2 + y_5 \geq 0 \tag{21}$$
$$y_3 + y_4 \geq 0. \tag{22}$$

The inequalities (18) and (19) imply $y_5 \leq -1$, and (20-21) imply $y_1 \geq -y_5 \geq 1$, $y_2 \geq -y_5 \geq 1$. Using these observations and inequality (22) we get

$$-1 \geq y_1 + y_2 + y_3 + y_4 + y_5 \geq y_2 + y_3 + y_4 \geq y_2 \geq 1,$$

which is a contradiction. Thus, H does not have the multiplied Hall property.

Hall Property → Multiplied Hall Property

If H has the Hall property, then $d^*(H^{(r-1)}) = 0$ implies $d(H) = 0$, as $d^*(H^{(r-1)}) \geq d(H)$. Furthermore, $d(H) = 0$ implies $\mathrm{def}(H) = 0$, thus $d^*(H^{(r-1)}) = 0 \Rightarrow \mathrm{def}(H) = 0$. It is easy to see that the converse implication $\mathrm{def}(H) = 0 \Rightarrow d^*(H^{(r-1)}) = 0$ always holds. In particular, H has the multiplied Hall property.

Hall Property ↛ Kőnig Property

See Figure 7b or 6.

Hall Property ↛ Strong Multiplied Hall Property

The hypergraph H depicted in Figure 7b has the Hall property but $d^*(H^{(2)}) = 1 < 3 = \mathrm{def}(H)$.

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