

A Configuration Model for the Line Planning Problem

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Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Berlin, den 10. Juni 2014

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Zusammenfassung

Das Linienplanungsproblem ist ein wichtiges Teilproblem der Angebotsplanung im öffentlichen Nahverkehr. Dabei werden Routen und Betriebsfrequenzen von Linien in einem gegebenen Infrastrukturnetzwerk gesucht unter der Voraussetzung, dass ein gegebener Beförderungsbedarf gedeckt wird und die Kosten minimal sind. Aufgrund von Anforderungen in realen Anwendungen steht nur eine diskrete Menge an möglichen Frequenzen für die Linien zur Verfügung. Praktische Ansätze zum Lösen dieser Probleme beruhen auf ganzzahliger Programmierung. Ein Schwachpunkt klassischer Ansätze ist, dass schon kleine Änderungen der Eingabeparameter eine Vergrößerung oder Verkleinerung der Menge der zulässigen fraktionalen Lösungen bewirken können, auch wenn die Menge der ganzzahligen Lösungen unverändert bleibt. Wir betrachten in dieser Arbeit einen neuen kombinatorischen Ansatz, welcher die Möglichkeiten den Bedarf auf einer Verbindung im Netzwerk mit Frequenzen zu überdecken mit Hilfe von diskreten "Konfigurationen" beschreibt und dieses Problem eliminiert. Um die Vorteile dieses Konfigurationsmodells aufzuzeigen, vergleichen wir es mit einem klassischen Ansatz, den wir Standardmodell nennen. Wir zeigen, dass das Konfigurationsmodell eine schärfere LP-Relaxierung als das Standardmodell besitzt. Weitere polyedrische Untersuchungen ergeben, dass die LP-Relaxierung des Konfigurationsmodells mehrere facettendefinierende Ungleichungen des Standardmodells impliziert. Unsere Rechenergebnisse bestätigen ebenfalls, dass wir das Standardmodell durch die Einführung von Konfigurationsvariablen verbessern. Um die Anzahl dieser Variablen auch für große Instanzen zu beschränken, betrachten wir ein weiteres Modell, welches nur eine Teilmenge der Konfigurationsvariablen enthält. Dieses stellt sich als ein sehr guter Kompromiss heraus und ist den anderen Modellen vor allem auf Instanzen, welche auf realen Personennahverkehrsnetzwerken basieren, überlegen. Durch die Verwendung von Preprocessing-Techniken und Heuristiken können wir den Lösungsprozess zudem beschleunigen beziehungsweise die Qualität der gefundenen Lösungen verbessern. Da das Linienplanungsproblem eine Spezialisierung des Netz-

werkdesignproblems ist, können unsere Methoden auch auf andere Problemklassen übertragen werden.

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Chapter 1

Introduction

1.1 Motivation and Outline

Numerous infrastructure planning problems such as telecommunication network design and line planning in public transportation are specializations of the *discrete capacitated network design problem*: The task is to provide sufficient capacity on the links of a given network such that a certain demand can be routed without exceeding the installed capacity and such that the costs are minimal. For most of these problems there exists no polynomial time algorithm. Due to restrictions in real world applications the set of available capacities is often assumed to be discrete, which makes finding an optimal solution more difficult. One weakness of classical integer programming models presented in the literature is the following: Small changes in the input data can cause an enlargement of the set of fractional solutions even though the set of feasible integral solutions remains unchanged. In this thesis we present a novel combinatorial approach that addresses this problem by modeling the limited number of possible combinations of installed capacities by so called *configurations*. We focus on the line planning problem and compare the resulting *configuration model* both in theory and in practice with a classical approach to demonstrate its strength. We remark that configuration models have also been used successfully in railway vehicle rotation planning [7] and railway track allocation applications [8].

The strategic planning process in public transportation usually consists of several consecutive stages: network design, line planning, timetabling and vehicle and crew scheduling, for an overview see for instance Bussieck, Winter and Zimmermann [13]. We focus on the *line planning problem* (LPP) in

this thesis. The problem is to find a set of lines defined by their paths and frequencies in a public transportation network such that a given travel demand can be routed and a certain objective is minimized. In the following section we give a brief review on the LPP literature and list assumptions on the problem setting we make throughout this thesis. All models in the literature employ some type of capacity or frequency demand constraints in order to cover a given demand. In this thesis we discuss a concept to strengthen such constraints by means of a novel extended formulation. The idea is to enumerate the set of possible configurations of line frequencies for each capacity constraint. In Chapter 2 the line planning problem is formulated as an integer program, the *standard model*, and the associated polytope is investigated. The *extended configuration model* is introduced in Chapter 3. We compare it with the standard model in terms of polyhedral aspects and show that it implies facet-defining inequalities for the standard model. In Chapter 4 we report on our computational studies to compare the models. We close with some final remarks in Chapter 5.

We remark that both the standard model and the configuration model have also been described by Meirich in [21]. His polyhedral investigations refer solely to the standard model. We deliver an elaborate comparison of the two models in terms of polyhedral aspects and computational studies and develop several new classes of valid inequalities for both models in this thesis.

The results in this thesis have been developed within the project “Service Design in Public Transport” supported by the DFG Research Center MATH-EON *Mathematics for key technologies* at the Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB). Parts of this thesis have been published by Borndörfer, Hoppmann and Karbstein in [5], which won the *Best Paper Award* at the 13th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2013).

1.2 Preliminaries and Literature Overview

The links in the LPP infrastructure network correspond to different modes of transportation, e.g., streets (bus) or tracks (railway, tram, subway). The passenger demand data is generally given by a so-called *origin-destination matrix (OD-matrix)*; the OD-matrix gives for each pair of stations in the network the number of passengers that want to travel from one station to the other within a fixed *time horizon*, e.g., one hour. A common approach in the literature to deal with multi-modal transportation systems is based

on the so-called *system split* procedure, see Oltrogge [22]. It is assumed that passengers change to faster train types as early as possible and leave them as late as possible in order to minimize their traveling time. With the additional assumption that passengers travel on shortest paths, one can determine the traveling paths of the passengers and, hence, the number of passengers for each link in the network before the line plan is known. Therefore, we can decompose the LPP into separate problems for each transportation mode. Moreover, we are given a set of possible lines, which correspond to paths in the transportation network. We assume that all lines are operated in both directions. There are usually two main competing objectives. One is to minimize the induced operational costs of the line plan and the other is to minimize the passenger inconvenience, usually measured by the total passenger traveling time or the number of transfers. In this thesis we focus on a model that aims at minimizing the operational costs and is based on system split.

Since the late nineteen-nineties, the line planning literature has developed a variety of integer programming approaches that capture different aspects. Schöbel [24] gives an overview of the different models and mathematical approaches in the line planning literature.

Cost-oriented models. Serving as a basis for many following publications, Claessens, van Dijk and Kroon discussed in [15] a nonlinear programming model to minimize operational costs subject to service constraints and capacity requirements, including line types and train lengths in terms of numbers of cars. Their model is transformed into an integer linear program by introducing binary variables, which is solved by a branch-and-bound approach. Claessens et al. [15] showed that the problem of finding a cost optimal line plan subject to lower frequency bounds on the edges of the network is \mathcal{NP} -hard. Bussieck developed in his thesis [10] a branch-and-bound algorithm to compare the linearization proposed by Claessens et al. [15] with a different linearization of the same nonlinear formulation using general integer variables. His computational studies show that his linearization provides better upper bounds, while the linearization of Claessens et al. seems to be superior in generating lower bounds in order to prove optimality. Bussieck also proved that the problem of finding a line plan with fixed passenger routes that fulfills lower and upper frequency requirements for each edge is \mathcal{NP} -hard. Goosens, van Hoesel, and Kroon [17] propose a branch-and-cut approach based on a model similar to the linearization of Claessens et al. They do not consider upper frequency requirements and develop several valid inequalities for their

model. In [12] Bussieck, Lindner, and Lübbecke propose a fast procedure to find cost optimal line plans. They use the nonlinear formulation of Claessens et al. to determine a set of promising lines and solve the linear model only for these lines to obtain a good primal solution within a rather small computation time.

Passenger convenience models. Bussieck, Kreuzer, and Zimmermann [11] (see also the thesis of Bussieck [10]) propose an integer programming model to maximize the number of direct travelers. Schöbel and Scholl [25] presented the first model that focuses on minimizing the travel time and includes penalties for transfers. Their approach includes routing of passengers, i.e., it does not rely on an assignment of passengers to paths a priori. Borndörfer, Grötschel and Pfetsch [4] also propose an integrated line planning and passenger routing model that additionally allows to generate lines dynamically. They propose a multicommodity flow model that minimizes a combination of the total traveling time and operational costs, which they solve by a column-generation approach. Borndörfer and Karbstein [6, 20] develop a direct connection approach and further integrated line planning and passenger routing in their model. Karbstein compares in her thesis [20] the direct connection approach with the approaches of Borndörfer et al. [4] and of Schöbel and Scholl [25]. She concludes that the direct connection approach is currently the only computationally tractable integrated line planning and passenger routing method that provides good estimates of transfer times.

1.3 Notation

Throughout this thesis it is assumed that the reader is familiar with the fundamental concepts of graph theory, polyhedral theory, and linear and integer programming. For a detailed introduction we refer to the books [26, 18]; we follow the notation of [18]. Whenever needed further notation is introduced.

We will consider frequently the polyhedra associated with integer programs. For an integer program $\text{IP} = \min\{c^T x : Ax \geq b, x \in \mathbb{Z}^n\}$ we denote by $P(\text{IP})$ the polyhedron defined by the convex hull of all feasible solutions of IP and by $P_{LP}(\text{IP})$ the set of feasible solutions of the LP relaxation of IP, i.e., $P(\text{IP}) = \text{conv}\{x \in \mathbb{Z}^n : Ax \geq b\}$ and $P_{LP}(\text{IP}) = \{x \in \mathbb{R}^n : Ax \geq b\}$. For a polyhedron $P = \{(x, y) \in \mathbb{R}^{n+m} : Ax + By \geq b\}$ we denote by $P|_x :=$

$\{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ s.t. } (x, y) \in P\}$ the projection of P onto the space of x -variables.

In this thesis we resort for the line planning problem to the following notation. We are given an undirected graph $G = (V, E)$ representing the *transportation network*. A *line* is a simple path in G and by $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$, $n \in \mathbb{N}$, we denote the set of predefined lines. For an edge $e \in E$ let $\mathcal{L}(e) := \{\ell \in \mathcal{L} : e \in \ell\}$ be the set of lines that contain e . Furthermore, we are given an ordered set of available *frequencies* $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq \mathbb{N}$, $m \in \mathbb{N}$, such that $0 < f_1 < \dots < f_m$. All lines are of the same transportation mode and have capacity $\kappa > 0$, i.e., κ passengers can be transported by any line. The number of passengers traveling on an edge e is given by the *transportation demand* $d(e) \in \mathbb{N}$. Hence, at least $\left\lceil \frac{d(e)}{\kappa} \right\rceil$ lines have to pass edge e to cover its demand. We call this value the *frequency demand of e* and denote it by $F(e) \in \mathbb{N}$. The cost of operating line $\ell \in \mathcal{L}$ at frequency $f \in \mathcal{F}$ is given by $c_{\ell, f} \in \mathbb{Q}_+$. A *line plan* $(\bar{\mathcal{L}}, \bar{f})$ consists of a subset $\bar{\mathcal{L}} \subseteq \mathcal{L}$ of lines and an assignment $\bar{f} : \bar{\mathcal{L}} \rightarrow \mathcal{F}$ of frequencies to these lines. A line plan is *feasible* if the frequencies of its lines ensure that all passengers that travel on each edge can be transported, that is, the line plan satisfies the frequency demand requirement

$$\sum_{\ell \in \bar{\mathcal{L}}(e)} \bar{f}(\ell) \geq F(e) \tag{1.1}$$

for each edge $e \in E$ in the network. We define the cost of a line plan $(\bar{\mathcal{L}}, \bar{f})$ as $c(\bar{\mathcal{L}}, \bar{f}) := \sum_{\ell \in \bar{\mathcal{L}}} c_{\ell, \bar{f}(\ell)}$. The *line planning problem* is to find a feasible line plan of minimal cost.

Chapter 2

Standard Model

In this chapter we investigate the so-called standard model for the line planning problem. We are given a transportation network, a set of available frequencies, and a set of lines with their operational costs. Further, a transportation demand is given for each edge. The goal is to determine a cost optimal line plan, i.e., a set of lines that covers the frequency demand of each edge and minimizes the total operating costs. We introduce an integer programming formulation for this problem and investigate the associated polyhedron. We will show that this formulation can be strengthened by band inequalities, which were introduced in the context of network design, and the mixed integer rounding technique. Moreover, we present a new class of improving inequalities for the line planning problem and show that it contains a knapsack substructure.

After describing the line planning problem in mathematical terms, we introduce an integer programming model for this problem in Section 2.1. We investigate the polyhedron associated with this model and analyze several classes of valid inequalities in Section 2.2. In Section 2.3 we discuss the relation of the presented model with two others from the literature.

2.1 Integer Programming Formulation for the Standard Model

The line planning problem permits an intuitive integer programming formulation. It is based on the natural idea to introduce binary variables $x_{\ell,f}$ that

indicate whether line $\ell \in \mathcal{L}$ is operated at frequency $f \in \mathcal{F}$. The resulting *standard model* is the following:

$$\begin{aligned}
 (\text{SLP}) \quad & \min && \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f} \\
 & \text{s.t.} && \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \geq F(e) \quad \forall e \in E \quad (2.1)
 \end{aligned}$$

$$\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L} \quad (2.2)$$

$$x_{\ell,f} \in \{0, 1\} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F}. \quad (2.3)$$

Model (SLP) minimizes the cost of a line plan. The *frequency demand constraints* (2.1) ensure that the frequency demand is covered on each edge in the network. The *assignment constraints* (2.2) make sure that every line is operated with at most one frequency. Hence, the feasible solutions of (SLP) correspond to the set of feasible line plans.

Example 2.1. Consider the line planning problem defined in Figure 2.1. The standard model for this example reads as follows:

$$\begin{aligned}
 (\text{SLP}) \quad & \min && 4x_{\ell_1,2} + 16x_{\ell_1,8} + 4x_{\ell_2,2} + 16x_{\ell_2,8} + 2x_{\ell_3,2} + 8x_{\ell_3,8} \\
 & \text{s.t.} && 2x_{\ell_1,2} + 8x_{\ell_1,8} + 2x_{\ell_2,2} + 8x_{\ell_2,8} + 2x_{\ell_3,2} + 8x_{\ell_3,8} \geq 9 \\
 & && 2x_{\ell_1,2} + 8x_{\ell_1,8} + 2x_{\ell_2,2} + 8x_{\ell_2,8} \geq 1 \\
 & && x_{\ell_1,2} + x_{\ell_1,8} \leq 1 \\
 & && \quad + x_{\ell_2,2} + x_{\ell_2,8} \leq 1 \\
 & && \quad \quad + x_{\ell_3,2} + x_{\ell_3,8} \leq 1 \\
 & && x_{\ell_i,f} \in \{0, 1\}.
 \end{aligned}$$

An optimal solution to (SLP) is given by x^* , where $x_{\ell_3,8}^* = 1$, $x_{\ell_2,2}^* = 1$, and $x_{\ell,f}^* = 0$ otherwise, with an objective function value of 12.

2.2 Polyhedral Aspects

In this section we focus on the *standard line planning polytope* $P(\text{SLP})$, which corresponds to the convex hull of all feasible solutions of (SLP). We investigate its basic properties in Section 2.2.1. In Section 2.2.2 we derive band

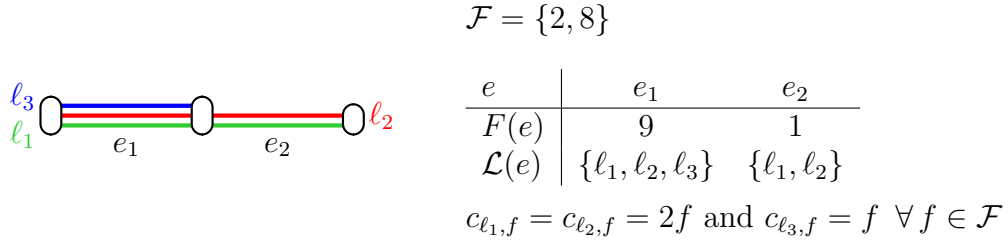


Figure 2.1: An instance of the line planning problem. *Left*: A transportation network consisting of two edges and three lines. *Right*: A given set of frequencies, frequency demands, and line costs.

inequalities for the line planning problem as a class of facet-defining inequalities. In Section 2.2.3 we study the mixed integer rounding procedure to derive a class of strong valid inequalities. We consider several classes of cuts that can be derived by considering subsets of edges in Section 2.2.4. Finally, in Section 2.2.5 we introduce a relaxation of $P(\text{SLP})$ which we obtain by considering a single edge.

2.2.1 Basic Properties of the Line Planning Polytope

In this section we analyze the dimension and trivial facets of the polytope $P(\text{SLP})$.

First we determine the dimension of $P(\text{SLP})$. The polytope $P(\text{SLP})$ is full-dimensional if and only if for each line there is a feasible line plan in which this line is not operated, which we show in the following proposition. The proof is an adaption of a proof by Karbstein [20] for the dimension of the line planning polytope with integrated passenger routing. We compare this polytope with the polytope $P(\text{SLP})$ at the end of this chapter in Section 2.3.2.

Proposition 2.1. *The standard line planning polytope is full-dimensional, i.e.,*

$$\dim(P(\text{SLP})) = |\mathcal{L}| \cdot |\mathcal{F}|$$

if and only if the polytope

$$P_\ell := \{x \in P(\text{SLP}) : x_{\ell, f} = 0 \quad \forall f \in \mathcal{F}\}$$

is non-empty for all $\ell \in \mathcal{L}$.

Proof. In order to prove the proposition, we use the property that $P(\text{SLP})$ is full-dimensional if and only if there is no non-trivial linear equality in $\mathbb{R}^{\mathcal{L} \times \mathcal{F}}$ satisfied by all points in $P(\text{SLP})$.

“ \Rightarrow ” Let $P(\text{SLP})$ be full-dimensional. Assume P_ℓ is empty for some line $\ell \in \mathcal{L}$. This implies that $\sum_{f \in \mathcal{F}} x_{\ell,f} = 1$ holds for every point x in $P(\text{SLP})$. This is a contradiction, since $P(\text{SLP})$ is full-dimensional. Hence, P_ℓ is non-empty for all $\ell \in \mathcal{L}$.

“ \Leftarrow ” Now, let P_ℓ be non-empty for every $\ell \in \mathcal{L}$. Assume there exists an equality $a^T x = \lambda$ with non-zero coefficient vector $a \in \mathbb{R}^{\mathcal{L} \times \mathcal{F}}$ and $\lambda \in \mathbb{R}$, which is valid for all x in $P(\text{SLP})$. Let $\tilde{\ell} \in \mathcal{L}$ be an arbitrary line and $x^{\tilde{\ell}} \in P_{\tilde{\ell}}$, i.e., $x_{\tilde{\ell},f}^{\tilde{\ell}} = 0$ for all $f \in \mathcal{F}$. Let $\tilde{f} \in \mathcal{F}$ be an arbitrary frequency and define $e^{\tilde{\ell},\tilde{f}}$ as the unit vector in $\mathbb{R}^{\mathcal{L} \times \mathcal{F}}$, where $e_{\tilde{\ell},\tilde{f}}^{\tilde{\ell},\tilde{f}} = 1$ and $e_{\tilde{\ell},f}^{\tilde{\ell},\tilde{f}} = 0$ otherwise. Since $x^{\tilde{\ell}} \in P(\text{SLP})$ and $x_{\tilde{\ell},f}^{\tilde{\ell}} = 0$ for all $f \in \mathcal{F}$, it follows $x^{\tilde{\ell}} + e^{\tilde{\ell},\tilde{f}} \in P(\text{SLP})$ and hence $a^T x^{\tilde{\ell}} = \lambda = a^T (x^{\tilde{\ell}} + e^{\tilde{\ell},\tilde{f}})$. This implies $a_{\tilde{\ell},\tilde{f}} = 0$. Since we chose $\tilde{\ell}$ and \tilde{f} arbitrarily, we can derive $a = 0$. This contradicts our assumption and thus the polyhedron $P(\text{SLP})$ is full-dimensional. \square

Remark 2.2. From now on, we assume that $P(\text{SLP})$ is full-dimensional and in particular non-empty. Proposition 2.1 directly implies that for every edge $e \in E$ and every line $\ell \in \mathcal{L}(e)$

$$\sum_{\ell \in \mathcal{L}(e) \setminus \{\tilde{\ell}\}} f_m = (|\mathcal{L}(e)| - 1) f_m \geq F(e)$$

holds. This means that setting the frequency of all lines except one to the maximum frequency f_m and setting the frequency of the remaining line to an arbitrary frequency or zero yields a feasible line plan.

Next we will investigate the trivial facets of $P(\text{SLP})$. In particular, we state under which conditions the assignment constraints (2.2) and non-negativity inequalities implied by the integrality constraints (2.3) are facet-defining. The proof of the following proposition follows the corresponding proof presented by Meirich [21] and is given for the sake of completeness.

Proposition 2.3. *For every $\ell \in \mathcal{L}$ the inequality $\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1$ is facet-defining for $P(\text{SLP})$.*

Proof. Let $\ell' \in \mathcal{L}$ be an arbitrary line. We prove the proposition by giving $|\mathcal{L}| \cdot |\mathcal{F}|$ affinely independent points in $P(\text{SLP})$ that satisfy $\sum_{f \in \mathcal{F}} x_{\ell',f} \leq 1$ with equality:

x^{f_m} , which corresponds to operating all lines at the highest frequency, i.e.,

$$x_{\ell',f}^{f_m} = \begin{cases} 1 & \text{if } f = f_m \\ 0 & \text{else} \end{cases} \quad \forall \ell' \in \mathcal{L}, \forall f \in \mathcal{F},$$

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$x^{\tilde{\ell}, \tilde{f}}$ for all $\tilde{\ell} \in \mathcal{L}$ and $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$, which corresponds to operating line $\tilde{\ell}$ at frequency \tilde{f} and all other lines at the highest frequency, i.e.,

$$x_{\ell, f}^{\tilde{\ell}, \tilde{f}} = \begin{cases} 1 & \text{if } \ell \neq \tilde{\ell}, f = f_m \\ 1 & \text{if } \ell = \tilde{\ell}, f = \tilde{f} \\ 0 & \text{else} \end{cases} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F},$$

$x^{\tilde{\ell}, 0}$ for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$ which corresponds to not operating $\tilde{\ell}$ and operating all other lines at the highest frequency, i.e.,

$$x_{\ell, f}^{\tilde{\ell}, 0} = \begin{cases} 1 & \text{if } \ell \neq \tilde{\ell}, f = f_m \\ 0 & \text{else} \end{cases} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F}.$$

We defined in total $1 + |\mathcal{L}| \cdot (|\mathcal{F}| - 1) + |\mathcal{L}| - 1 = |\mathcal{L}| \cdot |\mathcal{F}|$ points. They satisfy by definition $\sum_{f \in \mathcal{F}} x_{\ell, f} = 1$ and the assignment constraints (2.2). We also observe that they satisfy the frequency demand constraints (2.1), since we assume that $P(\text{SLP})$ is full-dimensional, see Remark 2.2. Hence they are feasible for $P(\text{SLP})$ and it is left to show that they are affinely independent.

Assume they are not affinely independent. Then there is a vector $\lambda \neq 0$ such that

$$\begin{aligned} \lambda^{f_m} x_{\ell, f}^{f_m} + \sum_{\tilde{\ell} \in \mathcal{L}} \sum_{\tilde{f} \in \mathcal{F} \setminus \{f_m\}} \lambda^{\tilde{\ell}, \tilde{f}} x_{\ell, f}^{\tilde{\ell}, \tilde{f}} + \sum_{\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}} \lambda^{\tilde{\ell}, 0} x_{\ell, f}^{\tilde{\ell}, 0} &= 0 \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \\ \text{and } \lambda^{f_m} + \sum_{\tilde{\ell} \in \mathcal{L}} \sum_{\tilde{f} \in \mathcal{F} \setminus \{f_m\}} \lambda^{\tilde{\ell}, \tilde{f}} + \sum_{\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}} \lambda^{\tilde{\ell}, 0} &= 0. \end{aligned}$$

Let $\tilde{\ell} \in \mathcal{L}$ and $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$, then $x_{\tilde{\ell}, \tilde{f}}^{\tilde{\ell}, \tilde{f}} = 1$ and for every other defined point the entry at position $\tilde{\ell}, \tilde{f}$ is equal to zero. Hence $\lambda^{\tilde{\ell}, \tilde{f}} = 0$ for all $\tilde{\ell} \in \mathcal{L}$ and $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$. This implies

$$\begin{aligned} \lambda^{f_m} x_{\ell, f}^{f_m} + \sum_{\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}} \lambda^{\tilde{\ell}, 0} x_{\ell, f}^{\tilde{\ell}, 0} &= 0 \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \\ \text{and } \lambda^{f_m} + \sum_{\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}} \lambda^{\tilde{\ell}, 0} &= 0. \end{aligned}$$

When we consider the entries of these points at position ℓ, f_m for all $\ell \in \mathcal{L}$, we can derive the following:

$$\begin{aligned} \lambda^{f_m} + \sum_{\tilde{\ell} \in \mathcal{L} \setminus \{\ell, \ell'\}} \lambda^{\tilde{\ell}, 0} &= 0 \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \\ \text{and } \lambda^{f_m} + \sum_{\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}} \lambda^{\tilde{\ell}, 0} &= 0. \end{aligned}$$

We observe that this implies $\lambda = 0$. This is a contradiction to our assumption $\lambda \neq 0$. Hence, the above defined points are affinely independent. \square

Next, we show that the non-negativity inequality $x_{\ell,f} \geq 0$ for a line $\ell \in \mathcal{L}$ and frequency $f \in \mathcal{F}$ defines a facet if and only if for each other line $\tilde{\ell} \in \mathcal{L} \setminus \{\ell\}$ there exists a line plan such that ℓ is not operated at frequency f and $\tilde{\ell}$ is not operated at all. We remark that Meirich [21] also presents a proof to show that $x_{\ell,f} \geq 0$ is facet-defining for line ℓ and frequency f , but he does not define enough affinely independent points in $P(\text{SLP})$ satisfying $x_{\ell,f} = 0$. Furthermore, his proof requires that for every two lines $\ell_1, \ell_2 \in \mathcal{L}$ there exists a point in $P(\text{SLP})$ satisfying $x_{\ell_1, \tilde{f}} = x_{\ell_2, \tilde{f}} = 0$ for all $\tilde{f} \in \mathcal{F}$. We provide a more general proof, for which it suffices to require that for every line $\tilde{\ell} \in \mathcal{L}$ there exists a point in $P(\text{SLP})$ such that $x_{\ell,f} = 0$ and $x_{\tilde{\ell}, \tilde{f}} = 0$ for all $\tilde{f} \in \mathcal{F}$.

Proposition 2.4. *For every line $\ell \in \mathcal{L}$ and frequency $f \in \mathcal{F}$ the inequality $x_{\ell,f} \geq 0$ is facet-defining for $P(\text{SLP})$ if and only if the polyhedron*

$$P_{\ell,f}^{\tilde{\ell}} := \{x \in P(\text{SLP}) : x_{\ell,f} = 0 \text{ and } x_{\tilde{\ell}, \tilde{f}} = 0 \forall \tilde{f} \in \mathcal{F}\}$$

is non-empty for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell\}$.

Proof. “ \Leftarrow ” Let $\ell' \in \mathcal{L}$, $f' \in \mathcal{F}$ such that $P_{\ell',f'}^{\tilde{\ell}} \neq \emptyset$ for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$. We prove the proposition by giving $|\mathcal{L}| \cdot |\mathcal{F}|$ affinely independent points in $P(\text{SLP})$ that satisfy $x_{\ell',f'} = 0$. In order to define these points, we distinguish between two cases.

1. case: $f' \neq f_m$: For this case, let x^{f_m} , $x^{\tilde{\ell}, \tilde{f}}$ for all $\tilde{\ell} \in \mathcal{L}$, $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$ and $x^{\tilde{\ell}, 0}$ for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$ be as in the proof of Proposition 2.3 with one exception. We replace $x^{\ell', f'}$ by $x^{\ell', 0}$, which is defined by

$$x_{\ell', f'}^{\ell', 0} = \begin{cases} 1 & \text{if } \ell \neq \ell', f = f_m \\ 0 & \text{else} \end{cases} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F}.$$

2. case: $f' = f_m$: We have to differentiate between two cases again.

- If $|\mathcal{F}| = 1$, i.e., $\mathcal{F} = \{f_m\}$, then define the following points:

$x^{\tilde{\ell}, \ell', 0}$ for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$ be defined by

$$x_{\ell', f_m}^{\tilde{\ell}, \ell', 0} = \begin{cases} 1 & \text{if } \ell \notin \{\ell', \tilde{\ell}\} \\ 0 & \text{else} \end{cases} \quad \forall \ell \in \mathcal{L}$$

and $x^{\ell',0}$ defined by

$$x_{\ell',f_m}^{\ell',0} = \begin{cases} 1 & \text{if } \ell \neq \ell' \\ 0 & \text{else} \end{cases} \quad \forall \ell \in \mathcal{L}.$$

For this case, we defined in total $|\mathcal{L}| = |\mathcal{L}| \cdot |\mathcal{F}|$ points.

- Otherwise, if $|\mathcal{F}| \geq 2$, we can define the following points:

$x^{\tilde{\ell},\tilde{f}}$ for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$, $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$, which corresponds to operating ℓ' at the second highest frequency, $\tilde{\ell}$ at \tilde{f} , and all other lines at the highest frequency, i.e.,

$$x_{\ell',f}^{\tilde{\ell},\tilde{f}} = \begin{cases} 1 & \text{if } \ell \notin \{\ell', \tilde{\ell}\}, f = f_m \\ 1 & \text{if } \ell = \ell', f = f_{m-1} \\ 1 & \text{if } \ell = \tilde{\ell}, f = \tilde{f} \\ 0 & \text{else,} \end{cases} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F},$$

$x^{\tilde{\ell},0}$ for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$, which corresponds to operating ℓ' at the second highest frequency, not operating $\tilde{\ell}$, and operating all other lines at the highest frequency, i.e.,

$$x_{\ell',f}^{\tilde{\ell},0} = \begin{cases} 1 & \text{if } \ell \notin \{\ell', \tilde{\ell}\}, f = f_m \\ 1 & \text{if } \ell = \ell', f = f_{m-1} \\ 0 & \text{else,} \end{cases} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F},$$

$x^{\ell',\tilde{f}}$ for all $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$, which corresponds to operating ℓ' at \tilde{f} and all other lines at the highest frequency, i.e.,

$$x_{\ell',f}^{\ell',\tilde{f}} = \begin{cases} 1 & \text{if } \ell \neq \ell', f = f_m \\ 1 & \text{if } \ell = \ell', f = \tilde{f} \\ 0 & \text{else} \end{cases} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F},$$

$x^{\ell',0}$, which corresponds to not operating ℓ' and operating all other lines at the highest frequency, i.e.,

$$x_{\ell',f}^{\ell',0} = \begin{cases} 1 & \text{if } \ell \neq \ell', f = f_m \\ 0 & \text{else,} \end{cases} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F}.$$

For this case, we defined in total $(|\mathcal{L}| - 1)(|\mathcal{F}| - 1) + |\mathcal{L}| - 1 + |\mathcal{F}| - 1 + 1 = |\mathcal{L}| \cdot |\mathcal{F}|$ points.

We found for every case $|\mathcal{L}| \cdot |\mathcal{F}|$ points. They satisfy by definition $x_{\ell', f'} = 0$ and are in $P(\text{SLP})$, since $P_{\ell', f'}^{\tilde{\ell}} \neq \emptyset$ for all $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$. The affine independence can be shown in all cases analogously to the proof of Proposition 2.3.

“ \Rightarrow ” Now let $\ell' \in \mathcal{L}$, $f' \in \mathcal{F}$ such that there is a line $\tilde{\ell} \in \mathcal{L} \setminus \{\ell'\}$ with $P_{\ell', f'}^{\tilde{\ell}} = \emptyset$. Thus, either the polytope $P_{\ell', f'} := \{x \in P(\text{SLP}) : x_{\ell', f'} = 0\}$ is empty or the equality $\sum_{\tilde{f} \in \mathcal{F}} x_{\tilde{\ell}, \tilde{f}} = 1$ holds for every $x \in P_{\ell', f'}$. In both cases, we have $\dim(P_{\ell', f'}) < |\mathcal{L}| \cdot |\mathcal{F}|$. This implies that $x_{\ell', f'} \geq 0$ is not facet-defining. \square

2.2.2 Band Inequalities

In this section we study *band inequalities*, which were proposed by Stoer and Dahl in [30] in the context of telecommunication network design. They introduced this class of valid inequalities in order to tighten the LP relaxation of their formulation by exploiting a knapsack substructure. The inequalities are closely related to the cover inequalities for the knapsack polytope with GUB constraints, see Wolsey [32]. We will show that the line planning problem has a knapsack substructure late in this chapter in Section 2.2.5.

We adapt the definition of a *band* to our case.

Definition 2.5. Let $e \in E$. A *band* $f_{\mathcal{B}} : \mathcal{L}(e) \rightarrow \mathcal{F} \cup \{0\}$ assigns to each line containing e a frequency or zero. We call $f_{\mathcal{B}}$ a *valid band of e* if

$$\sum_{\ell \in \mathcal{L}(e)} f_{\mathcal{B}}(\ell) < F(e).$$

We call the band $f_{\mathcal{B}}$ *maximal* if $f_{\mathcal{B}}$ is valid and there is no valid band $f_{\mathcal{B}'}$ with $f_{\mathcal{B}}(\ell) \leq f_{\mathcal{B}'}(\ell)$ for every line $\ell \in \mathcal{L}(e)$ and $f_{\mathcal{B}}(\ell) < f_{\mathcal{B}'}(\ell)$ for at least one line $\ell \in \mathcal{L}(e)$.

A valid band can be interpreted as follows. If all lines on an edge are operated at the frequencies of the band, then the frequency demand of this edge is not covered. Hence, at least one line needs to be operated at a higher frequency. This leads to the following proposition.

Proposition 2.6. Let $f_{\mathcal{B}}$ be a valid band of $e \in E$, then

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > f_{\mathcal{B}}(\ell)}} x_{\ell, f} \geq 1 \quad (2.4)$$

is a valid inequality for $P(\text{SLP})$.

It is easy to see that all non-redundant band inequalities (2.4) are induced by maximal bands. Band inequalities play a special role for the polytope $P(\text{SLP}|e)$, which results from reducing the line planning problem to a single edge $e \in E$. We will give a formal definition of $P(\text{SLP}|e)$ in Section 2.2.5. Applying the results of Stoer and Dahl [30] to our setting yields:

Proposition 2.7. *Let $e \in E$ and let $f_{\mathcal{B}}$ be a valid band of e . Let $|\mathcal{L}(e)| \geq 2$ and $f_m \geq F(e)$, i.e., operating any line in $\mathcal{L}(e)$ at the highest frequency covers the frequency demand of e . Then the band inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > f_{\mathcal{B}}(\ell)}} x_{\ell,f} \geq 1$$

is facet-defining for $P(\text{SLP}|e)$ if and only if $f_{\mathcal{B}}$ is maximal.

If $f_m < F(e)$, then the statement of Proposition 2.7 does not hold. Karbstein [20] gives an example, with $f_m < F(e)$, of a band inequality for a maximal band that is not facet-defining. She also shows that $f_m \geq F(e)$ is not a necessary requirement for a maximal band to induce a facet-defining inequality. An example for a facet-defining band inequality is given in Example 2.2.

Since we assume that every edge has a positive demand, there must be at least one line operated on every edge in any feasible line plan. This observation corresponds to the band inequalities implied by the valid band $f_{\mathcal{B}}(\ell) = 0$ for all $\ell \in \mathcal{L}(e)$ and implies the following proposition.

Proposition 2.8. *The set cover inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} x_{\ell,f} \geq 1 \tag{2.5}$$

is valid for $P(\text{SLP})$ for all $e \in E$.

In the following example, we see that band inequalities improve the LP relaxation of the standard model.

Example 2.2. Consider the line planning instance given in Figure 2.2 and the band $f_{\mathcal{B}}$ of e_1 , where $f_{\mathcal{B}}(\ell_1) = 0$, $f_{\mathcal{B}}(\ell_2) = 1$, and $f_{\mathcal{B}}(\ell_3) = 3$. The associated band inequality

$$x_{\ell_1,1} + x_{\ell_1,3} + x_{\ell_1,4} + x_{\ell_2,3} + x_{\ell_2,4} + x_{\ell_3,4} \geq 1$$

is facet-defining for $P(\text{SLP})$ in this example. The inequality is violated by $\bar{x} \in P_{LP}(\text{SLP})$, where $\bar{x}_{\ell_1,3} = \frac{1}{3}$, $\bar{x}_{\ell_2,1} = \bar{x}_{\ell_3,3} = 1$, and $\bar{x}_{\ell,f} = 0$ otherwise.

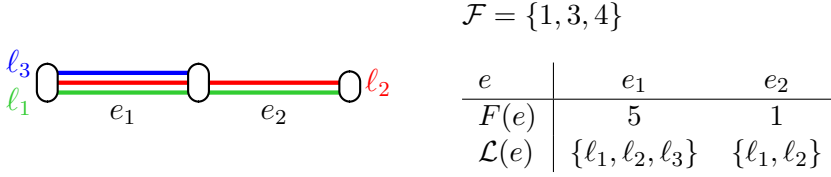


Figure 2.2: An instance of the line planning problem. *Left:* A transportation network consisting of two edges and three lines. *Right:* A given set of frequencies and frequency demands of the two edges.

2.2.3 MIR Inequalities

In this section we study the *mixed integer rounding (MIR) procedure* to strengthen the frequency demand constraints (2.1). The presented class of inequalities dominates the multicover inequalities proposed by Goosens et al. in [17], which is shown at the end of this section. For an introduction to MIR we refer to Raack [23] and Dash et al. [16]. To derive the *MIR inequalities* we introduce the basic MIR inequality defined by Wolsey [33]:

Lemma 2.9 (Wolsey [33]). *Let $Q_I := \{(x, y) \in \mathbb{Z} \times \mathbb{R} : x + y \geq \beta, y \geq 0\}$ with $\beta \in \mathbb{R}$. The basic MIR inequality*

$$rx + y \geq r \lceil \beta \rceil$$

with $r := r(\beta) = \beta - \lfloor \beta \rfloor$ is valid for Q_I and defines a facet of $\text{conv}(Q_I)$ if $r > 0$.

We can generate new inequalities by multiplying the frequency demand constraint for a given edge e with an arbitrary $\lambda > 0$. This new constraint can be strengthened by mixed integer rounding, i.e., by applying Lemma 2.9.

Proposition 2.10. *Let $\lambda \in \mathbb{R}_+ \setminus \{0\}$, $e \in E$, and define $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor$ and $r_f = \lambda f - \lfloor \lambda f \rfloor$. The MIR inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} (r \lfloor \lambda f \rfloor + \min(r_f, r)) x_{\ell, f} \geq r \lceil \lambda F(e) \rceil \quad (2.6)$$

induced by the demand inequality (2.1) scaled by λ is valid for $P(\text{SLP})$.

Proof. Scaling inequality (2.1) by $\lambda > 0$ yields

$$\begin{aligned}
 \lambda \cdot F(e) &\leq \lambda \cdot \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \\
 &= \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} \lambda \cdot f \cdot x_{\ell,f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} \lambda \cdot f \cdot x_{\ell,f} \\
 &\leq \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} (\lfloor \lambda \cdot f \rfloor + r_f) \cdot x_{\ell,f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} (\lfloor \lambda \cdot f \rfloor + 1) \cdot x_{\ell,f} \\
 &= \underbrace{\sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot x_{\ell,f}}_{\geq 0} + \underbrace{\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot x_{\ell,f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} x_{\ell,f}}_{\in \mathbb{Z}}.
 \end{aligned}$$

Applying Lemma 2.9 yields

$$\begin{aligned}
 r \cdot \lceil \lambda \cdot F(e) \rceil &\leq \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot x_{\ell,f} + r \cdot \left(\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot x_{\ell,f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} x_{\ell,f} \right) \\
 &= \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} (r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)) \cdot x_{\ell,f}.
 \end{aligned}$$

□

Notice that $\lambda > 0$ only produces a non-trivial MIR inequality (2.6) if $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor > 0$. Dash, Günlük and Lodi [16] analyze for which λ the MIR inequality (2.6) is non-redundant.

Proposition 2.11 (Dash, Günlük and Lodi [16]). *Each non-redundant MIR inequality (2.6) is defined by $\lambda \in (0, 1)$, where λ is a rational number with denominator equal to some $f \in \mathcal{F}$.*

Again, we can give an example where a MIR inequality improves the LP relaxation of the standard model.

Example 2.3. Consider the line planning instance given in Figure 2.3. The frequency demand constraint for e_1 reads as follows:

$$2x_{\ell_1,2} + 5x_{\ell_1,5} + 9x_{\ell_1,9} + 2x_{\ell_2,2} + 5x_{\ell_2,5} + 9x_{\ell_2,9} + 2x_{\ell_3,2} + 5x_{\ell_3,5} + 9x_{\ell_3,9} \geq 8.$$

Let $\lambda = \frac{1}{5}$, then we have $r = \frac{8}{5} - 1 = \frac{3}{5}$ and

$$r_2 = \frac{2}{5} - 0 = \frac{2}{5}, \quad r_5 = \frac{5}{5} - 1 = 0, \quad r_9 = \frac{9}{5} - 1 = \frac{4}{5}.$$

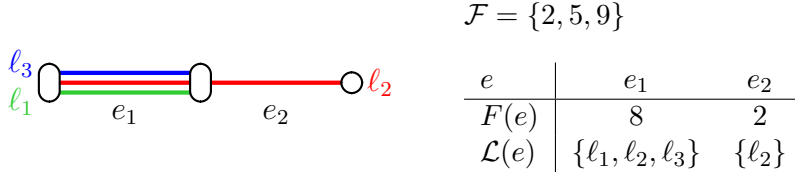


Figure 2.3: An instance of the line planning problem. *Left*: A transportation network consisting of two edges and three lines. *Right*: A given set of frequencies and frequency demands.

And hence we get

$$r \left\lfloor \frac{2}{5} \right\rfloor + \min(r_2, r) = \frac{2}{5}, \quad r \left\lfloor \frac{5}{5} \right\rfloor + \min(r_5, r) = \frac{3}{5}, \quad r \left\lfloor \frac{9}{5} \right\rfloor + \min(r_9, r) = \frac{6}{5}.$$

Thus, the MIR inequality for e_1 and $\lambda = \frac{1}{5}$ reads as follows

$$\frac{2}{5}x_{\ell_1,2} + \frac{3}{5}x_{\ell_1,5} + \frac{6}{5}x_{\ell_1,9} + \frac{2}{5}x_{\ell_2,2} + \frac{3}{5}x_{\ell_2,5} + \frac{6}{5}x_{\ell_2,9} + \frac{2}{5}x_{\ell_3,2} + \frac{3}{5}x_{\ell_3,5} + \frac{6}{5}x_{\ell_3,9} \geq \frac{6}{5}$$

which can be scaled to

$$2x_{\ell_1,2} + 3x_{\ell_1,5} + 6x_{\ell_1,9} + 2x_{\ell_2,2} + 3x_{\ell_2,5} + 6x_{\ell_2,9} + 2x_{\ell_3,2} + 3x_{\ell_3,5} + 6x_{\ell_3,9} \geq 6. \quad (2.7)$$

Inequality (2.7) is violated by $\bar{x} \in P_{LP}(\text{SLP})$, where $\bar{x}_{\ell_1,5} = \frac{1}{5}$, $\bar{x}_{\ell_2,2} = 1$, $\bar{x}_{\ell_3,5} = 1$, and $\bar{x}_{\ell,f} = 0$ otherwise. One can show that (2.7) is even facet-defining for $P(\text{SLP})$ in this example.

Multicover Cuts

Goosens, van Hoesel, and Kroon propose in [17] a class of valid inequalities for $P(\text{SLP})$, which are also based on the frequency demand constraints. The inequalities are motivated by the following observation: Let e be an edge, then every feasible solution contains either more than one line passing e at a frequency smaller than $F(e)$ or a line passing e at a frequency higher than or equal to $F(e)$.

Proposition 2.12 (Goosens et al. [17]). *Let $e \in E$ and define $\mathcal{F}^< := \{f \in \mathcal{F} : f < F(e)\}$, then the two-cover inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}^<} x_{\ell,f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F} \setminus \mathcal{F}^<} 2x_{\ell,f} \geq 2$$

is valid for $P(\text{SLP})$.

For an example see Example 2.4. The two-cover inequalities described in Proposition 2.12 can be generalized by dividing the set of frequencies into several subsets.

Proposition 2.13 (Goosens et al. [17]). *Let $e \in E$ and $s \in \mathbb{N}$ such that $2 \leq s + 1 < F(e)$. Define $\mathcal{F}^k := \{f \in \mathcal{F} : (k - 1)F(e) \leq s \cdot f < kF(e)\}$ for $k = 1, \dots, s$ and $\mathcal{F}^C := \mathcal{F} \setminus \bigcup_{k=1}^s \mathcal{F}^k$. Then the multicover inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{k=1}^s \sum_{f \in \mathcal{F}^k} kx_{\ell,f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}^C} (s + 1)x_{\ell,f} \geq s + 1 \quad (2.8)$$

is valid for $P(\text{SLP})$.

Proof. Let $e \in E$ and $\epsilon > 0$ small enough such that $\mathcal{F}^k = \mathcal{F}_\epsilon^k$ for all $k = 1, \dots, s$, where $\mathcal{F}_\epsilon^k := \{f \in \mathcal{F} : (k - 1)(F(e) - \epsilon) \leq s \cdot f < k(F(e) - \epsilon)\}$ for $k = 1, \dots, s$. And define $\lambda := \frac{s}{F(e) - \epsilon}$. Scaling inequality (2.1) by λ yields

$$\begin{aligned} \lambda F(e) &\leq \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lambda f x_{\ell,f} \\ &\leq \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lceil \lambda f \rceil x_{\ell,f}. \end{aligned}$$

Since the right-hand side of this inequality is always integral, we can round up the left-hand side and obtain

$$\lceil \lambda F(e) \rceil \leq \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lceil \lambda f \rceil x_{\ell,f}. \quad (*)$$

By the choice of ϵ we can derive $\lceil \lambda f \rceil = \left\lceil \frac{s \cdot f}{F(e) - \epsilon} \right\rceil = \left\lceil \frac{s \cdot f}{F(e)} \right\rceil \leq k$ for all $f \in \mathcal{F}^k$, $k = 1, \dots, s$. For every $f \in \mathcal{F}^C$ we can set the coefficient to the value of the left-hand side in (*), since it suffices to operate one line at this frequency in any solution. Further, we have $\lceil \lambda F(e) \rceil = \left\lceil \frac{s \cdot F(e)}{F(e) - \epsilon} \right\rceil \geq s + 1$ and

$$\lceil \lambda F(e) \rceil = \left\lceil \frac{sF(e)}{F(e) - \epsilon} \right\rceil \leq \left\lceil \frac{sF(e)}{F(e) - 1} \right\rceil \leq \left\lceil \frac{(s + 1)(F(e) - 1)}{F(e) - 1} \right\rceil = s + 1.$$

Hence, $\lceil \lambda F(e) \rceil = s + 1$ and we obtain from (*):

$$s + 1 \leq \sum_{\ell \in \mathcal{L}(e)} \sum_{k=1}^s \sum_{f \in \mathcal{F}^k} kx_{\ell,f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}^C} (s + 1)x_{\ell,f}.$$

□

In the proof of the following theorem we see that the multicover inequalities are dominated by the MIR inequalities. For a definition of dominated inequalities we refer to Wolsey [33], see also Raack [23]. However, we can show that they strengthen the LP relaxation of $P(\text{SLP})$, see Example 2.4.

Theorem 2.14. *The multicover inequalities (2.8) are dominated by the MIR inequalities.*

Proof. Let $e \in E$ and $\lambda := \frac{s}{F(e)-\epsilon}$ as in the proof of Proposition 2.13. Then the MIR inequality

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} (r \lfloor \lambda f \rfloor + \min(r_f, r)) x_{\ell, f} \geq r \lceil \lambda F(e) \rceil \quad (2.9)$$

is valid for $P(\text{SLP})$ by Proposition 2.10, where $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor$ and $r_f = \lambda f - \lfloor \lambda f \rfloor$. Since $r \lfloor \lambda f \rfloor + \min(r_f, r) \leq r \lceil \lambda f \rceil$, the corresponding multicover inequality

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lceil \lambda f \rceil x_{\ell, f} \geq \lceil \lambda F(e) \rceil$$

is dominated by (2.9). \square

Example 2.4 (Example 2.3 continued). Let $e = e_1$ and recall $\mathcal{F} = \{2, 5, 9\}$, $F(e_1) = 8$. Then the two-cover inequality from Proposition 2.12 for e_1 reads as follows

$$x_{\ell_1, 2} + x_{\ell_1, 5} + 2x_{\ell_1, 9} + x_{\ell_2, 2} + x_{\ell_2, 5} + 2x_{\ell_2, 9} + x_{\ell_3, 2} + x_{\ell_3, 5} + 2x_{\ell_3, 9} \geq 2.$$

Now let $s = 5$, then we have $\mathcal{F}^1 = \emptyset$, $\mathcal{F}^2 = \{2\}$, $\mathcal{F}^3 = \emptyset$, $\mathcal{F}^4 = \{5\}$, and $\mathcal{F}^5 = \emptyset$ by the definition in Proposition 2.13. Hence, the multicover inequality for e_1 and $s = 5$ reads as follows

$$2x_{\ell_1, 2} + 4x_{\ell_1, 5} + 6x_{\ell_1, 9} + 2x_{\ell_2, 2} + 4x_{\ell_2, 5} + 6x_{\ell_2, 9} + 2x_{\ell_3, 2} + 4x_{\ell_3, 5} + 6x_{\ell_3, 9} \geq 6. \quad (2.10)$$

Note that (2.10) is strictly dominated by the MIR inequality (2.7).

2.2.4 Improving Inequalities from Edge Subsets

The inequalities studied in Section 2.2.2 and Section 2.2.3 are all derived by strengthening the frequency demand constraint (2.1) of a single edge. In this section we investigate cuts that can be derived by aggregating the frequency demand constraints for a subset of edges. We apply two classes of valid

inequalities that were presented by Bussieck [10] to our problem setting and develop a new class of valid inequalities.

We need some further notation in this section. For a subset $E' \subseteq E$ of edges denote the total demand by $F(E') := \sum_{e \in E'} F(e)$ and the set of lines passing an edge in E' by $\mathcal{L}(E') := \bigcup_{e \in E'} \mathcal{L}(e)$. For a line $\ell \in \mathcal{L}(E')$ we define by $\alpha_{E'}^\ell := |\ell \cap E'|$ the number of edges in ℓ that are contained in E' . And define $\alpha_{E'}^{\max} := \max\{\alpha_{E'}^\ell : \ell \in \mathcal{L}(E')\}$.

The first inequality, we present in this section, was introduced by Bussieck in Section 6.5.2 in [10] and can be motivated by the small line planning problem depicted in Figure 2.4. A closer inspection of this instance shows that the sum of frequencies at which the lines are operated in any feasible solution must be at least 11. This conclusion is generalized in Proposition 2.15.

Proposition 2.15 (Bussieck [10]). *Let $E' \subseteq E$ be a subset of edges. The aggregated frequency inequality*

$$\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \geq \left\lceil \frac{F(E')}{\alpha_{E'}^{\max}} \right\rceil \quad (2.11)$$

is valid for $P(\text{SLP})$, if $\alpha_{E'}^{\max} > 0$.

Proof. Let $x \in P(\text{SLP})$ and $E' \subseteq E$ such that $\alpha_{E'}^{\max} > 0$. By (2.1) we get

$$F(e) \leq \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \quad \forall e \in E'.$$

This implies

$$\begin{aligned} F(E') &\leq \sum_{e \in E'} \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \\ &= \sum_{\ell \in \mathcal{L}(E')} \alpha_{E'}^\ell \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \\ &\leq \sum_{\ell \in \mathcal{L}(E')} \alpha_{E'}^{\max} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f}. \end{aligned}$$

Dividing by $\alpha_{E'}^{\max}$ gives

$$\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \geq \frac{F(E')}{\alpha_{E'}^{\max}}. \quad (*)$$

Since the left-hand side of (*) is always integral, we can round up the right-hand side and obtain (2.11). \square

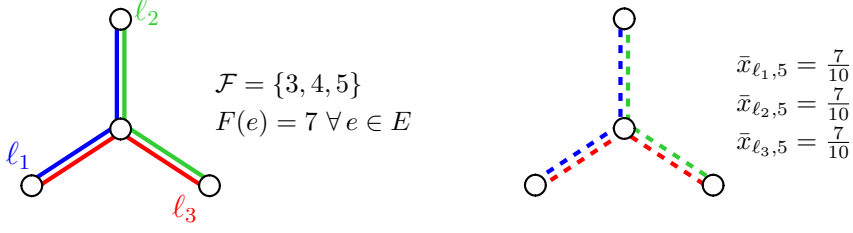


Figure 2.4: *Left*: An instance of the line planning problem, where all edges have frequency demand 7. *Right*: A fractional solution.

We show in Example 2.5 that the aggregated frequency inequalities (2.11) are not in general valid for all points in $P_{LP}(\text{SLP})$, i.e., these inequalities strengthen the LP relaxation of $P(\text{SLP})$.

Example 2.5. Consider the line planning problem instance depicted in Figure 2.4. When we define $E' := \{e_1, e_2, e_3\}$, the corresponding aggregated frequency inequality (2.11) from Proposition 2.15 reads as follows:

$$3x_{\ell_1,3} + 4x_{\ell_1,4} + 5x_{\ell_1,5} + 3x_{\ell_2,3} + 4x_{\ell_2,4} + 5x_{\ell_2,5} + 3x_{\ell_3,3} + 4x_{\ell_3,4} + 5x_{\ell_3,5} \geq 11. \quad (2.12)$$

Inequality (2.12) is violated by $\bar{x} \in P_{LP}(\text{SLP})$, where $\bar{x}_{\ell_1,5} = \bar{x}_{\ell_2,5} = \bar{x}_{\ell_3,5} = \frac{7}{10}$ and $\bar{x}_{\ell,f} = 0$ otherwise.

We consider the example depicted in Figure 2.5 to derive a new class of valid inequalities. To cover the demand of all three edges at least two lines have to be operated in any solution. This observation is generalized in the following proposition.

Proposition 2.16. *Let $E' \subseteq E$, then the aggregated cardinality inequality*

$$\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} x_{\ell,f} \geq \left\lceil \frac{|E'|}{\alpha_{E'}^{\max}} \right\rceil \quad (2.13)$$

is valid for $P(\text{SLP})$, if $\alpha_{E'}^{\max} > 0$.

Proof. Let $x \in P(\text{SLP})$ and $E' \subseteq E$ such that $\alpha_{E'}^{\max} > 0$. By (2.5) we get

$$1 \leq \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} x_{\ell,f} \quad \forall e \in E'.$$

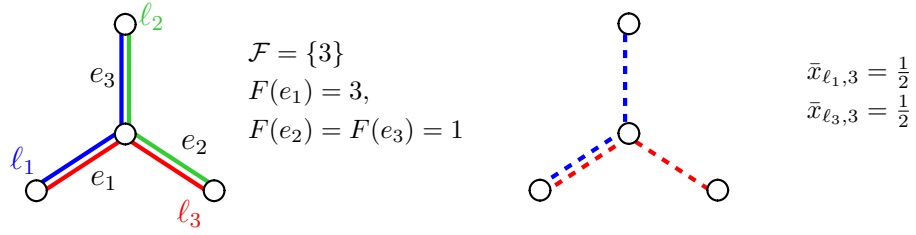


Figure 2.5: *Left*: An instance of the line planning problem. *Right*: A fractional solution.

This implies

$$\begin{aligned} |E'| &\leq \sum_{e \in E'} \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} x_{\ell,f} \\ &= \sum_{\ell \in \mathcal{L}(E')} \alpha_{E'}^\ell \sum_{f \in \mathcal{F}} x_{\ell,f} \\ &\leq \sum_{\ell \in \mathcal{L}(E')} \alpha_{E'}^{\max} \sum_{f \in \mathcal{F}} x_{\ell,f}. \end{aligned}$$

Dividing by $\alpha_{E'}^{\max}$ gives

$$\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} x_{\ell,f} \geq \frac{|E'|}{\alpha_{E'}^{\max}}. \quad (*)$$

Since the left-hand side of (*) is always integral, we can round up the right-hand side and obtain (2.13). \square

In Example 2.6 we show that the aggregated frequency inequalities given in Proposition 2.15 do not dominate the aggregated cardinality inequalities given in Proposition 2.16 and vice versa.

Example 2.6 (Example 2.5 continued). First consider the line planning problem depicted in Figure 2.4 and recall the point $\bar{x} \in P_{LP}(\text{SLP})$ from Example 2.5. The aggregated cardinality inequality (2.13) for $E' := \{e_1, e_2, e_3\}$ reads as follows:

$$x_{\ell_1,3} + x_{\ell_1,4} + x_{\ell_1,5} + x_{\ell_2,3} + x_{\ell_2,4} + x_{\ell_2,5} + x_{\ell_3,3} + x_{\ell_3,4} + x_{\ell_3,5} \geq 2. \quad (2.14)$$

Note that \bar{x} satisfies (2.14), but violates the aggregated frequency inequality (2.12).

Now consider the line planning problem depicted in Figure 2.5. Define $E' := \{e_1, e_2, e_3\}$, the corresponding aggregated frequency inequality (2.11) is given by

$$3x_{\ell_1,3} + 3x_{\ell_2,3} + 3x_{\ell_3,3} \geq 3. \quad (2.15)$$

and the aggregated cardinality inequality (2.13) is given by

$$x_{\ell_1,3} + x_{\ell_2,3} + x_{\ell_3,3} \geq 2. \quad (2.16)$$

The point $\tilde{x} \in P_{LP}(\text{SLP})$, where $\tilde{x}_{\ell_1,3} = \tilde{x}_{\ell_3,3} = \frac{1}{2}$ and $\tilde{x}_{\ell_2,3} = 0$, violates (2.16) but satisfies (2.15).

The third class of inequalities is motivated by the example illustrated in Figure 2.6 and the following observation. In any feasible line plan for this instance there exists either a line with positive frequency containing e_1 and terminating at v or the lines containing e_2 and e_3 have to cover a demand of $F(e_1) = 8$.

Proposition 2.17 (Bussieck [10]). *Let $E' \subset E$ and $\tilde{e} \in E \setminus E'$. The inequality*

$$(F(\tilde{e}) - F(E')) \sum_{\substack{\ell \in \mathcal{L}(\tilde{e}) \\ \ell \notin \mathcal{L}(E')}} \sum_{f \in \mathcal{F}} x_{\ell,f} + \sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} \alpha_{E'}^\ell \cdot f \cdot x_{\ell,f} \geq F(\tilde{e}) \quad (2.17)$$

is valid for $P(\text{SLP})$, if $F(\tilde{e}) > F(E')$.

Proof. Let $x \in P(\text{SLP})$, $E' \subset E$ and $\tilde{e} \in E \setminus E'$ such that $F(\tilde{e}) > F(E')$. Since $x \in \{0, 1\}^{\mathcal{L} \times \mathcal{F}}$ we know that the sum $\sum_{\substack{\ell \in \mathcal{L}(\tilde{e}) \\ \ell \notin \mathcal{L}(E')}} \sum_{f \in \mathcal{F}} x_{\ell,f}$ is either equal to zero or greater than or equal to one. First, let $\sum_{\substack{\ell \in \mathcal{L}(\tilde{e}) \\ \ell \notin \mathcal{L}(E')}} \sum_{f \in \mathcal{F}} x_{\ell,f} \geq 1$.

This implies

$$(F(\tilde{e}) - F(E')) \sum_{\substack{\ell \in \mathcal{L}(\tilde{e}) \\ \ell \notin \mathcal{L}(E')}} \sum_{f \in \mathcal{F}} x_{\ell,f} \geq F(\tilde{e}) - F(E')$$

since $F(\tilde{e}) - F(E') > 0$. Combining this with the equation

$$\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} \alpha_{E'}^\ell \cdot f \cdot x_{\ell,f} \geq F(E')$$

from the proof of Proposition 2.15 gives (2.17).

Now let $\sum_{\substack{\ell \in \mathcal{L}(\tilde{e}) \\ \ell \notin \mathcal{L}(E')}} \sum_{f \in \mathcal{F}} x_{\ell,f} = 0$, i.e., all lines in the line plan corresponding

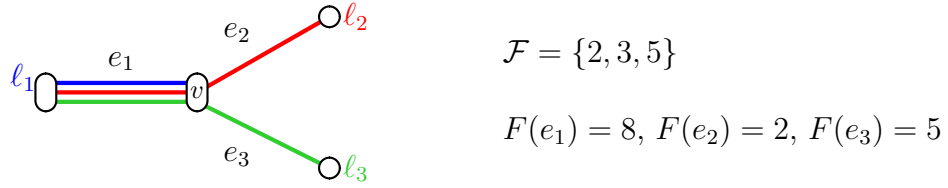


Figure 2.6: An instance of the line planning problem. *Left*: A transportation network consisting of three edges and three lines. *Right*: A given set of frequencies and frequency demands.

to x containing \tilde{e} contain at least one edge in E' . Hence, starting with (2.1) we get

$$F(\tilde{e}) \leq \sum_{\ell \in \mathcal{L}(\tilde{e})} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \leq \sum_{e \in E'} \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \leq \sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} \alpha_{E'}^{\ell} \cdot f \cdot x_{\ell,f},$$

which implies (2.17). We considered both cases and the claim is proved. \square

Again, we give an example to show that the inequalities (2.17) improve the LP relaxation of (SLP).

Example 2.7. Consider the line planning problem in Figure 2.6. When we define $\tilde{e} := e_1$ and $E' := \{e_2, e_3\}$, the corresponding inequality (2.17) from Proposition 2.17 reads as follows:

$$x_{\ell_1,2} + x_{\ell_1,3} + x_{\ell_1,5} + 2x_{\ell_2,2} + 3x_{\ell_2,3} + 5x_{\ell_2,5} + 2x_{\ell_3,2} + 3x_{\ell_3,3} + 5x_{\ell_3,5} \geq 8$$

This inequality is violated by $\bar{x} \in P_{LP}(\text{SLP})$, where $\bar{x}_{\ell_1,3} = \frac{1}{2}$, $\bar{x}_{\ell_2,2} = 1$, $\bar{x}_{\ell_3,5} = 1$, and $\bar{x}_{\ell,f} = 0$ otherwise.

2.2.5 Single Edge Relaxation

In this section we introduce and study a relaxation of the polytope $P(\text{SLP})$, which results from reducing the line planning problem to a single edge. For an edge $e \in E$ we denote by

$$P(\text{SLP}|e) := \text{conv} \left\{ x \in \{0, 1\}^{\mathcal{L}(e) \times \mathcal{F}} : \begin{array}{l} \text{(SLP) (2.1) for } e, \\ \text{(SLP) (2.2) for all } \ell \in \mathcal{L}(e) \end{array} \right\}$$

the polytope of the single edge relaxation of the line planning problem. Obviously, any inequality that is valid for $P(\text{SLP}|e)$ is also valid for $P(\text{SLP})$.

The polytope $P(\text{SLP}|e)$ can be transformed into a knapsack polytope with generalized upper bound (GUB) constraints, which is shown in Lemma 2.19 below. The knapsack polytope with GUB constraints is studied in [19, 28, 32] and is defined in the standard form, see Johnson and Padberg [19], as follows:

Definition 2.18. Let $N = \{1, \dots, n\}$, $K := \{1, \dots, k\}$ and $N_i \subseteq N$ for all $i \in K$ such that $\bigcup_{i \in K} N_i = N$ and $N_i \cap N_j = \emptyset$ for all $i, j \in K$, $i \neq j$. Let $b > 0$ and $0 < w_j \leq b$ for $j \in N$ be positive integers. The knapsack polytope with GUB constraints is defined by

$$P(\text{KPGUB}) := \text{conv} \left\{ x \in \{0, 1\}^n : \sum_{i \in K} \sum_{j \in N_i} w_j x_j \leq b, \sum_{j \in N_i} x_j \leq 1 \forall i \in K \right\}.$$

We show that $P(\text{SLP}|e)$ can be viewed as a knapsack polytope with GUB constraints by applying a linear transformation which was presented by Johnson and Padberg in [19].

Lemma 2.19. *Let $e \in E$, then there is a knapsack polytope with GUB constraints $P_K(\text{SLP}|e)$ such that there is a one-to-one mapping between the points in $P(\text{SLP}|e)$ and $P_K(\text{SLP}|e)$, respectively.*

Proof. For a given edge $e \in E$ we define the polytope $P_K(\text{SLP}|e)$ as follows:

$$P_K(\text{SLP}|e) := \text{conv} \left\{ z \in \{0, 1\}^{\mathcal{L}(e) \times \mathcal{F}} : \begin{array}{l} \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} w_{\ell, f} z_{\ell, f} \leq b, \\ \sum_{f \in \mathcal{F}} z_{\ell, f} \leq 1 \forall \ell \in \mathcal{L}(e) \end{array} \right\}$$

where we define $w_{\ell, f_m} := f_m$ and $w_{\ell, f} := f_m - f$ for all $f \in \mathcal{F} \setminus \{f_m\}$ and $b := |\mathcal{L}(e)| \cdot f_m - F(e)$. Note that $w_{\ell, f} > 0$ for all $f \in \mathcal{F}$ and $b > 0$, since $|\mathcal{L}(e)| \cdot f_m > F(e)$ by the assumption in Remark 2.2. The mapping is provided by the following variable transformation:

$$z_{\ell, f} := \begin{cases} x_{\ell, f} & \text{for } f \neq f_m, \\ 1 - \sum_{f \in \mathcal{F}} x_{\ell, f} & \text{for } f = f_m. \end{cases}$$

□

Lemma 2.19 implies that all results for the knapsack polytope with GUB constraints apply also to the polytope $P(\text{SLP}|e)$.

Using results from Sherali and Lee in [28], we can determine an equation for the dimension and the trivial facets of $P(\text{SLP}|e)$.

Proposition 2.20 (Sherali and Lee [28]). *Let $e \in E$ and define*

$$L_0 := \begin{cases} 0 & \text{if } (|\mathcal{L}(e)| - 1) \cdot f_m \geq F(e), \\ 1 & \text{else.} \end{cases}$$

Then the following holds

$$\dim(P(\text{SLP}|e)) = (|\mathcal{F}| - L_0) \cdot |\mathcal{L}(e)|.$$

Proof. First let $L_0 = 0$. Obviously $\dim(P(\text{SLP}|e)) \leq |\mathcal{L}(e)| \cdot |\mathcal{F}|$. We construct $|\mathcal{L}(e)| \cdot |\mathcal{F}| + 1$ many affinely independent points in $P(\text{SLP}|e)$ to prove the statement. For each $\ell \in \mathcal{L}(e)$ construct the following set of points:

$x^{\tilde{\ell}, \tilde{f}}$ for all $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$, which corresponds to operating $\tilde{\ell}$ at \tilde{f} and all other lines passing e at the highest frequency, i.e.,

$$x_{\ell, f}^{\tilde{\ell}, \tilde{f}} = \begin{cases} 1 & \text{if } \ell = \tilde{\ell}, f = \tilde{f} \\ 1 & \text{if } \ell \neq \tilde{\ell}, f = f_m \\ 0 & \text{else.} \end{cases} \quad \forall \ell \in \mathcal{L}(e), \forall f \in \mathcal{F},$$

$x^{\tilde{\ell}, 0}$, which corresponds to not operating $\tilde{\ell}$ and operating all other lines passing e at the highest frequency, i.e.,

$$x_{\ell, f}^{\tilde{\ell}, 0} = \begin{cases} 1 & \text{if } \ell \neq \tilde{\ell}, f = f_m \\ 0 & \text{else.} \end{cases} \quad \forall \ell \in \mathcal{L}(e), \forall f \in \mathcal{F},$$

x^{f_m} , which corresponds to operating all lines passing e at the highest frequency, i.e.,

$$x_{\ell, f}^{f_m} = \begin{cases} 1 & \text{if } \ell \in \mathcal{L}(e), f = f_m \\ 0 & \text{else.} \end{cases} \quad \forall \ell \in \mathcal{L}(e), \forall f \in \mathcal{F}.$$

These $|\mathcal{L}(e)| \cdot |\mathcal{F}| + 1$ affinely independent points are all feasible for $P(\text{SLP}|e)$, since $L_0 = 0$.

Now let $L_0 = 1$. We must have $\sum_{f \in \mathcal{F}} x_{\ell, f} = 1$ for all $\ell \in \mathcal{L}(e)$, and hence $\dim(P(\text{SLP}|e)) \leq (|\mathcal{F}| - 1) \cdot |\mathcal{L}(e)|$. Note that x^{f_m} and for each $\tilde{\ell} \in \mathcal{L}(e)$ and $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$ the point $x^{\tilde{\ell}, \tilde{f}}$ are feasible for $P(\text{SLP}|e)$ by the assumption in Remark 2.2. Hence, $P(\text{SLP}|e)$ contains $(|\mathcal{F}| - 1) \cdot |\mathcal{L}(e)| + 1$ affinely independent points. This completes the proof of this proposition. \square

Applying further results of Sherali and Lee [28] to our setting, we can conclude that the non-negativity inequality $x_{\ell, f} \geq 0$ defines a facet of the polytope $P(\text{SLP}|e)$ for all lines $\ell \in \mathcal{L}(e)$ and every frequency in $f \in \mathcal{F} \setminus \{f_m\}$.

2.3 Relation to Models in the Literature

In this section we compare the standard model (SLP) with two models in the literature, which integrate certain additional aspects. First, we compare it with a model which integrates the possibility to operated lines with additional cars to increase their capacity. Then, we consider a model that allows to freely route the passengers along paths and includes the total passenger traveling time in the objective function.

2.3.1 Including Line Capacities

We first compare the standard model (SLP) with the integer program proposed by Bussieck [10] for the cost optimal line planning problem including line capacities and the number of cars of a line. We extend our notation by the following: For each edge $e \in E$ we are given lower $\underline{F}(e)$ and upper $\overline{F}(e)$ frequency bounds. That is the sum of the frequencies of lines operating on edge e is bounded from below and above by $\underline{F}(e)$ and $\overline{F}(e)$, respectively. In contrast to the standard model with a fixed train capacity, the number of coaches per train is determined in Bussieck's model. The train of an operating line consists of at least \underline{k} cars and at most \overline{k} cars. The capacity of a single car is denoted by $\kappa_k > 0$. The total capacity of all lines operating on edge e has to cover the transportation demand $d(e)$ of e . For each line $\ell \in \mathcal{L}$ and each frequency $f \in \mathcal{F}$ we introduce integer variables $z_{\ell,f} \in \mathbb{Z}_+$ representing the number of additional cars of line ℓ at frequency f . Then using our notation Bussieck's model (COSTILP) reads as follows:

$$\begin{aligned}
 (\text{COSTILP}) \quad \min \quad & \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f}^t x_{\ell,f} + c_{\ell,f}^k (\underline{k} x_{\ell,f} + z_{\ell,f}) \\
 & \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \geq \underline{F}(e) \quad \forall e \in E \quad (2.18)
 \end{aligned}$$

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \leq \overline{F}(e) \quad \forall e \in E \quad (2.19)$$

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \kappa_k \cdot f \cdot (\underline{k} x_{\ell,f} + z_{\ell,f}) \geq d(e) \quad \forall e \in E \quad (2.20)$$

$$\underline{k} x_{\ell,f} + z_{\ell,f} \leq \overline{k} x_{\ell,f} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \quad (2.21)$$

$$\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L} \quad (2.22)$$

$$x_{\ell,f} \in \{0, 1\} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \quad (2.23)$$

$$z_{\ell,f} \in \mathbb{Z}_+ \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F}, \quad (2.24)$$

where $c_{\ell,f}^t > 0$ is the cost of a train serving line ℓ at frequency f and $c_{\ell,f}^k > 0$ is the cost per car. Both $c_{\ell,f}^t$ and $c_{\ell,f}^k$ depend on the length of ℓ and f . The frequency constraints (2.18) and (2.19) require that the lower and upper frequency bounds are satisfied for each edge in the network. The capacity constraint (2.20) ensure that sufficient capacity is available. The constraints (2.21) bound the number of cars for each line and frequency. The assignment constraints (2.22) make sure that every line is operated with at most one frequency.

As mentioned above, the difference of (SLP) and (COSTILP) is that model (COSTILP) includes the number of cars of a line. We will show in the following that the two models are equivalent, if no line can be operated with additional cars and the upper frequency bounds are satisfied by every point in $P(\text{SLP})$.

Proposition 2.21. *Let $\underline{k} = \overline{k}$ and the frequency demand for the standard model be given by $F(e) = \max \left\{ \underline{F}(e), \left\lceil \frac{d(e)}{\kappa_k \cdot \underline{k}} \right\rceil \right\}$ for every edge $e \in E$. Further, let the upper frequency bound constraints (2.19) be satisfied by every point in $P(\text{SLP})$. Then the following holds*

$$P(\text{COSTILP})|_x = P(\text{SLP}).$$

Proof. If the maximum number of cars \overline{k} equals the minimum number of cars \underline{k} , then $z \equiv 0$ holds for every feasible solution (x, z) of (COSTILP). Hence, to

prove the proposition it suffices to show that for all $(x, z) \in \{0, 1\}^{(\mathcal{L} \times \mathcal{F}) \times (\mathcal{L} \times \mathcal{F})}$ with $z \equiv 0$ the following holds:

$$x \text{ satisfies (2.1)} \Leftrightarrow (x, z) \text{ satisfies (2.18) and (2.20)}.$$

Let $(x, z) \in \{0, 1\}^{(\mathcal{L} \times \mathcal{F}) \times (\mathcal{L} \times \mathcal{F})}$ with $z \equiv 0$ and assume (x, z) satisfies the capacity constraints (2.20). From this we get

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \geq \frac{d(e)}{\kappa_k \cdot \underline{k}} \quad \forall e \in E. \quad (*)$$

Since the left-hand side of (*) is integral, we can round up the right-hand side and obtain

$$(x, z) \text{ satisfies (2.20)} \Leftrightarrow \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \geq \left\lceil \frac{d(e)}{\kappa_k \cdot \underline{k}} \right\rceil \quad \forall e \in E.$$

In addition, we observe that the left-hand side of (*) is equivalent to the left-hand side of the lower frequency bound constraints (2.18). Therefore, we conclude that the following holds:

$$\begin{aligned} \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \geq F(e) &= \max \left\{ \underline{F}(e), \left\lceil \frac{d(e)}{\kappa_k \cdot \underline{k}} \right\rceil \right\} \quad \forall e \in E \\ \Leftrightarrow & \quad x \text{ satisfies (2.18) and (2.20)}. \end{aligned}$$

This completes the proof of the proposition. \square

Proposition 2.21 implies that under the preconditions of this proposition solving (COSTILP) is equivalent to solving (SLP), if the operational costs are defined by $c_{\ell, f} = c_{\ell, f}^t + \underline{k} \cdot c_{\ell, f}^k$ for all $\ell \in \mathcal{L}$ and all $f \in \mathcal{F}$. Note that operating every line at the highest frequency f_m yields a feasible solution of (SLP). Therefore, the upper frequency bound constraints (2.19) are satisfied by every point in $P(\text{SLP})$ if and only if $\overline{F}(e) \geq |\mathcal{L}(e)| \cdot f_m$ holds for every edge $e \in E$.

2.3.2 Integrated Passenger Routing

Now we consider a multi-commodity flow model for the line planning problem that allows a dynamic generation of lines and passenger paths by a column-generation approach. This model was originally proposed by Borndörfer, Grötschel and Pfetsch [4]; we consider here a “discrete frequency variant”

that was described by Karbstein in [20]. The objective of this model is a combination of operating costs and passenger travel time.

Again, we extend our notation. Let us consider more in detail the OD-matrix. Points where passengers start or end their trips are called *OD-nodes* and are denoted by $V_O \subseteq V$. For each pair $(s, t) \in V_O \times V_O$ we denote the corresponding entry of the OD-matrix by $d_{st} \in \mathbb{Q}_+$, i.e., d_{st} is the number of passengers that want to travel from $s \in V_O$ to $t \in V_O$. Let $D = \{(s, t) \in V_O \times V_O : d_{st} > 0\}$ be the set of all OD-pairs with positive transportation demand. Passengers travel along paths in a directed *passenger routing graph* $D(G) = (V, A)$ that arises from $G = (V, E)$ by replacing each edge $e \in E$ with two antiparallel arcs $a(e)$ and $\bar{a}(e)$; conversely let $e(a) \in E$ be the undirected edge corresponding to $a \in A$. For each arc $a \in A$ we are given a travel time $\tau_a \in \mathbb{Q}_+$. Furthermore, we are given for every line $\ell \in \mathcal{L}$ a capacity $\kappa_\ell > 0$ and define $\kappa_{\ell, f} = f \cdot \kappa_\ell$ for all $f \in \mathcal{F}$, i.e., $\kappa_{\ell, f}$ is the number of passengers that can be transported by line ℓ when operated at frequency f . We require for every edge $e \in E$ that all lines passing this edge have the same capacity, i.e., we are given κ_e and $\kappa_\ell = \kappa_e$ holds for all $\ell \in \mathcal{L}(e)$.

Denote by \mathcal{P}_{st} the set of all directed paths from $s \in V_O$ to $t \in V_O$ in $D(G)$, by $\mathcal{P} := \bigcup_{(s,t) \in D} \mathcal{P}_{st}$ all *passenger paths*, and by $\mathcal{P}(a) := \{p \in \mathcal{P} : a \in p\}$ the set of all passenger paths that use arc a . The *traveling time* of a passenger path $p \in \mathcal{P}$ is defined as $\tau_p = \sum_{a \in p} \tau_a$.

Now, we can define the variables for this model. For every line $\ell \in \mathcal{L}$ and every frequency $f \in \mathcal{F}$ we introduce a decision variable $x_{\ell, f} \in \{0, 1\}$, which indicates whether ℓ is operated at f . For every $p \in \mathcal{P}$ we introduce a continuous variable $\pi_p \geq 0$ for the number of passengers that travel on path p . Let a parameter $0 \leq \lambda \leq 1$ be given, then the *basic dynamic model* (BD) reads as follows:

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$$\begin{aligned}
 \text{(BD)} \quad \min \quad & \lambda \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f} + (1 - \lambda) \sum_{p \in \mathcal{P}} \tau_p \pi_p \\
 & \sum_{p \in \mathcal{P}_{st}} \pi_p = d_{st} \quad \forall (s, t) \in D \quad (2.25)
 \end{aligned}$$

$$\sum_{p \in \mathcal{P}(a)} \pi_p \leq \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} \kappa_{\ell,f} x_{\ell,f} \quad \forall a \in A \quad (2.26)$$

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \leq \bar{F}(e) \quad \forall e \in E \quad (2.27)$$

$$\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L} \quad (2.28)$$

$$x_{\ell,f} \in \{0, 1\} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \quad (2.29)$$

$$\pi_p \geq 0 \quad \forall p \in \mathcal{P} \quad (2.30)$$

Model (BD) minimizes a weighted sum of line operating costs and the total passenger travel time. Equations (2.25) enforce the passenger flow. The capacity constraints (2.26) ensure that there is sufficient total transportation capacity on each arc. The constraints (2.27) ensure that the upper frequency bounds are satisfied for each edge in the network. The assignment constraints (2.28) make sure that every line is operated with at most one frequency.

Next, we want to elaborate the relation of the models (BD) and (SLP). We show that we obtain (SLP) from (BD) by fixing the passengers paths and setting the parameter λ to 1.

Proposition 2.22. *Let $\bar{\pi} \in \mathbb{Q}_+^{\mathcal{P}}$ be a vector corresponding to a fixed passenger routing, i.e., $\sum_{p \in \mathcal{P}_{st}} \bar{\pi}_p = d_{st}$ for all $(s, t) \in D$, define the polytope*

$$P_{\bar{\pi}}(\text{BD}) := \left\{ x \in P(\text{BD})|_x : \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} \kappa_{\ell,f} x_{\ell,f} \geq \sum_{p \in \mathcal{P}(a)} \bar{\pi}_p \quad \forall a \in A \right\}.$$

and denote for each arc $a \in A$ by $d(a, \bar{\pi}) := \sum_{p \in \mathcal{P}(a)} \bar{\pi}_p$ the transportation demand induced by $\bar{\pi}$. Furthermore, let the frequency demand for the standard model be given by

$$F(e) = \max \left\{ \left\lceil \frac{d(a(e), \bar{\pi})}{\kappa_e} \right\rceil, \left\lceil \frac{d(\bar{a}(e), \bar{\pi})}{\kappa_e} \right\rceil \right\} \quad \forall e \in E$$

and let the upper frequency bound constraints (2.27) be satisfied by every point in $P(\text{SLP})$. Then the following holds

$$P(\text{SLP}) = P_{\bar{\pi}}(\text{BD}) \subseteq P(\text{BD})|_x.$$

Proof. $P_{\bar{\pi}}(\text{BD}) \subseteq P(\text{BD})|_x$ clearly holds, since $(x, \bar{\pi}) \in P(\text{BD})$ for all $x \in P_{\bar{\pi}}(\text{BD})$. Note that in order to prove that $P(\text{SLP}) = P_{\bar{\pi}}(\text{BD})$ holds it suffices to show for all $x \in \{0, 1\}^{\mathcal{L} \times \mathcal{F}}$

$$x \text{ satisfies (2.1)} \Leftrightarrow (x, \bar{\pi}) \text{ satisfies (2.26)}.$$

Remember that for every edge $e \in E$ for all lines in $\mathcal{L}(e)$ we have $\kappa_\ell = \kappa_e$. Hence, starting with inequality (2.26) we get

$$\begin{aligned} & \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} \kappa_\ell \cdot f \cdot x_{\ell, f} \geq \sum_{p \in \mathcal{P}(a)} \bar{\pi}_p \quad \forall a \in A \\ \Leftrightarrow & \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} \kappa_{e(a)} \cdot f \cdot x_{\ell, f} \geq \sum_{p \in \mathcal{P}(a)} \bar{\pi}_p \quad \forall a \in A \\ \Leftrightarrow & \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} \kappa_{e(a)} \cdot f \cdot x_{\ell, f} \geq d(a, \bar{\pi}) \quad \forall a \in A \\ \Leftrightarrow & \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \geq \frac{d(a, \bar{\pi})}{\kappa_{e(a)}} \quad \forall a \in A. \quad (2.31) \end{aligned}$$

$$\Leftrightarrow \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \geq \left\lceil \frac{d(a, \bar{\pi})}{\kappa_{e(a)}} \right\rceil \quad \forall a \in A. \quad (2.32)$$

The last equivalence holds, since the left-hand side of (2.31) is integral. The left-hand side of (2.32) is equivalent for two antiparallel arcs $a(e)$ and $\bar{a}(e)$, $e \in E$. Therefore, we conclude that for all $x \in \{0, 1\}^{\mathcal{L} \times \mathcal{F}}$ the following holds:

$$\begin{aligned} & \sum_{\ell \in \mathcal{L}(e(a))} \sum_{f \in \mathcal{F}} \kappa_{\ell, f} \cdot x_{\ell, f} \geq \sum_{p \in \mathcal{P}(a)} \bar{\pi}_p \quad \forall a \in A \\ \Leftrightarrow & \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \geq \max \left\{ \left\lceil \frac{d(a(e), \bar{\pi})}{\kappa_e} \right\rceil, \left\lceil \frac{d(\bar{a}(e), \bar{\pi})}{\kappa_e} \right\rceil \right\} \quad \forall e \in E \\ \Leftrightarrow & \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \geq F(e) \quad \forall e \in E. \end{aligned}$$

□

Proposition 2.22 can be interpreted as follows. The polytope $P_{\text{LPP}} := P(\text{BD})|_x$ describes the convex hull of all line plans that support any passenger routing for the transportation demand given by the OD-matrix, while the polytope $P(\text{SLP})$ describes the convex hull of all line plans that provide sufficient capacity for a fixed passenger routing. Karbstein [20] investigates polyhedral aspects of P_{LPP} and derives several classes of facet-defining inequalities for P_{LPP} .

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Chapter 3

Configuration Model

In this chapter we discuss a novel extended formulation for the line planning problem. The aim of the extended formulation is to tighten the frequency demand constraints of the standard model. A small increase or decrease of the frequency demand of an edge causes a small change in the right hand side of the corresponding frequency demand constraint. This in turn can enlarge or reduce the set of feasible fractional solutions of (SLP), respectively, while the set of integer solutions remains unchanged. The idea is to enumerate the set of possible *frequency configurations* for each edge. We show that such an extended formulation implies general facet-defining inequalities for the standard model and implies a tighter LP relaxation. In this section we prove that the set cover, symmetric band, MIR, and multicover inequalities are implied by the LP relaxation of the configuration model. As for the standard model in the previous chapter, we consider the polytope associated with the configuration model for a relaxation of the line planning to a single edge and can derive a large class of valid inequalities for the configuration model. At the end of this chapter we generalize the configuration model to obtain a new model, which LP relaxation turns out to be even tighter.

We describe the extended configuration model in the following section. An algorithm to compute the set of *minimal configurations* for an edge is given in Section 3.2. We compare the configuration model with the standard model in terms of polyhedral aspects in Section 3.3. In Section 3.4, we describe a generalization of the configuration model and compare this new model with the standard and configuration model.

3.1 Extended Model

We introduce the configuration model in the following. Since the extended formulation aims at strengthening the frequency demand constraints (2.1) of the standard model, we start to explain the underlying idea by considering an alternative method to ensure the frequency demand requirement (1.1) of a given edge $e \in E$. The set of *configurations of e* , denoted by $\bar{\mathcal{Q}}(e)$, is a relaxed description of all possibilities to cover the demand of e . To be more precise, a configuration $q \in \bar{\mathcal{Q}}(e)$ states the number q_f of lines that are operated at frequency f on e . It does not specify which lines in $\mathcal{L}(e)$ are operated at frequency $f \in \mathcal{F}$. A formal definition reads as follows.

Definition 3.1. For $e \in E$ we denote by

$$\bar{\mathcal{Q}}(e) := \left\{ q = (q_{f_1}, \dots, q_{f_m}) \in \mathbb{Z}_+^{\mathcal{F}} : \sum_{f \in \mathcal{F}} q_f \leq |\mathcal{L}(e)|, \sum_{f \in \mathcal{F}} f \cdot q_f \geq F(e) \right\}$$

the set of *feasible frequency configurations of e* or shortly *configurations of e* .

We consider an example. Let $e \in E$ be an edge with frequency demand of $F(e) = 9$ and three lines containing this edge. Let the available frequencies be given by $\mathcal{F} = \{2, 8\}$. There are in total five configurations, namely $\bar{\mathcal{Q}}(e) = \{(0, 2), (0, 3), (1, 1), (1, 2), (2, 1)\}$. The first coordinate of a configuration in $\bar{\mathcal{Q}}$ gives the number of lines that are operated at frequency 2 and the second coordinate the number of lines that are operated at frequency 8. For instance, in the last configuration $(2, 1)$ there are two lines at frequency 2 and one line at frequency 8.

Our example shows that some configurations might be dominated by others component-wise. That is $\bar{q} \in \bar{\mathcal{Q}}(e)$ is *dominated* by $q \in \bar{\mathcal{Q}}$, if $q_f \leq \bar{q}_f$ for all $f \in \mathcal{F}$ and $q_{\tilde{f}} < \bar{q}_{\tilde{f}}$ for at least one $\tilde{f} \in \mathcal{F}$. This observation gives rise to the following definition.

Definition 3.2. For $e \in E$ denote by

$$\mathcal{Q}(e) := \{q \in \bar{\mathcal{Q}}(e) : (q_{f_1}, \dots, q_{f_i} - 1, \dots, q_{f_m}) \notin \bar{\mathcal{Q}}(e) \quad \forall i = 1, \dots, m\}$$

the set of *minimal configurations of e* .

A configuration is *minimal* if and only if it is not dominated by any other configuration. In the above example the set of minimal configurations is $\mathcal{Q}(e) = \{(0, 2), (1, 1)\}$.

Remark 3.3. The minimal configurations can be considered as a lower bound on the set of all configurations. For every edge $e \in E$ and for each

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configuration $\bar{q} \in \bar{\mathcal{Q}}(e)$ there is a minimal configuration $q \in \mathcal{Q}(e)$ such that $\bar{q}_f \geq q_f$ for all $f \in \mathcal{F}$. This follows directly from Definition 3.2.

Now, we will argue that satisfying the frequency demand is equivalent to choosing a minimal configuration for each edge. Let $(q(e))_{e \in E}$ be a vector of minimal configurations, i.e., $q(e) \in \mathcal{Q}(e)$ for all $e \in E$. And let $(\bar{\mathcal{L}}, \bar{f})$ be a line plan such that $|\{\ell \in \bar{\mathcal{L}}(e) : \bar{f}(\ell) = f\}| \geq q(e)_f$ for all $e \in E, f \in \mathcal{F}$. Then we know by Definition 3.2 and Definition 3.1 that the following holds for each edge $e \in E$:

$$\sum_{\ell \in \bar{\mathcal{L}}(e)} \bar{f}(\ell) \geq \sum_{f \in \mathcal{F}} q(e)_f \cdot f \geq F(e).$$

Therefore, $(\bar{\mathcal{L}}, \bar{f})$ is a feasible line plan. On the other hand, let $(\bar{\mathcal{L}}, \bar{f})$ be a feasible line plan. Then $\bar{q}(e)$ with $\bar{q}(e)_f := |\{\ell \in \bar{\mathcal{L}}(e) : \bar{f}(\ell) = f\}|$ for all $f \in \mathcal{F}$ defines for every edge e a feasible configuration. There is a minimal configuration $q(e)$ for each edge such that $\bar{q}(e)_f \geq q(e)_f$ for all $f \in \mathcal{F}$, cf. Remark 3.3. This means, a feasible line plan can be found by choosing a minimal configuration for each edge and finding a line plan that satisfies the lower bounds given by these configurations. This leads to the following formulation.

We extend the standard model (SLP) with binary variables $y_{e,q}$ indicating for each edge $e \in E$ which configuration $q \in \mathcal{Q}(e)$ is chosen. This results in the following integer program:

$$\text{(QLP)} \quad \min \quad \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f}$$

$$\text{s.t.} \quad \sum_{\ell \in \mathcal{L}(e)} x_{\ell,f} \geq \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_{e,q} \quad \forall e \in E, \forall f \in \mathcal{F} \quad (3.1)$$

$$\sum_{q \in \mathcal{Q}(e)} y_{e,q} = 1 \quad \forall e \in E \quad (3.2)$$

$$\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L} \quad (3.3)$$

$$x_{\ell,f} \in \{0, 1\} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \quad (3.4)$$

$$y_{e,q} \in \{0, 1\} \quad \forall e \in E, \forall q \in \mathcal{Q}(e). \quad (3.5)$$

The (*extended*) *configuration model* (QLP) also minimizes the cost of a line plan. The *configuration assignment constraints* (3.2) ensure that exactly one configuration is chosen for each edge. The *coupling constraints* (3.1) force

enough lines to be operated on each edge at every frequency for the chosen configurations.

Example 3.1 (Example 2.1 continued). We reconsider the line planning problem defined in Figure 2.1 and recall that $\mathcal{F} = \{2, 8\}$, $F(e_1) = 9$, $F(e_2) = 1$, $\mathcal{L}(e_1) = \{\ell_1, \ell_2, \ell_3\}$, and $\mathcal{L}(e_2) = \{\ell_1, \ell_2\}$. The minimal configurations for the edges are given by $\mathcal{Q}(e_1) = \{q_1 = (0, 2), q_2 = (1, 1)\}$ and $\mathcal{Q}(e_2) = \{q_1 = (0, 1), q_2 = (1, 0)\}$. The configuration model for this example is:

$$\begin{aligned}
 (\text{QLP}) \quad \min \quad & 4x_{\ell_1,2} + 16x_{\ell_1,8} + 4x_{\ell_2,2} + 16x_{\ell_2,8} + 2x_{\ell_3,2} + 8x_{\ell_3,8} \\
 \text{s.t.} \quad & x_{\ell_1,2} + x_{\ell_2,2} + x_{\ell_3,2} - y_{e_1,q_2} \geq 0 \\
 & \quad + x_{\ell_1,8} + x_{\ell_2,8} + x_{\ell_3,8} - 2y_{e_1,q_1} - y_{e_1,q_2} \geq 0 \\
 & x_{\ell_1,2} + x_{\ell_2,2} - y_{e_2,q_2} \geq 0 \\
 & \quad + x_{\ell_1,8} + x_{\ell_2,8} - y_{e_2,q_1} \geq 0 \\
 & x_{\ell_1,2} + x_{\ell_1,8} \leq 1 \\
 & \quad + x_{\ell_2,2} + x_{\ell_2,8} \leq 1 \\
 & \quad + x_{\ell_3,2} + x_{\ell_3,8} \leq 1 \\
 & \quad + y_{e_1,q_1} + y_{e_1,q_2} = 1 \\
 & \quad + y_{e_2,q_1} + y_{e_2,q_2} = 1 \\
 & x_{\ell_i,f} \in \{0, 1\} \\
 & y_{e,q} \in \{0, 1\}.
 \end{aligned}$$

An optimal solution to (QLP) is given by (x^*, y^*) , where $x_{\ell_3,8}^* = 1$, $x_{\ell_2,2}^* = 1$, $y_{e_1,q_2}^* = 1$, $y_{e_2,q_2}^* = 1$, and $x_{\ell,f}^* = 0$ and $y_{e,q}^* = 0$ otherwise. The objective function value of this solution is 12.

3.2 Computing Configurations

In this section we want to give an iterative algorithm that computes for a given edge $e \in E$ the set of minimal configurations of e . Let us recall for reasons of convenience that by Definition 3.1 any minimal configuration $q \in \mathcal{Q}(e)$ satisfies the *frequency demand condition* $\sum_{f \in \mathcal{F}} f \cdot q_f \geq F(e)$ and the *cardinality condition* $\sum_{f \in \mathcal{F}} q_f \leq |\mathcal{L}(e)|$.

Throughout the algorithm we maintain two sets. The set \mathcal{Q} contains already constructed minimal configurations and is initially empty. In the second set \mathcal{T} , initially containing the zero vector, we keep vectors $q \in \mathbb{Z}_+^{\mathcal{F}}$ which satisfy the cardinality condition but do not yet satisfy the frequency demand condition. We iterate over \mathcal{F} in decreasing order, i.e., starting with the highest frequency f_m . And for each $f' \in \mathcal{F}$ in turn we iterate over all vectors in \mathcal{T} . In the beginning of an iteration for $f' \in \mathcal{F}$ and $q \in \mathcal{T}$ we calculate the smallest number $r_{f'}^{\max} \in \mathbb{N}$ that is needed such that $\sum_{f \in \mathcal{F}} f \cdot q_f + f' \cdot r_{f'}^{\max} \geq F(e)$. If $\sum_{f \in \mathcal{F}} q_f + r_{f'}^{\max} > |\mathcal{L}(e)|$, i.e., increasing the f' -coordinate of q to $r_{f'}^{\max}$

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would violate the cardinality condition, then we remove q from \mathcal{T} . Otherwise, we calculate the smallest number $r_{f'}^{\min} \in \mathbb{Z}_+$ to which we need to increase the f' -coordinate of q , such that a minimal configuration can be constructed from q in any subsequent iteration corresponding to a smaller frequency, i.e.,

$$r_{f'}^{\min} := \begin{cases} \min \{j \in \mathbb{Z}_+ : \sum_{f \in \mathcal{F}} q_f + j + \left\lceil \frac{F^r - j \cdot f'}{f_{i-1}} \right\rceil \leq |\mathcal{L}(e)|\} & \text{if } f' = f_i, i > 1 \\ r_{f'}^{\max} & \text{else,} \end{cases}$$

where $F^r := F(e) - \sum_{f \in \mathcal{F}} f \cdot q_f$. For a more detailed explanation of $r_{f'}^{\min}$ see Example 3.2. For each $j \in \{r_{f'}^{\min}, \dots, r_{f'}^{\max}\}$ we create a copy $q(j)$ of q and increase the f' -coordinate, that is we set $q(j)_{f'} = j$. Since $\sum_{f \in \mathcal{F}} f \cdot q(r_{f'}^{\max})_f \geq F(e)$ holds, the vector $q(r_{f'}^{\max})$ is added to \mathcal{Q} . All other copies are added to \mathcal{T} in the end of iteration f' . The algorithm is given in detail in Algorithm 1. The correctness of this algorithm is proved in the next proposition.

Proposition 3.4. *Algorithm 1 computes for an edge $e \in E$ the set of minimal configurations.*

Proof. Let $e \in E$ and $\tilde{\mathcal{Q}}$ be the set returned by Algorithm 1 for e . We will first show that $\tilde{\mathcal{Q}} \subseteq \mathcal{Q}(e)$ holds. By construction, for every $q \in \mathcal{T}$ we have $\sum_{f \in \mathcal{F}} q_f \leq |\mathcal{L}(e)|$ throughout the algorithm. When $q \in \mathcal{T}$ is moved into $\tilde{\mathcal{Q}}$, $\sum_{f \in \mathcal{F}} f \cdot q_f \geq F(e)$ additionally holds. Hence we get $\tilde{\mathcal{Q}} \subseteq \bar{\mathcal{Q}}(e)$. To argue that every configuration $\tilde{q} \in \tilde{\mathcal{Q}}$ is minimal, we have a look at the iteration f' when \tilde{q} is constructed from $q \in \mathcal{T}$. The number $r_{f'}^{\max} = \left\lceil \frac{F(e) - \sum_{f \in \mathcal{F}} f \cdot q_f}{f'} \right\rceil$ is the smallest integer such that $\sum_{f \in \mathcal{F}} f \cdot q_f + f' \cdot r_{f'}^{\max} \geq F(e)$. Since $q_f = 0$ for all $f \leq f'$, this implies

$$f' \cdot (r_{f'}^{\max} - 1) + \sum_{\substack{f \in \mathcal{F} \\ f > f'}} f \cdot q_f < F(e).$$

We have $\tilde{q}_f = 0$ for all $f < f'$, $\tilde{q}_{f'} \leq r_{f'}^{\max}$ and $\tilde{q}_f = q_f$ for all $f > f'$ and therefore

$$\tilde{f} \cdot (\tilde{q}_{\tilde{f}} - 1) + \sum_{\substack{f \in \mathcal{F} \\ f \neq \tilde{f}}} f \cdot \tilde{q}_f < F(e) \text{ for } \tilde{f} \in \mathcal{F}.$$

Thus, $\tilde{\mathcal{Q}} \subseteq \mathcal{Q}(e)$.

It is left to show that $\mathcal{Q}(e) \subseteq \tilde{\mathcal{Q}}$ holds. First note that if $q \in \mathcal{T}$ is removed from \mathcal{T} in iteration f' , since $\sum_{f \in \mathcal{F}} q_f + r_{f'}^{\max} > |\mathcal{L}(e)|$, then in no subsequent iteration a minimal configuration can be created from q . Moreover, we only

Algorithm 1: Computing minimal configurations for one edge

Input : Set of frequencies $\mathcal{F} = \{f_1, \dots, f_m\}$, an edge e , the frequency demand $F(e)$ of e and the number $|\mathcal{L}(e)|$ of lines containing e .

Output: The set $\mathcal{Q}(e)$ of minimal feasible configurations of e .

```

1  $\mathcal{Q} := \emptyset$ 
2  $q := (0, \dots, 0) \in \mathbb{Z}_+^k$ 
3  $\mathcal{T} := \{q\}$ 
4 for  $f' := f_m, \dots, f_1$  do
5    $\mathcal{T}' := \emptyset$ 
6   foreach  $q \in \mathcal{T}$  do
7      $F^r := F(e) - \sum_{f \in \mathcal{F}} f \cdot q_f$ 
8      $r_{f'}^{\max} := \left\lceil \frac{F^r}{f'} \right\rceil$ 
9     if  $\sum_{f \in \mathcal{F}} q_f + r_{f'}^{\max} < |\mathcal{L}(e)|$  then
10       $\mathcal{T} := \mathcal{T} \setminus \{q\}$ 
11    else
12      if  $f' = f_i$  for some  $i > 1$  then
13         $r_{f'}^{\min} := \min \left\{ j \in \mathbb{Z}_+ : \sum_{f \in \mathcal{F}} q_f + j + \left\lceil \frac{F^r - j \cdot f'}{f_{i-1}} \right\rceil \leq |\mathcal{L}(e)| \right\}$ 
14      else
15         $r_{f'}^{\min} := r_{f'}^{\max}$ 
16      end
17      for  $j := r_{f'}^{\min}, \dots, r_{f'}^{\max}$  do
18         $\tilde{q} := q$ 
19         $\tilde{q}_{f'} := j$ 
20        if  $j = r_{f'}^{\max}$  then
21           $\mathcal{Q} := \mathcal{Q} \cup \{\tilde{q}\}$ 
22        else
23           $\mathcal{T}' := \mathcal{T}' \cup \{\tilde{q}\}$ 
24        end
25      end
26    end
27  end
28   $\mathcal{T} := \mathcal{T} \cup \mathcal{T}'$ 
29 end
30 return  $\mathcal{Q}$ 

```

create necessary copies of $q \in \mathcal{T}$, i.e., the definition of $r_{f'}^{\min}$ is correct: If we created for $j < r_{f'}^{\min}$ a copy $\tilde{q}(j)$ of \tilde{q} and set $\tilde{q}(j)_{f'}$, then $\tilde{q}(j)$ would be removed from \mathcal{T} in the next iteration due to the cardinality condition. Assume there is a minimal configuration $q \in \mathcal{Q}(e)$ such that $q \notin \tilde{\mathcal{Q}}$ and, hence, $q \notin \mathcal{T}$ holds throughout the algorithm. Let $f' \in \mathcal{F}$ be the smallest frequency such that $q_{f'} > 0$ and q' the vector with $q'_f = q_f$ for all $f \in \mathcal{F} \setminus \{f'\}$ and $q'_{f'} = 0$. Since q was not created from a copy of q' in iteration f' , we can conclude that q' was not added to \mathcal{T} in any iteration. When we carry on with this argument, we get the contradiction that $0 \notin \mathcal{T}$ holds throughout the algorithm. Hence, it follows $\mathcal{Q}(e) \subseteq \tilde{\mathcal{Q}}$. \square

We remark that Algorithm 1 can be seen as an improved version of an algorithm given in [21]. We obtain a significantly shorter running time by introducing the bound $r_{f'}^{\min}$ in each iteration f' , which avoids adding needless vectors to \mathcal{T} . We generated the set of minimal configurations with Algorithm 1 for a set of test instances. Our computations show that we can compute the set of minimal configurations with this algorithm in a short amount of time. Even for the largest instance, which has in total more than 2.3 million minimal configurations, the total CPU time to generate the set of minimal configurations for all edges was significantly less than one second. For more details on the test instances see Chapter 4.

Example 3.2. Let $\mathcal{F} = \{1, 2, 5\}$ and $F(e) = 9$ of an edge e and let there be four edges passing e , i.e., $|\mathcal{L}(e)| = 4$. Assume we are in an iteration of Algorithm 1 for $f' = 2$ and $q = (0, 0, 1)$. Then the residual demand is $F^r = 9 - 5 \cdot 1 = 4$ and we have $r_2^{\max} = \lceil \frac{4}{2} \rceil = 2$. That is, at most two lines have to be operated at frequency 2 to cover the residual demand. Assume we create $\tilde{q} = (0, j, 1)$ for some $j \in \mathbb{Z}_+$ and add \tilde{q} to \mathcal{T} . In the next iteration for frequency 1 in that \tilde{q} is considered the residual demand is $F^r = 9 - 5 \cdot 1 - j \cdot 2$ and hence $r_1^{\max} = \lceil \frac{9-5 \cdot 1 - j \cdot 2}{1} \rceil$. If $1 + j + r_1^{\max} > |\mathcal{L}(e)|$, then \tilde{q} will be removed from \mathcal{T} , since there could not be created a configuration satisfying the frequency condition without violating the cardinality condition. By only choosing j such that $1 + j + \lceil \frac{9-5 \cdot 1 - j \cdot 2}{1} \rceil \leq |\mathcal{L}(e)|$ we would prevent this; hence, we choose only $j \geq 1$. Note that this observation is reflected by the definition of $r_2^{\min} := \min \{j \in \mathbb{Z}_+ : 1 + j + \lceil \frac{9-5 \cdot 1 - j \cdot 2}{1} \rceil \leq |\mathcal{L}(e)|\}$.

3.3 Polyhedral Aspects

In the following subsections we compare the configuration model with the standard model in terms of polyhedral aspects and investigate the polytope

of the single edge relaxation for the configuration model.

3.3.1 Model Comparison

In this section we compare the standard with the extend configuration model for the line planning problem. We show that although the two formulations are equivalent, the configuration model yields a stronger LP relaxation. For this purpose we will reconsider the classes of valid inequalities discussed in Section 2.2.2 and Section 2.2.3 and show that they are implied by the LP relaxation of the configuration model.

Remark 3.5. The three classes of valid inequalities presented in Section 2.2.4 are not in general valid for every point in $P_{LP}(\text{QLP})$, i.e., these inequalities improve the LP relaxation of the configuration model as well. For an example we refer to Example 3.7 in Section 3.4.

When we introduced the set of configurations of an edge, we already argued that the configuration model describes the same set of feasible line plans as the standard model. In the following lemma we give a formal proof by comparing the associated polytopes.

Lemma 3.6. *(QLP) provides an extended formulation for (SLP), i.e.,*

$$P(\text{QLP})|_x = P(\text{SLP}).$$

Proof. $P(\text{SLP})$ and $P(\text{QLP})$ are the convex hull of all feasible solutions of (SLP) and (QLP) respectively. Hence, it suffices to show that all integer points in $P(\text{SLP})$ lie in $P(\text{QLP})|_x$ and vice versa.

We show first $P(\text{QLP})|_x \subseteq P(\text{SLP})$. Let (\bar{x}, \bar{y}) be an integer point in $P(\text{QLP})$. Obviously, \bar{x} satisfies (2.2) and (2.3). We further get

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot \bar{x}_{\ell, f} \stackrel{(3.1)}{\geq} \sum_{f \in \mathcal{F}} \left(f \cdot \sum_{q \in \mathcal{Q}(e)} q_f \cdot \bar{y}_{e, q} \right) = \underbrace{\sum_{q \in \mathcal{Q}(e)} \bar{y}_{e, q}}_{\stackrel{(3.2)}{=} 1} \cdot \underbrace{\sum_{f \in \mathcal{F}} f \cdot q_f}_{\geq F(e) \forall q \in \mathcal{Q}(e)} \geq F(e)$$

Hence, \bar{x} satisfies (2.1) as well and is contained in $P(\text{SLP})$.

Now, let \bar{x} be an integer point in $P(\text{SLP})$. We want to find \bar{y} such that $(\bar{x}, \bar{y}) \in P(\text{QLP})$. We define for each $e \in E$ the configuration \bar{q}^e by:

$$\bar{q}_f^e := \sum_{\ell \in \mathcal{L}(e)} \bar{x}_{\ell, f} \text{ for all } f \in \mathcal{F}.$$

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It follows directly from (2.1), (2.2), and (2.3) that $\bar{q}^e \in \bar{\mathcal{Q}}(e)$ for all $e \in E$. For every $e \in E$ let $q^e \in \mathcal{Q}(e)$ be a minimal configuration such that $\bar{q}_f^e \geq q_f^e$ for all $f \in \mathcal{F}$, cf. Remark 3.3. Now we can define \bar{y} :

$$\bar{y}_{e,q} := \begin{cases} 1 & \text{if } q = q^e \\ 0 & \text{else} \end{cases} \quad \text{for all } e \in E, q \in \mathcal{Q}(e).$$

Let $f \in \mathcal{F}$ and $e \in E$, the following holds:

$$\sum_{\ell \in \mathcal{L}(e)} \bar{x}_{\ell,f} = \bar{q}_f^e \geq q_f^e = \sum_{q \in \mathcal{Q}(e)} q_f \cdot \bar{y}_{e,q}.$$

Therefore (\bar{x}, \bar{y}) satisfies (3.1)–(3.5) and $(\bar{x}, \bar{y}) \in P(\text{QLP})$. \square

Lemma 3.6 implies that the integer programs (SLP) and (QLP) minimize the same objective function over an identical set of points in $\{0, 1\}^{\mathcal{L} \times \mathcal{F}}$. Hence, solving (SLP) is equivalent to solving (QLP).

However, this is not true for the LP relaxations (SLP)_{LP} and (QLP)_{LP}, which we obtain by relaxing the binary constraints (2.3), (3.4), and (3.5) to $0 \leq x_{\ell,f} \leq 1 \ \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F}$ and $0 \leq y_{e,q} \leq 1 \ \forall e \in E, \forall q \in \mathcal{Q}(e)$, respectively. For instance x^* , defined by $x_{\ell_3,8}^* = 1$, $x_{\ell_2,2}^* = \frac{1}{2}$, and $x_{\ell,f}^* = 0$ otherwise, is an optimal fractional solution to the standard model in Example 2.1. This solution has an objective function value of 10. However, the optimal fractional solutions to the configuration model have an objective function value of 12. We will see in the following that the configuration model provides in general a stronger LP relaxation than the standard model.

Theorem 3.7. *The LP relaxation of $P(\text{QLP})|_x$ is tighter than the LP relaxation of $P(\text{SLP})$, i.e.,*

$$P_{LP}(\text{QLP})|_x \subseteq P_{LP}(\text{SLP}).$$

Proof. Let $(\bar{x}, \bar{y}) \in P_{LP}(\text{QLP})$. Obviously, \bar{x} satisfies (2.2) and (2.3). We further get

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot \bar{x}_{\ell,f} \stackrel{(3.1)}{\geq} \sum_{f \in \mathcal{F}} \left(f \cdot \sum_{q \in \mathcal{Q}(e)} q_f \cdot \bar{y}_{e,q} \right) = \underbrace{\sum_{q \in \mathcal{Q}(e)} \bar{y}_{e,q}}_{\stackrel{(3.2)}{=} 1} \cdot \underbrace{\sum_{f \in \mathcal{F}} f \cdot q_f}_{\geq F(e) \ \forall q \in \mathcal{Q}(e)} \geq F(e).$$

Hence, \bar{x} satisfies (2.1) as well and is contained in $P_{LP}(\text{SLP})$. \square

The converse, i.e., $P_{LP}(\text{SLP}) \subseteq P_{LP}(\text{QLP})|_x$, does not hold in general as we have already seen in the above example. Indeed, the ratio of the optimal objectives of the two LP relaxations can be arbitrarily large.

Example 3.3. Consider an instance of the line planning problem involving only one edge $E = \{e\}$, one line $\mathcal{L}(e) = \{l\}$, a frequency demand $F(e) = 6$, and one frequency $\mathcal{F} = \{M\}$ such that $M > 6$ with cost function $c_{l,M} = M$. The only minimal configuration for e is $q = (1)$.

$$\begin{array}{ll}
 (\text{QLP})_{LP}: \min & M \cdot x_{\ell,M} \\
 \text{s.t.} & x_{\ell,M} - y_q \geq 0 \\
 & y_q = 1 \\
 & y_q, x_{\ell,M} \geq 0
 \end{array}
 \qquad
 \begin{array}{ll}
 (\text{SLP})_{LP}: \min & M \cdot x_{\ell,M} \\
 \text{s.t.} & M \cdot x_{\ell,M} \geq 6 \\
 & x_{\ell,M} \geq 0
 \end{array}$$

Obviously, $x_{l,M} = 1$ is the only and hence optimal solution to $(\text{QLP})_{LP}$ with objective value M and $x_{l,M} = \frac{6}{M}$ is an optimal solution to $(\text{SLP})_{LP}$ with objective value 6.

In the following we show that the LP relaxation of the configuration model implies general classes of facet-defining inequalities for the line planning polytope $P(\text{SLP})$, which we presented in Section 2.2 and are discussed in the literature.

Band Inequalities

In this section we take another look at the band inequalities, which we introduced in Section 2.2.2. We proved that they define a class of valid inequalities of $P(\text{SLP})$ and gave examples to show that they improve the LP relaxation of $P(\text{SLP})$. Now we consider *symmetric bands*.

Definition 3.8. Let $e \in E$, $f \in \mathcal{F}$, and $f_{\mathcal{B}} : \mathcal{L}(e) \rightarrow \mathcal{F} \cup \{0\}$ be a valid band of e . We call $f_{\mathcal{B}}$ *symmetric* if $f_{\mathcal{B}}(\ell) = f$ for all $\ell \in \mathcal{L}(e)$.

Symmetric bands are of particular interest, as they are implied by the LP relaxation of the configuration model.

Theorem 3.9. *Let $e \in E$ and $f_{\mathcal{B}}$ be a valid symmetric band of e . Then the band inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > f_{\mathcal{B}}(\ell)}} x_{\ell,f} \geq 1$$

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is implied by the LP relaxation of the configuration model, i.e., all band inequalities (2.4) that are induced by a valid symmetric band are valid for $P_{LP}(\text{QLP})$.

Proof. Assume $f_{\mathcal{B}}$ is a valid symmetric band of some edge e with $f_{\mathcal{B}}(\ell) = \tilde{f}$ for all $\ell \in \mathcal{L}(e)$ with $\tilde{f} \in \mathcal{F}$. Thus $\sum_{\ell \in \mathcal{L}(e)} f_{\mathcal{B}}(\ell) = |\mathcal{L}(e)| \cdot \tilde{f} < F(e)$. Note that $\tilde{f} < f_m$ holds by the assumption in Remark 2.2. Hence, in every minimal configuration $q \in \mathcal{Q}(e)$ there is a frequency $f > \tilde{f}$ such that $q_f \geq 1$. Starting from the coupling constraints (3.1), we get:

$$\begin{aligned}
 & \sum_{\ell \in \mathcal{L}(e)} x_{\ell, f} \geq \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_q && \forall f \in \mathcal{F} \\
 \Rightarrow & \sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} \sum_{\ell \in \mathcal{L}(e)} x_{\ell, f} \geq \sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_q \\
 \Rightarrow & \sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} x_{\ell, f} \geq \sum_{q \in \mathcal{Q}(e)} y_q \cdot \underbrace{\sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} q_f}_{\geq 1} \\
 & \geq \sum_{q \in \mathcal{Q}(e)} y_q \\
 & = 1.
 \end{aligned}$$

□

We observe that the set cover inequalities are induced by the valid symmetric band $f_{\mathcal{B}}(\ell) \equiv 0$. This implies that they are also valid for the LP relaxation of the configuration model.

Corollary 3.10. *The set cover inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} x_{\ell, f} \geq 1$$

is valid for $P_{LP}(\text{QLP})$ for all $e \in E$.

The same does not hold for the standard model, which we showed in Example 3.3. In the following we give another example that symmetric band inequalities strengthen the LP relaxation of the standard model. Furthermore, band inequalities induced by non-symmetric bands are not implied by the LP relaxation of the configuration model, see also the following example.

Example 3.4 (Example 2.2 continued). Again, we consider the line planning instance given in Figure 2.2 and recall $\mathcal{F} = \{1, 3, 4\}$, $F(e_1) = 5$, $\mathcal{L}(e_1) = \{\ell_1, \ell_2, \ell_3\}$. First, consider the symmetric maximal band $f_{\mathcal{B}}$ of e_1 given by $f_{\mathcal{B}}(\ell) = 1$ for all $\ell \in \mathcal{L}(e_1)$. One can show that the corresponding band inequality

$$x_{\ell_1,3} + x_{\ell_1,4} + x_{\ell_2,3} + x_{\ell_2,4} + x_{\ell_3,3} + x_{\ell_3,4} \geq 1 \quad (3.6)$$

is facet-defining for $P(\text{SLP})$ in this particular example. Inequality (3.6) is valid for $P_{LP}(\text{QLP})$ by Theorem 3.9. However, the same does not hold for $P_{LP}(\text{SLP})$. Inequality (3.6) is violated by $\tilde{x} \in P_{LP}(\text{SLP})$, where $\tilde{x}_{\ell_1,4} = \frac{3}{4}$, $\tilde{x}_{\ell_2,1} = \tilde{x}_{\ell_3,1} = 1$, and $\tilde{x}_{\ell,f} = 0$ otherwise.

Now, reconsider the band inequality

$$x_{\ell_1,1} + x_{\ell_1,3} + x_{\ell_1,4} + x_{\ell_2,3} + x_{\ell_2,4} + x_{\ell_3,4} \geq 1$$

from Example 2.2. The inequality is violated by $(\bar{x}, \bar{y}) \in P_{LP}(\text{QLP})$, where $\bar{x}_{\ell_1,3} = \frac{1}{2}$, $\bar{x}_{\ell_2,1} = \bar{x}_{\ell_3,3} = 1$, and $\bar{x}_{\ell,f} = 0$ otherwise. For e_1 we set $\bar{y}_{e_1,q_{12}} = \bar{y}_{e_1,q_{15}} = \frac{1}{2}$ and $\bar{y}_{e_1,q} = 0$ otherwise, where $q_{12} := (0, 2, 0)$ and $q_{15} := (2, 1, 0)$. For e_2 we set $\bar{y}_{e_2,q_{23}} = 1$ and $\bar{y}_{e_2,q} = 0$ otherwise, where $q_{23} := (0, 1, 0)$.

MIR Inequalities

We study in this section the MIR inequalities in the context of the configuration model. Again, we can show that these inequalities are implied by the LP relaxation of the configuration model. The proof is based on the following lemma, a configuration version of Proposition 2.10.

Lemma 3.11. *For $e \in E$, $q \in \mathcal{Q}(e)$, and $\lambda \in (0, 1)$, it holds*

$$\sum_{f \in \mathcal{F}} (r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)) q_f \geq r \cdot \lceil \lambda \cdot F(e) \rceil,$$

where $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor$ and $r_f = \lambda f - \lfloor \lambda f \rfloor$.

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Proof. $q \in \mathcal{Q}(e)$ implies $\sum_{f \in \mathcal{F}} f \cdot q_f \geq F(e)$ and hence we get for $\lambda \in (0, 1)$

$$\begin{aligned}
 \lambda \cdot F(e) &\leq \lambda \cdot \sum_{f \in \mathcal{F}} f \cdot q_f = \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} \lambda \cdot f \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} \lambda \cdot f \cdot q_f \\
 &\leq \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} (\lfloor \lambda \cdot f \rfloor + r_f) \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} (\lfloor \lambda \cdot f \rfloor + 1) \cdot q_f \\
 &= \underbrace{\sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot q_f}_{\geq 0} + \underbrace{\sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} q_f}_{\in \mathbb{Z}}.
 \end{aligned}$$

Applying Lemma 2.9 yields

$$\begin{aligned}
 r \cdot \lceil \lambda \cdot F(e) \rceil &\leq \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot q_f + r \cdot \left(\sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} q_f \right) \\
 &= \sum_{f \in \mathcal{F}} (r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)) \cdot q_f.
 \end{aligned}$$

□

Theorem 3.12. *Let $\lambda \in (0, 1)$, $e \in E$, $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor$ and $r_f = \lambda f - \lfloor \lambda f \rfloor$. Then the MIR inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} (r \lfloor \lambda f \rfloor + \min(r_f, r)) x_{\ell, f} \geq r \lceil \lambda F(e) \rceil$$

is implied by the LP relaxation of the configuration model, i.e., the MIR inequalities (2.6) are valid for $P_{LP}(\text{QLP})|_x$.

Proof. Let $(x, y) \in P_{LP}(\text{QLP})$. Then by (3.1)

$$\sum_{\ell \in \mathcal{L}(e)} x_{\ell, f} \geq \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_q \quad \forall f \in \mathcal{F}.$$

Scaling this inequality by $\lambda_f^r := r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)$ yields

$$\begin{aligned}
 \sum_{\ell \in \mathcal{L}(e)} \lambda_f^r \cdot x_{\ell, f} &\geq \sum_{q \in \mathcal{Q}(e)} \lambda_f^r \cdot q_f \cdot y_q && \forall f \in \mathcal{F} \\
 \Rightarrow \sum_{f \in \mathcal{F}} \sum_{\ell \in \mathcal{L}(e)} \lambda_f^r \cdot x_{\ell, f} &\geq \sum_{f \in \mathcal{F}} \sum_{q \in \mathcal{Q}(e)} \lambda_f^r \cdot q_f \cdot y_q \\
 \Leftrightarrow \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lambda_f^r \cdot x_{\ell, f} &\geq \sum_{q \in \mathcal{Q}(e)} \sum_{f \in \mathcal{F}} \lambda_f^r \cdot q_f \cdot y_q \\
 &\stackrel{(*)}{\geq} \sum_{q \in \mathcal{Q}(e)} r \cdot \lfloor \lambda \cdot F(e) \rfloor \cdot y_q \\
 &= r \cdot \lfloor \lambda \cdot F(e) \rfloor \cdot \sum_{q \in \mathcal{Q}(e)} y_q \\
 &\stackrel{(3.2)}{=} r \cdot \lfloor \lambda \cdot F(e) \rfloor.
 \end{aligned}$$

(*) apply Lemma 3.11 here. □

We proved in Theorem 2.14 that the multicover inequalities proposed by Goosens et al. [17] are dominated by the MIR inequalities. Hence, it follows with the previous theorem the following.

Corollary 3.13. *The multicover inequalities (2.8) are implied by the LP relaxation of the configuration model. I.e., let $e \in E$ and $s \in \mathbb{N}$ such that $2 \leq s + 1 < F(e)$. Define $\mathcal{F}^k := \{f \in \mathcal{F} : (k - 1)F(e) \leq s \cdot f < kF(e)\}$ for $k = 1, \dots, s$ and $\mathcal{F}^C := \mathcal{F} \setminus \bigcup_{k=1}^s \mathcal{F}^k$, then the multicover inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{k=1}^s \sum_{f \in \mathcal{F}^k} kx_{\ell, f} + \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}^C} (s + 1)x_{\ell, f} \geq s + 1$$

is valid for $P_{LP}(\text{QLP})$.

Remember that we showed in Example 2.3 and Example 2.4 that the MIR as well as the multicover inequalities strengthen the LP relaxation of the standard model.

3.3.2 Single Edge Relaxation

In this section we analyze the polytope of the single edge relaxation for the configuration model to derive classes of facet-defining inequalities for

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$P(\text{QLP})$. Further, we describe a large class of valid inequalities, which we believe to complete the description of the polytope of this relaxation.

We introduced the polytope of the single edge relaxation in Section 2.2.5 for the standard model. Analogously, we define for an edge $e \in E$ the polytope

$$P(\text{QLP}|e) := \text{conv} \left\{ (x, y) \in \{0, 1\}^{\mathcal{L}(e) \times \mathcal{F} \times \mathcal{Q}(e)} : \begin{array}{l} (\text{QLP}) \text{ (3.1) for } e, \\ (\text{QLP}) \text{ (3.2) for } e, \\ (\text{QLP}) \text{ (3.3) for all } \ell \in \mathcal{L}(e) \end{array} \right\}$$

for the configuration model. We want to emphasize that we can apply Lemma 3.6 and Theorem 3.7 also to $P(\text{SLP}|e)$ and $P(\text{QLP}|e)$ and derive that $P(\text{QLP}|e)|_x = P(\text{SLP}|e)$ and $P_{LP}(\text{QLP}|e)|_x \subseteq P_{LP}(\text{SLP}|e)$ holds for every $e \in E$. Obviously, any inequality that is valid for $P(\text{SLP}|e)$ is also valid for $P(\text{SLP})$ and $P(\text{QLP})$. Inequalities that are valid for $P(\text{QLP}|e)$ are in general only valid for $P(\text{QLP})$.

We start by determining the dimension of $P(\text{QLP}|e)$.

Proposition 3.14. *Let $e \in E$, then*

$$\dim(P(\text{QLP}|e)) = |\mathcal{L}(e)| \cdot |\mathcal{F}| + |\mathcal{Q}(e)| - 1$$

holds if and only if $P(\text{SLP}|e)$ is full-dimensional.

Proof. “ \Rightarrow ” Let $P(\text{SLP}|e)$ not be full-dimensional. Since equation (3.2) is valid for $P(\text{QLP}|e)$, we have $\dim(P(\text{QLP}|e)) \leq |\mathcal{L}(e)| \cdot |\mathcal{F}| + |\mathcal{Q}(e)| - 1$. Since $P(\text{SLP}|e)$ is not full-dimensional, we know from Proposition 2.20 that $\sum_{f \in \mathcal{F}} x_{\ell, f} = 1$ holds for every $\ell \in \mathcal{L}(e)$ for every point in $P(\text{QLP}|e)$. Hence, we get $\dim(P(\text{QLP}|e)) < |\mathcal{L}(e)| \cdot |\mathcal{F}| + |\mathcal{Q}(e)| - 1$.

“ \Leftarrow ” Now, let $P(\text{SLP}|e)$ be full-dimensional. We define $q^m \in \mathbb{Z}_+^{\mathcal{F}}$ with $q_f^m = 0$ for all $f \in \mathcal{F} \setminus \{f_m\}$ and $q_{f_m}^m = \left\lceil \frac{F(e)}{f_m} \right\rceil$. Since $P(\text{SLP}|e)$ is full-dimensional, we know that $(|\mathcal{L}(e)| - 1) \cdot f_m \geq F(e)$ and therefore $q_{f_m}^m \leq |\mathcal{L}(e)| - 1$. Hence $q^m \in \mathcal{Q}(e)$ and we define y^m with $y_{e, q}^m = 0 \forall q \in \mathcal{Q}(e) \setminus \{q^m\}$ and $y_{e, q^m}^m = 1$. For each $\tilde{\ell} \in \mathcal{L}(e)$ and $\tilde{f} \in \mathcal{F}$ the points $(x^{\tilde{\ell}, \tilde{f}}, y^m)$, $(x^{\tilde{\ell}, 0}, y^m)$ and (x^{f_m}, y^m) are feasible for $P(\text{QLP}|e)$, with $x^{\tilde{\ell}, \tilde{f}}$, $x^{\tilde{\ell}, 0}$, x^{f_m} from the proof of Proposition 2.20. Note, for every $\tilde{q} \in \mathcal{Q}(e)$ there exists by definition a vector $x \in P(\text{SLP}|e)$ such that $\sum_{\ell \in \mathcal{L}(e)} x_{\ell, f} = \tilde{q}_f$ for all $f \in \mathcal{F}$. For every $\tilde{q} \in \mathcal{Q}(e) \setminus \{q^m\}$ let $x^{\tilde{q}} \in P(\text{SLP}|e)$ be such a vector, chosen arbitrarily. Define $y^{\tilde{q}}$ with $y_{e, \tilde{q}}^{\tilde{q}} = 1$ and $y_{e, q}^{\tilde{q}} = 0$ for all $q \in \mathcal{Q}(e) \setminus \{\tilde{q}\}$. Obviously, $(x^{\tilde{q}}, y^{\tilde{q}}) \in P(\text{QLP}|e)$ for every $\tilde{q} \in \mathcal{Q}(e) \setminus \{q^m\}$. Hence, we found $|\mathcal{Q}(e) - 1|$ new affinely independent points in $P(\text{QLP}|e)$. We defined in total $|\mathcal{L}(e)| \cdot |\mathcal{F}| + |\mathcal{Q}(e)|$ affinely independent points in $P(\text{QLP}|e)$, which proves the proposition. \square

Now we can prove that the coupling constraint (3.1) for f_m is facet-defining.

Proposition 3.15. *For every edge $e \in E$ the inequality*

$$\sum_{l \in \mathcal{L}(e)} x_{l, f_m} \geq \sum_{q \in \mathcal{Q}(e)} q_{f_m} \cdot y_{e, q} \quad (3.7)$$

is facet-defining for $P(\text{QLP}|e)$, if $P(\text{SLP}|e)$ is full-dimensional.

Proof. Let $e \in E$ and let $P(\text{SLP}|e)$ be full-dimensional. Proposition 3.14 implies that in order to prove the statement it suffices to define $|\mathcal{L}(e)| \cdot |\mathcal{F}| + |\mathcal{Q}(e)| - 1$ many affinely independent points in $P(\text{QLP}|e)$ that satisfy (3.7) with equality. Recall from the proof of Proposition 3.14 the point y^m and the minimal configuration $q^m \in \mathbb{Z}_+^{\mathcal{F}}$, where $q_f^m = 0 \forall f \in \mathcal{F} \setminus \{f_m\}$ and $q_{f_m}^m = \left\lceil \frac{F(e)}{f_m} \right\rceil$, and remember that $q_{f_m}^m \leq |\mathcal{L}(e)| - 1$ holds. Now, for all $\tilde{\ell} \in \mathcal{L}(e)$ and $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$ define the following points: Choose an arbitrary set $\mathcal{L}' \subseteq \mathcal{L}(e) \setminus \{\tilde{\ell}\}$ such that $|\mathcal{L}'| = q_{f_m}^m$ and define

$$x_{\tilde{\ell}, \tilde{f}} = \begin{cases} 1 & \text{if } \ell = \tilde{\ell}, f = \tilde{f} \\ 1 & \text{if } \ell \in \mathcal{L}', f = f_m \\ 0 & \text{else} \end{cases}$$

and

$$x_{\tilde{\ell}, 0} = \begin{cases} 1 & \text{if } \ell \in \mathcal{L}', f = f_m \\ 0 & \text{else.} \end{cases}$$

It can be easily verified that $(x_{\tilde{\ell}, \tilde{f}}, y^m)$, $(x_{\tilde{\ell}, 0}, y^m) \in P(\text{QLP}|e)$. For every minimal configuration $\tilde{q} \in \mathcal{Q}(e) \setminus \{q^m\}$ let $(x^{\tilde{q}}, y^{\tilde{q}}) \in P(\text{QLP}|e)$ be as in the proof of Proposition 3.14.

Now we want to show that these points satisfy (3.7) with equality. Let $\tilde{\ell} \in \mathcal{L}(e)$ and $\tilde{f} \in \mathcal{F} \setminus \{f_m\}$, then the following holds:

$$\underbrace{\sum_{\ell \in \mathcal{L}(e)} x_{\ell, \tilde{f}}}_{=\left\lceil \frac{F(e)}{f_m} \right\rceil} = \underbrace{\sum_{q \in \mathcal{Q}(e)} q_{f_m} y_{e, q}^m}_{=q_{f_m}^m = \left\lceil \frac{F(e)}{f_m} \right\rceil}$$

The same holds for $(x_{\tilde{\ell}, 0}, y^m)$. Let $\tilde{q} \in \mathcal{Q}(e) \setminus \{q^m\}$, then by construction

$$\sum_{\ell \in \mathcal{L}(e)} x_{\ell, f_m}^{\tilde{q}} = \sum_{q \in \mathcal{Q}(e)} q_{f_m} y_{e, q}^{\tilde{q}}$$

Hence, we found a set of $|\mathcal{F}| \cdot |\mathcal{L}(e)| + |\mathcal{Q}(e)| - 1$ points with the required properties. \square

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In the following lemma we describe a class of valid inequalities for $P(\text{QLP})$. We will see later that they define facets for $P(\text{QLP}|e)$ and improve the LP relaxation of $P(\text{QLP})$. For a motivation look at the edge e_1 in Example 3.5 with $\mathcal{F} = \{2, 5, 9\}$. If the minimal configuration $q_3 = (2, 1, 0) \in \mathcal{Q}(e_1)$ is chosen for e_1 , then the line ℓ_1 has to be operated at frequency 2 or 5 regardless at which frequencies the lines ℓ_2 and ℓ_3 are operated.

Lemma 3.16. *Let $e \in E$ be an edge, $\mathcal{L}' \subseteq \mathcal{L}(e)$ be a subset of lines passing e , $\mathcal{F}' \subseteq \mathcal{F}$. The inequality*

$$\sum_{\ell \in \mathcal{L}'} \sum_{f \in \mathcal{F}'} x_{\ell, f} \geq \sum_{q \in \mathcal{Q}(e)} \max \left\{ 0, \sum_{f \in \mathcal{F}'} q_f - |\mathcal{L}(e) \setminus \mathcal{L}'| \right\} y_{e, q} \quad (3.8)$$

is valid for $P(\text{QLP})$.

Proof. Let $(x, y) \in P(\text{QLP})$ and $\bar{q} \in \mathcal{Q}(e)$ such that $y_{e, \bar{q}} = 1$. Then the following is implied by the coupling constraints (3.1)

$$\begin{aligned} \sum_{\ell \in \mathcal{L}'} \sum_{f \in \mathcal{F}'} x_{\ell, f} &\geq \sum_{q \in \mathcal{Q}(e)} \sum_{f \in \mathcal{F}'} q_f \cdot y_{e, q} - \underbrace{\sum_{\ell \in \mathcal{L}(e) \setminus \mathcal{L}'} \sum_{f \in \mathcal{F}'} x_{\ell, f}}_{\substack{(2.2) \\ \leq 1}} \\ &\geq \sum_{f \in \mathcal{F}'} \bar{q}_f - |\mathcal{L}(e) \setminus \mathcal{L}'|. \end{aligned}$$

Since $x_{\ell, f} \geq 0 \forall \ell \in \mathcal{L}, f \in \mathcal{F}$ and $y_{e, q} = 0$ for all $q \in \mathcal{Q}(e) \setminus \{\bar{q}\}$, it follows

$$\begin{aligned} \sum_{\ell \in \mathcal{L}'} \sum_{f \in \mathcal{F}'} x_{\ell, f} &\geq \max \left(0, \sum_{f \in \mathcal{F}'} \bar{q}_f - |\mathcal{L}(e) \setminus \mathcal{L}'| \right) \\ &= \sum_{q \in \mathcal{Q}(e)} \max \left(0, \sum_{f \in \mathcal{F}'} q_f - |\mathcal{L}(e) \setminus \mathcal{L}'| \right) y_{e, q}. \end{aligned}$$

□

We like to point out that inequality (3.8) is for the case $\mathcal{L}' = \mathcal{L}(e)$ and $\mathcal{F}' = \{f\}$ equivalent to the coupling constraint (3.1) for $e \in E$ and $f \in \mathcal{F}$. That is inequalities (3.8) generalize the coupling constraints. In the following proposition we show that we can generalize these inequalities even further to obtain a larger class.

Proposition 3.17. *Let $e \in E$ and let $\mathcal{F}_B : \mathcal{L}(e) \rightarrow 2^{\mathcal{F}}$ be a map that assigns each line in $\mathcal{L}(e)$ to a subset of frequencies. And define for $q \in \mathcal{Q}(e)$*

$$\alpha(\mathcal{F}_B, q) := \max \left\{ \sum_{f \in \mathcal{F}'} q_f - |\mathcal{L}(e) \setminus \{\ell \in \mathcal{L}(e) : \mathcal{F}' \subseteq \mathcal{F}_B(\ell)\}| : \mathcal{F}' \subseteq \mathcal{F} \right\}.$$

Then the configuration band inequality

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}_{\mathcal{B}}(\ell)} x_{\ell,f} \geq \sum_{q \in \mathcal{Q}(e)} \max \{0, \alpha(\mathcal{F}_{\mathcal{B}}, q)\} y_{e,q} \quad (3.9)$$

is valid for $P(\text{QLP})$.

Proof. Let $(x, y) \in P(\text{QLP})$ and $\bar{q} \in \mathcal{Q}(e)$ such that $y_{\bar{q}} = 1$. Let $\mathcal{F}' \subseteq \mathcal{F}$ be an arbitrary subset of frequencies. Then we get for all $f \in \mathcal{F}'$

$$\sum_{\substack{\ell \in \mathcal{L}(e) \\ \mathcal{F}' \subseteq \mathcal{F}_{\mathcal{B}}(\ell)}} x_{\ell,f} + \sum_{\substack{\ell \in \mathcal{L}(e) \\ \mathcal{F}' \not\subseteq \mathcal{F}_{\mathcal{B}}(\ell)}} x_{\ell,f} = \sum_{\ell \in \mathcal{L}(e)} x_{\ell,f} \stackrel{(3.1)}{\geq} \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_{e,q} = \bar{q}_f.$$

By summing this inequality over all $f \in \mathcal{F}'$ and rearranging we get

$$\begin{aligned} \sum_{\substack{\ell \in \mathcal{L}(e) \\ \mathcal{F}' \subseteq \mathcal{F}_{\mathcal{B}}(\ell)}} \sum_{f \in \mathcal{F}'} x_{\ell,f} &\geq \sum_{f \in \mathcal{F}'} \bar{q}_f - \sum_{\substack{\ell \in \mathcal{L}(e) \\ \mathcal{F}' \not\subseteq \mathcal{F}_{\mathcal{B}}(\ell)}} \underbrace{\sum_{f \in \mathcal{F}'} x_{\ell,f}}_{\stackrel{(2.2)}{\leq} 1} \\ &\geq \sum_{f \in \mathcal{F}'} \bar{q}_f - |\{\ell \in \mathcal{L}(e) : \mathcal{F}' \not\subseteq \mathcal{F}_{\mathcal{B}}(\ell)\}| \\ &= \sum_{f \in \mathcal{F}'} \bar{q}_f - |\mathcal{L}(e) \setminus \{\ell \in \mathcal{L}(e) : \mathcal{F}' \subseteq \mathcal{F}_{\mathcal{B}}(\ell)\}|. \end{aligned}$$

Since $y_q = 0$ for all $q \in \mathcal{Q}(e) \setminus \{\bar{q}\}$, it follows

$$\begin{aligned} \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}_{\mathcal{B}}(\ell)} x_{\ell,f} &\geq \sum_{\substack{\ell \in \mathcal{L}(e) \\ \mathcal{F}' \subseteq \mathcal{F}_{\mathcal{B}}(\ell)}} \sum_{f \in \mathcal{F}'} x_{\ell,f} \\ &\geq \sum_{q \in \mathcal{Q}(e)} \left(\sum_{f \in \mathcal{F}'} q_f - |\mathcal{L}(e) \setminus \{\ell \in \mathcal{L}(e) : \mathcal{F}' \subseteq \mathcal{F}_{\mathcal{B}}(\ell)\}| \right) y_{e,q}. \end{aligned}$$

We chose $\mathcal{F}' \subseteq \mathcal{F}$ arbitrarily and $x_{\ell,f} \geq 0$ holds for all $f \in \mathcal{F}$ and all $\ell \in \mathcal{L}$, therefore we get

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}_{\mathcal{B}}(\ell)} x_{\ell,f} \geq \sum_{q \in \mathcal{Q}(e)} \max \{0, \alpha(\mathcal{F}_{\mathcal{B}}, q)\} y_{e,q}.$$

□

As indicated by the term *configuration band inequality*, the inequalities (3.9) are closely related to the band inequalities (2.4). In particular, we show that the configuration band inequalities imply the band inequalities. An example is given at the end of this section. We start the proof with the following lemma.

Lemma 3.18. *Let $e \in E$ be an edge, $f_{\mathcal{B}} : \mathcal{L}(e) \rightarrow \mathcal{F} \cup \{0\}$ a band of e , and $q \in \mathcal{Q}(e)$ be a minimal configuration of e such that*

$$\sum_{\substack{f \in \mathcal{F} \\ f \geq f'}} |\{\ell \in \mathcal{L}(e) : f_{\mathcal{B}}(\ell) = f\}| \geq \sum_{\substack{f \in \mathcal{F} \\ f \geq f'}} q_f \quad \forall f' \in \mathcal{F},$$

then the following holds

$$\sum_{f \in \mathcal{F}} f_{\mathcal{B}}(\ell) \geq \sum_{f \in \mathcal{F}} f \cdot q_f.$$

Proof. If $|\mathcal{F}| = 1$, then the Lemma is obviously true. Assume $|\mathcal{F}| \geq 1$ and define for a simplified notation $k(f) := |\{\ell \in \mathcal{L}(e) : f_{\mathcal{B}}(\ell) = f\}|$ for all $f \in \mathcal{F}$. Then the assumption in the lemma implies:

$$\sum_{f \in \mathcal{F}} k(f) \geq \sum_{f \in \mathcal{F}} q_f. \quad (3.10)$$

We can derive the following:

$$\begin{aligned} \sum_{f \in \mathcal{F}} f_{\mathcal{B}}(\ell) &= \sum_{f \in \mathcal{F}} f \cdot |\{\ell \in \mathcal{L}(e) : f_{\mathcal{B}}(\ell) = f\}| = \sum_{f \in \mathcal{F}} f \cdot k(f) \\ &= f_1 \cdot k(f_1) + \sum_{f \in \mathcal{F} \setminus \{f_1\}} f \cdot k(f) \\ &= f_1 \cdot k(f_1) + \sum_{f \in \mathcal{F} \setminus \{f_1\}} f (k(f) - q_f) + \sum_{f \in \mathcal{F} \setminus \{f_1\}} f \cdot q_f \\ &\geq f_1 \left(\underbrace{\sum_{f \in \mathcal{F}} k(f) - \sum_{f \in \mathcal{F} \setminus \{f_1\}} q_f}_{\substack{(3.10) \\ \geq q_{f_1}}} \right) + \sum_{f \in \mathcal{F} \setminus \{f_1\}} f \cdot q_f \\ &\geq \sum_{f \in \mathcal{F}} f \cdot q_f. \end{aligned}$$

This completes the proof. \square

Proposition 3.19. *The configuration band inequalities (3.9) imply the band inequalities (2.4). In particular, let $e \in E$ be an edge and $f_{\mathcal{B}}$ be a valid band of e . Define $\mathcal{F}_{\mathcal{B}}(\ell) := \{f \in \mathcal{F} : f > f_{\mathcal{B}}(\ell)\}$ for all $\ell \in \mathcal{L}(e)$ and let $(x, y) \in P_{LP}(\text{QLP})$ such that (x, y) satisfies the configuration band inequality*

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}_{\mathcal{B}}(\ell)} x_{\ell, f} \geq \sum_{q \in \mathcal{Q}(e)} \max \{0, \alpha(\mathcal{F}_{\mathcal{B}}, q)\} y_{e, q} \quad (3.11)$$

with $\alpha(\mathcal{F}_B, q)$ for all $q \in \mathcal{Q}(e)$ as defined in Proposition 3.17. Then x satisfies the band inequality

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > f_B(\ell)}} x_{\ell, f} \geq 1. \quad (3.12)$$

Proof. It suffices to show that $\alpha(\mathcal{F}_B, q) \geq 1$ holds for all $q \in \mathcal{Q}(e)$, since this would imply

$$\sum_{\ell \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > f_B(\ell)}} x_{\ell, f} \stackrel{(3.11)}{\geq} \sum_{q \in \mathcal{Q}(e)} \max\{0, \alpha(\mathcal{F}_B, q)\} y_{e, q} \geq \sum_{q \in \mathcal{Q}(e)} y_{e, q} \stackrel{(3.2)}{\geq} 1.$$

Recall the definition for all $q \in \mathcal{Q}(e)$

$$\alpha(\mathcal{F}_B, q) := \max \left\{ \sum_{f \in \mathcal{F}'} q_f - |\mathcal{L}(e) \setminus \{\ell \in \mathcal{L}(e) : \mathcal{F}' \subseteq \mathcal{F}_B(\ell)\}| : \mathcal{F}' \subseteq \mathcal{F} \right\}.$$

Assume there exists a minimal configuration $q \in \mathcal{Q}(e)$ such that $\alpha(\mathcal{F}_B, q) \geq 1$ does not hold. Since $\alpha(\mathcal{F}_B, q) \in \mathbb{Z}$, this implies $\alpha(\mathcal{F}_B, q) \leq 0$. This in turn implies the following

$$\sum_{f \in \mathcal{F}'} q_f \leq |\mathcal{L}(e) \setminus \{\ell \in \mathcal{L}(e) : \mathcal{F}' \subseteq \mathcal{F}_B(\ell)\}| \quad \forall \mathcal{F}' \subseteq \mathcal{F}.$$

Hence, for $f' \in \mathcal{F}$ and $\mathcal{F}' = \{f \in \mathcal{F} : f \geq f'\}$ we have

$$\begin{aligned} \sum_{\substack{f \in \mathcal{F} \\ f \geq f'}} q_f &\leq |\mathcal{L}(e) \setminus \{\ell \in \mathcal{L}(e) : f' > f_B(\ell)\}| \\ &= |\{\ell \in \mathcal{L}(e) : f' \leq f_B(\ell)\}| \\ &= \sum_{\substack{f \in \mathcal{F} \\ f \geq f'}} |\{\ell \in \mathcal{L}(e) : f = f_B(\ell)\}|. \end{aligned}$$

Applying Lemma 3.18 yields

$$\sum_{f \in \mathcal{F}} f_B(\ell) \geq \sum_{f \in \mathcal{F}} f \cdot q_f \geq F(e).$$

But this contradicts the assumption that f_B is a valid band, see Definition 2.5. Hence, it follows $\alpha(\mathcal{F}_B, q) \geq 1$ for every $q \in \mathcal{Q}(e)$ and the proposition is proved. \square

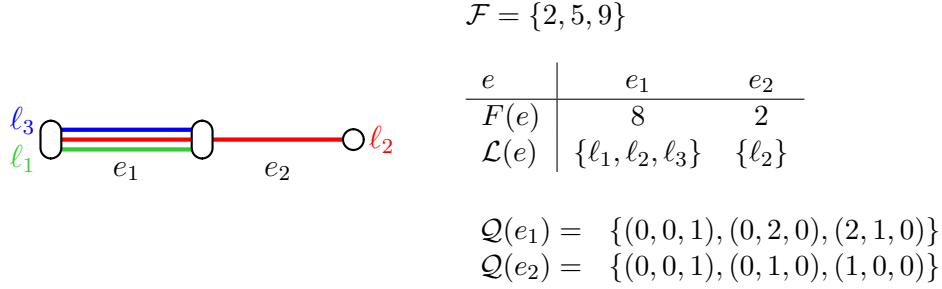


Figure 3.1: An instance of the line planning problem. *Left:* Transportation network consisting of two edges and three lines. *Right:* The given set of frequencies, the frequency demands and minimal configurations.

Our results can be summarized to the following. The configuration model implies MIR inequalities that are induced by the frequency demand constraints. When we extend the configuration model by the configuration band inequalities, then all band inequalities are additionally implied. These inequalities can be interpreted as follows. For any possible assignment \mathcal{F}_B of lines to frequencies the corresponding value $\max\{0, \alpha(\mathcal{F}_B, q)\}$ is for each minimal configuration $q \in \mathcal{Q}(e)$ a lower bound of the number of lines that need to be assigned to a frequency given by \mathcal{F}_B if this configuration is chosen. This leads to the following conjecture.

Conjecture 3.20. *Let $e \in E$ be an edge, then the configuration band inequalities together with the configuration assignment constraints, and the line frequency assignment constraints provide a complete description of $P(\text{QLP}|e)$. I.e., we define the polytope*

$$PQ := \left\{ (x, y) \in \mathbb{R}_+^{\mathcal{L}(e) \times \mathcal{F} \times \mathcal{Q}(e)} : \begin{array}{l} \text{(3.9) for } e, \\ \text{(QLP) (3.2) for } e, \\ \text{(QLP) (3.3) for all } \ell \in \mathcal{L}(e) \end{array} \right\},$$

then the following holds

$$PQ = P(\text{QLP}|e)$$

and

$$PQ|_x = P(\text{SLP}|e).$$

We close the section by giving an example for which (3.8) and (3.9) are facet-defining and cut off a fractional solution.

Example 3.5. Consider the line planning instance given in Figure 3.1. Let $\mathcal{L}' = \{\ell_1\}$ and $\mathcal{F}' = \{2, 5\}$, then the corresponding inequality (3.8) for e_1 reads as follows:

$$x_{\ell_1, 2} + x_{\ell_1, 5} \geq y_{e_1, q_3}. \tag{3.13}$$

Let $\mathcal{F}_{\mathcal{B}}(\ell_1) = \{2, 5\}$, $\mathcal{F}_{\mathcal{B}}(\ell_2) = \emptyset$, and $\mathcal{F}_{\mathcal{B}}(\ell_3) = \{5\}$, then the corresponding configuration band inequality (3.9) for e_1 reads as follows:

$$x_{\ell_1,2} + x_{\ell_1,5} + x_{\ell_3,5} \geq y_{e_1,q_2} + y_{e_1,q_3}. \quad (3.14)$$

One can show that (3.13) and (3.14) are facet-defining for $P(\text{QLP})$ in this example.

Now, consider the valid band $f_{\mathcal{B}}$ of e_1 , where $f_{\mathcal{B}}(\ell_1) = 0$, $f_{\mathcal{B}}(\ell_2) = 5$, and $f_{\mathcal{B}}(\ell_3) = 9$. The associated band inequality (2.4) reads as follows:

$$x_{\ell_1,2} + x_{\ell_1,5} + x_{\ell_1,9} + x_{\ell_2,9} + x_{\ell_3,5} + x_{\ell_3,9} \geq 1. \quad (3.15)$$

Let $\mathcal{F}_{\mathcal{B}'}(\ell_1) = \{2, 5, 9\}$, $\mathcal{F}_{\mathcal{B}'}(\ell_2) = \{9\}$, and $\mathcal{F}_{\mathcal{B}'}(\ell_3) = \{5, 9\}$, then the corresponding configuration band for e_1 reads as follows:

$$x_{\ell_1,2} + x_{\ell_1,5} + x_{\ell_1,9} + x_{\ell_2,9} + x_{\ell_3,5} + x_{\ell_3,9} \geq y_{e_1,q_1} + y_{e_1,q_2} + y_{e_1,q_3}. \quad (3.16)$$

Note that (3.16) implies (3.15), since $y_{e_1,q_1} + y_{e_1,q_2} + y_{e_1,q_3} = 1$ holds.

We define $(\bar{x}, \bar{y}) \in P_{LP}(\text{QLP}|e_1)$ by $\bar{x}_{\ell_1,9} = \frac{1}{2}$, $\bar{x}_{\ell_2,5} = \frac{1}{2}$, $\bar{x}_{\ell_3,2} = 1$, $\bar{y}_{e_1,q_1} = \frac{1}{2}$, $\bar{y}_{e_1,q_3} = \frac{1}{2}$, and $\bar{x}_{\ell,f} = \bar{y}_{e_1,q} = 0$ otherwise. Note that (3.13), (3.14), and (3.16) are violated by (\bar{x}, \bar{y}) .

3.4 Multi-Edge Configurations

In this section we generalize the concept of minimal configurations. We defined for the configuration model for each edge in the network a minimal configuration. The LP relaxation of this model is tighter than the LP relaxation of the standard model and implies several classes of facet-defining inequalities. These inequalities can all be obtained by strengthening the frequency demand constraint of a single edge. But the inequalities presented in Section 2.2.4, which we obtain by aggregating the frequency demand constraints for a subset of edges, are not implied by the LP relaxation of the configuration model. This gives rise to the underlying idea of the *multi-edge configuration model*, which is presented in this section. This model is computationally not tractable due to the enormous number of variables and will not be considered in our computational studies. However, we will show in this section that it provides a stronger LP relaxation than the configuration model.

The idea is to define a set of minimal configurations for a subset of edges $E' \subseteq E$. We divide the lines that pass any edge in E' into equivalence classes;

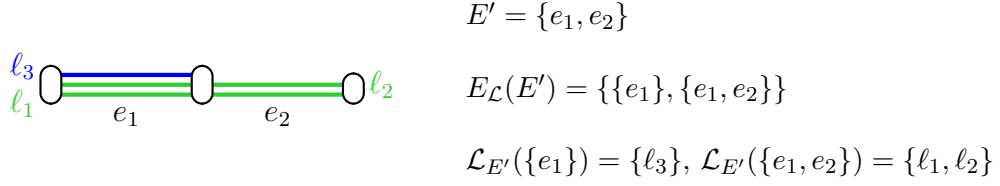


Figure 3.2: A subset of edges, the equivalence classes of lines.

we consider two lines as equivalent if they pass the same edges in E' . We need to introduce further notation.

Definition 3.21. Let $E', E'' \subseteq E$ be subsets of edges with $E'' \subseteq E'$, then we denote by

$$\mathcal{L}_{E'}(E'') := \{\ell \in \mathcal{L} : \ell \cap E' = E''\}$$

the set of all lines that pass all edges in E'' but no edge in $E' \setminus E''$. Further, we define for $E' \subseteq E$ the set

$$E_{\mathcal{L}}(E') := \{E'' \subseteq E' : \mathcal{L}_{E'}(E'') \neq \emptyset\},$$

which contains all subsets E'' of E' such that there exists a line in $\mathcal{L}_{E'}(E'')$.

Note that the set $\mathcal{L}_{E'}(E'')$ is an equivalence class according to the definition above and that the equivalence classes for all $E'' \notin E_{\mathcal{L}}(E')$ are empty. Also note that for every line ℓ in $\mathcal{L}(e)$ and for every edge in $e \in E'$ there exists a unique set E'' such that $\ell \in \mathcal{L}_{E'}(E'')$. We give an illustrated example in Figure 3.2.

Analogously to the configurations of a single edge, we define a *multi-edge configuration* of a subset of edges $E' \subseteq E$ by specifying for each set of edges $E'' \in E_{\mathcal{L}}(E')$ how many lines in $\mathcal{L}_{E'}(E'')$ are operated at a certain frequency. We denote a multi-edge configuration by a matrix in $\mathbb{Z}_+^{E_{\mathcal{L}}(E') \times \mathcal{F}}$. We give a formal definition in the following.

Definition 3.22. Let $E' \subseteq E$ be a subset of edges.

- We call $Q \in \mathbb{Z}_+^{E_{\mathcal{L}}(E') \times \mathcal{F}}$ a *multi-edge configuration* of E' if

$$\sum_{f \in \mathcal{F}} Q_{E'', f} \leq |\mathcal{L}_{E'}(E'')| \quad \forall E'' \in E_{\mathcal{L}}(E') \quad (3.17)$$

$$\text{and} \quad \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} \sum_{f \in \mathcal{F}} f \cdot Q_{E'', f} \geq F(e) \quad \forall e \in E'. \quad (3.18)$$

- We denote by

$$\bar{\mathcal{Q}}(E') := \left\{ Q \in \mathbb{Z}_+^{E_{\mathcal{L}}(E') \times \mathcal{F}} : Q \text{ satisfies (3.17) and (3.18)} \right\}$$

the set of *multi-edge configurations of E'* .

- We call a multi-edge configuration $Q \in \bar{\mathcal{Q}}(E')$ *minimal* if there is no multi-edge configuration $\bar{Q} \in \bar{\mathcal{Q}}(E')$ such that \bar{Q} dominates Q , i.e., $\bar{Q}_{E'',f} \leq Q_{E'',f}$ for all $E'' \in E_{\mathcal{L}}(E')$ and all $f \in \mathcal{F}$ and $\bar{Q}_{E'',f} < Q_{E'',f}$ for at least one $E'' \in E_{\mathcal{L}}(E')$ and $f \in \mathcal{F}$.
- We denote by

$$\mathcal{Q}(E') := \{ Q \in \bar{\mathcal{Q}}(E') : Q \text{ is minimal} \}$$

the set of *minimal multi-edge configurations of E'* .

We want to ensure that the frequency demand of every edge is satisfied by choosing minimal multi-edge configurations. Therefore, it is required that for every edge there is a minimal multi-edge configuration for a subset of edges in which this edge is contained. Let \mathcal{E} be a cover of E , i.e., $\mathcal{E} \subseteq 2^E$ such that $\bigcup_{E' \in \mathcal{E}} E' = E$. We extend the standard model (SLP) with binary variables $y_{E',Q}$ indicating for each subset of edges $E' \in \mathcal{E}$ which minimal multi-edge configuration $Q \in \mathcal{Q}(E')$ is chosen. The *multi-edge configuration model induced by the edge cover \mathcal{E}* is the following:

$$\begin{aligned} (\text{QLP}^{\mathcal{E}}) \quad \min \quad & \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f} \\ \sum_{\ell \in \mathcal{L}_{E'}(E'')} x_{\ell,f} \geq & \sum_{Q \in \mathcal{Q}(E')} Q_{E'',f} \cdot y_{E',Q} \quad \forall E' \in \mathcal{E}, \forall E'' \in E_{\mathcal{L}}(E'), \forall f \in \mathcal{F} \end{aligned} \quad (3.19)$$

$$\sum_{Q \in \mathcal{Q}(E')} y_{E',Q} = 1 \quad \forall E' \in \mathcal{E} \quad (3.20)$$

$$\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L} \quad (3.21)$$

$$x_{\ell,f} \in \{0, 1\} \quad \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \quad (3.22)$$

$$y_{E',Q} \in \{0, 1\} \quad \forall E' \in \mathcal{E}, \forall Q \in \mathcal{Q}(E'). \quad (3.23)$$

The multi-edge configuration model $(\text{QLP}^{\mathcal{E}})$ minimizes the cost of a line plan. The *coupling constraints* (3.19) force for each subset of edges of the

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given cover \mathcal{E} that enough lines are operated at every frequency for the chosen minimal multi-edge configuration. The *multi-edge configuration assignment constraints* (3.20) ensure that exactly one minimal multi-edge configuration is chosen for each subset of edges in the cover \mathcal{E} .

Example 3.6 (Example 2.1 continued). We consider the line planning problem defined in Figure 2.1 and also considered in Figure 3.2. Recall that $\mathcal{F} = \{2, 8\}$, $F(e_1) = 9$, and $F(e_2) = 1$. Define $E' = \{e_1, e_2\}$, then we have $E_{\mathcal{L}}(E') = \{\{e_1\}, \{e_1, e_2\}\}$, $\mathcal{L}_{E'}(\{e_1\}) = \{\ell_3\}$, and $\mathcal{L}_{E'}(\{e_1, e_2\}) = \{\ell_1, \ell_2\}$. The set of minimal multi-edge configurations of E' is

$$\mathcal{Q}(E') = \left\{ \begin{array}{l} \{e_1\} \\ \{e_1, e_2\} \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$$

Let $\mathcal{E} = \{E'\}$ be a cover of E . The multi-edge configuration model induced by \mathcal{E} for this example is:

$$\begin{array}{rcl} \text{(QLP}^{\mathcal{E}}\text{)} & & \\ \min & 4x_{\ell_1,2} + 16x_{\ell_1,8} + 4x_{\ell_2,2} + 16x_{\ell_2,8} + 2x_{\ell_3,2} + 8x_{\ell_3,8} & \\ & + x_{\ell_3,2} & -y_{E',Q_1} \geq 0 \\ & + x_{\ell_3,8} & -y_{E',Q_2} - y_{E',Q_3} \geq 0 \\ + x_{\ell_1,2} & + x_{\ell_2,2} & -y_{E',Q_2} \geq 0 \\ & + x_{\ell_1,8} + x_{\ell_2,8} & -y_{E',Q_1} - y_{E',Q_3} - 2y_{E',Q_4} \geq 0 \\ + x_{\ell_1,2} + x_{\ell_1,8} & & \leq 1 \\ & + x_{\ell_2,2} + x_{\ell_2,8} & \leq 1 \\ & + x_{\ell_3,2} + x_{\ell_3,8} & \leq 1 \\ & & + y_{E',Q_1} + y_{E',Q_2} + y_{E',Q_3} + y_{E',Q_4} = 1 \\ & & x_{\ell_i,f} \in \{0, 1\} \\ & & y_{E',Q_i} \in \{0, 1\}. \end{array}$$

An optimal solution to (QLP $^{\mathcal{E}}$) is given by (x^*, y^*) , where $x_{\ell_3,8}^* = 1$, $x_{\ell_2,2}^* = 1$, $y_{E',Q_2}^* = 1$, and $x_{\ell,f}^* = 0$ and $y_{E',Q}^* = 0$ otherwise. The objective function value of this solution is 12.

We like to remark that the multi-edge configuration model is indeed a generalization of the configuration model. When we consider the cover $\mathcal{E} = \bigcup_{e \in E} \{e\}$ of E that covers each edge exactly once by an one-element set, then every subset $E' \in \mathcal{E}$ contains only a single edge e and we have $E_{\mathcal{L}}(E') = \mathcal{E}_{\mathcal{L}}(\{e\}) = \{e\}$ and $\mathcal{L}_{\{e\}}(\{e\}) = \mathcal{L}(e)$. Hence, it follows $Q(\{e\}) = Q(e)$. Therefore, the multi-edge configuration model is equivalent to the configuration model in this case.

We show in the next lemma that the multi-edge configuration model describes the same set of line plans as the standard model and the configuration model.

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Lemma 3.23. *Let \mathcal{E} be a cover of E . Then $(\text{QLP}^\mathcal{E})$ provides an extended formulation for (SLP) , i.e.,*

$$P(\text{QLP}^\mathcal{E})|_x = P(\text{QLP})|_x = P(\text{SLP}).$$

Proof. We proved in Lemma 3.6 $P(\text{QLP})|_x = P(\text{SLP})$. We can prove in a similar fashion that $P(\text{QLP}^\mathcal{E})|_x = P(\text{SLP})$ holds. $P(\text{SLP})$ and $P(\text{QLP}^\mathcal{E})$ are the convex hull of all feasible solutions of (SLP) and $(\text{QLP}^\mathcal{E})$ respectively. Hence, it suffices to show that all integer points in $P(\text{SLP})$ lie in $P(\text{QLP}^\mathcal{E})|_x$ and vice versa.

We show first $P(\text{QLP}^\mathcal{E})|_x \subseteq P(\text{SLP})$. Let (\bar{x}, \bar{y}) be a point in $P(\text{QLP}^\mathcal{E})$. Obviously, \bar{x} satisfies (2.2) and (2.3). Let $e \in E$ and let $E' \in \mathcal{E}$ such that $e \in E'$. We get

$$\begin{aligned} \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot \bar{x}_{\ell,f} &= \sum_{f \in \mathcal{F}} f \sum_{\ell \in \mathcal{L}(e)} \bar{x}_{\ell,f} \\ &= \sum_{f \in \mathcal{F}} f \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} \sum_{\ell \in \mathcal{L}_{E'}(E'')} \bar{x}_{\ell,f} \\ &\stackrel{(3.19)}{\geq} \sum_{f \in \mathcal{F}} f \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} \sum_{Q \in \mathcal{Q}(E')} Q_{E'',f} \bar{y}_{E',Q} \\ &= \sum_{f \in \mathcal{F}} \sum_{Q \in \mathcal{Q}(E')} \bar{y}_{E',Q} \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} f \cdot Q_{E'',f} \\ &= \sum_{Q \in \mathcal{Q}(E')} \bar{y}_{E',Q} \underbrace{\sum_{f \in \mathcal{F}} \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} f \cdot Q_{E'',f}}_{\stackrel{(3.18)}{\geq} F(e)} \\ &\geq F(e) \underbrace{\sum_{Q \in \mathcal{Q}(E')} \bar{y}_{E',Q}}_{\stackrel{(3.20)_1}{=} 1} \\ &= F(e) \end{aligned}$$

Hence, \bar{x} satisfies (2.1) as well and is contained in $P(\text{SLP})$.

Now, let \bar{x} be an integer point in $P(\text{SLP})$. We want to define \bar{y} such that $(\bar{x}, \bar{y}) \in P(\text{QLP}^\mathcal{E})$. We define for each $E' \in \mathcal{E}$ a multi-edge configuration:

$$\bar{Q}^{E'} \in \mathbb{Z}_+^{E_{\mathcal{L}}(E') \times \mathcal{F}} \text{ with } \bar{Q}_{E'',f}^{E'} := \sum_{\ell \in \mathcal{L}_{E'}(E'')} \bar{x}_{\ell,f} \quad \forall E'' \in E_{\mathcal{L}}(E'), \forall f \in \mathcal{F}.$$

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Then we have for all $E' \in \mathcal{E}$ and all $E'' \in E_{\mathcal{L}}(E')$

$$\sum_{f \in \mathcal{F}} \bar{Q}_{E'',f}^{E'} = \sum_{\ell \in \mathcal{L}_{E'}(E'')} \sum_{f \in \mathcal{F}} \bar{x}_{\ell,f} \stackrel{(2.2)}{\leq} |\mathcal{L}_{E'}(E'')|$$

and for all $e \in E'$

$$\begin{aligned} \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} \sum_{f \in \mathcal{F}} f \cdot \bar{Q}_{E'',f}^{E'} &= \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} \sum_{f \in \mathcal{F}} f \sum_{\ell \in \mathcal{L}_{E'}(E'')} \bar{x}_{\ell,f} \\ &= \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} \sum_{\ell \in \mathcal{L}_{E'}(E'')} \sum_{f \in \mathcal{F}} f \cdot \bar{x}_{\ell,f} \\ &= \sum_{\ell \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot \bar{x}_{\ell,f} \\ &\stackrel{(2.1)}{\geq} F(e). \end{aligned}$$

Therefore, we can conclude $\bar{Q}^{E'} \in \bar{\mathcal{Q}}(E')$ for all $E' \in \mathcal{E}$. For every $E' \in \mathcal{E}$ let $Q^{E'} \in \mathcal{Q}(E')$ be a minimal multi-edge configuration such that $\bar{Q}_{E'',f}^{E'} \geq Q_{E'',f}^{E'}$ for all $E'' \in E_{\mathcal{L}}(E')$ and for all $f \in \mathcal{F}$, see Definition 3.22. Now we can define:

$$\bar{y}_{E',Q} := \begin{cases} 1 & \text{if } Q = Q^{E'} \\ 0 & \text{else} \end{cases} \quad \text{for all } E' \in \mathcal{E}, Q \in \mathcal{Q}(E').$$

Let $E' \in \mathcal{E}$, $E'' \in E_{\mathcal{L}}(E')$, and $f \in \mathcal{F}$, then the following holds:

$$\sum_{\ell \in \mathcal{L}_{E'}(E'')} \bar{x}_{\ell,f} = \bar{Q}_{E'',f}^{E'} \geq Q_{E'',f}^{E'} = \sum_{Q \in \mathcal{Q}(E')} Q_{E'',f} \cdot \bar{y}_{E',Q}.$$

Therefore (\bar{x}, \bar{y}) satisfies (3.19)–(3.23) and $(\bar{x}, \bar{y}) \in P(\text{QLP}^{\mathcal{E}})$ follows. \square

Lemma 3.23 implies that the integer programs (SLP), (QLP), and $(\text{QLP}^{\mathcal{E}})$ minimize the same objective function over an identical set of points. Further, we can show that the multi-edge configuration model $(\text{QLP}^{\mathcal{E}})$ provides a tighter LP relaxation than the standard model and the configuration model.

Theorem 3.24. *Let \mathcal{E} be a partition of E , i.e., $\mathcal{E} \subseteq 2^E$ such that for all $E_1, E_2 \in \mathcal{E}$ we have $E_1 \cap E_2 = \emptyset$ and $\bigcup_{E' \in \mathcal{E}} E' = E$ holds. The LP relaxation of $P(\text{QLP}^{\mathcal{E}})$ is tighter than the LP relaxation of $P(\text{SLP})$ and $P(\text{QLP})|_x$, i.e.,*

$$P_{LP}(\text{QLP}^{\mathcal{E}})|_x \subseteq P_{LP}(\text{QLP})|_x \subseteq P_{LP}(\text{SLP}).$$

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Proof. Theorem 3.7 gives $P_{LP}(\text{QLP})|_x \subseteq P_{LP}(\text{SLP})$. Hence we only need to show $P_{LP}(\text{QLP}^\mathcal{E})|_x \subseteq P_{LP}(\text{QLP})|_x$. Let $(x, y) \in P_{LP}(\text{QLP}^\mathcal{E})$, we will define \bar{y} such that $(x, \bar{y}) \in P_{LP}(\text{QLP})$ to prove the claim. We first need the following definition. Let $E' \in \mathcal{E}$ and $Q \in \mathcal{Q}(E')$ a minimal multi-edge configuration, then define for all $e \in E'$ the configuration $\bar{q}(Q, e)$ by

$$\bar{q}(Q, e)_f := \sum_{\substack{E'' \in \mathcal{E}_{\mathcal{L}}(E') \\ e \in E''}} Q_{E'', f}.$$

It follows directly from the Definition 3.22 that $\bar{q}(Q, e) \in \bar{\mathcal{Q}}(e)$ holds. Let $q(Q, e) \in \mathcal{Q}(e)$ be a minimal configuration of e such that $\bar{q}(Q, e)_f \geq q(Q, e)_f$ for all $f \in \mathcal{F}$, see Remark 3.3. Now define \bar{y} by

$$\bar{y}_{e, q} := \sum_{\substack{Q \in \mathcal{Q}(E') \\ q = q(Q, e)}} y_{E', Q} \text{ for all } e \in E \text{ and let } E' \in \mathcal{E} \text{ with } e \in E'.$$

We will show now that $(x, \bar{y}) \in P_{LP}(\text{QLP})$ holds. Let $e \in E$ be an edge and let $E' \in \mathcal{E}$ such that $e \in E'$, then we have for all $f \in \mathcal{F}$

$$\begin{aligned} \sum_{\ell \in \mathcal{L}(e)} x_{\ell, f} &= \sum_{\substack{E'' \in \mathcal{E}_{\mathcal{L}}(E') \\ e \in E''}} \sum_{\ell \in \mathcal{L}_{E'}(E'')} x_{\ell, f} \\ &\stackrel{(3.19)}{\geq} \sum_{\substack{E'' \in \mathcal{E}_{\mathcal{L}}(E') \\ e \in E''}} \sum_{Q \in \mathcal{Q}(E')} Q_{E'', f} \cdot y_{E', Q} \\ &= \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \sum_{\substack{E'' \in \mathcal{E}_{\mathcal{L}}(E') \\ e \in E''}} Q_{E'', f} \\ &= \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \cdot \bar{q}(Q, e)_f \\ &\geq \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \cdot q(Q, e)_f \\ &= \sum_{\substack{q \in \mathcal{Q}(e) \\ q = q(Q, e)}} \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \cdot q(Q, e)_f \\ &= \sum_{q \in \mathcal{Q}(e)} q_f \sum_{\substack{Q \in \mathcal{Q}(E') \\ q = q(Q, e)}} y_{E', Q} \\ &= \sum_{q \in \mathcal{Q}(e)} q_f \cdot \bar{y}_{e, q} \end{aligned}$$

and

$$\sum_{q \in \mathcal{Q}(e)} \bar{y}_{e,q} = \sum_{q \in \mathcal{Q}(e)} \sum_{\substack{Q \in \mathcal{Q}(E') \\ q=q(Q,e)}} y_{E',Q} = \sum_{Q \in \mathcal{Q}(E')} y_{E',Q} = 1.$$

Hence, (x, \bar{y}) satisfies (3.1)–(3.3) and we get $(x, \bar{y}) \in P_{LP}(\text{QLP})$. \square

Now, we reconsider the aggregated frequency and aggregated cardinality inequalities, presented in Section 2.2.4. We show that they are implied by $P_{LP}(\text{QLP}^\mathcal{E})$ for $E' \in \mathcal{E}$ and give an example to show that they strengthen the LP relaxation of $P(\text{QLP})$. We recall the following definitions. For a subset $E' \subseteq E$ of edges denote the total demand by $F(E') := \sum_{e \in E'} F(e)$ and the set of lines passing an edge in E' by $\mathcal{L}(E') := \bigcup_{e \in E'} \mathcal{L}(e)$. For a line $\ell \in \mathcal{L}(E')$ we define by $\alpha_{E'}^\ell := |\ell \cap E'|$ the number of edges in ℓ that are contained in E' . And define $\alpha_{E'}^{\max} := \max\{\alpha_{E'}^\ell : \ell \in \mathcal{L}(E')\}$. We need to prove first the following lemma.

Lemma 3.25. *Let \mathcal{E} be a cover of E , $E' \in \mathcal{E}$ a subset of edges, and $Q \in \mathcal{Q}(E')$ a multi-edge configuration of E' . Then*

$$\sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} f \cdot Q_{E'',f} \geq \left\lceil \frac{F(E')}{\alpha_{E'}^{\max}} \right\rceil$$

and

$$\sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} Q_{E'',f} \geq \left\lceil \frac{|E'|}{\alpha_{E'}^{\max}} \right\rceil$$

holds, if $\alpha_{E'}^{\max} > 0$.

Proof. First, notice that $\alpha_{E'}^{\max} = \max\{|E''| : E'' \in E_{\mathcal{L}}(E')\}$, cf. Definition 3.21. From (3.18) we immediately get for all $Q \in \mathcal{Q}(E')$:

$$\begin{aligned} F(E') &\leq \sum_{e \in E'} \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E''}} \sum_{f \in \mathcal{F}} f \cdot Q_{E'',f} \\ &= \sum_{E'' \in E_{\mathcal{L}}(E')} |E''| \sum_{f \in \mathcal{F}} f \cdot Q_{E'',f} \\ &\leq \sum_{E'' \in E_{\mathcal{L}}(E')} \alpha_{E'}^{\max} \sum_{f \in \mathcal{F}} f \cdot Q_{E'',f} \\ \Rightarrow \frac{F(E')}{\alpha_{E'}^{\max}} &\leq \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} f \cdot Q_{E'',f}. \end{aligned}$$

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Since the right-hand side of this inequality is always integral we can round up the left-hand side and obtain:

$$\sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} f \cdot Q_{E'',f} \geq \left\lceil \frac{F(E')}{\alpha_{E'}^{\max}} \right\rceil.$$

On the other hand, $F(e) > 0$ for all $e \in E$ implies

$$\sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E'}} \sum_{f \in \mathcal{F}} Q_{E'',f} \geq 1.$$

From this it follows:

$$\begin{aligned} |E'| &\leq \sum_{e \in E'} \sum_{\substack{E'' \in E_{\mathcal{L}}(E') \\ e \in E'}} \sum_{f \in \mathcal{F}} Q_{E'',f} \\ &= \sum_{E'' \in E_{\mathcal{L}}(E')} |E''| \sum_{f \in \mathcal{F}} Q_{E'',f} \\ &\leq \sum_{E'' \in E_{\mathcal{L}}(E')} \alpha_{E''}^{\max} \sum_{f \in \mathcal{F}} Q_{E'',f} \\ \Rightarrow \frac{F(E')}{\alpha_{E'}^{\max}} &\leq \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} Q_{E'',f}. \end{aligned}$$

Since the right-hand side of this inequality is always integral we can round up the left-hand side and obtain:

$$\sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} Q_{E'',f} \geq \left\lceil \frac{|E'|}{\alpha_{E'}^{\max}} \right\rceil.$$

This completes the proof of the lemma. □

Now we can show that the aggregated frequency inequalities (2.11) and aggregated cardinality inequalities (2.13) for every set $E' \in \mathcal{E}$ are implied by the LP relaxation of the multi-edge configuration model.

Proposition 3.26. *Let \mathcal{E} be a cover of E , $E' \in \mathcal{E}$ a subset of edges. The aggregated frequency inequality*

$$\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} f \cdot x_{\ell,f} \geq \left\lceil \frac{F(E')}{\alpha_{E'}^{\max}} \right\rceil \tag{3.24}$$

is valid for $P_{LP}(\text{QLP}^{\mathcal{E}})$, if $\alpha_{E'}^{\max} > 0$.

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Proof. Let $(x, y) \in P_{LP}(\text{QLP}^{\mathcal{E}})$, then we have with Lemma 3.25:

$$\begin{aligned}
\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} &= \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{\ell \in \mathcal{L}_{E'}(E'')} \sum_{f \in \mathcal{F}} f \cdot x_{\ell, f} \\
&= \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} f \sum_{\ell \in \mathcal{L}_{E'}(E'')} x_{\ell, f} \\
&\stackrel{(3.19)}{\geq} \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} f \sum_{Q \in \mathcal{Q}(E')} Q_{E'', f} \cdot y_{E', Q} \\
&= \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} f \cdot Q_{E'', f} \\
&\geq \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \left\lceil \frac{F(E')}{\alpha_{E'}^{\max}} \right\rceil \\
&\stackrel{(3.20)}{=} \left\lceil \frac{F(E')}{\alpha_{E'}^{\max}} \right\rceil.
\end{aligned}$$

□

Proposition 3.27. *Let \mathcal{E} be a cover of E , $E' \in \mathcal{E}$ a subset of edges. The aggregated cardinality inequality*

$$\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} x_{\ell, f} \geq \left\lceil \frac{|E'|}{\alpha_{E'}^{\max}} \right\rceil \tag{3.25}$$

is valid for $P_{LP}(\text{QLP}^{\mathcal{E}})$, if $\alpha_{E'}^{\max} > 0$.

Proof. Let $(x, y) \in P_{LP}(\text{QLP}^{\mathcal{E}})$, then we have with Lemma 3.25:

$$\begin{aligned}
\sum_{\ell \in \mathcal{L}(E')} \sum_{f \in \mathcal{F}} x_{\ell, f} &= \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{\ell \in \mathcal{L}_{E'}(E'')} \sum_{f \in \mathcal{F}} x_{\ell, f} = \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} \sum_{\ell \in \mathcal{L}_{E'}(E'')} x_{\ell, f} \\
&\stackrel{(3.19)}{\geq} \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} \sum_{Q \in \mathcal{Q}(E')} Q_{E'', f} \cdot y_{E', Q} \\
&= \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \sum_{E'' \in E_{\mathcal{L}}(E')} \sum_{f \in \mathcal{F}} Q_{E'', f} \\
&\geq \sum_{Q \in \mathcal{Q}(E')} y_{E', Q} \left\lceil \frac{|E'|}{\alpha_{E'}^{\max}} \right\rceil \\
&\stackrel{(3.20)}{=} \left\lceil \frac{|E'|}{\alpha_{E'}^{\max}} \right\rceil.
\end{aligned}$$

□

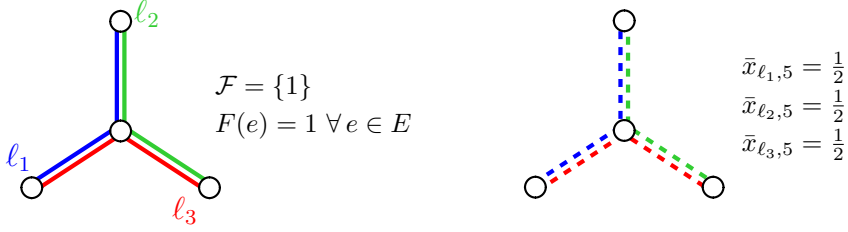


Figure 3.3: *Left*: An instance of the line planning problem, where all edges have frequency demand 7. *Right*: A fractional solution.

We show in Example 3.7 that the aggregated frequency inequalities and aggregated cardinality inequalities improve the LP relaxation of the configuration model.

Example 3.7. Consider the line planning problem instance depicted in Figure 3.3. When we define $E' := \{e_1, e_2, e_3\}$, the corresponding aggregated frequency inequality (3.24) and the aggregated cardinality inequality (3.25) are equivalent in this case and read as follows:

$$x_{\ell_1,1} + x_{\ell_2,1} + x_{\ell_3,1} \geq 2. \quad (3.26)$$

For every edge in this instance, the set of minimal configurations contains only the configuration $q = (1)$. Inequality (3.26) is violated by $(\bar{x}, \bar{y}) \in P_{LP}(\text{QLP})$, where $\bar{x}_{\ell_1,5} = \bar{x}_{\ell_2,1} = \bar{x}_{\ell_3,1} = \frac{1}{2}$ and $\bar{x}_{\ell,f} = 0$ otherwise and $y_{e,q} = 1$ for all $e \in E$.

Chapter 4

Computational Study

In this chapter we report about our computational experiments. In order to compare the configuration model (QLP) with the standard model (SLP), we apply them to a set of test instances. For a deeper understanding of the respective advantages of these two models we compare them with two additional models (SLP⁺) and (SLP^Q). Model (SLP⁺) is obtained by adding the set cover, symmetric band, and MIR inequalities for all edges to the standard model, see Sections 2.2.2 and 2.2.3. To obtain MIR inequalities we divide the frequency demand constraints by all $f \in \mathcal{F}$. Model (SLP^Q) has been developed to cut down the number of configuration variables, which can explode for large instances. We order the edges in non-decreasing order with respect to the number of minimal configurations and iteratively generate the configuration variables and their associated constraints as long as the number of generated configuration variables does not exceed 25% of the number of variables for lines and frequencies. For the remaining edges we add the set cover, symmetric band, and MIR inequalities.

The test instances are described in Section 4.1. We apply several techniques to accelerate the solving process. In Section 4.2 we present preprocessing routines that aim at reducing the problem size and report on computational studies to measure their impact. We also implement primal heuristics, which are described in Section 4.3. The results of our final computations are presented and discussed in Section 4.4.

All computations were performed on an Intel(R) Xeon(R) CPU E3-1290, 3.7 GHz computer (in 64 bit mode) with 8 MB cache, running Linux and 16 GB of memory. We used the constraint integer programming framework SCIP version 3.1.0 [27] with CPLEX 12.6 as LP-solver to solve all integer programs.

4.1 Test Instances

We use the same test instances as in [5]. Our test set consists of five transportation networks that we denote as China, Dutch, SiouxFalls, Chicago, and Potsdam. The instances SiouxFalls and Chicago use the graph and the demand of the street network with the same name from the Transportation Network Test Problems Library of Bar-Gera [31]. Instances China, Dutch, and Potsdam correspond to public transportation networks. The Dutch network was introduced by Bussieck in the context of line planning [9]. The China instance is artificial; we constructed it as a showcase example, connecting the twenty biggest cities in China by the 2009 high speed train network. The Potsdam instances are real multi-modal public transportation networks for 1998 and 2009. We constructed a line pool by generating for each pair of terminals all lines that satisfy a certain length restriction. To be more precise, the number of edges of a line between two terminals s and t must be less than or equal to k times the number of edges of the shortest path between s and t . For each network, we increased k in three steps to produce three instances with different line pool sizes. For Dutch and China instance number 3 contains all lines, i.e., all paths that are possible in the network. We omit the Potsdam1998a instance, since its line pool of size 207 is too small to generate a feasible solution. The Potsdam2010 instance arose within a project with the Verkehr in Potsdam GmbH (ViP) [29] to optimize the 2010 line plan [3]. The line pool contains all possible lines that fulfill the ViP requirements.

For all instances the lines can be operated at frequencies 3, 6, 9, 18, 36, and 72. This corresponds to a cycle time of 60, 30, 20, 10, 5, and 2.5 minutes in a time horizon of 3 hours. We set the line cost to be proportional to the line length and the frequency plus a fixed cost term that is used to reduce the number of lines. The costs and the capacities of the lines depend on the mode of transportation (e.g., bus, streetcar). The travelling paths for all passengers are generated as shortest path in the network. The transportation demand for all edges is then determined for each edge. In the instances each edge is associated with exactly one mode, i.e., all lines on an edge have the same capacity, see Karbstein [20] for more details. Hence, we can express the transportation demands in terms of frequency demands. Table 4.1 lists some statistics about the test instances. The second to fourth columns give the number of edges, nodes and lines in the transportation network, respectively. The last column lists the total number of minimal configurations of all edges. For each instance, the total CPU time to generate the set of minimal configurations with Algorithm 1, given in Chapter 3, for

Table 4.1: Statistics on the line planning instances. The columns list the instance name, the number of edges and the number of nodes of the transportation network, the number of lines and the total number of minimal configurations of all edges.

name	$ E $	$ V $	$ \mathcal{L} $	$\sum_{e \in E} \mathcal{Q}(e) $
China1	27	20	472	38 403
China2	27	20	4 869	38 405
China3	27	20	19 353	38 405
Dutch1	30	23	402	240
Dutch2	30	23	2 679	240
Dutch3	30	23	7 302	240
SiouxFalls1	37	24	865	748 652
SiouxFalls2	37	24	9 396	846 329
SiouxFalls3	37	24	15 364	846 329
Potsdam1998b	351	325	1 905	18 383
Potsdam1998c	351	325	4 340	18 383
Potsdam2010	517	486	3 432	1 074
Chicago	1 028	523	23 109	2 370 726

all edges is in total significantly less than one second and will be disregarded in the following. In the following section we describe how the size of the problem in terms of number of variables and constraints can be reduced.

4.2 Preprocessing

In this section we present several problem specific presolving techniques that aim at reducing the size of the initial problem in terms of variables and constraints. We transform the problem instances by removing redundant constraints and fixing variables. In order to measure the effect of our presolving routines we compare the size, in terms of number of variables and constraints, of the initial instances with the size of the transformed instances after presolving of SCIP. A description of the general purpose presolving algorithms implemented in SCIP can be found in [1].

The resulting numbers of variables and constraints for the different models and instances are displayed in Table 4.2. For most instances the impact of our preprocessing techniques is only small or non-existent. For the configuration models (SLP^Q) and (QLP), however, the number of constraints and

Table 4.2: Statistics on the preprocessing on the line planning instances. The columns list the instance and the number of constraints and variables after presolving of SCIP for the four models. The original instances, at which we did not apply our preprocessing techniques, are marked with asterisks (*). The cases when the number of variables or constraints of the transformed instance compared to the respective number of the original instance is decreased by more than 1%, 10% or more than 30%, are highlighted in **green**, **orange**, and **red**, respectively.

name	(SLP)		(SLP ⁺)		(SLP ^Q)		(QLP)	
	#vars	#cons	#vars	#cons	#vars	#cons	#vars	#cons
China1*	2 793	499	2 793	613	3 771	656	41 235	661
China1	2 793	499	2 793	613	3 732	656	41 196	661
China2*	29 170	4 896	29 170	5 008	36 801	5 058	67 619	5 058
China2	29 170	4 896	29 170	5 008	36 757	5 058	67 575	5 058
China3*	116 074	19 380	116 074	19 492	145 780	19 542	154 523	19 542
China3	116 074	19 380	116 074	19 492	145 736	19 542	154 479	19 542
Dutch1*	1 544	424	1 544	481	2 652	612	2 652	612
Dutch1	1 544	424	1 544	481	1 760	580	1 760	580
Dutch2*	11 779	2 701	11 779	2 758	16 314	2 889	16 314	2 889
Dutch2	11 779	2 701	11 779	2 758	11 997	2 859	11 997	2 859
Dutch3*	33 988	7 324	33 988	7 381	44 052	7 512	44 052	7 512
Dutch3	33 988	7 324	33 988	7 381	34 206	7 482	34 206	7 482
SiouxFalls1*	5 188	902	5 188	1 112	6 682	1 117	753 842	1 124
SiouxFalls1	5 188	902	5 188	1 112	6 680	1 117	753 840	1 124
SiouxFalls2*	56 374	9 433	56 374	9 643	73 533	9 648	902 705	9 655
SiouxFalls2	56 374	9 433	56 374	9 643	73 531	9 648	902 703	9 655
SiouxFalls3*	92 182	15 401	92 182	15 611	117 713	15 616	938 513	15 623
SiouxFalls3	92 182	15 401	92 182	15 611	117 711	15 616	938 511	15 623
Potsdam1998b*	10 754	1 980	10 763	2 609	13 989	3 960	38 861	4 120
Potsdam1998b	10 732	1 930	10 750	2 301	13 800	2 838	29 363	2 842
Potsdam1998c*	25 295	4 413	25 304	5 042	32 324	6 501	53 471	6 555
Potsdam1998c	25 272	4 363	25 290	4 729	32 327	5 274	43 908	5 276
Potsdam2010*	8 719	2 926	8 926	3 391	22 561	5 557	22 561	5 557
Potsdam2010	8 714	2 920	8 927	3 080	10 360	3 923	10 360	3 923
Chicago*	131 915	24 067	131 915	27 826	173 431	30 239	2 509 855	30 295
Chicago	131 910	24 053	131 910	27 773	165 194	30 117	2 502 699	30 173

variables are significantly lower for some instances when we apply our pre-processing techniques. Especially on the real world instances Potsdam1998b, Potsdam1998c and Potsdam2010 our techniques caused a further reduction of the number of constraints by more than 18% for (SLP^Q) and (QLP). In the instance Potsdam2010 the number of variables for the models (SLP^Q) and (QLP) were further reduced by more than 50%. The applied techniques to reduce the number of constraints and variables are presented in the following.

4.2.1 Reducing the Number of Constraints

First, we focus on decreasing the number of constraints by reducing the size of the given transportation network. Assume the network $G(V, E)$ contains two edges $e_1, e_2 \in E$, such that every line covering e_2 also covers e_1 and the frequency demand of e_2 is not less than the frequency demand of e_1 , i.e., $\mathcal{L}(e_2) \subseteq \mathcal{L}(e_1)$ and $F(e_2) \geq F(e_1)$. Then, any line plan that fulfills the frequency demand requirement (1.1) for e_2 also covers the frequency demand of e_1 and we call the edge e_1 *redundant*, cf. Figure 4.1. Hence, the frequency demand constraints (2.1) and the coupling (3.1) and configuration assignment constraints (3.2) are redundant for the description of all feasible solutions and can be removed from the respective model for every redundant edge.

We identify redundant edges in two steps. In the first step we iteratively consider all nodes $v \in V$ with $|\delta(v)| = 2$. If no line terminates at v , then we can compose the incident edges e_1 and e_2 of v to a single edge $e_{1,2}$ with frequency demand $F(e_{1,2}) = \max\{F(e_1), F(e_2)\}$, see Figure 4.2a. This reduction was proposed by Bussieck [10]. If only lines via e_2 terminate at v and $F(e_2) \geq F(e_1)$, then e_2 is redundant and we compose the two edges to $e_{1,2}$ with $F(e_{1,2}) = F(e_2)$ and $\mathcal{L}(e_{1,2}) = \mathcal{L}(e_2)$, cf. Figure 4.2b. In a second step we iterate over all nodes of degree higher than 2 and pairwise compare all outgoing edges to identify redundant ones. Since the maximum degree of our test instances is bounded by 9, this procedure takes reasonably short computation time.

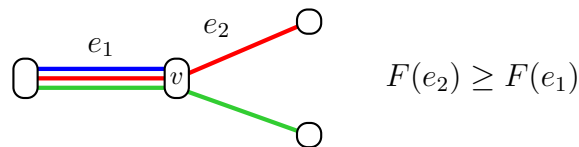


Figure 4.1: The edge e_1 is *redundant* since $\mathcal{L}(e_2) \subseteq \mathcal{L}(e_1)$ and $F(e_2) \geq F(e_1)$.



(a) If no line terminates at v , the edges e_1, e_2 can be composed to $e_{1,2}$ with $F(e_{1,2}) = \max\{F(e_1), F(e_2)\}$.



(b) If no line via e_2 terminates at v and $F(e_2) \geq F(e_1)$, the edges e_1, e_2 can be composed to $e_{1,2}$ with $F(e_{1,2}) = F(e_2)$ and $\mathcal{L}(e_{1,2}) = \mathcal{L}(e_2)$.

Figure 4.2: Composition of edges in two line problems.

4.2.2 Reducing the Number of Variables

We next focus on deleting variables, that are equal to zero in a cost optimal solution or even in any feasible solution.

Deleting and Fixing Line Variables. In order to describe our variable deleting routine for the line variables we need the following frequency bound definitions.

Definition 4.1. For an edge $e \in E$ we denote by

$$F^r(e) := \max\{0, F(e) - (|\mathcal{L}(e)| - 1) \cdot f_m\}$$

the residual demand that results from operating all lines except one at the highest frequency. For a line $\ell \in \mathcal{L}$ we denote by

$$f_\ell^{\min} := \max\{F^r(e) : e \in \ell\}$$

the highest residual demand over all edges in ℓ and by

$$f_\ell^{\max} := \min\{f \in \mathcal{F} \cup \{\infty\} : f \geq F(e) \forall e \in \ell\}$$

the smallest frequency that suffices to cover the frequency demand of all edges in ℓ .

We can interpret $F^r(e)$ for an edge $e \in E$ as follows. If all lines except one line in $\mathcal{L}(e)$ are operated at the highest frequency f_m , then $F^r(e)$ is the residual frequency demand that needs to be covered by the remaining line. This implies that every line containing e has to be operated at a frequency

greater than or equal to $F^r(e)$ in any feasible solution. Consequently, every line $\ell \in \mathcal{L}$ has to be operated at a frequency greater than or equal to f_ℓ^{\min} in any feasible line plan. Thus, we can fix all variables $x_{\ell,f}$ with $f < f_\ell^{\min}$ to zero. An example will be given later in this section.

Similarly, let $\ell \in \mathcal{L}$ be a line such that $f_\ell^{\max} < f_m$. I.e., the frequency f_ℓ^{\max} suffices to cover the demand of all edges contained in ℓ . Since the cost functions of the lines are proportional to the frequencies, the line ℓ is not operated at a frequency higher than f_ℓ^{\max} in any optimal solution and we can delete the corresponding x variables, i.e., fix them to zero.

Finally, if there is a line $\ell \in \mathcal{L}$ such that $|\{f \in \mathcal{F} : f_\ell^{\min} \leq f \leq f_\ell^{\max}\}| = 1$ we remove the corresponding frequency assignment constraint (2.2), since it is redundant. If additionally $f_\ell^{\min} > 0$ holds for this line, we can fix the remaining variable to one.

Deleting and Fixing Configuration Variables. We introduce some further notation before describing which configuration variables can be fixed to zero.

Definition 4.2. For an edge $e \in E$ and frequency $f \in \mathcal{F}$ we denote by

$$q_{e,f}^{\text{feas}} := |\mathcal{L}(e)| - |\{\ell \in \mathcal{L}(e) : f_\ell^{\min} > f\}|$$

the maximum number of lines that can be operated at f on e in a feasible line plan. And by

$$q_{e,f}^{\text{opt}} := \begin{cases} 0 & \text{if } f > f_\ell^{\max} \forall \ell \in \mathcal{L}(e) \\ |\mathcal{L}(e)| & \text{else.} \end{cases}$$

we indicate whether lines can be operated at frequency f in an optimal line plan. For notational convenience define

$$q_{e,f}^{\max} := \min \{q_{e,f}^{\text{feas}}, q_{e,f}^{\text{opt}}\}$$

for all $e \in E$ and $f \in \mathcal{F}$.

Let $e \in E$ and $f \in \mathcal{F}$, then we know by the observations in the previous paragraph that at least $|\{\ell \in \mathcal{L}(e) : f_\ell^{\min} > f\}|$ lines containing e have to be operated at a frequency higher than f . This implies that at most $q_{e,f}^{\text{feas}}$ lines on e can be operated at frequency f in any feasible line plan. Hence, we can conclude that the left-hand side of the frequency demand constraint

$$\sum_{\ell \in \mathcal{L}(e)} x_{\ell,f} \geq \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_{e,q}$$

is less than or equal to $q_{e,f}^{\text{feas}}$ in any feasible solution. This in turn means, that any minimal configuration $q \in \mathcal{Q}(e)$ with $q_f > q_{e,f}^{\text{feas}}$ cannot be chosen for e in any feasible solution of the configuration model and the corresponding y variable can be fixed to zero.

The argumentation in the previous paragraph also implies, that no line on e is operated at a frequency higher than $\max\{f_\ell^{\text{max}} : \ell \in \mathcal{L}(e)\}$ in any optimal solution. Hence, every variable $y_{e,q}$ for a configuration $q \in \mathcal{Q}(e)$ is equal to zero in any optimal solution, if $q_f > q_{e,f}^{\text{opt}}$ for some $f \in \mathcal{F}$. We can fix these variables to zero as well.

Finally, if there is an edge $e \in E$ such that $|\{q \in \mathcal{Q}(e) : q_f \leq q_{e,f}^{\text{max}} \forall f \in \mathcal{F}\}| = 1$, we fix the corresponding y variable to 1 and remove the corresponding configuration assignment constraint 3.2.

Example 4.1. Consider the small line planning problem given in Figure 4.3. Since the edge e_2 has a frequency demand of 9, the lines ℓ_1 and ℓ_2 both have to be operated at frequency 5 or higher. Hence, the minimal configuration $\tilde{q} = (3, 0, 0)$ of e_1 cannot be chosen in any feasible solution. These observations are reflected by the values $f_{\ell_1}^{\text{min}} = f_{\ell_2}^{\text{min}} = 2$, $q_{e_1,1}^{\text{feas}} = q_{e_1,1}^{\text{max}} = 1$, and $q_{e_2,1}^{\text{max}} = q_{e_2,1}^{\text{feas}} = 0$. The variables $x_{\ell_1,1}$, $x_{\ell_2,1}$, and $y_{e_1,\tilde{q}}$ can be fixed to zero. For all edges contained in line ℓ_3 frequency 5 or higher suffices to cover their demand. Hence, assuming the operational cost of this line is proportional to the frequency, ℓ_3 is not operated at frequency 7 and the minimal configuration $\hat{q} = (0, 0, 1)$ of e_3 is not chosen in any optimal solution. This observation is reflected by the values $f_{\ell_3}^{\text{max}} = 5$ and $q_{e_3,7}^{\text{opt}} = q_{e_3,7}^{\text{max}} = 0$. The variables $x_{\ell_3,7}$ and $y_{e_3,\hat{q}}$ can be fixed to zero.

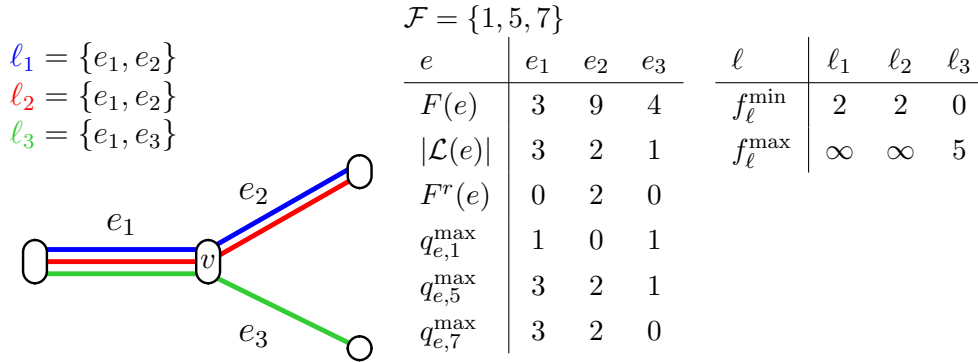


Figure 4.3: An instance of the line planning problem. *Left:* Transportation network consisting of three edges and three lines. *Right:* The given set of frequencies and frequency demands, and the values from Definition 4.1 and Definition 4.2 for all edges and lines, respectively.

4.3 Primal Heuristics

Given a mixed integer program, a primal heuristic is an algorithm that tries to find a feasible solution in a reasonably short amount of time. General-purpose primal heuristics play an indispensable part of state-of-the-art MIP solvers. For an brief overview about the primal heuristics implemented in SCIP see Achterberg [1]. A detailed description can be found in Berthold [2].

In this section we will introduce three primal heuristics for the line planning problem. For all these heuristics we take advantage of problem knowledge in order to transform a fractional solution into a feasible integer solution. Our computational studies, which can be found in the next section, show that they usefully complement the general-purpose heuristics of SCIP. We first give basic versions of the heuristics, that provide solutions for (SLP) and (SLP⁺), and describe how they can be adapted for the configuration models (QLP) and (SLP^Q) thereafter.

Rounding Heuristic

A rounding heuristic takes an LP-feasible solution and iteratively rounds all integer variables with a fractional value in order to obtain a feasible solution. The main idea of the rounding heuristic `roundResorted`, described in this section, is a problem specific strategy to choose the next variables to round.

The `roundResorted` heuristic takes as input an LP-feasible solution of (SLP) or (SLP⁺). In the initialization phase the variables of all lines with already integral values are fixed and for each edge the residual frequency demand is initialized, i.e., the frequency demand reduced by the frequencies provided by all already fixed lines. In the rounding phase we iteratively fix the variables for all lines with fractional values. In each iteration we choose among the remaining fractional lines the line with the highest total residual frequency demand of the edges contained in the line. We break ties by taking a line with lowest operational costs. The chosen line is fixed to the lowest frequency that either covers the residual frequency demand of all edges contained in the line or is at least as high as the total frequency assigned to the line in the given LP solution. Note that there always exists such a frequency, since the total frequency assigned to the line in the given solution cannot be greater than the highest frequency. In the end of each iteration the residual demands of all edges in the fixed line are updated. The heuristic is depicted in Algorithm 2. We only assign to a line a frequency less than the total frequency in the given LP solution, if this frequency suffices to covers the residual demand of

Algorithm 2: Primal Heuristic: roundResorted

Input : LP solution x' of (SLP) or (SLP⁺).**Output:** IP solution x^* of (SLP) or (SLP⁺) resp.

// initialization

1 $\mathcal{L}' := \{\ell \in \mathcal{L} : \exists f \in \mathcal{F} \text{ such that } x'_{\ell,f} \in (0, 1)\}$ 2 $\mathcal{L}^* := \{\ell \in \mathcal{L} : \forall f \in \mathcal{F} \text{ it holds } x'_{\ell,f} \in \{0, 1\}\}$ 3 $x^*_{\ell,f} := x'_{\ell,f}$ for all $\ell \in \mathcal{L}^*, f \in \mathcal{F}$ 4 $w(e) := \max\{0, F(e) - \sum_{\ell \in \mathcal{L}^*} \sum_{f \in \mathcal{F}} f \cdot x^*_{\ell,f}\}$ // residual demand

// round all fractional lines

5 **while** $\mathcal{L}' \neq \emptyset$ **do**6 $\ell' := \arg \max \{\sum_{e \in \ell} w(e) : \ell \in \mathcal{L}'\}$ 7 $f' := \min \{f \in \mathcal{F} \cup \{0\} : f \geq \max_{e \in \ell'} w(e) \text{ or } f \geq \sum_{f \in \mathcal{F}} f \cdot x'_{\ell',f}\}$ 8 $x^*_{\ell',f'} := 1$ and $x^*_{\ell',f} := 0$ for all $f \in \mathcal{F} \setminus \{f'\}$

// update

9 $w(e) := \max\{0, w(e) - f'\}$ for all $e \in \ell'$ 10 $\mathcal{L}' := \mathcal{L}' \setminus \{\ell'\}$ 11 **end**

all edges contained in the line. Hence, the computed solution is feasible for (SLP) and (SLP⁺).

The heuristic `roundResorted` is based on an adaption of the heuristic `roundSorted` in Karbstein [20, Section 8.2] to our problem setting. In `roundSorted` the lines are sorted only once before rounding all fractional lines. More specifically, the lines are sorted in non-increasing order of the sums of the pseudocosts of their variables.

Diving Heuristics

A diving heuristic also takes an LP-feasible solution as input to find a feasible integer solution. It iteratively bounds or fixes a subset of variables and resolves the LP, see Berthold [2]. The heuristic either terminates with a feasible solution or stops if the modified LP is infeasible or has an objective value not better than the best known IP solution.

The first diving heuristic we present for the models (SLP) and (SLP⁺) differs from `roundResorted` only in the rounding iterations. As in `roundResorted`, in the beginning of each iteration we fix the variables of the line, which

Algorithm 3: Primal Heuristic: roundReopt

Input : LP solution x' of (SLP) or (SLP⁺).
Output: IP solution x^* of (SLP) or (SLP⁺) resp.

```

// initialization
1  $\mathcal{L}' := \{\ell \in \mathcal{L} : \exists f \in \mathcal{F} \text{ such that } x'_{\ell,f} \in (0, 1)\}$ 
2  $\mathcal{L}^* := \{\ell \in \mathcal{L} : \forall f \in \mathcal{F} \text{ it holds } x'_{\ell,f} \in \{0, 1\}\}$ 
3  $x^*_{\ell,f} := x'_{\ell,f}$  for all  $\ell \in \mathcal{L}^*, f \in \mathcal{F}$ 
4  $w(e) := \max \{0, F(e) - \sum_{\substack{\ell \in \mathcal{L}^* \\ e \in \ell}} \sum_{f \in \mathcal{F}} f \cdot x^*_{\ell,f}\}$  // residual demand
// round all fractional lines
5 while  $\mathcal{L}' \neq \emptyset$  do
6    $\ell' := \arg \max \{\sum_{e \in \ell} w(e) : \ell \in \mathcal{L}'\}$ 
7    $f' := \min \{f \in \mathcal{F} : f \geq \sum_{f \in \mathcal{F}} f \cdot x'_{\ell',f}\}$ 
8    $x^*_{\ell',f'} := 1$  and  $x^*_{\ell',f} := 0$  for all  $f \in \mathcal{F} \setminus \{f'\}$ 
// update
9    $x' :=$  solution for (by  $x^*$ ) modified LP
10   $\mathcal{L}' := \{\ell \in \mathcal{L} : \exists f \in \mathcal{F} \text{ such that } x'_{\ell,f} \in (0, 1)\}$ 
11   $\mathcal{L}^* := \{\ell \in \mathcal{L} : \forall f \in \mathcal{F} \text{ it holds } x'_{\ell,f} \in \{0, 1\}\}$ 
12   $x^*_{\ell,f} := x'_{\ell,f}$  for all  $\ell \in \mathcal{L}^*, f \in \mathcal{F}$ 
13   $w(e) := \max \{0, F(e) - \sum_{\substack{\ell \in \mathcal{L}^* \\ e \in \ell}} \sum_{f \in \mathcal{F}} f \cdot x^*_{\ell,f}\}$ 
14 end

```

edges have the highest total residual frequency demand among all lines with fractional values. It then solves the modified LP relaxation and fixes all lines with integral values. The heuristic is depicted in detail in Algorithm 3 and is called **roundReopt**. Note that $\sum_{f \in \mathcal{F}} f \cdot x'_{\ell',f} \leq \max_{e \in \ell'} w(e)$ holds for all $\ell' \in \mathcal{L}'$ in every iteration, since the modified LP is solved after each fixing: Compare line 7 in Algorithm 3 with line 7 in Algorithm 2.

The second diving heuristic we present differs from **roundReopt** only in the choice of lines we fix in the initialization and in each rounding iteration. We fix only variables of lines, which have one variable equal to 1: We fix all variables of lines in $\mathcal{L}^* := \{\ell \in \mathcal{L} : \exists f \in \mathcal{F} \text{ such that } x'_{\ell,f} = 1\}$, cf. line 2 and line 11 in Algorithm 3. We call this heuristic **roundReopt2**.

Adjustments for Configuration Models. We simply extend the three heuristics to make them applicable to the configuration models (QLP) and (SLP^Q). After setting the values x^* for all line variables we also have to fix

the configuration variables for each configuration edge. For an edge $e \in E$ we consider the configuration $q^* \in \bar{\mathcal{Q}}(e)$ that is given by the values x^* of the line variables on this edge, i.e., $q_f^* = \sum_{\ell \in \mathcal{L}(e)} x_{\ell,f}^*$ for all $f \in \mathcal{F}$. We simply choose among the minimal configurations $\mathcal{Q}(e)$ a configuration that fits the bounds given by q^* , that is we choose a configuration $\tilde{q} \in \{q \in \mathcal{Q}(e) : q_f \leq q_f^* \text{ for all } f \in \mathcal{F}\}$ and set $y_{e,\tilde{q}}^* = 1$ and $y_{e,q}^* = 0$ for all $q \in \mathcal{Q}(e) \setminus \{\tilde{q}\}$. Note that such a minimal configuration \tilde{q} always exists (see Remark 3.3) and the choice of \tilde{q} does not influence the objective value of the resulting solution (x^*, y^*) to (QLP) and (SLP^Q) respectively.

4.4 Computations

In the following we report on our computational results. We solved all instances using the constraint integer programming framework SCIP version 3.1.0 [27] with CPLEX 12.6 as LP-solver. We used the default settings of SCIP apart from the time limit, which we set to one hour, and the primal heuristic “shiftandpropagate”, which we turned off. We implemented the preprocessing techniques as described in Section 4.2.

First, we report on our computational studies regarding the primal heuristics, which we presented in Section 4.3. Some statistics on our computational experiments for the presented heuristics are displayed in Table 4.6 – Table 4.5 for the different models, respectively. The tables list the time in seconds and the integrality gap after solving the root node with SCIP. The second column lists the respective values when only SCIP’s default heuristics are enabled, the third column when additionally the presented two diving heuristics are enabled. The next three columns list the respective values when we enable only each of the three presented heuristics. The last row in each table lists the arithmetic average of the integrality gap over all instances. When comparing the three heuristics, the computations show that the heuristic `roundReopt2` finds the best solutions for the China and Dutch instances while the heuristic `roundReopt` has the best average performance for the other instances. For the heuristic `roundResorted` there are only very few instances for which it finds better solutions than one of the diving heuristics. The computations also reveal that we can find on average the best solutions when the diving heuristics are applied in addition to the SCIP default heuristics. The integrality gap is less only for three instances for model (SLP⁺) and for the Chicago instance for model (QLP) if only the default heuristics are enabled. This can happen because of predefined execution characteristics of some of the

default heuristics of SCIP; they may be turned on or off depending on the number of already found primal solutions. When the heuristic `roundResorted` is additionally applied neither the integrality gap nor the solving time is improved, hence we disabled this heuristic. Further computations showed that on average we obtain the best results when applying the diving heuristics `roundReopt` and `roundReopt2` in the root node for all models. We will use this strategy in the following.

Now, we will discuss our computational results for solving all thirteen instances with the four models. The results are displayed in Table 4.8. The computations confirm that the configuration model (QLP) is superior to the standard model (SLP); except for China1, Potsdam2010, and Chicago, the configuration model finds a better solution within one hour or has a significantly shorter solving time, when both models find an optimal solution. Note that all models solve the instance Potsdam2010 within a few seconds and that the number of variables of Chicago is more than 18 times higher for the configuration model than the number of variables for the standard model, see Table 4.2. The configuration model has a better dual bound after solving the root node than the standard model for all instances except Potsdam1998b and Potsdam1998c. However, Potsdam1998c can only be solved to optimality with the models (SLP^Q) and (QLP). And all models find the optimal solution for Potsdam1998b within one hour, but optimality can only be proven by (QLP) and (SLP^Q). The computations also show that the set cover, symmetric band, and MIR cuts indeed improve the standard model: model (SLP⁺) performance better than model (SLP) on all instances except Dutch3 and Potsdam2010. For the Dutch instances and SiouxFalls1, which can be solved to optimality by all models, the solving time for (QLP) and (SLP^Q) is much shorter than for (SLP) and (SLP⁺). In Table 4.3a some statistics on the computational results over all instances are summarized. Comparing all four models we see that (SLP^Q) performances on average best with respect to solving time, integrality gap, primal bound and dual bound; but the difference in the average solving time between (SLP⁺) and (SLP^Q) is more significant than the difference in the average integrality gap. To get a better understanding of the strength of these models we divide the test instances in two groups: Group TN contains all instances that correspond to a public transportation network, i.e., China, Dutch, and Potsdam, and group SN contains all instances corresponding to a street network, i.e., SiouxFalls and Chicago. Some statistics on the computational results regarding these groups are summarized in Table 4.3b and Table 4.3c, respectively. (SLP^Q) clearly outperforms all other models with respect to the instances in the group TN, while (SLP⁺) seems to be better suited for the instances in SN.

4.4. COMPUTATIONS

Table 4.3: Statistics on the computations for models (SLP), (SLP⁺), (SLP^Q), and (QLP). The rows of the tables list the average computation time, average integrality gap, average primal bound, and average dual bound over all instances in the respective group; the best value in each row is highlighted in blue; ¹geometric mean, ²arithmetic mean.

(a) Statistics on all instances.

	(SLP)	(SLP ⁺)	(SLP ^Q)	(QLP)
time ¹	2314.12	2359.77	2083.04	2145.75
gap ¹	0.98	0.69	0.60	0.64
primal ²	76781.91	76634.17	76474.84	76594.26
dual ²	76055.34	76076.51	76110.95	76107.78

(b) Statistics on the group TN, i.e., China, Dutch, and Potsdam.

	(SLP)	(SLP ⁺)	(SLP ^Q)	(QLP)
time ¹	607.10	725.44	340.24	351.04
gap ¹	0.41	0.31	0.14	0.19
primal ²	107790.08	107602.16	107317.00	107486.26
dual ²	106843.17	106868.30	106919.42	106917.89

(c) Statistics on the group SN, i.e., SiouxFalls and Chicago.

	(SLP)	(SLP ⁺)	(SLP ^Q)	(QLP)
time ¹	2101.08	1956.05	1624.12	1884.99
gap ¹	3.30	1.96	2.40	2.34
primal ²	7013.53	6956.20	7079.98	7087.26
dual ²	3276.03	3294.62	3293.96	3292.98

The average solving time for (SLP^Q) and also for (QLP) is less than half the average solving time for (SLP⁺) on the instances in TN. The better performance of (SLP^Q) and (QLP) on group TN can not solely be explained by the larger size of the instances in SN; the number of variables and constraints for China3 is greater than for SiouxFalls3 and SiouxFalls2, see Table 4.2, but the solution computed by (SLP^Q) for China3 has a much better integrality gap than the solution computed by (SLP⁺).

The deviation of computational results from the results published in [5] is due to the improved performance of the newer SCIP version as well as to the use of enhanced preprocessing techniques and the two new diving heuristics.

Table 4.4: Statistics on the heuristics for the model (SLP^Q). The smallest integrality gap is highlighted in blue in each row.

name	SCIP		+ div. heur.		roundReopt		roundReopt2		roundResorted	
	time	gap	time	gap	time	gap	time	gap	time	gap
China1	8s	7.14 %	8s	1.49 %	7s	11.55 %	7s	4.73 %	7s	17.96 %
China2	61s	15.64 %	65s	3.61 %	55s	3.65 %	61s	3.61 %	56s	17.57 %
China3	276s	21.47 %	242s	1.75 %	212s	15.79 %	211s	1.75 %	262s	23.67 %
Dutch1	<1s	7.23 %	<1s	1.45 %	<1s	4.34 %	<1s	1.45 %	<1s	11.14 %
Dutch2	4s	10.55 %	4s	1.31 %	3s	6.44 %	4s	1.31 %	3s	11.58 %
Dutch3	9s	15.53 %	9s	1.16 %	6s	3.04 %	8s	1.16 %	8s	24.43 %
SiouxFalls1	6s	2.81 %	6s	2.81 %	4s	27.96 %	4s	61.55 %	4s	45.33 %
SiouxFalls2	62s	10.59 %	90s	10.59 %	59s	58.71 %	47s	226.90 %	59s	83.62 %
SiouxFalls3	177s	12.55 %	260s	12.55 %	171s	26.48 %	133s	265.44 %	171s	56.54 %
Potsdam1998b	27s	7.52 %	28s	6.58 %	37s	9.75 %	28s	10.71 %	26s	20.64 %
Potsdam1998c	98s	5.59 %	90s	3.98 %	79s	7.01 %	88s	8.97 %	96s	11.13 %
Potsdam2010	2s	0.22 %	2s	0.22 %	2s	1.90 %	2s	2.35 %	2s	0.09 %
Chicago	475s	6.45 %	770s	6.03 %	286s	24.97 %	589s	43.06 %	331s	48.00 %
average gap		9.48 %		4.12 %		15.51 %		48.69 %		28.59 %

Table 4.5: Statistics on the heuristics for the model (QLP). The smallest integrality gap is highlighted in blue in each row.

name	SCIP		+ div. heur.		roundReopt		roundReopt2		roundResorted	
	time	gap	time	gap	time	gap	time	gap	time	gap
China1	21s	10.82 %	19s	2.10 %	18s	7.67 %	16s	2.10 %	18s	14.74 %
China2	37s	19.22 %	40s	2.25 %	27s	9.70 %	36s	2.25 %	33s	16.48 %
China3	192s	28.72 %	147s	2.46 %	180s	10.57 %	137s	2.46 %	180s	26.57 %
Dutch1	<1s	7.23 %	<1s	1.45 %	<1s	4.34 %	<1s	1.45 %	<1s	11.14 %
Dutch2	4s	10.55 %	4s	1.31 %	3s	6.44 %	4s	1.31 %	3s	11.58 %
Dutch3	9s	15.53 %	9s	1.16 %	6s	3.04 %	9s	1.16 %	8s	24.43 %
SiouxFalls1	62s	2.82 %	76s	2.82 %	64s	40.71 %	72s	95.36 %	58s	40.99 %
SiouxFalls2	110s	10.28 %	140s	10.28 %	102s	58.92 %	133s	154.45 %	98s	65.90 %
SiouxFalls3	115s	12.51 %	126s	12.51 %	109s	77.08 %	164s	221.79 %	100s	62.24 %
Potsdam1998b	41s	22.07 %	33s	7.19 %	39s	7.35 %	37s	3.92 %	33s	23.69 %
Potsdam1998c	53s	10.74 %	55s	5.26 %	52s	6.71 %	46s	6.40 %	47s	12.82 %
Potsdam2010	2s	0.22 %	2s	0.22 %	2s	1.90 %	2s	2.35 %	2s	0.09 %
Chicago	1925s	7.62 %	1h	24.10 %	1475s	21.16 %	1h	59.59 %	1701s	51.35 %
average gap		12.18 %		5.62 %		19.66 %		42.66 %		27.85 %

Table 4.6: Statistics on the heuristics for the model (SLP). The smallest integrality gap is highlighted in blue in each row.

name	SCIP		+ div. heur.		roundReopt		roundReopt2		roundResorted	
	time	gap	time	gap	time	gap	time	gap	time	gap
China1	3s	5.96 %	4s	3.90 %	3s	8.25 %	3s	3.90 %	3s	10.70 %
China2	22s	20.60 %	19s	2.09 %	18s	10.74 %	18s	2.09 %	17s	19.20 %
China3	491s	21.68 %	416s	2.03 %	414s	10.86 %	410s	2.03 %	484s	30.92 %
Dutch1	1s	7.24 %	1s	0.78 %	1s	1.80 %	1s	0.78 %	1s	1.80 %
Dutch2	5s	9.37 %	4s	1.32 %	4s	8.17 %	4s	1.32 %	4s	11.25 %
Dutch3	50s	16.73 %	43s	1.16 %	44s	5.09 %	42s	1.16 %	44s	11.25 %
SiouxFalls1	2s	2.67 %	2s	2.67 %	2s	19.33 %	2s	107.71 %	2s	36.44 %
SiouxFalls2	90s	11.33 %	94s	11.33 %	84s	41.23 %	88s	186.73 %	84s	66.24 %
SiouxFalls3	528s	13.89 %	537s	13.89 %	507s	57.61 %	515s	240.61 %	514s	71.79 %
Potsdam1998b	16s	3.43 %	20s	1.52 %	18s	8.43 %	18s	8.72 %	18s	14.44 %
Potsdam1998c	61s	3.76 %	64s	2.46 %	59s	7.19 %	61s	7.41 %	59s	12.04 %
Potsdam2010	1s	0.81 %	1s	0.81 %	1s	2.06 %	1s	1.86 %	1s	4.65 %
Chicago	247s	5.48 %	262s	4.44 %	182s	23.51 %	230s	61.46 %	177s	51.61 %
average gap		9.46 %		3.72 %		15.71 %		48.14 %		26.33 %

Table 4.7: Statistics on the heuristics for the model (SLP⁺). The smallest integrality gap is highlighted in blue in each row.

name	SCIP		+ div. heur.		roundReopt		roundReopt2		roundResorted	
	time	gap	time	gap	time	gap	time	gap	time	gap
China1	13s	7.45 %	12s	1.75 %	9s	7.82 %	11s	1.75 %	11s	18.51 %
China2	51s	10.83 %	39s	2.09 %	39s	11.31 %	36s	2.09 %	39s	22.16 %
China3	643s	13.06 %	640s	3.01 %	621s	12.30 %	624s	3.01 %	624s	24.75 %
Dutch1	1s	5.20 %	1s	1.08 %	1s	1.75 %	1s	1.08 %	1s	14.03 %
Dutch2	8s	15.71 %	7s	1.33 %	7s	4.24 %	6s	1.33 %	7s	25.98 %
Dutch3	58s	14.71 %	65s	1.19 %	47s	9.23 %	64s	1.19 %	47s	22.93 %
SiouxFalls1	9s	2.59 %	7s	2.63 %	7s	7.01 %	7s	57.05 %	9s	23.87 %
SiouxFalls2	82s	10.63 %	85s	10.63 %	51s	46.69 %	53s	233.09 %	51s	76.70 %
SiouxFalls3	266s	12.38 %	272s	12.38 %	157s	41.37 %	162s	302.02 %	156s	63.51 %
Potsdam1998b	47s	3.94 %	48s	1.31 %	44s	6.76 %	42s	9.14 %	42s	17.42 %
Potsdam1998c	170s	2.98 %	180s	3.56 %	173s	6.27 %	173s	6.70 %	172s	8.97 %
Potsdam2010	2s	1.86 %	2s	1.07 %	2s	2.34 %	2s	1.07 %	2s	3.57 %
Chicago	613s	3.69 %	667s	5.09 %	468s	22.99 %	480s	39.47 %	430s	49.11 %
average gap		8.08 %		3.62 %		13.85 %		50.69 %		28.58 %

Table 4.8: Statistics on the computations for the models (SLP), (SLP⁺), (SLP^Q), and (QLP). The columns list the instance name, model, computation time, number of branching nodes, the integrality gap, the primal bound, the dual bound, and the dual bound after solving the root node.

name	model	time	nodes	gap	primal	dual	root dual
China1	(SLP)	1h	1 086 084	0.22 %	235 649.00	235 122.59	234 928.08
	(SLP ⁺)	1h	723 287	0.14 %	235 449.00	235 119.75	234 908.95
	(SLP ^Q)	1h	876 159	0.23 %	235 649.00	235 109.37	234 946.04
	(QLP)	1h	22 511	0.42 %	236 025.80	235 049.92	234 977.93
China2	(SLP)	1h	123 300	1.19 %	236 689.20	233 912.28	233 796.77
	(SLP ⁺)	1h	32 452	0.84 %	235 977.40	234 014.88	233 905.19
	(SLP ^Q)	1h	19 029	0.43 %	235 149.00	234 147.43	234 078.07
	(QLP)	1h	10 411	0.65 %	235 695.60	234 167.09	234 102.59
China3	(SLP)	1h	17 070	2.01 %	238 387.40	233 680.06	233 648.25
	(SLP ⁺)	1h	2 632	1.70 %	237 747.40	233 763.22	233 735.48
	(SLP ^Q)	1h	4 097	0.87 %	235 949.00	233 911.99	233 867.69
	(QLP)	1h	4 160	1.12 %	236 549.00	233 938.01	233 895.49
Dutch1	(SLP)	3.61s	1 546	0.00 %	59 000.00	59 000.00	58 839.16
	(SLP ⁺)	2.49s	193	0.00 %	59 000.00	59 000.00	58 867.09
	(SLP ^Q)	2.14s	114	0.00 %	59 000.00	59 000.00	58 851.85
	(QLP)	2.14s	114	0.00 %	59 000.00	59 000.00	58 851.85
Dutch2	(SLP)	688.34s	234 649	0.00 %	58 600.00	58 600.00	58 426.27
	(SLP ⁺)	748.68s	162 964	0.00 %	58 600.00	58 600.00	58 425.15
	(SLP ^Q)	36.08s	4 593	0.00 %	58 600.00	58 600.00	58 435.12
	(QLP)	35.96s	4 593	0.00 %	58 600.00	58 600.00	58 435.12
Dutch3	(SLP)	177.46s	7 902	0.00 %	58 500.00	58 500.00	58 425.00
	(SLP ⁺)	815.12s	41 710	0.00 %	58 500.00	58 500.00	58 406.25
	(SLP ^Q)	25.26s	42	0.00 %	58 500.00	58 500.00	58 425.00
	(QLP)	25.13s	42	0.00 %	58 500.00	58 500.00	58 425.00
SiouxFalls1	(SLP)	412.17s	362 959	0.00 %	2 409.84	2 409.84	2 364.70
	(SLP ⁺)	307.58s	113 569	0.00 %	2 409.84	2 409.84	2 360.83
	(SLP ^Q)	141.57s	47 109	0.00 %	2 409.84	2 409.84	2 360.69
	(QLP)	264.09s	53 060	0.00 %	2 409.84	2 409.84	2 364.97
SiouxFalls2	(SLP)	1h	20 978	5.00 %	1 714.78	1 633.14	1 633.07
	(SLP ⁺)	1h	35 469	3.57 %	1 705.75	1 646.95	1 646.95
	(SLP ^Q)	1h	22 846	3.51 %	1 704.76	1 646.95	1 646.95
	(QLP)	1h	5 055	3.61 %	1 706.45	1 646.95	1 646.95
SiouxFalls3	(SLP)	1h	1 363	13.01 %	1 521.27	1 346.13	1 345.56
	(SLP ⁺)	1h	5 753	3.90 %	1 417.60	1 364.33	1 363.93
	(SLP ^Q)	1h	6 639	4.16 %	1 420.71	1 363.98	1 363.93
	(QLP)	1h	2 284	3.43 %	1 410.83	1 364.09	1 363.93
Potsdam1998b	(SLP)	1h	642 191	0.22 %	36 067.04	35 986.98	35 910.31
	(SLP ⁺)	1h	269 321	0.20 %	36 067.04	35 995.58	35 885.10
	(SLP ^Q)	1 983.64s	103 309	0.00 %	36 067.04	36 067.04	35 872.81
	(QLP)	3 495.89s	80 224	0.00 %	36 067.04	36 067.04	35 906.74
Potsdam1998c	(SLP)	1h	150 341	1.21 %	36 151.46	35 719.99	35 632.87
	(SLP ⁺)	1h	81 532	0.72 %	36 011.96	35 754.66	35 670.87
	(SLP ^Q)	3 287.74s	65 856	0.00 %	35 872.32	35 872.32	35 724.59
	(QLP)	2 468.43s	32 373	0.00 %	35 872.32	35 872.32	35 727.29
Potsdam2010	(SLP)	1.88s	1	0.00 %	11 066.60	11 066.60	11 066.60
	(SLP ⁺)	3.13s	6	0.00 %	11 066.60	11 066.60	11 044.12
	(SLP ^Q)	3.00s	1	0.00 %	11 066.60	11 066.60	11 066.60
	(QLP)	3.01s	1	0.00 %	11 066.60	11 066.60	11 066.60
Chicago	(SLP)	1h	1 500	3.07 %	22 408.23	21 741.77	21 708.41
	(SLP ⁺)	1h	455	2.45 %	22 291.60	21 758.79	21 722.97
	(SLP ^Q)	1h	82	4.77 %	22 784.61	21 746.86	21 731.98
	(QLP)	1h	1	5.08 %	22 821.90	21 719.27	21 719.27

Chapter 5

Summary

We presented an integer program, the standard model (SLP), and a novel extended formulation, the configuration model (QLP), for the line planning problem. We compared these two models in terms of polyhedral aspects and proved that the configuration model provides a tighter LP relaxation. In particular, we showed that the LP relaxation of the configuration model implies several facet-defining inequalities for the standard model: set cover, symmetric band, multicover, and MIR inequalities. For our computational studies we proposed two additional models, (SLP⁺) and (SLP^Q). Applying the results from our polyhedral studies to the four models we obtain that the associated polytopes have the following relations

$$P(\text{QLP})|_x = P(\text{SLP}^{\text{Q}})|_x = P(\text{SLP}^+) = P(\text{SLP})$$

and

$$P_{LP}(\text{QLP})|_x \subseteq P_{LP}(\text{SLP}^{\text{Q}})|_x \subseteq P_{LP}(\text{SLP}^+) \subseteq P_{LP}(\text{SLP}).$$

Hence, from a theoretical point of view the configuration model is the tightest. Furthermore, our computational studies show that the standard model can be strengthened by including configuration variables. However, the enormous number of configuration variables for some instances is a drawback. Model (SLP^Q), which includes the configuration variables only for a subset of edges, provides a good compromise between improving the formulation with configuration variables and keeping the size of the formulation small. Especially on instances corresponding to real public transportation networks (SLP^Q) outperforms the three other models. The instance Potsdam1998c can only be solved to optimality with the models (SLP^Q) and (QLP). For the instance Potsdam1998b on the other hand, all models find the optimal solution within one hour, but optimality can only be proven by (QLP) and (SLP^Q).

The comparison of the computational results of (SLP⁺), (SLP^Q), and (QLP) shows that we obtain a better model by extending the standard model with configuration variables than by adding set cover, symmetric band, and MIR inequalities. Applying several preprocessing techniques and including primal heuristics we can speed up the solving process and improve the quality of the solutions found by all models, respectively. Finally, we like to remark that we focused on the line planning problem in this thesis, but the concept of configurations can also be adapted to other capacitated network design problems.

Notation

General

\mathbb{R}, \mathbb{R}_+	set of (non-negative) real numbers
\mathbb{Q}, \mathbb{Q}_+	set of (non-negative) rational numbers
\mathbb{Z}, \mathbb{Z}_+	set of (non-negative) integral numbers
\mathbb{N}	set of natural numbers
$\lfloor \lambda \rfloor$	floor of a real number λ , $\lfloor \lambda \rfloor \in \mathbb{Z}$
$\lceil \lambda \rceil$	ceil of a real number λ , $\lceil \lambda \rceil \in \mathbb{Z}$
2^M	power set of M
P	polyhedron
$P(\text{IP})$	polyhedron defined by the convex hull of all feasible solutions of the integer program IP
$P_{LP}(\text{IP})$	polyhedron defined by the convex hull of all feasible solutions of the LP relaxation of the integer program IP
$P _x$	projection of P onto the space of x -variables

Graphs

$G = (V, E)$	undirected graph
e, \tilde{e}	single edges
v	single node
$\delta(v)$	set of edges with v as an endnode
$D(G) = (V, A)$	directed graph that arises from G by replacing each edge by two antiparallel arcs

a	single directed arc
$e(a)$	edge corresponding to arc a
$a(e), \bar{a}(e)$	antiparallel arcs corresponding to edge e
$\mathcal{P}, \mathcal{P}_{st}$	set of directed paths, set of directed (s, t) -paths
$\mathcal{P}(a)$	set of directed paths passing arc a

Line Planning

$(\bar{\mathcal{L}}, \bar{f})$	line plan, \bar{f} assigns to each line in $\bar{\mathcal{L}}$ a frequency
\mathcal{L}	set of lines
$\mathcal{L}(e)$	set of lines passing edge e
$\mathcal{L}(E')$	set of lines passing any edge in E'
$\ell, \ell', \tilde{\ell}$	single lines
\mathcal{F}	set of frequencies, $\mathcal{F} \subset \mathbb{N}$
f, f', \tilde{f}	single frequencies
f_1, f_m	lowest and highest available frequency
$d(e)$	transportation demand of e , $d(e) \in \mathbb{N}$
d_{st}	transportation demand from node s to node t , $d_{st} \in \mathbb{Q}_+$
D	set of node-pairs (s, t) with positive transportation demand d_{st}
$F(e)$	frequency demand of e , $F(e) \in \mathbb{N}$
$F(E')$	total frequency demand of edges in E'
$\underline{F}(e), \bar{F}(e)$	lower and upper bound on the total frequency of lines passing edge e
τ_a, τ_p	travel time of arc a , travel time of path p , $\tau_a, \tau_p \in \mathbb{Q}_+$
$c_{\ell, f}$	cost of operating line ℓ at frequency f , $c_{\ell, f} \in \mathbb{Q}_+$
$c_{\ell, f}^t$	cost of operating a train on line ℓ at frequency f , $c_{\ell, f}^t \in \mathbb{Q}_+$
$c_{\ell, f}^k$	cost per car for line ℓ at frequency f , $c_{\ell, f}^k \in \mathbb{Q}_+$
κ, κ_ℓ	capacity of a line, capacity of line ℓ , $\kappa, \kappa_\ell > 0$
κ_e	capacity of an line passing edge e , $\kappa_e > 0$
κ_k	capacity of a single car, $\kappa_k > 0$

NOTATION

$\bar{Q}(e), Q(e)$	set of (minimal) configurations of edge e
\bar{q}, q	single (minimal) configuration, $\bar{q}, q \in \mathbb{Z}_+^{\mathcal{F}}$
\mathcal{E}	cover (partition) of E , $\mathcal{E} \subseteq 2^E$
$\mathcal{L}_{E'}(E'')$	set of all lines that pass all edges in E'' but no edge in $E' \setminus E''$
$E_{\mathcal{L}}(E')$	set of subsets of edges such that $\mathcal{L}_{E'}(E'')$ is not empty
$\bar{Q}(E'), Q(E')$	set of (minimal) multi-edge configurations of E'
\bar{Q}, Q	single (minimal) multi-edge configuration of E' , $\bar{Q}, Q \in \mathbb{Z}_+^{E_{\mathcal{L}}(E') \times \mathcal{F}}$
f_B	band of an edge e assigning each line in $\mathcal{L}(e)$ to a frequency or zero
\mathcal{F}_B	configuration band of an edge e assigning each line in $\mathcal{L}(e)$ to a subset of frequencies
$\alpha_{E'}^{\ell}$	number of edges line ℓ passes in E'
$\alpha_{E'}^{\max}$	maximum number of edges in E' any line in $\mathcal{L}(E')$ passes

Variables

x	line and frequency variables
y	configuration variables, multi-edge configuration variables
z	variables for the number of additional cars of a line
π	passenger flow variables

Example

We give here for an example with one edge and three lines a list of all non-trivial facets of $P(\text{SLP})$ and $P(\text{QLP})$, which we computed with PORTA [14].

Let $\mathcal{L}(e) = \{\ell_1, \ell_2, \ell_3\}$, $\mathcal{F} = \{1, 3, 4\}$, and $F(e) = 5$. The non-trivial facets of $P(\text{SLP})$ are given by the following inequalities:

$$\begin{aligned}
& +x_{l_1,1}+2x_{l_1,3}+3x_{l_1,4}+x_{l_2,1}+2x_{l_2,3}+3x_{l_2,4}+x_{l_3,1}+2x_{l_3,3}+3x_{l_3,4} \geq 4 & (MIR_1) \\
& +x_{l_1,1}+x_{l_1,3} +x_{l_1,4} +x_{l_2,1}+x_{l_2,3} +x_{l_2,4} +x_{l_3,1}+x_{l_3,3} +x_{l_3,4} \geq 2 & (MIR_2) \\
& \quad +x_{l_1,3} +x_{l_1,4} \quad +x_{l_2,3} +x_{l_2,4} \quad +x_{l_3,3} +x_{l_3,4} \geq 1 & (B_1) \\
& +x_{l_1,1}+x_{l_1,3} +x_{l_1,4} \quad +x_{l_2,3} +x_{l_2,4} \quad +x_{l_3,4} \geq 1 & (B_2) \\
& +x_{l_1,1}+x_{l_1,3} +x_{l_1,4} \quad +x_{l_2,4} \quad +x_{l_3,3} +x_{l_3,4} \geq 1 & (B_3) \\
& \quad +x_{l_1,3} +x_{l_1,4} +x_{l_2,1}+x_{l_2,3} +x_{l_2,4} \quad +x_{l_3,4} \geq 1 & (B_4) \\
& \quad +x_{l_1,4} +x_{l_2,1}+x_{l_2,3} +x_{l_2,4} \quad +x_{l_3,3} +x_{l_3,4} \geq 1 & (B_5) \\
& \quad +x_{l_1,3} +x_{l_1,4} \quad +x_{l_2,4} +x_{l_3,1}+x_{l_3,3} +x_{l_3,4} \geq 1 & (B_6) \\
& \quad +x_{l_1,4} \quad +x_{l_2,3} +x_{l_2,4} +x_{l_3,1}+x_{l_3,3} +x_{l_3,4} \geq 1 & (B_7) \\
& +x_{l_1,1}+x_{l_1,3} +2x_{l_1,4}+x_{l_2,1}+x_{l_2,3} +2x_{l_2,4} \quad +x_{l_3,3} +x_{l_3,4} \geq 2 & (LB_1) \\
& +x_{l_1,1}+x_{l_1,3} +2x_{l_1,4} \quad +x_{l_2,3} +x_{l_2,4} +x_{l_3,1}+x_{l_3,3} +2x_{l_3,4} \geq 2 & (LB_2) \\
& \quad +x_{l_1,3} +x_{l_1,4} +x_{l_2,1}+x_{l_2,3} +2x_{l_2,4}+x_{l_3,1}+x_{l_3,3} +2x_{l_3,4} \geq 2 & (LB_3) \\
& +x_{l_1,1}+2x_{l_1,3}+2x_{l_1,4} \quad +x_{l_2,3} +2x_{l_2,4} \quad +x_{l_3,3} +2x_{l_3,4} \geq 2 & (LB_4) \\
& \quad +x_{l_1,3} +2x_{l_1,4}+x_{l_2,1}+2x_{l_2,3}+2x_{l_2,4} \quad +x_{l_3,3} +2x_{l_3,4} \geq 2 & (LB_5) \\
& \quad +x_{l_1,3} +2x_{l_1,4} \quad +x_{l_2,3} +2x_{l_2,4}+x_{l_3,1}+2x_{l_3,3}+2x_{l_3,4} \geq 2 & (LB_6) \\
& +x_{l_1,1}+2x_{l_1,3}+2x_{l_1,4}+x_{l_2,1}+2x_{l_2,3}+2x_{l_2,4}+x_{l_3,1}+x_{l_3,3} +2x_{l_3,4} \geq 3 & (LB_7) \\
& +x_{l_1,1}+2x_{l_1,3}+2x_{l_1,4}+x_{l_2,1}+x_{l_2,3} +2x_{l_2,4}+x_{l_3,1}+2x_{l_3,3}+2x_{l_3,4} \geq 3 & (LB_8) \\
& +x_{l_1,1}+x_{l_1,3} +2x_{l_1,4}+x_{l_2,1}+2x_{l_2,3}+2x_{l_2,4}+x_{l_3,1}+2x_{l_3,3}+2x_{l_3,4} \geq 3 & (LB_9)
\end{aligned}$$

- Inequality (MIR_1) is a MIR inequality for $\lambda = \frac{1}{3}$, see Proposition 2.10.

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- Inequality (MIR_2) is a MIR inequality for $\lambda = \frac{1}{4}$.
 - Inequality (B_1) is a band inequality for the band $f_{\mathcal{B}}(\ell_1) = 1$, $f_{\mathcal{B}}(\ell_2) = 1$, $f_{\mathcal{B}}(\ell_3) = 1$, see Proposition 2.6.
 - Inequalities (B_2)–(B_7) are also band inequalities for bands, which are a permutation of $f_{\mathcal{B}}$, where $f_{\mathcal{B}}(\ell_1) = 0$, $f_{\mathcal{B}}(\ell_2) = 1$, $f_{\mathcal{B}}(\ell_3) = 3$.

The minimal configurations of e are

$$\mathcal{Q}(e) = \{q_1 = (0, 0, 2), q_2 = (0, 1, 1), q_3 = (1, 0, 1), q_4 = (0, 2, 0), q_5 = (2, 1, 0)\}.$$

All non-trivial facets of $P(\text{QLP})$ are configuration band inequalities in this example. We list now facet-defining configuration band inequalities, see Proposition 3.17. The remaining facets, which are not given in the following, are configuration band inequalities for a configuration band which can be obtained by a permutation of another configuration band, which is considered here. We first give the configuration band and then the corresponding configuration band inequality.

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{1, 4\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{1, 4\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \emptyset:$$

$$+x_{\ell_1,1} \quad +x_{\ell_1,4}+x_{\ell_2,1} \quad +x_{\ell_2,4} \quad -y_{q_1} \quad -y_{q_3} \quad -y_{q_5} \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{4\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{4\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \emptyset:$$

$$+x_{\ell_1,4} \quad +x_{\ell_2,4} \quad -y_{q_1} \quad \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{1, 3\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \emptyset, \mathcal{F}_{\mathcal{B}}(\ell_3) = \emptyset:$$

$$+x_{\ell_1,1}+x_{\ell_1,3} \quad -y_{q_5} \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{1\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{1\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \emptyset:$$

$$+x_{\ell_1,1} \quad +x_{\ell_2,1} \quad -y_{q_5} \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{3\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{3\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \emptyset:$$

$$+x_{\ell_1,3} \quad +x_{\ell_2,3} \quad -y_{q_4} \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{1, 3\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{3\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \emptyset:$$

$$+x_{\ell_1,1}+x_{\ell_1,3} \quad +x_{\ell_2,3} \quad -y_{q_4} -y_{q_5} \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{1, 3\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{1, 3\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \emptyset:$$

$$+x_{\ell_1,1}+x_{\ell_1,3} \quad +x_{\ell_2,1}+x_{\ell_2,3} \quad -y_{q_4} -2y_{q_5} \geq 0$$

EXAMPLE

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{1\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{1\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \{1\}:$$

$$+x_{\ell_1,1} \quad +x_{\ell_2,1} \quad +x_{\ell_3,1} \quad -y_{q_3} \quad -2y_{q_5} \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{3\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{3\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \{3\}:$$

$$+x_{\ell_1,3} \quad +x_{\ell_2,3} \quad +x_{\ell_3,3} \quad -y_{q_2} \quad -2y_{q_4} - y_{q_5} \geq 0$$

$$\mathcal{F}_{\mathcal{B}}(\ell_1) = \{4\}, \mathcal{F}_{\mathcal{B}}(\ell_2) = \{4\}, \mathcal{F}_{\mathcal{B}}(\ell_3) = \{4\}:$$

$$+x_{\ell_1,4} \quad +x_{\ell_2,4} \quad +x_{\ell_3,4} - 2y_{q_1} - y_{q_2} - y_{q_3} \geq 0$$

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