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The Cycle Embedding Problem

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The Cycle Embedding Problem

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Abstract Given two hypergraphs, representing a fine and a coarse “layer”, and a cycle cover of the nodes of the coarse layer, the cycle embedding problem (CEP) asks for an embedding of the coarse cycles into the fine layer. The CEP is NP-hard for general hypergraphs, but it can be solved in polynomial time for graphs. We propose an integer programming formulation for the CEP that provides a complete description of the CEP polytope for the graphical case. The CEP comes up in railway vehicle rotation scheduling. We present computational results for problem instances of DB Fernverkehr AG that justify a sequential coarse-first-fine-second planning approach.

1 The Cycle Embedding Problem (CEP)

Let $G = (V, A, H)$ be a directed hypergraph with node set $V \subseteq E \times S$, i.e., a node $v = (e, s) \in V$ is a pair of an *event* $e \in E$ and *state* $s \in S$, arc set $A \subseteq V \times V$, and hyperarc set $H \subseteq 2^A$, i.e., a hyperarc consists of a set of arcs (this is different from most of the hypergraph literature). Each hyperarc $h \in H$ has cost $c_h \in \mathbb{Q}$. The following projections discard the state information:

$$\begin{aligned} [v] &:= e && \text{for } v = (e, s) \in V && ([v]^{-1} = \{w \in V \mid [w] = [v]\}), \\ [a] &:= ([u], [v]) && \text{for } a = (u, v) \in A, \\ [h] &:= \{[a_i] \mid i = 1, \dots, k\} && \text{for } h = \{a_i \mid i = 1, \dots, k\} \in H. \end{aligned}$$

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We call $G = (V, A, H)$ the *fine (composition) layer* and $[G] := ([V], [A], [H])$ with $[V] := \{[v] \mid v \in V\}$, $[A] := \{[a] \mid a \in A\}$, and $[H] := \{[h] \mid h \in H\}$ the *coarse (configuration) layer*. W.l.o.g., we assume $[V] = E$. If $A = H$ then we equate $G = (V, A, H)$ with the standard graph $G = (V, A)$.

A set $K \subseteq A$ is a *cycle packing (partition)* in G if

1. $|\delta^-(v) \cap K| = |\delta^+(v) \cap K| \leq (=) 1$, i.e., each node has at most (exactly) one incoming and at most (exactly) one outgoing arc and
2. there exists $H' \subseteq H$ such that $K = \bigcup_{h \in H'} h$ and $\forall a \in K \exists! h \in H' : a \in h$, i.e., the arc set K can be partitioned into hyperarcs; we say that $H(K) = H'$ is supported by K (there may be several supports).

K decomposes into a set of cycles C^1, \dots, C^k . Let $C \in K \subseteq A$ be a cycle in G . We denote by $l(C) = |C|$ the *length* of cycle C . These definitions carry over to cycles and sets of cycles in $[G]$. It is easy to see that a cycle packing (partition) can only support hyperarcs h with $|h \cap \delta^-(v)| \leq 1$ and $|h \cap \delta^+(v)| \leq 1$ for all $v \in V$ and we henceforth assume that every $h \in H$ satisfies this property. We say that $[h] \in [H]$ is *embedded into* $h \in H$ and $h \in H$ *embeds* $[h]$. Our aim is to embed a coarse cycle partition into the fine layer.

Definition 1. Let $M \subseteq [A]$ be a cycle partition in $[G]$. The *CEP* is to find a cost minimal cycle packing $K \subseteq A$ in G such that

1. $|[v]^{-1} \cap V(K)| = 1$ for $[v] \in [V]$, i.e., the cycle packing M visits every event in exactly one state and
2. there exist fine and coarse supports $H(K)$ and $H(M)$ such that $[H(K)] = H(M)$, i.e., every hyperarc of $H(M)$ is embedded into a hyperarc of $H(K)$.

We call **1** the *uniqueness-condition* and **2** the *embedding-condition* and refer to the data of the cycle embedding problem as $(G, H, c, [G], M)$. Note that the embedding-condition **2** implies $[K] = M$. It further follows that the decomposition of K into cycles C^1, \dots, C^k gives rise to a decomposition of cycles $[C^1], \dots, [C^k]$ for M .

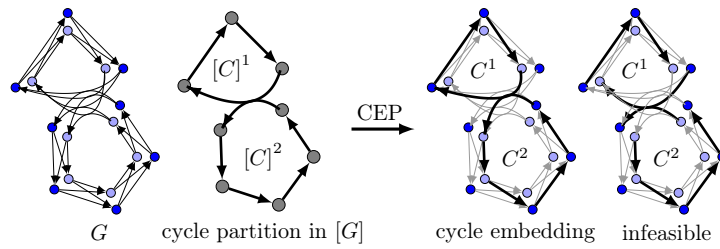


Fig. 1 Example of the cycle embedding problem. *Left:* A fine graph G and a hypercycle in $[G]$. *Right:* A feasible cycle embedding and an infeasible set of hyperarcs which does not satisfy the uniqueness-condition **1**.

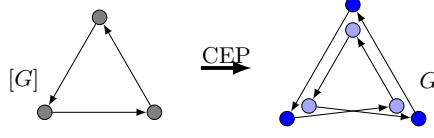


Fig. 2 (CEP) instance without an integer solution

An example for the CEP is illustrated in Figure 1. We refer the reader to the master thesis of Mehrgardt [1] for further details and for proofs of the following results. For CEPs on standard graphs (the case $A = H$) the problem can be decomposed by considering each cycle of the coarse cycle partition M individually. For each such cycle one can define a “start node”. Solving a shortest path problem for each state of the start node in the fine layer yields a polynomial time algorithm. In general, however, the problem is hard.

Theorem 1. *The cycle embedding problem can be solved in polynomial time for standard graphs; for hypergraphs, it is NP-hard.*

The CEP defined by the data $(G, H, c, [G], M)$ can be formulated as the following integer program. (Note that $h \in \delta^{+/-}(v) \Leftrightarrow \exists! a \in h : a \in \delta^{+/-}(v)$.)

$$\begin{aligned}
 \min \quad & c^T x && \text{(CEP)} \\
 \text{s.t.} \quad & \sum_{h \in \delta^-(v)} x_h - \sum_{h \in \delta^+(v)} x_h = 0, && \forall v \in V && \text{(flow)} \\
 & \sum_{h \in H: [h]=b} x_h = 1, && \forall b \in [H](M) && \text{(embedding)} \\
 & x_h \in \{0, 1\} && \forall h \in H.
 \end{aligned}$$

There is a binary variable x_h for each hyperarc $h \in H$ indicating whether all arcs of h are contained in the cycle packing K . The (embedding)-constraints together with the integrality constraints define an assignment of every hyperarc in $[H](M)$ to exactly one hyperarc in H w.r.t. the given projection. The (flow)-constraints ensure that the solution is a cycle packing. Both conditions together ensure a unique assignment of events to states, i.e., the cycle packing K visits every event in exactly one state.

Figure 2 shows an instance of the CEP in the graph case for which the LP relaxation of this IP formulation is not integral. Consider the coarse graph $[G]$ on the left of this figure. It contains exactly one cycle with three nodes; we have $M := [A]$ and $[H] = [A]$. The fine graph G is shown on the right. The only cycle in G contains six arcs while the cycle in $[G]$ contains three arcs. Therefore, there exists no feasible solution for the CEP. However, setting $x_h = \frac{1}{2}$ for all $h \in H$ is a feasible solution of the LP relaxation of (CEP). This suggests to consider some kind of length constraints. Indeed, the uniqueness-condition 1 can be reformulated as follows.

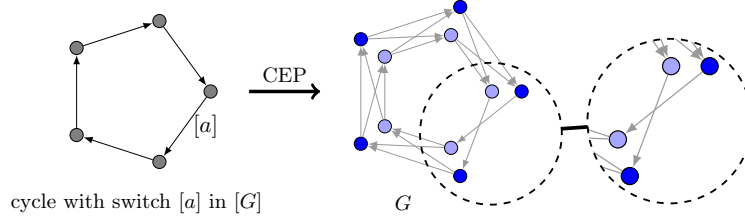


Fig. 3 Switch in a basic CEP

The uniqueness-condition 1 can be reformulated as follows.

Lemma 1. *Let K be a cycle packing in G with cycles $\{C^1, \dots, C^k\}$ and let $[K]$ be a cycle partition in $[G]$. Then*

$$|[v]^{-1} \cap V(K)| = 1 \forall [v] \in [V] \Leftrightarrow \ell([C^i]) = \ell(C^i) \forall i \in \{1, \dots, k\}. \quad (1)$$

Proof. “ \Rightarrow ” Let $[C]$ be a cycle in the coarse graph. Since the $[C]$ cover the same $l(C) = l([C])$.

“ \Leftarrow ” Suppose $[v]^{-1} \cap V(K) = \emptyset$, i.e., all $v \in [v]^{-1}$ are not covered by any hyperarc $h \in K$. Thus $[v]$ is not covered by $[K]$, which contradicts with the embedding-condition 2. Suppose $|[v]^{-1} \cap V(K)| > 1$ for one $[v]$ and $|[w]^{-1} \cap V(K)| = 1$ for all other $[w] \in [V] \setminus [v]$. Then K covers $\sum_i \ell(C_i) + 1$ nodes, but K covers $\sum_i \ell([C^i]) = \sum_i \ell(C^i)$ by assumption. \square

Using this observation, we can come up with inequalities that prohibit cycles in G with different lengths than the corresponding cycles in $[G]$. Consider for some cycle $C \in M$ the set $\mathcal{U}(C) := \{\tilde{C} \text{ cycle in } G \mid [\tilde{C}] = C, \ell(\tilde{C}) \neq \ell(C)\}$ of cycles that project to the cycle C in $[G]$ but have a different length than C . Then, the cycles in $\mathcal{U}(C)$ can be eliminated as follows:

$$\sum_{h: h \cap \tilde{C} \neq \emptyset} x_h \leq \ell(C) - 1 \quad \forall C \in M, \tilde{C} \in \mathcal{U}(C). \quad (2)$$

We call (2) *infeasible cycle constraints*. For the *basic CEP* these inequalities are all that it needed. More precisely, the CEP is a *basic CEP* if $H = A$ and $|S| = 2$, this is the simplest non-trivial problem variant.

Theorem 2. *The LP relaxation of (CEP) plus all infeasible-cycle constraints (2) provide a complete description for the basic CEP.*

W.l.o.g. we can restrict G to the set of arcs and hyperarcs which projections are contained in the cycle partition M of $[G]$. We can further assume that every node $v \in V$ of G has at least one incoming and one outgoing arc (otherwise those nodes can be deleted).

The feasibility of a basic CEP can also be characterized combinatorially in terms of *switches*. A coarse arc $[(u, v)] \in [A]$ is a *switch w.r.t. state s* if $\delta^-(v) = \{(u, v)\}$, $u = (e, \tilde{s})$, $v = (f, s)$, $\tilde{s} \neq s$, i.e., each fine cycle containing node $v = (f, s)$ has to use arc b that switches from state \tilde{s} to state s , see Figure 3. The following theorem gives a complete characterization of the feasibility of the basic CEP that is easy to check:

Theorem 3. *A basic CEP has a feasible cycle embedding if and only if every coarse cycle has a state with an even number of switches.*

Proof. “ \Rightarrow ” Let C be a cycle in G . Assume the number of switches for $[C]$ w.r.t. both states are odd. Let $v = (e, s)$ be covered by C . Traversing the cycle in forward direction until we again reaches event e has to end in state $s' \neq s$, i.e., C is not an embedding of cycle $[C]$.

“ \Leftarrow ” Let C be a cycle in G and assume the number of switches for $[C]$ for state s is even. We construct a feasible cycle embedding by stating at a node $v = (e, s) \in V$ with a state s for that the number of switches is even. We only move to nodes of different states if it is mandatory. After $|[C]|$ steps we end up at the starting node v again since we changed the state a even number of times. \square

2 Application to Rolling Stock Rotation Planning

We aim at embedding a set of cycles, representing railway vehicle rotations computed in a coarse graph layer, into a finer graph layer with a higher level of detail. Our exposition resorts to a hypergraph based model of the *rolling stock rotation problem* (RSRP) proposed in our previous paper [2]. For ease of exposition, we discuss a simplified setting without maintenance and capacity constraints. In the following we define the (RSRP), introduce aspects of vehicle composition, and show how the results of Section 1 can be utilized in a two-step approach for the RSRP. Let V a set of *nodes*, $A \subseteq V \times V$ a set of directed *standard arcs*, and $H \subseteq 2^A$ a set of *hyperarcs*, forming an RSRP *hypergraph* that we denote by $G = (V, A, H)$. The nodes represent departures and arrivals of vehicles operating a set of timetabled passenger *trips* T , the arcs represent different ways to operate a timetabled or a deadhead trip by a single vehicle, the hyperarcs represent vehicle compositions to form trains. The hyperarc $h \in H$ covers trip $t \in T$ if every standard arc $a \in h$ represents t . We denote the set of all hyperarcs that cover $t \in T$ by $H(t) \subseteq H$. There are costs associated with the hyperarcs. The RSRP is to find a cost minimal set of hyperarcs $H_0 \subseteq H$ such that each timetabled trip $t \in T$ is covered by exactly one hyperarc $h \in H_0$ and $\bigcup_{h \in H_0} h \subseteq A$ is a set of *rotations*, i.e., a packing of cycles. The RSRP is NP-hard [2]. A key concept is the *orientation*, that describes the two options ($O = \{Tick, Tack\}$) how vehicles can be placed on a railway track. Deutsche Bahn Fernverkehr AG distinguishes the position of the first class carriage of the vehicle w.r.t. the driving direction. Tick (Tack) means that the first class carriage is located at the head (tail)

of the vehicle. Every node of $v \in V$ has the form $v = (e, o)$, where e refers to an arrival or departure event of a vehicle operating a trip with an orientation $o \in \mathcal{O}$. A hyperarc $h \in H$ models the connection of the involved tail and head nodes by a set of vehicles considering technical rules w.r.t. changes of the orientation. The hyperarcs of H are distinguished into those which implement a change of orientation caused by the network topology and those which implement an additional *turn around trip* that is necessary to establish dedicated orientations. The idea of our two step approach for the RSRP is to discard the orientations of the nodes in a first coarse planning step and to subsequently solve a CEP that arises from the solution of the first step in order to arrive at a solution in the fine layer including orientations. Evaluating this two-step approach investigates the question whether the topology of a railway network offers enough degrees of freedom to plan turn around trips subordinately. Let $[G] = ([V], [A], [H])$ be the hypergraph that arises if we discard the orientation by the projection procedure as it was defined for the CEP in Section 1. If we prevent ourselves from producing infeasible cycles in the first (coarse) step we increase the chance to end up with a CEP with a feasible solution in the second (fine) step. Forbidding such cycles is the main idea of our two-step approach. Let $[C]$ be the set of all cycles in $([V], [A])$ that cannot be embedded if we consider the cycles as input for the basic CEP. Using a binary decision variable for every hyperarc $[h] \in [H]$, the model that we solve in the first step is the following coarse integer program:

$$\min \sum_{[h] \in [H]} c_{[h]} x_{[h]}, \quad (\text{MP})$$

$$\sum_{[h] \in [H](t)} x_{[h]} = 1 \quad \forall t \in T, \quad (3)$$

$$\sum_{[h] \in [H](v)^{\text{in}}} x_{[h]} - \sum_{[h] \in [H](v)^{\text{out}}} x_{[h]} = 0 \quad \forall [v] \in [V], \quad (4)$$

$$\sum_{[h] \in [H](c)} x_{[h]} \leq |[C]| - 1 \quad \forall [C] \in [C], \quad (5)$$

$$x_{[h]} \in \{0, 1\} \quad \forall [h] \in [H]. \quad (6)$$

The objective function of model (MP) minimizes the total cost. For each trip $t \in T$ the covering constraints (3) assign one hyperarc of $[H](t)$ to t . The equations (4) are flow conservation constraints for each node $[v] \in [V]$ that define a set of cycles of arcs of $[A]$. Inequalities (5) forbid all cycles of $[C]$. Finally, (6) states the integrality constraints for our decision variables. We solve model (MP) with the commercial solver for mixed integer programs `Cplex 12.4`. Since the number of inequalities (5) is exponential we handle these constraints dynamically by a separation routine that is based on an evaluation of integer solutions using Theorem 3 from the previous section, i.e., we check the switches in every cycle to see if it can be embedded or not. We evaluate this two-step approach by comparing two algorithmic variants to verify whether inequalities (5) are necessary and/or useful. The first variant “with ICS” is

instance	trips	with ICS				without ICS		
		$ H $	cuts	fine / coarse	slacks	$ H $	fine / coarse	slacks
1	267	1434	16	2 / 2	0	1434	2 / 4	199
2	617	3296	5	4 / 4	0	3292	4 / 5	208
3	617	3296	2	3 / 3	0	3292	4 / 5	14
4	617	3302	13	6 / 6	0	3292	2 / 5	285
5	617	3296	2	4 / 4	0	3294	3 / 4	29
6 a	884	4779	2	5 / 5	0	4775	5 / 6	15
7	1443	78809	6	38 / 38	1	78807	39 / 44	63
8 a	1443	29321	0	23 / 23	0	29321	23 / 23	0
9	1443	25779	2	33 / 33	1	25777	31 / 33	30
10	1443	14427	1	20 / 20	0	14421	21 / 22	12
11	1443	11738	2	17 / 17	0	11728	15 / 16	29
12	1713	14084	99	8 / 11	0	14074	1 / 11	1392
14	2319	15807	2	16 / 17	0	15787	17 / 17	43
15	2421	15829	4	18 / 19	0	15789	16 / 18	61
16	3101	40707	4240	40 / 42	74	40705	48 / 59	259
17	15	80	1	2 / 2	0	76	- / 1	15

Table 1 Comparison between embedding results of CEP models with or without separation of cycles that can not be embedded.

to solve the model [MP](#) as described and to solve the arising CEP in the second step. The only difference to the second variant “without ICS” is that we solve the model without constraints [\(5\)](#). [Table 1](#) reports computational results for 16 instances with different numbers of trips (second column). Column “cuts” denotes the number of constraints [\(5\)](#) that were separated in the first step, while column “ $|H|$ ” denotes the number of hyperarcs that appeared in the CEP of the second step. The columns “fine / coarse” and “slacks” report about the number of cycles that could be embedded vs. the number of cycles that were given to the CEP of the second step as well as the number of trips that could not be covered by the cycle packing in G . The instances were created to provoke that the arising CEPs are infeasible. Namely, we increased the time necessary to perform a turn around trip by approximately ten times the times in the real world data. This worst-case scenario substantially constrains the number of possible turn around trips. Our computational results show that by utilizing constraints [\(5\)](#) the RSRP can still be tackled by a two-step approach to produce feasible solutions. This gives evidence that railway vehicle rotation planning can indeed be done in two steps, embedding a coarse initial solution into a fine model layer.

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