



Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

JESCO HUMPOLA

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Konrad-Zuse-Zentrum für Informationstechnik Berlin
Takustraße 7
D-14195 Berlin-Dahlem

Telefon: 030-84185-0
Telefax: 030-84185-125

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Sufficient Pruning Conditions for MINLP in Gas Network Design

Jesco Humpola

Zuse Institute Berlin,
Takustr. 7, 14195 Berlin, Germany
humpola@zib.de
<http://www.zib.de>

Abstract. One quarter of Europe’s energy demand is provided by natural gas distributed through a vast pipeline network covering the whole of Europe. At a cost of 1 million Euros per kilometer the extension of the European pipeline network is already a multi billion Euro business. Therefore, automatic planning tools that support the decision process are desired. We model the topology extension problem in gas networks by a mixed-integer nonlinear program (MINLP). This gives rise to a so-called active transmission problem, a continuous nonlinear non-convex feasibility problem which emerges from the MINLP model by fixing all integral variables. In this article we offer novel sufficient conditions for proving the infeasibility of this active transmission problem. These conditions can be expressed in the form of a mixed-integer program (MILP), i.e., the infeasibility of a non-convex NLP can be certified by solving an MILP. These results provide an efficient bounding procedure in a branch-and-bound algorithm. Our computational results demonstrate a substantial speed-up for the necessary computations.

Keywords: Network Design; Mixed-Integer Nonlinear Programming; Infeasibility Detection.

1 Introduction

Natural gas usage represents one quarter of the world’s energy demand [5]. The gas is distributed in huge pipeline systems. The gas transport network of Germany alone has a size of over 30,000 km. Given a cost of about one million Euro per km the network constitutes a substantial investment. Naturally, there is a high demand to minimize the building costs in case the network has to be extended [27].

A gas network mainly consists of *pipelines, compressors, valves, and control valves*. It can be extended by adding one of these elements in order to increase the transportation capacity [7]. Several approaches to improve the topology of a gas network are reported in the literature. Boyd et al. [4] apply a genetic algorithm to solve a pipe-sizing problem for a network with 25 nodes and 25 pipes, each of which could have one out of six possible diameters. Castillo and González [6] also apply a genetic algorithm for finding a tree topology solution for a network problem with up to 21 nodes and 20 arcs. Mariani et al. [16] describe the design problem of a natural gas pipeline. They present a set of parameters to evaluate the quality of the transportation system. Based on these ones they evaluate a number of potential network topologies to identify the best among them. Osiadacz and Górecki [20] formulate a network design problem for a given topology as a nonlinear optimization problem, for which they iteratively compute a local optimum. For a given topology the diameter of the pipes is a free design variable. Their method is applied to a network with up to 108 pipes and 83 nodes. De Wolf and Smeers [9] also use a nonlinear formulation and apply a local solver. They distinguish the operational problem (running the network) from the strategical investment problem (extending the network). For a given topology with up to 30 arcs and nodes they can determine (locally) optimized pipe diameters. Bonnans and André [2, 3] consider the optimal design problem of a straight pipeline system and derive some theoretical properties of an optimal design. In [14] we describe a primal heuristic based on dual information from KKT solutions of the gas network model formulation.

In the following we will model the extension problem in gas networks by a mixed-integer nonlinear program (MINLP) called *topology optimization problem*. Computationally it is solved by applying a branch-and-cut approach. For more details on cutting planes and branch-and-bound for MILP we refer to Nemhauser and Wolsey [18], and for an application of this framework to global mixed-integer nonlinear programming to Smith and Pantelides [24], and Tawarmalani and Sahinidis [25, 26]. Information on the MINLP framework SCIP which we apply is given by Achterberg [1], and in particular on nonlinear aspects of SCIP such as spatial branching on continuous variables is given by Vigerske [28].

This article is organized as follows: The topology optimization problem modeling the task of extending a gas network by selecting from a finite set of additional network elements is presented in Section 2. In Section 3 we focus on the **active transmission problem** which is a non-convex nonlinear feasibility problem. It emerges from the topology optimization problem by fixing all integral variables. We present sufficient conditions for proving the infeasibility of the active transmission problem. They are expressed in the form of an MILP. This allows to certify the infeasibility of the non-convex active transmission problem by solving this MILP. An illustrative explanation for the definition of this MILP follows in Section 4. Our solution approach for solving the topology optimization problem and computational results are presented in Section 5. Finally, we conclude in Section 6 and give some ideas for future research directions.

2 Topology Optimization of Gas Networks

A gas transport network is modeled by a directed graph $G = (V, A)$ where V denotes the set of nodes and $A \subseteq V \times V$ the set of arcs. For an introduction to the notions of graph theory we refer to [15]. Each arc represents either an *active* or a *passive* network element as described below. More details can be found in [11].

The majority of the arcs in a gas transport network are passive pipelines. The flow q_a through a pipeline $a = (v, w) \in A$ is induced by a difference of pressures p_v and p_w at the end nodes v and w of the pipe. Note that a positive value of q_a means a flow from v to w , and a negative value is a flow in the opposite direction from w to v . The Weymouth equation [30] is an old but still used equation to approximate the flow of gas in long pipelines:

$$q_a |q_a| = C_a (p_v^2 - p_w^2).$$

Here C_a is computed by the following formula:

$$C_a := 96074.830 \frac{d_a^5}{\lambda_a z T L_a \delta},$$

where

$$\frac{1}{\lambda_a} = \left(2 \log \left(\frac{3.7 d_a}{\varepsilon} \right) \right)^2,$$

with L_a being the length of the pipe (m), d_a the inner diameter of the pipe (m), T the gas temperature (K), ε the absolute roughness of the pipe (m), δ the density of the gas relative to air, and z the gas compressibility factor [10, 19, 22]. After substituting $\pi_v = p_v^2$ and $\pi_w = p_w^2$ Weymouth's equation takes the general form

$$\alpha_a q_a |q_a|^{k_a} = \pi_v - \gamma_a \pi_w, \quad (2.1)$$

with $\gamma_a = 1$, $\alpha_a = C_a^{-1}$ and $k_a = 1$. The variables $(\pi_v)_{v \in V}$ are called the *node potential values*. The slightly more general version of Weymouth's equation with $\gamma_a \neq 1$ (for different heights of the pipe's end nodes) is given in [21]. In this general case it holds $\gamma_a = \exp(S_a)$ where $S_a \sim s_a L_a$ and s_a denotes the slope of pipe a . This means that S_a represents the height difference of the end nodes of pipe a . Hence, if some pipelines a_1, \dots, a_n form a directed cycle, it is assumed that $\gamma_{a_1} \cdot \dots \cdot \gamma_{a_n} = 1$. This reflects that there is no height difference when traversing a circuit completely. If $a = (v, w)$ is an arc and $a' = (w, v)$ is its anti-parallel counterpart, then we assume that the constants γ_a are such that $\gamma_a = \gamma_{a'}^{-1}$. (Note that the constant on the left-hand side of (2.1) changes from α_a to $\alpha_a \gamma_a^{-1}$ when considering a' instead of a .)

Apart from the pipelines that constitute the network there are active elements that allow to control the flow of gas. In particular these are valves to adapt the topology, compressors to increase pressure, and control valves to reduce pressure.

A valve is installed in the network to separate or join two different pipes. For a discrete decision they only allow either being open or closed. The spatial dimension of a valve is assumed to be small in comparison to the pipes. Hence in our model the pressures are identified when the valve is open. If the valve is closed then they are decoupled. Mathematically a valve is an arc $a = (v, w) \in A$ with the following description:

$$x_a = 1 \implies \pi_v - \pi_w = 0,$$

$$x_a = 0 \implies q_a = 0,$$

where $x_a \in \{0, 1\}$ is a binary decision variable. Recall the relation of the pressure and node potential $p_v^2 = \pi_v$ for every node $v \in V$.

A compressor is used to increase the pressure. We follow the approach of De Wolf and Smeers [10], and make use of the following formulation for a compressor $a = (v, w) \in A$:

$$\alpha_a q_a |q_a|^{k_a} \geq \pi_v - \pi_w, \quad (2.2)$$

which allows a flow larger than the one corresponding to the pressure decrease in a pipe. We rewrite this inequality as equality by introducing a weighted slack variable y_a as

$$\alpha_a q_a |q_a|^{k_a} - \beta_a y_a = \pi_v - \pi_w, \quad (2.3)$$

with constant $\beta_a \in \mathbb{R}$ and $\underline{y}_a \leq y_a \leq \bar{y}_a$.

In practice we need further restrictions such as a minimal and a maximal pressure difference or a restriction of the pressure difference that depends on the flow. Therefore we allow a linear inequality system coupling the flow q_a and the pressures p_v, p_w such that

$$A_a (q_a, p_v, p_w)^T \leq b_a. \quad (2.4)$$

Here it holds $A_a \in \mathbb{R}^{\nu_a \times 3}$ and $b_a \in \mathbb{R}^{\nu_a}$ for some value $\nu_a \in \mathbb{N}$. When the slack variable y_a is unbounded, then the only restrictions for the compressor are given by constraint (2.4). Hence (2.4) specifies the **operating range** of the compressor. The flow can only go in positive direction through a compressor, hence a corresponding lower bound needs to be set by this linear inequality system, i.e., $q_a \geq 0$.

A control valve allows to reduce the pressure along an arc $a = (v, w) \in A$ in the network, for example, to protect parts of the network from too high pressures. A control valve $a = (v, w)$ is inverse to a compressor. Hence we adapt inequality (2.2) as follows:

$$\alpha_a q_a |q_a|^{k_a} \leq \pi_v - \pi_w, \quad (2.5)$$

in order to decrease the pressure in w more than the flow and the input pressure will actually require. After introducing weighting slack variables y_a equation (2.5) appears similar to equation (2.3). (The only difference between a compressor and a control valve is either the sign of β_a or the bounds on y_a .) Similar to a compressor, the feasible region of a control valve is restricted by additional constraints like (2.4). For instance we set a minimal and maximal pressure difference between the end nodes v and w . Note that the flow direction through a control valve is also fixed by setting the lower bound to zero, i.e., $q_a \geq 0$.

2.1 An MINLP Model

A directed graph is used for the MINLP model of the extension problem in gas networks: We define an extended set of arcs $A_X \subseteq V \times V$. This set A_X contains all ‘‘original’’ arcs from A , that is, $A \subseteq A_X$. Furthermore the extended arc set A_X contains possible new network elements (pipes, valves, compressors, or control valves), where in principle a new element can be built between any pair of existing nodes $v, w \in V, v \neq w$. By (V, A_X) we denote the gas transport network together with its possible extensions.

We assume the following data to be given as parameters. For each node $v \in V$ we have lower and upper bounds on the node potential, $\underline{\pi}_v, \bar{\pi}_v \in \mathbb{R}$ with $\underline{\pi}_v \leq \bar{\pi}_v$. For each arc $a \in A_X$ we have lower and upper bounds on the flow, $\underline{q}_a, \bar{q}_a \in \mathbb{R}$ with $\underline{q}_a \leq \bar{q}_a$. For each node $v \in V$ the value $s_v \in \mathbb{R}$ denotes the amount of flow that is either led into the network (for $s_v > 0$), or taken out of the network (for $s_v < 0$). A node with $s_v > 0$ is also called source or entry node, and nodes with $s_v < 0$ are sinks or exit nodes. All the other nodes with $s_v = 0$ are inner or transmission nodes. Vector s is called *nomination*. For each arc $a \in A_X$ we have a transmission coefficient $\alpha_a \in \mathbb{R}_+$, bounds on the weighted slack variable $\underline{y}_a, \bar{y}_a \in \mathbb{R}$ with $\underline{y}_a \leq \bar{y}_a$, a scaling factor $\beta_a \in \mathbb{R}$ for the range coefficient, a coefficient $\gamma_a \in \mathbb{R}_+ \setminus \{0\}$, and a cost coefficient $c_a \in \mathbb{R}_+$. Note that $c_a = 0$ for all existing arcs $a \in A$.

Let us introduce the following variables. The flow on arc $a \in A_X$ is denoted by $q_a \in \mathbb{R}$. The potential value of a vertex $v \in V$ is given by $\pi_v \in \mathbb{R}$ representing the squared pressure at v . The pressure itself is modeled by $p_v \in \mathbb{R}$. The variable $y_a \in \mathbb{R}$ specifies the additive component of the pressure loss term in

(2.3). For pipelines and valves this variable is fixed to zero, whereas it is bounded for compressors and control valves.

We introduce a binary decision variable $x_a \in \{0, 1\}$ for each arc $a \in A_X$, where $x_a = 1$ represents the decision that arc a is used (i.e., a necessary condition for a non-zero flow). The other case $x_a = 0$ represents the decision that arc a is not contained in the network (V, A_X) . Pre-existing arcs $a \in A$ usually cannot be deactivated which implies that x_a is fixed to 1. This means that only a subset of all possible discrete settings is allowed. This subset is given by $\mathcal{X} \subseteq \{0, 1\}^{A_X}$. The set \mathcal{X} is described by linear inequalities in general, i.e.,

$$\mathcal{X} = \{x \in \{0, 1\}^{A_X} \mid Lx \leq t\},$$

where L is a matrix of integers with $|A_X|$ columns and t is a vector of integers such that the length of t is equal to the number of rows of L .

The following mixed-integer nonlinear program with indicator constraints is called **topology optimization problem**:

$$\min \sum_{a \in A_X} c_a x_a \quad (2.6a)$$

$$\text{s. t. } x_a = 1 \implies \alpha_a q_a |q_a|^{k_a} - \beta_a y_a - (\pi_v - \gamma_a \pi_w) = 0 \quad \forall a \in A_X, \quad (2.6b)$$

$$x_a = 1 \implies A_a (q_a, p_v, p_w)^T \leq b_a \quad \forall a \in A_X, \quad (2.6c)$$

$$x_a = 0 \implies q_a = 0 \quad \forall a \in A_X, \quad (2.6d)$$

$$\sum_{w:(v,w) \in A_X} q_{v,w} - \sum_{w:(w,v) \in A_X} q_{w,v} = s_v \quad \forall v \in V, \quad (2.6e)$$

$$p_v |p_v| - \pi_v = 0 \quad \forall v \in V, \quad (2.6f)$$

$$\underline{p}_v \leq p_v \leq \bar{p}_v \quad \forall v \in V, \quad (2.6g)$$

$$\underline{q}_a \leq q_a \leq \bar{q}_a \quad \forall a \in A_X, \quad (2.6h)$$

$$\underline{y}_a \leq y_a \leq \bar{y}_a \quad \forall a \in A_X, \quad (2.6i)$$

$$p_v, \pi_v \in \mathbb{R} \quad \forall v \in V, \quad (2.6j)$$

$$q_a, y_a \in \mathbb{R} \quad \forall a \in A_X, \quad (2.6k)$$

$$x_a \in \{0, 1\} \quad \forall a \in A_X, \quad (2.6l)$$

$$x \in \mathcal{X}. \quad (2.6m)$$

The objective function (2.6a) calculates the extension costs for those new pipes that are actually built. The indicator constraints (2.6b) are switching on only those pressure-flow coupling constraints for arcs that are actually used. Constraints (2.6c) represent the operating range of compressors and control valves. The indicator constraints (2.6d) forbid flow on those arcs that are not used. These indicator constraints are handled by our numerical solver SCIP by a special purpose constraint handler. Hence it is not necessary to reformulate by, for example, big- M -constraints and further binary variables. The node flow conservation constraints (also called Kirchhoff's constraints) are defined in (2.6e). The pressure and squared pressure coupling are modeled by constraints (2.6f). Constraints (2.6g) – (2.6i) define the trivial bounds on the variables, and constraints (2.6j) – (2.6l) specify the continuous or discrete range of the variables. Constraint (2.6m) ensures that only a subset of all discrete decisions is feasible.

For a given nomination s , the topology optimization problem (2.6) is to find a cost optimal selection of new network elements such that the transmission of the specific flow s in the transport network (V, A_X) is feasible. Recall that we only consider a finite number of extensions throughout this paper. Otherwise, if this transport is not possible, the nomination s is infeasible.

3 Sufficient Pruning Conditions for the Topology Optimization Problem

The topology optimization problem (2.6) can be solved with a branch-and-cut framework as implemented in MINLP solvers such as ANTIGONE [17], BARON [26] or SCIP [23]. The solver will iteratively branch on binary and integral decision variables, while also spatial branching on continuous variables is performed [28]. After several subsequent branches, all integer and binary variables are fixed. The remaining problem is of special interest in this section. Let A' contain all arcs where the flow is not fixed to zero, i.e., $A' = \{a \in A_X \mid x_a = 1\}$. We set $\delta_{A'}^+(v) := \{(v, w) \in A' \mid w \in V\}$ and $\delta_{A'}^-(v) := \{(w, v) \in A' \mid w \in V\}$ for $v \in V$. The remaining problem, which might still be infeasible with respect to the nonlinear constraints, then is as follows:

$$\exists q, \pi, p, y \quad (3.1a)$$

$$\text{s. t.} \quad \alpha_a q_a |q_a|^{k_a} - \beta_a y_a - (\pi_v - \gamma_a \pi_w) = 0 \quad \forall a = (v, w) \in A', \quad (3.1b)$$

$$\sum_{a \in \delta_{A'}^+(v)} q_a - \sum_{a \in \delta_{A'}^-(v)} q_a = d_v \quad \forall v \in V, \quad (3.1c)$$

$$A_a (q_a, p_v, p_w)^T \leq b_a \quad \forall a = (v, w) \in A', \quad (3.1d)$$

$$p_v |p_v| - \pi_v = 0 \quad \forall v \in V, \quad (3.1e)$$

$$\pi_v \leq \bar{\pi}_v \quad \forall v \in V, \quad (3.1f)$$

$$\pi_v \geq \underline{\pi}_v \quad \forall v \in V, \quad (3.1g)$$

$$q_a \leq \bar{q}_a \quad \forall a \in A', \quad (3.1h)$$

$$q_a \geq \underline{q}_a \quad \forall a \in A', \quad (3.1i)$$

$$y_a \leq \bar{y}_a \quad \forall a \in A', \quad (3.1j)$$

$$y_a \geq \underline{y}_a \quad \forall a \in A', \quad (3.1k)$$

$$p_v, \pi_v \in \mathbb{R} \quad \forall v \in V, \quad (3.1l)$$

$$q_a, y_a \in \mathbb{R} \quad \forall a \in A'. \quad (3.1m)$$

This problem (3.1) will be referred to as **active transmission problem** in the following. It is a non-convex feasibility problem due to constraints (3.1b). For simplicity we assume w.l.o.g. that (V, A') is a connected graph. If this is not the case, we split the active transmission problem such that we obtain one problem for each connected component.

The main Theorem 1 (see Section 3.2) of this paper states: The active transmission problem (3.1) is infeasible if the **infeasibility detection MILP** (3.8) (as defined in Section 3.1) is infeasible or has optimal objective value zero. Hence the infeasibility of the non-convex active transmission problem can be certified by solving an MILP. This result provides an efficient bounding procedure for the aforementioned branch-and-cut framework for solving (2.6): Whenever a node of the branch-and-bound tree is considered with all integral variables being fixed we consider the corresponding active transmission problem. We solve the infeasibility detection MILP (3.8) and prune the current node in case that infeasibility of the active transmission problem is certified.

3.1 Definition of the Infeasibility Detection MILP

We give the following definition in order to obtain the relation (3.3) before focussing on the infeasibility detection MILP (3.8):

Definition 1. Let (V, E) be the undirected version of (V, A') obtained by removing the orientation of the arcs $a \in A'$. This way each arc $a \in A'$ uniquely corresponds to an edge in $e \in E$ and vice versa. Let r be any node in V . For a node $v \in V$ denote by $P_r(v)$ an undirected path from r to v . Let v_1, \dots, v_n be the

nodes of this path, e_1, \dots, e_{n-1} the edges and a_1, \dots, a_{n-1} the corresponding arcs. Recall our assumption that (V, A') is connected which means that (V, E) is connected, too. We define

$$\gamma_{r,v} := \left(\prod_{i:a_i=(v_i, v_{i+1})} \gamma_{a_i} \right) \left(\prod_{i:a_i=(v_{i+1}, v_i)} \gamma_{a_i}^{-1} \right).$$

We have $\gamma_{r,v} > 0$ as $\gamma_a > 0$ for every arc $a \in A'$. The definition is actually independent of the path $P_r(v)$. To see this, let P' be a different r - v -path in (V, E) . Consider the cycle C from r to v on path P , and back from v to r on path P' in reverse order. Denote the reverse path of P' by Q' . Denote the nodes of this path Q' by $\tilde{v}_1, \dots, \tilde{v}_m$ with $\tilde{v}_1 = v$ and $\tilde{v}_m = r$, the edges by $\tilde{e}_1, \dots, \tilde{e}_{m-1}$ and the arcs by $\tilde{a}_1, \dots, \tilde{a}_{m-1}$. According to our assumption on $\gamma_a, a \in A$, in Section 2 we have

$$1 = \left(\prod_{i:a_i=(v_i, v_{i+1})} \gamma_{a_i} \right) \left(\prod_{i:a_i=(v_{i+1}, v_i)} \gamma_{a_i}^{-1} \right) \left(\prod_{i:\tilde{a}_i=(\tilde{v}_{i+1}, \tilde{v}_i)} \gamma_{\tilde{a}_i} \right) \left(\prod_{i:\tilde{a}_i=(\tilde{v}_i, \tilde{v}_{i+1})} \gamma_{\tilde{a}_i}^{-1} \right).$$

Hence $\gamma_{r,v}$ is uniquely defined.

Using this value $\gamma_{r,v}$ we define the function π'_v by

$$\pi'_v(\pi) := \gamma_{r,v} \pi_v \quad (3.2)$$

for every node $v \in V$. As a consequence of (3.2) we obtain lower and upper bounds of $\pi'_v(\pi)$ from $\underline{\pi}'_v := \pi'_v(\underline{\pi})$ and $\bar{\pi}'_v := \pi'_v(\bar{\pi})$, respectively, for each node $v \in V$.

From elementary calculations it follows that

$$\pi'_v(\pi) - \pi'_w(\pi) = \gamma_{r,v} \pi_v - \underbrace{\gamma_{r,w}}_{=\gamma_{r,v} \gamma_a} \pi_w = \gamma_{r,v} (\pi_v - \gamma_a \pi_w) \quad (3.3)$$

holds for each arc $a = (v, w) \in A'$. Throughout this section we use $\gamma_{r,v}$ from Definition 1 and the function $\pi'_v(\pi)$.

Now we turn to the definition of the infeasibility detection MILP. The flow conservation constraint (3.1c), constraint (3.7) and (in)equality (3.5) will form the main part of this MILP. For a motivation we consider a feasible solution (q^*, π^*, p^*, y^*) for the active transmission problem (3.1) and another (possibly infeasible) solution $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ fulfilling at least constraint (3.1b), (3.1c) and (3.1e). From constraint (3.1b) we obtain

$$\begin{aligned} \alpha_a q_a^* |q_a^*|^{k_a} - \beta_a y_a^* - (\pi_v^* - \gamma_a \pi_w^*) &= 0, \\ \alpha_a \tilde{q}_a |\tilde{q}_a|^{k_a} - \beta_a \tilde{y}_a - (\tilde{\pi}_v - \gamma_a \tilde{\pi}_w) &= 0, \end{aligned} \quad (3.4)$$

for every arc $a = (v, w) \in A'$. Using π' we derive from (3.4):

$$\begin{aligned} &(\pi'_v(\pi^*) - \pi'_v(\tilde{\pi})) - (\pi'_w(\pi^*) - \pi'_w(\tilde{\pi})) + \gamma_{r,v} \beta_a (y_a^* - \tilde{y}_a) \\ &= \gamma_{r,v} (\pi_v^* - \gamma_a \pi_w^* + \beta_a y_a^* - (\tilde{\pi}_v - \gamma_a \tilde{\pi}_w + \beta_a \tilde{y}_a)) \\ &= \gamma_{r,v} \alpha_a (q_a^* |q_a^*|^{k_a} - \tilde{q}_a |\tilde{q}_a|^{k_a}) \\ &\begin{cases} > 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) > 0, \\ = 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) = 0, \\ < 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) < 0, \end{cases} \end{aligned}$$

for every arc $a = (v, w) \in A'$. We write this inequality for short as

$$s_v - s_w + s_a \begin{cases} > 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) > 0, \\ = 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) = 0, \\ < 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) < 0. \end{cases} \quad (3.5)$$

Here s_a corresponds to $\gamma_{r,v} \beta_a (y_a^* - \tilde{y}_a)$ and s_v to $\pi'_v(\pi^*) - \pi'_v(\tilde{\pi})$.

For the node potential π' we observe after identifying x_v with $\pi'_v(\pi^*) - \pi'_v(\tilde{\pi})$ the conditions

$$\begin{aligned}
x_v \in \mathbb{R} & \quad \text{if } \underline{\pi}_v < \tilde{\pi}_v < \bar{\pi}_v, \\
x_v \leq 0 & \quad \text{if } \tilde{\pi}_v = \bar{\pi}_v, \\
x_v < 0 & \quad \text{if } \tilde{\pi}_v > \bar{\pi}_v, \\
x_v \geq 0 & \quad \text{if } \underline{\pi}_v = \tilde{\pi}_v, \\
x_v > 0 & \quad \text{if } \underline{\pi}_v > \tilde{\pi}_v.
\end{aligned} \tag{3.6}$$

We write these conditions (3.6) as a single constraint

$$x_v^+ - x_v^- - x_v - \kappa_v z = 0 \quad \text{with} \quad 0 \leq x_v^+ \leq \bar{x}_v^+, 0 \leq x_v^- \leq \bar{x}_v^-, z > 0 \tag{3.7}$$

where the variable bounds and κ_v are defined as

$$\bar{x}_v^+ := \begin{cases} 0 & \text{if } \tilde{\pi}_v \geq \bar{\pi}_v, \\ \infty & \text{else,} \end{cases} \quad \bar{x}_v^- := \begin{cases} 0 & \text{if } \tilde{\pi}_v \leq \underline{\pi}_v, \\ \infty & \text{else,} \end{cases} \quad \kappa_v := \begin{cases} 1 & \text{if } \tilde{\pi}_v > \bar{\pi}_v, \\ -1 & \text{if } \tilde{\pi}_v < \underline{\pi}_v, \\ 0 & \text{else,} \end{cases}$$

for each node $v \in V$.

As stated earlier, the flow conservation constraint (3.1c), constraint (3.7) and (in)equality (3.5) form the main part of the infeasibility detection MILP. Keeping this idea in mind, the MILP, which contains indicator constraints, is defined as follows:

Definition 2. Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution of the active transmission problem (3.1) fulfilling at least constraint (3.1b), (3.1c) and (3.1e). The **infeasibility detection MILP** is defined as follows:

$$\max z \tag{3.8a}$$

$$s. t. \quad \sum_{a \in \delta_{A'}^+(v)} q_a - \sum_{a \in \delta_{A'}^-(v)} q_a = d_v \quad \forall v \in V, \tag{3.8b}$$

$$x_v^+ - x_v^- - x_v - \kappa_v z = 0 \quad \forall v \in V, \tag{3.8c}$$

$$\tilde{\kappa}_a (q_a - \tilde{q}_a) > 0 \implies x_v - x_w + x_a \geq 0 \quad \forall a = (v, w) \in A', \tag{3.8d}$$

$$\tilde{\kappa}_a (q_a - \tilde{q}_a) = 0 \implies x_v - x_w + x_a = 0 \quad \forall a = (v, w) \in A', \tag{3.8e}$$

$$\tilde{\kappa}_a (q_a - \tilde{q}_a) < 0 \implies x_v - x_w + x_a \leq 0 \quad \forall a = (v, w) \in A', \tag{3.8f}$$

$$\alpha_a (q_a - \tilde{q}_a) > 0 \implies s_v - s_w + s_a \geq \kappa_a z \quad \forall a = (v, w) \in A', \tag{3.8g}$$

$$\alpha_a (q_a - \tilde{q}_a) = 0 \implies s_v - s_w + s_a = 0 \quad \forall a = (v, w) \in A', \tag{3.8h}$$

$$\alpha_a (q_a - \tilde{q}_a) < 0 \implies s_v - s_w + s_a \leq \kappa_a z \quad \forall a = (v, w) \in A', \tag{3.8i}$$

$$\underline{q}_a \leq q_a \leq \bar{q}_a \quad \forall a \in A', \tag{3.8j}$$

$$\underline{s}_a \leq s_a \leq \bar{s}_a \quad \forall a \in A', \tag{3.8k}$$

$$\underline{x}_a \leq x_a \leq \bar{x}_a \quad \forall a \in A', \tag{3.8l}$$

$$x_v^+ \leq \bar{x}_v^+ \quad \forall v \in V, \tag{3.8m}$$

$$x_v^- \leq \bar{x}_v^- \quad \forall v \in V, \tag{3.8n}$$

$$x_v, s_v \in \mathbb{R} \quad \forall v \in V, \tag{3.8o}$$

$$x_v^+, x_v^- \in \mathbb{R}_{\geq 0} \quad \forall v \in V, \tag{3.8p}$$

$$x_a, s_a, q_a \in \mathbb{R} \quad \forall a \in A', \tag{3.8q}$$

$$z \in \mathbb{R}_{\geq 0}. \tag{3.8r}$$

Note that this problem is not a mixed-integer nonlinear optimization problem due to strict inequality in the constraints (3.8d), (3.8f), (3.8g) and (3.8i). Nevertheless, for simplicity, we state the infeasibility detection MILP in the form of (3.8) and give an equivalent reformulation in Remark 1.

Constraint (3.8c) originates from (3.7) by expressing $z > 0$ as objective. Similarly (3.8g)–(3.8i) originate from (3.5). Constraints (3.8d)–(3.8f) form a weaker version of (3.5).

For the definition of this MILP we make use of different constants which are defined below. Especially the bounds on s_a for an arc $a = (v, w) \in A'$ originate from the previously described relation that s_a corresponds to $\gamma_{r,v}\beta_a(y_a^* - \tilde{y}_a)$:

$$\bar{s}_a := \begin{cases} 0 & \text{if } \beta_a \tilde{y}_a = \max\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\}, \\ \infty & \text{else} \end{cases} \quad \begin{array}{l} \forall a \in A', \\ a = (v, w), \end{array} \quad (3.9a)$$

$$\underline{s}_a := \begin{cases} 0 & \text{if } \beta_a \tilde{y}_a = \min\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\}, \\ -\infty & \text{else} \end{cases} \quad \begin{array}{l} \forall a \in A', \\ a = (v, w), \end{array} \quad (3.9b)$$

$$\bar{x}_a := \begin{cases} 0 & \text{if } \beta_a \tilde{y}_a = \max\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\}, \\ 0 & \text{if } \exists k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with} \\ & (A_a)_{(k,1)} \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0, \\ \infty & \text{else} \end{cases} \quad \begin{array}{l} \forall a \in A', \\ a = (v, w), \end{array} \quad (3.9c)$$

$$\underline{x}_a := \begin{cases} 0 & \text{if } \beta_a \tilde{y}_a = \min\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\}, \\ 0 & \text{if } \exists k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with} \\ & (A_a)_{(k,1)} \leq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0, \\ -\infty & \text{else} \end{cases} \quad \begin{array}{l} \forall a \in A', \\ a = (v, w), \end{array} \quad (3.9d)$$

$$\bar{x}_v^+ := \begin{cases} 0 & \text{if } \tilde{\pi}_v \geq \bar{\pi}_v, \\ \infty & \text{else} \end{cases} \quad \forall v \in V, \quad (3.9e)$$

$$\bar{x}_v^- := \begin{cases} 0 & \text{if } \tilde{\pi}_v \leq \underline{\pi}_v, \\ \infty & \text{else} \end{cases} \quad \forall v \in V, \quad (3.9f)$$

$$\kappa_v := \begin{cases} 1 & \text{if } \tilde{\pi}_v > \bar{\pi}_v, \\ -1 & \text{if } \tilde{\pi}_v < \underline{\pi}_v, \\ 0 & \text{else} \end{cases} \quad \forall v \in V, \quad (3.9g)$$

$$\kappa_a := \begin{cases} -1 & \text{if } \alpha_a > 0, \tilde{q}_a > \bar{q}_a, \\ 1 & \text{if } \alpha_a > 0, \tilde{q}_a < \underline{q}_a, \\ 0 & \text{else} \end{cases} \quad \forall a \in A', \quad (3.9h)$$

$$\tilde{\kappa}_a := \begin{cases} 1 & \text{if } \alpha_a > 0, \\ 1 & \text{if } \exists k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with} \\ & (A_a)_{(k,1)} \neq 0, \\ 0 & \text{else} \end{cases} \quad \forall a \in A'. \quad (3.9i)$$

Remark 1. We roughly describe how the MILP formulation of (3.8) is obtained. At first we replace $(q_a - \tilde{q}_a)$ by Δ_a for every arc $a \in A'$. As \tilde{q} fulfills the flow conservation (3.1c) we obtain that Δ is a circulation, i.e., we replace the flow conservation (3.8b) by

$$\sum_{a \in \delta_{A'}^+(v)} \Delta_a - \sum_{a \in \delta_{A'}^-(v)} \Delta_a = 0 \quad \forall v \in V.$$

We define bounds for Δ by

$$\bar{\Delta}_a := \begin{cases} \infty & \text{if } \tilde{q}_a < \bar{q}_a, \\ 0 & \text{if } \tilde{q}_a = \bar{q}_a, \end{cases} \quad \underline{\Delta}_a := \begin{cases} 0 & \text{if } \tilde{q}_a = \underline{q}_a, \\ -\infty & \text{if } \tilde{q}_a > \underline{q}_a, \end{cases}$$

and replace (3.8j) by $\underline{\Delta} \leq \Delta \leq \bar{\Delta}$. Then it is easy to see, as Δ is a circulation, that Δ can be chosen such that either $\Delta_a = 0$ or $|\Delta_a| \geq 1$ holds. We introduce binary variables $x_a^{FW}, x_a^{BW}, s_a^{FW}, s_a^{BW} \in \{0, 1\}$

in combination with indicator constraints as follows:

$$\begin{aligned}
x_a^{FW} = 1 &\implies \tilde{\kappa}_a \Delta_a \geq 1 && \forall a \in A', \\
x_a^{FW} = 0 &\implies \tilde{\kappa}_a \Delta_a \leq 0 && \forall a \in A', \\
x_a^{BW} = 1 &\implies \tilde{\kappa}_a \Delta_a \leq -1 && \forall a \in A', \\
x_a^{BW} = 0 &\implies \tilde{\kappa}_a \Delta_a \geq 0 && \forall a \in A', \\
s_a^{FW} = 1 &\implies \alpha_a \Delta_a \geq 1 && \forall a \in A', \\
s_a^{FW} = 0 &\implies \alpha_a \Delta_a \leq 0 && \forall a \in A', \\
s_a^{BW} = 1 &\implies \alpha_a \Delta_a \leq -1 && \forall a \in A', \\
s_a^{BW} = 0 &\implies \alpha_a \Delta_a \geq 0 && \forall a \in A'.
\end{aligned}$$

Then (3.8d)–(3.8i) are replaced by

$$\begin{aligned}
x_a^{FW} = 1 &\implies x_v - x_w + x_a \geq 0 && \forall a = (v, w) \in A', \\
x_a^{FW} = 0, x_a^{BW} = 0 &\implies x_v - x_w + x_a = 0 && \forall a = (v, w) \in A', \\
x_a^{BW} = 1 &\implies x_v - x_w + x_a \leq 0 && \forall a = (v, w) \in A', \\
s_a^{FW} = 1 &\implies s_v - s_w + s_a \geq \kappa_a z && \forall a = (v, w) \in A', \\
s_a^{FW} = 0, s_a^{BW} = 0 &\implies s_v - s_w + s_a = 0 && \forall a = (v, w) \in A', \\
s_a^{BW} = 1 &\implies s_v - s_w + s_a \leq \kappa_a z && \forall a = (v, w) \in A'.
\end{aligned}$$

All these reformulations yield an MILP with indicator constraints which is equivalent to (3.8).

3.2 Certifying Infeasibility of the Active Transmission Problem by Solving an MILP

This section addresses exclusively the proof of the following theorem:

Theorem 1. *Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution for the active transmission problem (3.1) fulfilling at least constraint (3.1b), (3.1c) and (3.1e). If the infeasibility detection MILP (3.8) is infeasible or has optimal objective value zero, then the active transmission problem (3.1) is infeasible.*

Proof: Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution for the active transmission problem (3.1) fulfilling at least constraint (3.1b), (3.1c) and (3.1e). If the infeasibility detection MILP (3.8), which depends on $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$, is infeasible, then there does not exist a flow vector $q' \in \mathbb{R}^{A'}$ which fulfills

$$\sum_{a \in \delta_{A'}^+(v)} q_a - \sum_{a \in \delta_{A'}^-(v)} q_a = d_v \quad \forall v \in V, \quad \underline{q}_a \leq q_a \leq \bar{q}_a \quad \forall a \in A'. \quad (3.10)$$

This can be seen as follows: Otherwise, if there exists a vector q' fulfilling (3.10), then $(q', 0)$ is a feasible solution for (3.8). We conclude that the active transmission problem (3.1) is infeasible if MILP (3.8) is infeasible.

Now assume that the MILP (3.8) has an optimal solution with objective value zero. We prove that this implies that the active transmission problem (3.1) is infeasible. Therefore we assume that the active transmission problem has a feasible solution (q^*, π^*, p^*, y^*) and show that there exists a feasible solution (q^*, x^*, s^*, z^*) to MILP (3.8) with positive objective, i.e., $z^* > 0$. In the following we describe how this solution (q^*, x^*, s^*, z^*) is defined. First we give the definition of s^* and z^* and show that (3.8g)–(3.8i) and (3.8k) are fulfilled. Then we turn to the definition of x^* and prove that (3.8c)–(3.8f) and (3.8l)–(3.8n) and (3.8p) are fulfilled. As the flow vector q^* is feasible for the flow conservation (3.8b) and the bound constraints (3.8j) we conclude that (q^*, x^*, s^*, z^*) is feasible for MILP (3.8).

Recall that r was used for the definition of $\gamma_{r,v}$ for every node $v \in V$. The vector (s^*, z^*) is defined as follows:

$$s_v^* := \pi'_v(\pi^*) - \pi'_v(\tilde{\pi}) \quad \forall v \in V,$$

$$\begin{aligned}
s_a^* &:= \gamma_{r,v} \beta_a (y_a^* - \tilde{y}_a) & \forall a = (v, w) \in A', \\
z^* &:= \min \{1, \min \{|\gamma_{r,v} \alpha_a (q_a^* |q_a^*|^{k_a} - \tilde{q}_a |\tilde{q}_a|^{k_a})| \mid a = (v, w) \in A' : \alpha_a q_a^* \neq \alpha_a \tilde{q}_a\}\}.
\end{aligned}$$

Let us now prove that (s^*, z^*) is feasible for (3.8g)–(3.8i) and (3.8k). First we prove that $s_a^* \leq \bar{s}_a$ holds for every arc $a \in A'$. Therefor let $a \in A'$. By definition (3.9a) we have to show $s_a^* \leq 0$ if $\beta_a \tilde{y}_a = \max\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\}$. This means that one of the following three cases applies:

1. $\tilde{y}_a = \bar{y}_a, \beta_a > 0 \Rightarrow y_a^* \leq \tilde{y}_a \Rightarrow s_a^* \leq 0,$
2. $\tilde{y}_a = \underline{y}_a, \beta_a < 0 \Rightarrow y_a^* \geq \tilde{y}_a \Rightarrow s_a^* \leq 0,$
3. $\beta_a = 0 \Rightarrow s_a^* = 0.$

Hence $s_a^* \leq 0$ if $\beta_a \tilde{y}_a = \max\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\}$. Similarly we prove that $s_a^* \geq \underline{s}_a$ holds for every arc $a \in A'$. We conclude that s^* is feasible for (3.8k). Now we turn to the constraints (3.8g)–(3.8i). We consider an arc $a = (v, w) \in A'$ and obtain:

$$\begin{aligned}
s_v^* - s_w^* + s_a^* &= \pi'_v(\pi^*) - \pi'_w(\pi^*) + \gamma_{r,v} \beta_a y_a^* - (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi}) + \gamma_{r,v} \beta_a \tilde{y}_a) \\
&= \gamma_{r,v} \alpha_a (q_a^* |q_a^*|^{k_a} - \tilde{q}_a |\tilde{q}_a|^{k_a}) \\
&\begin{cases} > 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) > 0, \\ = 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) = 0, \\ < 0 & \text{if } \alpha_a (q_a^* - \tilde{q}_a) < 0. \end{cases}
\end{aligned}$$

We conclude that (s^*, z^*) is feasible for (3.8g)–(3.8i) and (3.8k).

By Lemma 2 (stated in the remaining part of this section) there exists a vector x^* for z^* such that (x^*, z^*) is feasible for (3.8c)–(3.8f) and (3.8l)–(3.8n) and (3.8p). Furthermore the flow conservation constraint (3.8b) and the bound constraints (3.8j) are fulfilled by q^* as (q^*, π^*, p^*, y^*) is a feasible solution for the active transmission problem (3.1). Hence (q^*, x^*, s^*, z^*) is a feasible solution for MILP (3.8).

We finally show that $z^* > 0$. Because of $\gamma_{r,v} > 0$ for all nodes $v \in V$ we have that $\alpha_a q_a^* \neq \alpha_a \tilde{q}_a$ for an arc $a = (v, w) \in A'$ implies $\gamma_{r,v} \alpha_a (q_a^* |q_a^*|^{k_a} - \tilde{q}_a |\tilde{q}_a|^{k_a}) \neq 0$. This proves $z^* > 0$. \square

In the remaining part of this section we prove Lemma 2 which was used in the previous proof of Theorem 1. Therefor start with proving an auxiliary lemma.

Lemma 1. *Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution for the active transmission problem (3.1) fulfilling at least constraints (3.1e). Furthermore let (q^*, π^*, p^*, y^*) be a feasible solution for (3.1). There exists a partition of the node set $V = M_1 \dot{\cup} M_2 \dot{\cup} M_3$ fulfilling the following conditions:*

$$\begin{aligned}
- \forall a = (v, w) \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_3): \\
&(\pi'_v(\pi^*) - \pi'_w(\pi^*)) < (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})), \\
\#k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, (A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0,
\end{aligned}$$

$$\begin{aligned}
- \forall a = (v, w) \in \delta_{A'}^-(M_1) \cup \delta_{A'}^+(M_3): \\
&(\pi'_v(\pi^*) - \pi'_w(\pi^*)) > (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})), \\
\#k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, (A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0.
\end{aligned}$$

Furthermore it holds

$$\begin{aligned}
\{v \in V \mid \tilde{\pi}_v > \bar{\pi}_v\} \subseteq M_1, & \quad \{v \in V \mid \tilde{\pi}_v \leq \underline{\pi}_v\} \cap M_1 = \emptyset, \\
\{v \in V \mid \tilde{\pi}_v < \underline{\pi}_v\} \subseteq M_3, & \quad \{v \in V \mid \tilde{\pi}_v \geq \bar{\pi}_v\} \cap M_3 = \emptyset.
\end{aligned}$$

Proof: Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution for the active transmission problem (3.1) fulfilling at least constraint (3.1b), (3.1c) and (3.1e). Furthermore let (q^*, π^*, p^*, y^*) be a feasible solution for (3.1). We iteratively construct the sets M_1, M_2, M_3 with $V = M_1 \dot{\cup} M_2 \dot{\cup} M_3$ as follows:

1. Initially we set $M_1 := \{v \in V \mid \tilde{\pi}_v > \bar{\pi}_v\}$. Then we iteratively extend this set by considering every arc $a = (v, w) \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_1)$. If this arc does not fulfill one of the following cases, then we either add v to M_1 if $v \notin M_1$ and set the predecessor $p(v) := w$ or we add w to M_1 if $w \notin M_1$ and set $p(w) := v$.

Case $a = (v, w) \in \delta_{A'}^+(M_1)$:

$$(\pi'_v(\pi^*) - \pi'_w(\pi^*)) < (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})), \quad (3.11a)$$

$$\begin{aligned} \nexists k \in \{1, \dots, \nu_a\} : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with} \\ (A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0. \end{aligned} \quad (3.11b)$$

Case $a = (v, w) \in \delta_{A'}^-(M_1)$:

$$(\pi'_v(\pi^*) - \pi'_w(\pi^*)) > (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})), \quad (3.12a)$$

$$\begin{aligned} \nexists k \in \{1, \dots, \nu_a\} : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with} \\ (A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0. \end{aligned} \quad (3.12b)$$

This way we obtain the node set M_1 such that every arc $a = (v, w) \in \delta_{A'}^+(M_1)$ fulfills (3.11) and every arc $a = (v, w) \in \delta_{A'}^-(M_1)$ fulfills (3.12). Furthermore it holds

$$M_1 \cap \{v \in V \mid \tilde{\pi}_v \leq \underline{\pi}_v\} = \emptyset.$$

This can be seen as follows: If $\tilde{\pi}_v \leq \bar{\pi}_v$ holds for every node $v \in V$, then $M_1 = \emptyset$ by construction. Otherwise assume that M_1 contains a node t with $\tilde{\pi}_t \leq \underline{\pi}_t$. Then we consider the nodes $t, p(t), p(p(t)), \dots, s$ where $s \in M_1$ has no predecessor. These nodes define the nodes of an edge-disjoint s - t -path P in the undirected graph $(M_1, E'[M_1])$ which originates from $(M_1, A'[M_1])$ by removing the orientation of each arc $a \in A'[M_1]$. Note that $(M_1, E'[M_1])$ might contain multiple parallel edges. This way each arc $a \in A'[M_1]$ corresponds uniquely to an edge $e \in E'[M_1]$ and vice versa. Let v_1, \dots, v_{n+1} be the nodes and e_1, \dots, e_n with $e_i = \{v_i, v_{i+1}\}$ be the ordered edges of P and a_1, \dots, a_n be the corresponding arcs in $(V, A'[M_1])$. We have that for every arc $a_i, i = 1, \dots, n$ neither (3.11) nor (3.12) applies because otherwise t would not be contained in M_1 by construction of M_1 . This means that one of the following cases holds for every arc $a_i, i = 1, \dots, n$:

Case (3.11a) and (3.12a) do not apply: Node potential loss estimation derived as (3.11a) and (3.12a) do not apply, hence:

$$\pi'_{v_i}(\pi^*) - \pi'_{v_{i+1}}(\pi^*) \geq \pi'_{v_i}(\tilde{\pi}) - \pi'_{v_{i+1}}(\tilde{\pi}).$$

Case (3.11b) and (3.12b) do not apply: In this case we differentiate between the orientation of arc a_i .

- If arc $a_i = (v_i, v_{i+1})$ then, as (3.11b) does not apply, there exists an index $k \in \{1, \dots, \nu_{a_i}\}$ such that $[A_{a_i}(\tilde{q}_{a_i}, \tilde{p}_{v_i}, \tilde{p}_{v_{i+1}})]_k \geq [b_{a_i}]_k$ holds with $(A_{a_i})_{(k,1)}(q_{a_i}^* - \tilde{q}_{a_i}) \geq 0, (A_{a_i})_{(k,2)} < 0, (A_{a_i})_{(k,3)} > 0$. We rewrite this inequality as $a_1 \tilde{q}_{a_i} - a_3 \geq \tilde{p}_{v_i} - a_2 \tilde{p}_{v_{i+1}}$ with $a_2 \in \mathbb{R}_{\geq 0}$ and $a_1, a_3 \in \mathbb{R}$. Then we derive the estimation

$$\tilde{p}_{v_i} - a_2 \tilde{p}_{v_{i+1}} \leq a_1 \tilde{q}_{a_i} - a_3 \leq a_1 q_{a_i}^* - a_3 \leq p_{v_i}^* - a_2 p_{v_{i+1}}^*.$$

- If arc $a_i = (v_{i+1}, v_i)$ then, as (3.12b) does not apply, there exists an index $k \in \{1, \dots, \nu_{a_i}\}$ such that $[A_{a_i}(\tilde{q}_{a_i}, \tilde{p}_{v_{i+1}}, \tilde{p}_{v_i})]_k \geq [b_{a_i}]_k$ holds with $(A_{a_i})_{(k,1)}(q_{a_i}^* - \tilde{q}_{a_i}) \geq 0, (A_{a_i})_{(k,2)} > 0, (A_{a_i})_{(k,3)} < 0$. We rewrite this inequality as $a_1 \tilde{q}_{a_i} - a_3 \geq \tilde{p}_{v_i} - a_2 \tilde{p}_{v_{i+1}}$ with $a_2 \in \mathbb{R}_{\geq 0}$ and $a_1, a_3 \in \mathbb{R}$. Again we derive the estimation

$$\tilde{p}_{v_i} - a_2 \tilde{p}_{v_{i+1}} \leq a_1 \tilde{q}_{a_i} - a_3 \leq a_1 q_{a_i}^* - a_3 \leq p_{v_i}^* - a_2 p_{v_{i+1}}^*.$$

Because of the coupling constraint $\tilde{p}_v |\tilde{p}_v| = \tilde{\pi}_v$ relating the pressure and node potential variables for each node $v \in V$, by using the previous estimations, we obtain:

$$\begin{aligned} \tilde{\pi}_{v_1} > \pi_{v_1}^* &\Rightarrow \tilde{\pi}_{v_2} > \pi_{v_2}^*, \dots, \tilde{\pi}_{v_{n+1}} > \pi_{v_{n+1}}^*, \\ \tilde{\pi}_{v_{n+1}} < \pi_{v_{n+1}}^* &\Rightarrow \tilde{\pi}_{v_n} < \pi_{v_n}^*, \dots, \tilde{\pi}_{v_1} < \pi_{v_1}^*. \end{aligned}$$

The path P is chosen such that the start node v_1 violates its upper node potential bound, i.e., $\tilde{\pi}_{v_1} > \bar{\pi}_{v_1}$ and for the end node v_{n+1} it holds $\tilde{\pi}_{v_{n+1}} \leq \bar{\pi}_{v_{n+1}}$. Hence the first of the above cases applies. We conclude that π^* violates a node potential bound in v_{n+1} which is a contradiction to the assumption that (q^*, π^*, p^*, y^*) is feasible for the active transmission problem.

2. In a second step we initially set $M_3 := \{v \in V \mid \pi_v < \bar{\pi}_v\}$. We now concentrate on the graph $(V \setminus M_1, A' \setminus [V \setminus M_1])$. Again we iteratively consider each arc $a = (v, w) \in \delta_{A'}^+(M_3) \cup \delta_{A'}^-(M_3)$. If neither (3.11) applies for an ingoing arc nor (3.12) applies for an outgoing arc, then we add v to M_3 if $v \notin M_3$ and w if $w \notin M_3$. By definition it follows $M_1 \cap M_3 = \emptyset$. By a similar reasoning as in the previous item we conclude

$$M_3 \cap \{v \in V \mid \tilde{\pi}_v \geq \bar{\pi}_v\} = \emptyset.$$

3. In a third step we define $M_2 := \{v \in V \mid v \notin M_1 \cup M_3\}$. The previously defined sets then have the property $V = M_1 \dot{\cup} M_2 \dot{\cup} M_3$.

□

Lemma 2. *Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution for the active transmission problem (3.1) fulfilling at least constraint (3.1b) and (3.1e). Furthermore let (q^*, π^*, p^*, y^*) be a feasible solution for (3.1) and $0 \leq z^* \leq 1$. There exists a vector $x^* = (x_v^+, x_v^-, x_v, x_a)_{v \in V, a \in A'}$ with $x_v^{*+}, x_v^{*-} \in \mathbb{R}_{\geq 0}, x_v^*, x_a^* \in \mathbb{R}$ for $v \in V$ and $a \in A'$ with $x^* \neq 0$ which is feasible for (3.8c)–(3.8f) and (3.8l)–(3.8n).*

Proof: Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution for the active transmission problem (3.1) fulfilling at least constraint (3.1b) and (3.1e). Furthermore let (q^*, π^*, p^*, y^*) be a feasible solution for the active transmission problem (3.1). For $0 \leq z^* \leq 1$ we describe how to define $x^* = (x_v, x_a)_{v \in V, a \in A'}$ which is feasible for (3.8d)–(3.8f) and (3.8l).

By Lemma 1 there exists a partition $V = M_1 \dot{\cup} M_2 \dot{\cup} M_3$ such that the following holds:

$$- \forall a = (v, w) \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_3):$$

$$\begin{aligned} & (\pi'_v(\pi^*) - \pi'_w(\pi^*)) < (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})), \\ & \nexists k \in \{1, \dots, \nu_a\} : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with} \\ & (A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0. \end{aligned} \quad (3.13)$$

$$- \forall a = (v, w) \in \delta_{A'}^-(M_1) \cup \delta_{A'}^+(M_3):$$

$$\begin{aligned} & (\pi'_v(\pi^*) - \pi'_w(\pi^*)) > (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})), \\ & \nexists k \in \{1, \dots, \nu_a\} : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with} \\ & (A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0. \end{aligned} \quad (3.14)$$

From this we obtain an estimation which is needed in the following:

$$\begin{aligned} a = (v, w) \in \delta^+(M_1) \cup \delta_{A'}^-(M_3) \\ \Rightarrow \alpha_a q_a^* |q_a^*|^{k_a} - \beta_a y_a^* &= \gamma_{r,v}^{-1} (\pi'_v(\pi^*) - \pi'_w(\pi^*)) \\ &< \gamma_{r,v}^{-1} (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})) = \alpha_a \tilde{q}_a |\tilde{q}_a|^{k_a} - \beta_a \tilde{y}_a \end{aligned} \quad (3.15a)$$

$$\begin{aligned} a = (v, w) \in \delta^-(M_1) \cup \delta_{A'}^+(M_3) \\ \Rightarrow \alpha_a q_a^* |q_a^*|^{k_a} - \beta_a y_a^* &= \gamma_{r,v}^{-1} (\pi'_v(\pi^*) - \pi'_w(\pi^*)) \\ &> \gamma_{r,v}^{-1} (\pi'_v(\tilde{\pi}) - \pi'_w(\tilde{\pi})) = \alpha_a \tilde{q}_a |\tilde{q}_a|^{k_a} - \beta_a \tilde{y}_a \end{aligned} \quad (3.15b)$$

We use the sets M_1, M_2, M_3 to define the values $x_v^*, v \in V$ and $x_a^*, a = (v, w) \in A'$ as follows:

$$x_v^* := \begin{cases} -1 & \text{if } v \in M_1, \\ 0 & \text{if } v \in M_2, \\ 1 & \text{if } v \in M_3, \end{cases} \quad (3.16a)$$

$$x_a^* := \begin{cases} 0 & \text{if } a \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_3) : \tilde{\kappa}_a(q_a^* - \tilde{q}_a) < 0, \\ 0 & \text{if } a \in \delta_{A'}^-(M_1) \cup \delta_{A'}^+(M_3) : \tilde{\kappa}_a(q_a^* - \tilde{q}_a) > 0, \\ x_w^* - x_v^* & \text{else.} \end{cases} \quad (3.16b)$$

We proceed by showing that this definition is feasible for constraints (3.8d)–(3.8f) and (3.8l). We have $x_v^* = x_w^*$ and $x_a^* = 0$ for all arcs $a = (v, w) \in A'(M_1 : M_1) \cup A'(M_2 : M_2) \cup A'(M_3 : M_3)$. Thus (3.8d)–(3.8f) and (3.8l) are fulfilled for these arcs. Let us now turn to the remaining arcs. At first we observe:

- For every arc $a \in A'(M_1 : M_2) \cup A'(M_2 : M_3) \cup A'(M_1 : M_3)$ it holds $a \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_3)$.
- For every arc $a \in A'(M_2 : M_1) \cup A'(M_3 : M_1) \cup A'(M_3 : M_2)$ it holds $a \in \delta_{A'}^-(M_1) \cup \delta_{A'}^+(M_3)$.

These are the two cases that we distinguish in the following:

Case $q_a^* < \tilde{q}_a$: We distinguish two cases.

Case $a = (v, w) \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_3)$: In this case we have $x_v^* - x_w^* \leq -1$.

- By (3.8f) $\tilde{\kappa}_a \neq 0$ means $x_v - x_w + x_a \leq 0$ must be fulfilled by x^* . We have $\tilde{\kappa}_a \neq 0 \Rightarrow \tilde{\kappa}_a > 0 \Rightarrow \tilde{\kappa}_a(q_a^* - \tilde{q}_a) < 0 \Rightarrow x_a^* = 0$ by (3.16b). Hence it holds $x_v^* - x_w^* + x_a^* \leq 0$ and $\underline{x}_a \leq x_a^* \leq \bar{x}_a$.
- By (3.8e) $\tilde{\kappa}_a = 0$ means $x_v - x_w + x_a = 0$ must be fulfilled by x^* . We have $x_a^* = -(x_v^* - x_w^*) > 0$ by (3.16b). Hence it holds $x_v^* - x_w^* + x_a^* = 0$. $\tilde{\kappa}_a = 0$ means $\alpha_a = 0$ by (3.9i). This in combination with (3.15a) implies $\beta_a y_a^* > \beta_a \tilde{y}_a$. By (3.13) there exists no index k so that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0$. This and the conclusions that there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)} \neq 0$ (because of $\tilde{\kappa}_a = 0$) especially implies that there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)} \geq 0, (A_a)_{(k,2)} < 0$ and $(A_a)_{(k,3)} > 0$. From this we conclude $\bar{x}_a = \infty$ by (3.9c). This yields $\underline{x}_a \leq 0 < x_a^* < \bar{x}_a = \infty$.

Case $a = (v, w) \in \delta_{A'}^-(M_1) \cup \delta_{A'}^+(M_3)$: In this case we have $x_v^* - x_w^* \geq 1$.

- From $\tilde{\kappa}_a \geq 0$ it follows $\tilde{\kappa}_a(q_a^* - \tilde{q}_a) \leq 0$. Hence we obtain from (3.16b) that $x_a^* = -(x_v^* - x_w^*) \leq -1 < 0$ holds. By (3.8e) and (3.8f) $x_v - x_w + x_a \leq 0$ or $x_v - x_w + x_a = 0$ must be fulfilled by x^* , which is obviously true. By (3.15b) we have $\beta_a y_a^* < \beta_a \tilde{y}_a$. By (3.14) there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0$. This especially means that there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)} \leq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0$. Hence we have $\underline{x}_a = -\infty$ by (3.9d). This yields $\underline{x}_a < x_a^* \leq 0 \leq \bar{x}_a$.

Case $q_a^* = \tilde{q}_a$: We distinguish two cases.

Case $a = (v, w) \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_3)$: In this case we have $x_v^* - x_w^* \leq -1$. By (3.15a) we have $\beta_a y_a^* > \beta_a \tilde{y}_a$.

- By (3.8e) $x_v - x_w + x_a = 0$ must be fulfilled by x^* . Because of $\tilde{\kappa}_a(q_a^* - \tilde{q}_a) = 0$ we have $x_a^* = -(x_v^* - x_w^*) \geq 1 > 0$ by (3.16b). Hence $x_v^* - x_w^* + x_a^* = 0$. By (3.13) there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0$. This especially implies that there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)} \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0$. Hence $\bar{x}_a = \infty$ by (3.9c). So we obtain $\underline{x}_a \leq 0 < x_a^* < \bar{x}_a$.

Case $a = (v, w) \in \delta_{A'}^-(M_1) \cup \delta_{A'}^+(M_3)$: In this case we have $x_v^* - x_w^* \geq 1$. By (3.15b) we have $\beta_a y_a^* < \beta_a \tilde{y}_a$.

- By (3.8e) $x_v - x_w + x_a = 0$ must be fulfilled by x^* . Because of $\tilde{\kappa}_a(q_a^* - \tilde{q}_a) = 0$ we have $x_a^* = -(x_v^* - x_w^*) \leq -1 < 0$ by (3.16b). Hence $x_v^* - x_w^* + x_a^* = 0$. By (3.14) there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0$. This especially implies that there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)} \leq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0$. Hence $\underline{x}_a = -\infty$ by (3.9d). This yields $\underline{x}_a < x_a^* < 0 \leq \bar{x}_a$.

Case $q_a^* > \tilde{q}_a$: We distinguish two cases:

Case $a = (v, w) \in \delta_{A'}^+(M_1) \cup \delta_{A'}^-(M_3)$: In this case we have $x_v^* - x_w^* \leq -1$. By (3.15a) we have $\beta_a y_a^* > \beta_a \tilde{y}_a$.

- From $\tilde{\kappa}_a \geq 0$ it follows $\tilde{\kappa}_a(q_a^* - \tilde{q}_a) \geq 0$. Hence we obtain from (3.16b) that $x_a^* = -(x_v^* - x_w^*) \geq 1 > 0$ holds. By (3.8d) and (3.8e) $x_v - x_w + x_a \geq 0$ or $x_v - x_w + x_a = 0$ must be fulfilled by x^* , which is obviously true. By (3.13) there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0$. This especially implies that there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)} \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0$. Hence $\bar{x}_a = \infty$ by (3.9c). This yields $\underline{x}_a \leq 0 < x_a^* < \infty = \bar{x}_a$.

Case $a = (v, w) \in \delta_{A'}^-(M_1) \cup \delta_{A'}^+(M_3)$: In this case we have $x_v^* - x_w^* \geq 1$.

- By (3.8d) $\tilde{\kappa}_a \neq 0$ means $x_v - x_w + x_a \geq 0$ must be fulfilled by x^* . We have $\tilde{\kappa}_a \neq 0 \Rightarrow \tilde{\kappa}_a > 0 \Rightarrow \tilde{\kappa}_a(q_a^* - \tilde{q}_a) > 0 \Rightarrow x_a^* = 0$ by (3.16b). Hence it holds $x_v^* - x_w^* + x_a^* = x_v^* - x_w^* \geq 1 \geq 0$ and $\underline{x}_a \leq x_a^* \leq \bar{x}_a$.
- By (3.8e) $\tilde{\kappa}_a = 0$ means $x_v - x_w + x_a = 0$ must be fulfilled by x^* . By (3.16b) we have $x_a^* = -(x_v^* - x_w^*) < 0$ which implies $x_v^* - x_w^* + x_a^* = 0$. $\tilde{\kappa}_a = 0$ means $\alpha_a = 0$. This implies in combination with (3.15b) that $\beta_a y_a^* < \beta_a \tilde{y}_a$ holds. By (3.14) there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)}(q_a^* - \tilde{q}_a) \geq 0$, $(A_a)_{(k,2)} > 0$, $(A_a)_{(k,3)} < 0$. Additionally $\tilde{\kappa}_a = 0$ yields that $\nexists k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$ with $(A_a)_{(k,1)} \neq 0$ by (3.9i). This especially implies that there exists no index k such that $[A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k$, with $(A_a)_{(k,1)} \leq 0$, $(A_a)_{(k,2)} > 0$, $(A_a)_{(k,3)} < 0$. Hence $\underline{x}_a = -\infty$ by (3.9d). This yields $\underline{x}_a < x_a^* < 0 \leq \bar{x}_a$.

This case discussion proves that $(x_v^*, x_a^*)_{v \in V, a \in A'}$ is feasible for (3.8d)–(3.8f) and (3.8l). We set

$$\begin{aligned} x_v^{*+} &:= \max\{0, x_v^* + \kappa_v z^*\} & \forall v \in V, \\ x_v^{*-} &:= \max\{0, -x_v^* - \kappa_v z^*\} & \forall v \in V. \end{aligned} \quad (3.17)$$

From this definition it follows that (3.8c) is fulfilled. We prove that this definition is feasible for (3.8m) and (3.8n). Therefor we make use of

$$\{v \in V \mid \tilde{\pi}_v > \bar{\pi}_v\} \subseteq M_1 \quad \{v \in V \mid \tilde{\pi}_v \leq \underline{\pi}_v\} \cap M_1 = \emptyset, \quad (3.18a)$$

$$\{v \in V \mid \tilde{\pi}_v < \underline{\pi}_v\} \subseteq M_3 \quad \{v \in V \mid \tilde{\pi}_v \geq \bar{\pi}_v\} \cap M_3 = \emptyset, \quad (3.18b)$$

which holds by Lemma 1. We consider the bound constraints (3.8m) and (3.8n) separately:

- We show that it holds $x_v^{*+} \leq \bar{x}_v^+$ for every node $v \in V$: Let $v \in V$. According to (3.9e) we have to show $x_v^{*+} = 0$ if $\tilde{\pi}_v \geq \bar{\pi}_v$. In this case it holds $v \in M_1 \cup M_2$ by (3.18b). We distinguish two cases:

$$\begin{aligned} \tilde{\pi}_v = \bar{\pi}_v &\stackrel{(3.9g)}{\Rightarrow} \kappa_v = 0 \\ \Rightarrow x_v^* + \kappa_v z^* &= x_v^* \stackrel{(3.16a), (3.18b)}{\leq} 0 \Rightarrow x_v^{*+} = \max\{0, x_v^* + \kappa_v z^*\} = 0. \\ \tilde{\pi}_v > \bar{\pi}_v &\stackrel{(3.9g), (3.18a), (3.16a)}{\Rightarrow} \kappa_v = 1, x_v^* = -1 \\ \Rightarrow x_v^* + \kappa_v z^* &= -1 + z^* \stackrel{z^* \in [0,1]}{\leq} 0 \Rightarrow x_v^{*+} = \max\{0, x_v^* + \kappa_v z^*\} = 0. \end{aligned}$$

- We show that it holds $x_v^{*-} \leq \bar{x}_v^-$ for every node $v \in V$: Let $v \in V$. According to (3.9f) we have to show $x_v^{*-} = 0$ if $\tilde{\pi}_v \leq \underline{\pi}_v$. In this case it holds $v \in M_3 \cup M_2$ by (3.18a). We distinguish two cases:

$$\begin{aligned} \tilde{\pi}_v = \underline{\pi}_v &\stackrel{(3.9g)}{\Rightarrow} \kappa_v = 0 \\ \Rightarrow x_v^* + \kappa_v z^* &= x_v^* \stackrel{(3.16a), (3.18a)}{\geq} 0 \Rightarrow x_v^{*-} = \min\{0, x_v^* + \kappa_v z^*\} = 0. \\ \tilde{\pi}_v < \underline{\pi}_v &\stackrel{(3.9g), (3.18b), (3.16a)}{\Rightarrow} \kappa_v = -1, x_v^* = 1 \\ \Rightarrow x_v^* + \kappa_v z^* &= 1 - z^* \stackrel{z^* \in [0,1]}{\geq} 0 \Rightarrow x_v^{*-} = \min\{0, x_v^* + \kappa_v z^*\} = 0. \end{aligned}$$

We conclude that the definition (3.16a), (3.16b), and (3.17) yield a vector x^* which is feasible for (3.8c)–(3.8f) and (3.8l)–(3.8n). \square

4 Interpretation of the Infeasibility Detection MILP

In this section we give an illustrative explanation of the formulation of the infeasibility detection MILP (3.8). Therefor we look at the dual problem of (3.8) for a fixed flow vector $q' \in \mathbb{R}^{A'}$ which fulfills the flow conservation and bound constraints (3.10), i.e.,

$$\sum_{a \in \delta_{A'}^+(v)} q'_a - \sum_{a \in \delta_{A'}^-(v)} q'_a = d_v \quad \forall v \in V, \quad \underline{q}_a \leq q'_a \leq \bar{q}_a \quad \forall a \in A'.$$

Let $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ be a solution of the active transmission problem (3.1) fulfilling at least constraint (3.1b), (3.1c) and (3.1e). Assume that the MILP (3.8) has optimal objective value zero. By Theorem 1 we conclude that there do not exist vectors $\pi' \in \mathbb{R}^V$, $p' \in \mathbb{R}^V$, $y' \in \mathbb{R}^{A'}$ such that (q', π', p', y') is feasible for the active transmission problem (3.1). In the following we show an example demonstrating that especially $\pi' \in \mathbb{R}^V$ with $\underline{\pi} \leq \pi' \leq \bar{\pi}$ cannot exist. Therefor we assume that (q', π', p', y') is a feasible solution for (3.1) and derive a contradiction by comparing (q', π', p', y') and $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$.

As mentioned before we assume that MILP (3.8) has optimal objective value zero. This implies that the following linear optimization problem is bounded because the feasible solution space forms a cone (recall that q' and \tilde{q} are fixed):

$$\begin{aligned}
& \max z && (4.1) \\
\text{s. t.} & && \\
\begin{bmatrix} \lambda_v \\ \mu_a \\ \nu_a \end{bmatrix} & && \\
& x_v^+ - x_v^- - x_v - \kappa_v z = 0 && \forall v \in V, \\
& x_v - x_w + x_a - x_a^+ \tilde{\kappa}_a (q'_a - \tilde{q}_a) = 0 && \forall a = (v, w) \in A', \\
& s_v - s_w + s_a - s_a^+ \alpha_a (q'_a - \tilde{q}_a) - \kappa_a z = 0 && \forall a = (v, w) \in A', \\
& \underline{s}_a \leq s_a \leq \bar{s}_a && \forall a \in A', \\
& \underline{x}_a \leq x_a \leq \bar{x}_a && \forall a \in A', \\
& x_v^+ \leq \bar{x}_v^+ && \forall v \in V, \\
& x_v^- \leq \bar{x}_v^- && \forall v \in V, \\
& x_v, s_v \in \mathbb{R} && \forall v \in V, \\
& x_v^+, x_v^- \in \mathbb{R}_{\geq 0} && \forall v \in V, \\
& x_a, s_a \in \mathbb{R} && \forall a \in A', \\
& x_a^+, s_a^+ \in \mathbb{R}_{\geq 0} && \forall a \in A', \\
& z \in \mathbb{R}_{\geq 0}.
\end{aligned}$$

In this notation we associated dual variables λ_v for each node $v \in V$ and μ_a, ν_a for each arc $a \in A'$. As (4.1) is bounded it follows that its dual is feasible. This dual is as follows:

$$\begin{aligned}
& \exists \lambda, \mu, \nu && (4.2a) \\
\text{s. t.} & && \\
& \sum_{a \in \delta_{A'}^+(v)} \nu_a - \sum_{a \in \delta_{A'}^-(v)} \nu_a = 0 && \forall v \in V, && (4.2b) \\
& \sum_{a \in \delta_{A'}^+(v)} \mu_a - \sum_{a \in \delta_{A'}^-(v)} \mu_a - \lambda_v = 0 && \forall v \in V, && (4.2c) \\
& \sum_{v \in V: \tilde{\pi}_v > \bar{\pi}_v} \lambda_v - \sum_{v \in V: \tilde{\pi}_v < \underline{\pi}_v} \lambda_v + \sum_{a \in A'} \kappa_a \nu_a \geq 1, && && (4.2d) \\
& \tilde{\kappa}_a (q'_a - \tilde{q}_a) \mu_a \geq 0 && \forall a \in A', && (4.2e) \\
& \alpha_a (q'_a - \tilde{q}_a) \nu_a \geq 0 && \forall a \in A', && (4.2f) \\
& \underline{\lambda}_v \leq \lambda_v \leq \bar{\lambda}_v && \forall v \in V, && (4.2g) \\
& \underline{\mu}_a \leq \mu_a \leq \bar{\mu}_a && \forall a \in A', && (4.2h) \\
& \underline{\nu}_a \leq \nu_a \leq \bar{\nu}_a && \forall a \in A', && (4.2i) \\
& \mu_a, \nu_a \in \mathbb{R} && \forall a \in A', && (4.2j) \\
& \lambda_v \in \mathbb{R} && \forall v \in V. && (4.2k)
\end{aligned}$$

Here the variable bounds are defined as

$$\bar{\lambda}_v := \begin{cases} \infty & \text{if } \tilde{\pi}_v \geq \bar{\pi}_v, \\ 0 & \text{else} \end{cases} \quad \forall v \in V, \quad (4.3a)$$

$$\underline{\lambda}_v := \begin{cases} -\infty & \text{if } \tilde{\pi}_v \leq \underline{\pi}_v, \\ 0 & \text{else} \end{cases} \quad \forall v \in V, \quad (4.3b)$$

$$\bar{\mu}_a := \begin{cases} \infty & \text{if } \beta_a \tilde{y}_a = \max\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\} \\ \infty & \text{if } \exists k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with } \forall a \in A', \\ & (A_a)_{(k,1)} \geq 0, (A_a)_{(k,2)} < 0, (A_a)_{(k,3)} > 0, \quad a = (v, w), \\ 0 & \text{else} \end{cases} \quad (4.3c)$$

$$\underline{\mu}_a := \begin{cases} -\infty & \text{if } \beta_a \tilde{y}_a = \min\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\} \\ -\infty & \text{if } \exists k : [A_a(\tilde{q}_a, \tilde{p}_v, \tilde{p}_w)]_k \geq [b_a]_k, \text{ with } \forall a \in A', \\ & (A_a)_{(k,1)} \leq 0, (A_a)_{(k,2)} > 0, (A_a)_{(k,3)} < 0, \quad a = (v, w), \\ 0 & \text{else} \end{cases} \quad (4.3d)$$

$$\bar{\nu}_a := \begin{cases} \infty & \text{if } \beta_a \tilde{y}_a = \max\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\} & \forall a \in A', \\ 0 & \text{else} & a = (v, w), \end{cases} \quad (4.3e)$$

$$\underline{\nu}_a := \begin{cases} -\infty & \text{if } \beta_a \tilde{y}_a = \min\{\beta_a \underline{y}_a, \beta_a \bar{y}_a\} & \forall a \in A', \\ 0 & \text{else} & a = (v, w). \end{cases} \quad (4.3f)$$

Now let $(\lambda^*, \mu^*, \nu^*)$ be a feasible solution for (4.2). The vectors ν^* and μ^* form a network flow by constraints (4.2b) and (4.2c). For the following discussion we focus on the case that either $\mu^* \geq 0$ if $\lambda^* \neq 0$ or $\nu^* \geq 0$ holds. The case where these assumptions are not fulfilled can be led back to the case fulfilling the assumptions by changing the orientation of some arcs. Our initial motivation for the definition of MILP (3.8) was to look either for a path or a circuit as discussed in the following two cases:

Case $\lambda^* \neq 0$: We split the network flow μ^* into sets of flow along paths P_1, \dots, P_m and flow along circuits C_1, \dots, C_n . This way we obtain from $\mu^* \geq 0$ that there exist flow values $\mu_{P_i} > 0, i = 1, \dots, m$ and $\mu_{C_i} > 0, i = 1, \dots, n$ such that

$$\mu_a^* = \sum_{\substack{i=1, \dots, m: \\ a \in A'(P_i)}} \mu_{P_i} + \sum_{\substack{i=1, \dots, n: \\ a \in A'(C_i)}} \mu_{C_i} \quad \forall a \in A'.$$

Consider a path P_ℓ that starts in node v and ends in node w . Because of constraint (4.2d) the index ℓ can be chosen such that either $\tilde{\pi}_v > \bar{\pi}_v$ and $\tilde{\pi}_w \leq \bar{\pi}_w$ or $\tilde{\pi}_v \geq \bar{\pi}_v$ and $\tilde{\pi}_w < \bar{\pi}_w$ holds. Let the nodes of P_ℓ be given by v_1, \dots, v_{n+1} where $v_1 = v$ and $v_{n+1} = w$ and connecting arcs by a_1, \dots, a_n .

In order to show that (q', π', p', y') is not feasible for the active transmission problem (3.1) we distinguish two cases for each arc a_i of the path P_ℓ :

1. In the case that $\beta_{a_i} \tilde{y}_{a_i} = \max\{\beta_{a_i} \underline{y}_{a_i}, \beta_{a_i} \bar{y}_{a_i}\}$ we obtain the following estimation from $\tilde{q}_{a_i} \leq q'_{a_i}$ if $\alpha_{a_i} \neq 0$ (by (4.2e)) and $\beta_{a_i} y'_{a_i} \leq \beta_{a_i} \tilde{y}_{a_i}$:

$$\begin{aligned} \tilde{\pi}_{v_i} - \gamma_{a_i} \tilde{\pi}_{v_{i+1}} &= \alpha_{a_i} \tilde{q}_{a_i} |\tilde{q}_{a_i}|^{k_{a_i}} - \beta_{a_i} \tilde{y}_{a_i} \\ &\leq \alpha_{a_i} q'_{a_i} |q'_{a_i}|^{k_{a_i}} - \beta_{a_i} y'_{a_i} = \pi'_{v_i} - \gamma_{a_i} \pi'_{v_{i+1}}. \end{aligned}$$

2. In the case that $[A_{a_i}(\tilde{q}_{a_i}, \tilde{p}_{v_i}, \tilde{p}_{v_{i+1}})]_k \geq [b_{a_i}]_k$ holds for an index k with $(A_{a_i})_{(k,1)} \geq 0, (A_{a_i})_{(k,2)} < 0, (A_{a_i})_{(k,3)} > 0$ we rewrite this inequality as $a_1 \tilde{q}_{a_i} - a_3 \geq \tilde{p}_{v_i} - a_2 \tilde{p}_{v_{i+1}}$ with $a_1 \in \mathbb{R}_{\geq 0}, a_2 \in \mathbb{R}_{> 0}$ and $a_3 \in \mathbb{R}$. Then we derive the estimation (using (4.2e) and (3.9i)), which yields $a_1 > 0 \Rightarrow \tilde{q}_{a_i} \leq q'_{a_i}$:

$$\tilde{p}_{v_i} - a_2 \tilde{p}_{v_{i+1}} \leq a_1 \tilde{q}_{a_i} - a_3 \leq a_1 q'_{a_i} - a_3 \leq p'_{v_i} - a_2 p'_{v_{i+1}}.$$

We note that at least one of the previous cases applies because of $0 < \mu_{a_i}^* \leq \bar{\mu}_{a_i}$ and (4.3c). Because of the coupling constraints $\tilde{p}_v |\tilde{p}_v| = \tilde{\pi}_v$ and $p'_v |p'_v| = \pi'_v$ relating the pressure and node potential variables for each node $v \in V$, we obtain:

$$\tilde{\pi}_{v_1} > \pi'_{v_1} \quad \Rightarrow \quad \tilde{\pi}_{v_2} > \pi'_{v_2}, \dots, \tilde{\pi}_{v_{n+1}} > \pi'_{v_{n+1}}, \quad (4.4a)$$

$$\tilde{\pi}_{v_{n+1}} < \pi'_{v_{n+1}} \quad \Rightarrow \quad \tilde{\pi}_{v_n} < \pi'_{v_n}, \dots, \tilde{\pi}_{v_1} < \pi'_{v_1}. \quad (4.4b)$$

The path P_ℓ is chosen such that either the start node v_1 or the end node v_{n+1} of path P_ℓ violates its node potential bound, i.e., one of the following cases applies:

$$\tilde{\pi}_{v_1} > \bar{\pi}_{v_1} \text{ and } \tilde{\pi}_{v_{n+1}} \leq \bar{\pi}_{v_{n+1}} \quad \text{and} \quad \bar{\pi}_{v_1} \geq \pi'_{v_1} \quad \stackrel{(4.4a)}{\Rightarrow} \quad \pi'_{v_{n+1}} < \bar{\pi}_{v_{n+1}},$$

$$\tilde{\pi}_{v_1} \geq \bar{\pi}_{v_1} \text{ and } \tilde{\pi}_{v_{n+1}} < \underline{\pi}_{v_{n+1}} \text{ and } \underline{\pi}_{v_{n+1}} \leq \pi'_{v_{n+1}} \stackrel{(4.4b)}{\Rightarrow} \pi'_{v_1} > \bar{\pi}_{v_1}.$$

Hence π' violates the node potential bounds which implies that (q', π', p', y') is not feasible for the active transmission problem (3.1).

Case $\nu^* \neq 0, \lambda^* = 0$: Similar as in the previous case we split the network flow ν^* into sets of flow along circuits C_1, \dots, C_n . By constraint (4.2d) there exists an arc $a \in A'$ with $\nu_a^* \neq 0$ and $\alpha_a > 0$ by (3.9h). From our assumption we obtain $\nu_a^* > 0$ for this arc. Let ℓ be chosen such that C_ℓ contains this arc. Let the nodes of C_ℓ be given by v_1, \dots, v_{n+1} where $v_1 = v_{n+1}$ and connecting arcs by a_1, \dots, a_n . We note that $\beta_{a_i} \tilde{y}_{a_i} = \max\{\beta_{a_i} \underline{y}_{a_i}, \beta_{a_i} \bar{y}_{a_i}\}$ holds because of $0 < \nu_{a_i}^* \leq \bar{\nu}_{a_i}$ and (4.3e). From this observation we derive the following contradiction from $\tilde{q}_{a_i} \leq q'_{a_i}$ if $\alpha_{a_i} \neq 0$ (by (4.2f)):

$$\begin{aligned} 0 &= \sum_{i=1}^n \left(\prod_{j=1}^{i-1} \gamma_{a_j} \right) (\tilde{\pi}_{v_i} - \gamma_{a_i} \tilde{\pi}_{v_{i+1}}) \\ &= \sum_{i=1}^n \left(\prod_{j=1}^{i-1} \gamma_{a_j} \right) (\alpha_{a_i} \tilde{q}_{a_i} |\tilde{q}_{a_i}|^{k_{a_i}} - \beta_{a_i} \tilde{y}_{a_i}) \\ &< \sum_{i=1}^n \left(\prod_{j=1}^{i-1} \gamma_{a_j} \right) (\alpha_{a_i} q'_{a_i} |q'_{a_i}|^{k_{a_i}} - \beta_{a_i} y'_{a_i}) \\ &= \sum_{i=1}^n \left(\prod_{j=1}^{i-1} \gamma_{a_j} \right) (\pi'_{v_i} - \gamma_{a_i} \pi'_{v_{i+1}}) = 0. \end{aligned}$$

The inequality is strict because $\kappa_a \nu_a > 0$ by (4.2d) and $\tilde{q}_a < q_a \leq q'_a$ by (3.9h) and the feasibility of q' . This contradiction implies that our assumption was wrong and hence the solution (q', π', p', y') is not feasible for the active transmission problem (3.1).

We note that at least one of the above cases applies because of constraint (4.2d). This contradicts our assumption that (q', π', p', y') is feasible for the active transmission problem (3.1) and shows that this assumption was wrong.

5 Integration and Computational Results

Below we present our strategy for solving the topology optimization problem (2.6) as outlined in Section 3. Furthermore we present computational results that demonstrate the benefits of this solution approach.

5.1 Integration

Our solution approach is to solve the model (2.6) by SCIP [23] in combination with a specially tailored adaptation: Whenever the active transmission problem (3.1) arises from branching on integral variables we compute a local optimal solution for a relaxation of the active transmission problem. This relaxation is denoted as *domain relaxation*. It is defined as follows:

$$\min \sum_{v \in V} \Delta_v + \sum_{a \in A'} (\Delta_a + \|\Delta'_a\|) \quad (5.1a)$$

$$\text{s. t.} \quad \alpha_a q_a |q_a|^{k_a} - \beta_a y_a - (\pi_v - \gamma_a \pi_w) = 0 \quad \forall a = (v, w) \in A', \quad (5.1b)$$

$$\sum_{a \in \delta_{A'}^+(v)} q_a - \sum_{a \in \delta_{A'}^-(v)} q_a = d_v \quad \forall v \in V, \quad (5.1c)$$

$$A_a(q_a, p_v, p_w)^T - \Delta'_a \leq b_a \quad \forall a = (v, w) \in A', \quad (5.1d)$$

$$p_v |p_v| - \pi_v = 0 \quad \forall v \in V, \quad (5.1e)$$

$$\pi_v - \Delta_v \leq \bar{\pi}_v \quad \forall v \in V, \quad (5.1f)$$

$$\pi_v + \Delta_v \geq \underline{\pi}_v \quad \forall v \in V, \quad (5.1g)$$

$$q_a - \Delta_a \leq \bar{q}_a \quad \forall a \in A', \quad (5.1h)$$

$$q_a + \Delta_a \geq \underline{q}_a \quad \forall a \in A', \quad (5.1i)$$

$$y_a \leq \bar{y}_a \quad \forall a \in A', \quad (5.1j)$$

$$y_a \geq \underline{y}_a \quad \forall a \in A', \quad (5.1k)$$

$$p_v, \pi_v \in \mathbb{R} \quad \forall v \in V, \quad (5.1l)$$

$$q_a, y_a \in \mathbb{R} \quad \forall a \in A', \quad (5.1m)$$

$$\Delta_v \in \mathbb{R}_{\geq 0} \quad \forall v \in V, \quad (5.1n)$$

$$\Delta_a \in \mathbb{R}_{\geq 0} \quad \forall a \in A', \quad (5.1o)$$

$$\Delta'_a \in \mathbb{R}'_{\geq 0} \quad \forall a \in A'. \quad (5.1p)$$

This relaxation is a non-convex optimization problem due to the constraints (5.1b). We note that, if $\underline{y}_a = \bar{y}_a$, $A_a = 0$ and $b_a = 0$ holds for every arc $a \in A'$, then the domain relaxation turns into a convex optimization problem, see [13]. Nevertheless, we do not restrict to this special case.

For a feasible solution $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{\Delta}, \tilde{y})$ to the domain relaxation we differentiate between three cases:

Case 1: If $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ is feasible for the active transmission problem (3.1) then this solution is globally optimal. We add this feasible solution to the solution pool of SCIP. In this sense, solving the domain relaxation (5.1) is a primal heuristic for the active transmission problem.

Case 2: Otherwise, if $(\tilde{q}, \tilde{\pi}, \tilde{p}, \tilde{y})$ is not feasible for the active transmission problem, then the solution violates at most the constraints (3.1d) and (3.1f)-(3.1i). The next step is to solve the infeasibility detection MILP (3.8). If this problem turns out to be infeasible or has optimal objective value zero, then the infeasibility of the active transmission problem (3.1) is certified, see Theorem 1. In this case we manually prune the corresponding node of the branch-and-bound tree.

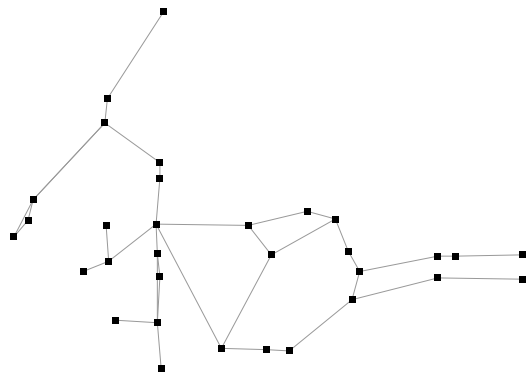
Case 3: Otherwise, if we cannot decide that the current active transmission problem is infeasible, we cannot interfere with the branching process. Neither a primal feasible solution can be provided nor a node of the branch-and-bound tree can be manually pruned.

We note that exactly one of the above cases applies. After having completed the above steps the solver continues with the branching process.

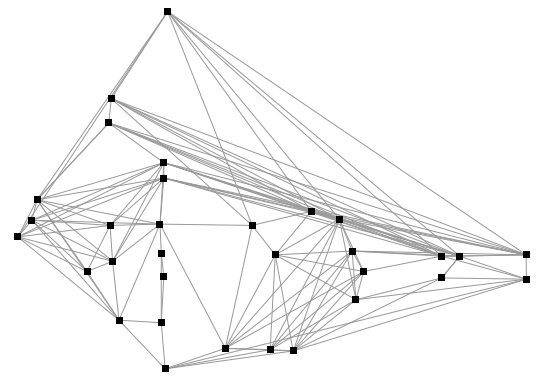
5.2 Computational Results

We compare four strategies for solving the topology optimization problem (2.6):

1. The first strategy is to use SCIP without any adaptations on the solver settings.
2. The second strategy is to enforce a certain branching priority rule, so that SCIP first branches on binary and discrete decision variables. Only after all discrete variables are fixed it is allowed to perform spatial branching on continuous variables.
3. The third strategy implements the domain relaxation (5.1) for computing a primal feasible solution for the active transmission problem (3.1). We apply SCIP as described for the second strategy. Whenever the active transmission problem arises from branching on integral variables we use the domain relaxation (5.1) as a heuristic for computing a feasible solution. For this computation we apply the nonlinear solver IPOpt [29]. If we obtain a primal feasible solution then we add it to the solution pool of the solver. Otherwise we cannot detect infeasibility of the active transmission problem and thus do not manually prune any node of the branch-and-bound tree.
4. The fourth strategy implements the solution process as described in Section 5.1. We use the nonlinear solver IPOpt [29] for solving the domain relaxation (5.1) and SCIP for solving the infeasibility detection MILP (3.8) in advance. For solving this MILP we impose a time limit of 15s. Additionally we set branching priorities according to the second strategy.

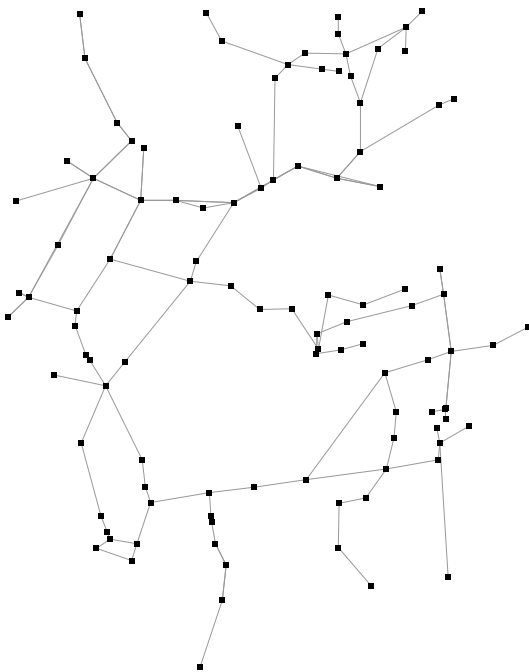


(a) Original test network.

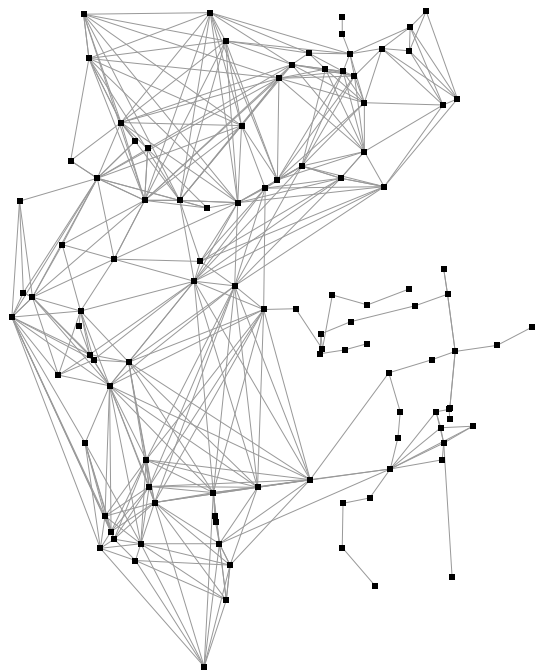


(b) Extended test network.

Fig. 5.1: The test network **net1**. It is an approximation of parts of the German gas network in the Rhine-Main-Ruhr area. More precisely the length and the diameters of the pipelines are real-world data while other parameters like roughness or compressor data are set to realistic mean values. Altered data of network **net1** with similar characteristic is publicly available at URL <http://gaslib.zib.de> under the name **gaslib-40**. The additional arcs were obtained manually. They represent each a pipeline in series with a valve. The length of these additional pipelines is set to the geographical distance between the end nodes. Costs associated with these pipelines reflect the building costs.



(a) Original test network.



(b) Extended test network.

Fig. 5.2: The test network **net2**. This network is an approximation of the German gas network for the high calorific gas. More precisely the length and the diameters of the pipelines are real-world data while other parameters like roughness or compressor data are set to realistic mean values. Altered data of the underlying network, which contains no extensions, is publicly available at URL <http://gaslib.zib.de> under the name **gaslib-135**. The additional arcs were obtained manually. They represent each a pipeline in series with a valve. The length of these additional pipelines is set to the geographical distance between the end nodes. Costs associated with these pipelines reflect the building costs.

We implemented the algorithm above in C, i.e., solving the domain relaxation (5.1) and the infeasibility detection MILP (3.8). We used a cluster of 64bit Intel Xeon X5672 CPUs at 3.20 GHz with 12 MByte cache and 48 GB main memory, running OpenSuse 12.1 Linux with a gcc 4.6.2 compiler. Furthermore, we used the following software packages: SCIP 3.0.1 [23] as mixed-integer nonlinear branch-and-cut framework, CPLEX 12.1 [8] as linear programming solver, IPOpt 3.10 [29] as nonlinear solver, and Lamatto++ [12] as framework for handling the input data. Hyperthreading and TurboBoost were disabled. In all experiments, we only ran one job per node.

We consider the networks `net1` and `net2` shown in Figure 5.1 and 5.2. For obtaining the extended networks we manually added pipelines in series with a valve. Network `net1` consists of 40 nodes, 6 compressors, 39 pipes and 103 extension pipes in series with a valve. Network `net2` consists of 135 nodes, 29 compressors, 141 pipes and 261 extension pipes in series with a valve. A summary of the computational results is shown in Tables 1-2 while the original results are available in Tables 3-4. The summary shows the number of globally solved instances and the shifted geometric mean of runtime and number of branch-and-bound nodes.

strategy	1	2	3	4	all
solved instances	25	26	26	39	52

Table 1: Summary of the Tables 3 and 4 showing the globally solved instances out of 52 nominations in total. The fourth strategy globally solves all instances which are solved to global optimality by the other strategies.

	(A,B) = (2,3)		(A,B) = (2,4)	
	solved(24) time [s]	nodes	solved(26) time [s]	nodes
strategy A	9.3	191	17.1	343
strategy B	17.2	212	16.3	192
shifted geom. mean	+86 %	+11 %	-5 %	-44 %

Table 2: Runtime and number of branch-and-bound nodes for the strategies 2 and 3 and additionally 2 and 4 (aggregated results). The columns contain mean values for those instances globally solved by both strategies A and B. The underlying data are available in Tables 3 and 4.

Table 1 shows that the fourth strategy clearly outperforms the other strategies in terms of number of solved instances. All instances that are globally solved by strategies 1 or 2 or 3 are also globally solved by the fourth strategy. Approximately 25% more instances of the test set (13 out of 52) are solved by the fourth strategy compared to the second one. The second strategy performs better than the first one because it solves one more instance within the time limit. Furthermore it shows better gap values, see Table 3. We conclude that branching priorities as imposed by the second strategy are a first step to improve the solving performance of SCIP.

The summary in Table 2 shows that approximately 50% of the instances of the test set (26 out of 52) are globally solved by the second strategy. Here the runtime decreases by approximately 5% on average following strategy 4 while the number of branch-and-bound nodes is decreased by approximately 44%. Clearly the reduction of nodes does not pay off compared to the comparatively slight decrease in runtime.

Comparing strategy 3 and 4 we observe that the third strategy allows to solve 50% of the instances of the test set (26 out of 52). Hence it performs nearly similar as the second strategy in terms of the number of globally solved instances. But the runtime increases by approximately 86% and the number of nodes by approximately 11%. We conclude that the primal heuristic (solving the domain relaxation (5.1)) as well as the verification of the infeasibility conditions (represented by the infeasibility detection MILP (3.8)) are both important for the performance of the fourth strategy.

nom	SCIP (strat. 1)			SCIP with priorities (strat. 2)			domain relaxation heuristic (strat. 3)			domain relaxation and check (strat. 4)					
	gap	primal	dual time	gap	primal	dual time	gap	primal	dual	time	nodes	gap	primal	dual time	nodes
1		996	0 limit												
2	5,499	2,270	40 limit												
3	1,972	1,908	92 limit	1,181	1,180	92 limit	888,972	4,725	4,444	92	limit	1,568,040	14,755	6,024	40 limit
4	63	1,148	701 limit	63	1,148	701 limit	376,552	873	6,832	701	limit	145,366	228	2,306	701 limit
5			726 limit												
6			772 limit												
7			1,038 limit												
8			1,019 limit												
9			1,099 limit												
10			1,252 limit												
11			1,187 limit												

Table 4: Results on network `net2` and 11 nominations. The instance marked with * had numerical troubles due to a CPLEX LP error. The time limit was set to 11 hours.

We observe from Table 4 that very few primal feasible solutions are computed for instances on network `net2`. For 6 out of 11 instances none of the strategies 1-4 terminated with a feasible solution available after 11 hours. Here primal heuristics are first choice for further investigations. Therefor we applied other state-of-the-art MINLP solvers than SCIP. More precisely, we used ANTIGONE [17] and BARON [26] for solving the topology optimization problem (2.6). Results for these solvers without any adaptations on the implementation are available in Tables 5-6. For `net2` we observe that ANTIGONE detects infeasibility for all instances while BARON does not compute any primal feasible solution. We conclude that ANTIGONE and BARON show even worse solving performances compared to SCIP (without any adaptations, strategy 1).

nom	ANTIGONE	BARON	nom	ANTIGONE	BARON
1	limit	39	21	limit	1,143
2	limit	115	22	limit	35,611
3	limit	76	23	limit	7,655
4	limit	46	24	limit	limit
5	limit	71	25	limit	limit
6	limit	78	26	limit	limit
7	37,098	21	27	limit	limit
8	limit	118	28	limit	limit
9	limit	22	29	limit	limit
10	limit	35	30	limit	limit
11	limit	limit	31	limit	limit
12	limit	*limit	32	limit	limit
13	limit	1,713	33	limit	limit
14	limit	4,433	34	limit	limit
15	limit	6,340	35	limit	limit
16	limit	85	36	limit	limit
17	limit	limit	37	limit	limit
18	limit	limit	38	limit	limit
19	limit	*limit	39	limit	limit
20	limit	32,072	40	limit	limit
			41	limit	limit

Table 5: Runtime results in seconds using ANTIGONE and BARON to solve the topology optimization problem (2.6) on 41 nominations on the network `net1`. The time limit was set to 11 hours. Those instances with a finite runtime were solved to global optimality. No primal solution was available for the other instances except those ones that are marked by *. Here a primal feasible solution was available.

6 Conclusions and Outlook

Regarding the literature known to the author, mainly various heuristic and local optimization methods are in use to solve network design problems in gas transport. This paper presents a mixed-integer non-linear programming formulation, which allows, at least in principle, to compute global optimal solutions. Using state-of-the-art solvers such as ANTIGONE, BARON or SCIP in their default modes, i.e., without any adaptations on the solver setting or implementation, about half of the instances of our test instances can be solved to global optimality. Using the proposed solution method we were able to solve about 25% more instances of our test set to global optimality. Furthermore the runtime is decreased by 5% on average.

So far, our method was applied to the case of a single nomination. In practice, however, one has to deal with a whole set of different infeasible nominations and needs to determine a topology extension that can cope with all of them simultaneously.

nom	ANTIGONE	BARON
1	3	limit
2	3	limit
3	3	limit
4	3	limit
5	3	limit
6	4	limit
7	3	limit
8	3	limit
9	4	limit
10	3	limit
11	3	limit

Table 6: Runtime results in seconds using ANTIGONE and BARON to solve the topology optimization problem (2.6) on 11 nominations on the network `net2`. The time limit was set to 11 hours. Those instances with a finite runtime were detected to be infeasible. No primal solution was available for the other instances.

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References

1. T. Achterberg. SCIP: Solving Constraint Integer Programs. *Mathematical Programming Computation*, 1(1):1–41, 2009.
2. J. André and J. F. Bonnans. Optimal features of gas transmission trunklines. In *EngOpt 2008 – International Conference on Engineering Optimization*, 2008. Rio de Janeiro, Brazil, June 1-5, 2008.
3. J. F. Bonnans and J. André. Optimal structure of gas transmission trunklines. Technical Report 6791, Institut National de Recherche en Informatique et en Automatique, January 2009.
4. I. D. Boyd, P. D. Surry, and N. J. Radcliffe. Constrained gas network pipe sizing with genetic algorithms. Technical Report EPCC-TR94-11, Edinburgh Parallel Computing Centre, 1994.
5. Bundesanstalt für Geowissenschaften und Rohstoffe. Energiestudie 2013 - Reserven, Ressourcen und Verfügbarkeit von Energierohstoffen, 2013. Downloaded on August 26, 2014.
6. L. Castillo and A. González. Distribution network optimization: Finding the most economic solution by using genetic algorithms. *European Journal of Operational Research*, 108(3):527–537, 1998.
7. G. Cerbe. *Grundlagen der Gastechnik: Gasbeschaffung – Gasverteilung – Gasverwendung*. Hanser Verlag, Leipzig, Germany, 2008.
8. CPLEX. *User’s Manual for CPLEX*. IBM Corporation, Armonk, USA, 12.1 edition, 2011.
9. D. De Wolf and Y. Smeers. Optimal dimensioning of pipe networks with application to gas transmission networks. *Operations Research*, 44(4):596–608, July 1996.
10. D. De Wolf and Y. Smeers. The gas transmission problem solved by an extension of the simplex algorithm. *Management Science*, 46(11):1454–1465, November 2000.
11. A. Fügenschuh, B. Geißler, R. Gollmer, A. Morsi, M. E. Pfetsch, J. Rövekamp, M. Schmidt, K. Spreckelsen, and M. C. Steinbach. Physical and technical fundamentals of gas networks. In T. Koch, B. Hiller, M. E. Pfetsch, and L. Schewe, editors, *Evaluating Gas Network Capacities*, MOS-SIAM Series on Optimization, chapter 2. SIAM, 2015. To appear.
12. B. Geißler, A. Martin, and A. Morsi. LaMaTTO++. Information available at URL <http://www.mso.math.fau.de/edom/projects/lamatto.html>, February 2013.
13. J. Humpola. *Gas Network Optimization by MINLP*. PhD thesis, Technical University Berlin, 2014.
14. J. Humpola, A. Fügenschuh, and T. Lehmann. A primal heuristic for optimizing the topology of gas networks based on dual information. *EURO Journal on Computational Optimization*, pages 1–26, 2014.
15. B. Korte and J. Vygen. *Combinatorial Optimization: Theory and Algorithms*. Springer Verlag, Berlin, 2007.
16. O. Mariani, F. Ancillai, and E. Donati. Design of a gas pipeline: Optimal configuration. Technical Report PSIG 9706, Pipeline Simulation Interest Group, 1997.
17. R. Misener and C. A. Floudas. A framework for globally optimizing mixed-integer signomial programs. *Journal of Optimization Theory and Applications*, 161(3):905–932, 2014.

18. G. L. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization*. John Wiley & Sons, New York, 1988.
19. R. P. O’Neill, M. Williard, B. Wilkins, and R. Pike. A mathematical programming model for allocation of natural gas. *Operations Research*, 27(5):857–873, 1979.
20. A. J. Osiadacz and M. Górecki. Optimization of pipe sizes for distribution gas network design. Technical Report PSIG 9511, Pipeline Simulation Interest Group, 1995.
21. M. E. Pfetsch, A. Fügenschuh, B. Geißler, N. Geißler, R. Gollmer, B. Hiller, J. Humpola, T. Koch, T. Lehmann, A. Martin, A. Morsi, J. Rövekamp, L. Schewe, M. Schmidt, R. Schultz, R. Schwarz, J. Schweiger, C. Stangl, M. C. Steinbach, S. Vigerske, and B. M. Willert. Validation of nominations in gas network optimization: Models, methods, and solutions. *Optimization Methods and Software*, 2014.
22. D. W. Schroeder. A tutorial on pipe flow equations. Technical Report PSIG 0112, Pipeline Simulation Interest Group, 2001.
23. SCIP: Solving constraint integer programs., 2013. <http://scip.zib.de/>.
24. E. M. B. Smith and C. C. Pantelides. A symbolic reformulation/spatial branch-and-bound algorithm for the global optimization of nonconvex MINLPs. *Computers & Chemical Engineering*, 23:457–478, 1999.
25. M. Tawarmalani and N. V. Sahinidis. Global optimization of mixed-integer nonlinear programs: A theoretical and computational study. *Mathematical Programming*, 99(3):563–591, 2004.
26. M. Tawarmalani and N. V. Sahinidis. A polyhedral branch-and-cut approach to global optimization. *Mathematical Programming*, 103(2):225–249, 2005.
27. Vereinigung der Fernleitungsnetzbetreiber Gas e. V. Netzentwicklungsplan gas 2013. Konsultationsdokument der deutschen Fernleitungsnetzbetreiber. Downloaded on August 26, 2014.
28. S. Vigerske. *Decomposition in Multistage Stochastic Programming and a Constraint Integer Programming Approach to Mixed-Integer Nonlinear Programming*. PhD thesis, Humboldt-University Berlin, 2012.
29. A. Wächter and L. T. Biegler. On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106(1):25–57, 2006.
30. T. R. Weymouth. Problems in natural gas engineering. *Transactions of the American Society of Mechanical Engineers*, 34(1349):185–231, 1912.