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# A Generalization of Odd Set Inequalities for the Set Packing Problem

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## Abstract

The set packing problem, sometimes also called the stable set problem, is a well-known NP-hard problem in combinatorial optimization with a wide range of applications and an interesting polyhedral structure, that has been the subject of intensive study. We contribute to this field by showing how, employing cliques, odd set inequalities for the matching problem can be generalized to valid inequalities for the set packing polytope with a clear combinatorial meaning.

## 1 Introduction and Terminology

The set packing problem, sometimes also called the stable set problem, is a well-known NP-hard (Garey and Johnson, 1979) problem in combinatorial optimization with a wide range of applications. Its weighted version can be formulated as follows. Given a finite set  $V$  and some set  $E \subseteq 2^V$  with weights assigned to each set in  $E$ , find a subset of pairwise disjoint sets from  $E$ , called a packing, with a maximum sum of weights. Although many classes of facets for the set packing problem polytope are known (see, e. g., (Borndörfer, 1998)), there is still no complete polyhedral description known and further facet classes have to be researched.

A polynomially solvable special case of the set packing problem, where all sets in  $E$  have size two, is the matching problem. For this problem, the polytope can be completely described by adding so-called odd set inequalities to the canonical description (Edmonds, 1965). In this paper, we show how, employing cliques, the odd set inequalities can be generalized to valid inequalities for the set packing problem polytope with a clear combinatorial meaning. For the hypergraph assignment problem (Borndörfer and Heismann, 2012), a partitioning problem on bipartite hypergraphs, inequalities from this class can be facet-defining. We also relate the presented inequality class to a different generalization of odd set inequalities for the stable set problem called general clique family inequalities (Pêcher and Wagler, 2006).

After summarizing the basic terminology needed in what follows, we present a combinatorial derivation of the inequality class. In the end, we show a comparison with the general clique family inequalities.

**Definition 1.1.** A *hypergraph*  $G = (V, E)$  is a pair of a vertex set  $V$  and a set  $E \subseteq 2^V \setminus \emptyset$  of subsets of  $V$  called *hyperedges*. A *packing*  $H \subseteq E$  in  $G$  is a subset of pairwise disjoint hyperedges, i. e.,  $e_1 \cap e_2 = \emptyset$  for all  $e_1, e_2 \in H$  with  $e_1 \neq e_2$ .

If all hyperedges have size  $k$ , i. e.,  $|e| = k$  for all  $e \in E$ ,  $G$  is called *k-uniform*. A two element hyperedge is also called an *edge*. If all hyperedges are edges, i. e., the hypergraph  $G$  is 2-uniform,  $G$  is also called a *graph*.

The set packing problem can then be stated as follows:

**Problem 1.2** (Set Packing Problem).

**Input:** A pair  $(G, c_E)$  consisting of a hypergraph  $G = (V, E)$  and a cost function  $c_E : E \rightarrow \mathbb{R}$ .

**Output:** A maximum cost packing in  $G$  w. r. t.  $c_E$ , i. e., a packing  $H^*$  in  $G$  such that

$$\sum_{e \in H^*} c_E(e) = \max \left\{ \sum_{e \in H} c_E(e) : H \text{ is a packing in } G \right\}.$$

The set packing problem can also be formulated as an integer linear program. The canonical formulations is the following.

$$\begin{aligned} & \underset{x \in \mathbb{R}^E}{\text{maximize}} && \sum_{e \in E} c_E(e) x_e && \text{(SSP)} \\ & \text{subject to} && \sum_{e \in \delta(v)} x_e \leq 1 && \forall v \in V && \text{(i)} \\ & && x \geq 0 && \text{(ii)} \\ & && x \in \mathbb{Z}^E. && \text{(iii)} \end{aligned}$$

Let  $P(\text{SSP}) := \text{conv}\{x \in \mathbb{R}^E : (\text{SSP}) \text{ (i)–(iii)}\}$  and  $P_{\text{LP}}(\text{SSP}) := \{x \in \mathbb{R}^E : (\text{SSP}) \text{ (i)–(ii)}\}$  be the polytopes associated with the integer linear program (SSP) and its LP relaxation, respectively.

At the end of our generalization procedure, we will substitute vertices by hyperedge cliques. They are defined as follows.

**Definition 1.3.** A *hyperedge clique* in a hypergraph  $G = (V, E)$  is a set  $Q \subseteq E$  of hyperedges such that every two hyperedges  $e_1, e_2 \in Q$  have at least one vertex in common, i. e.,  $e_1 \cap e_2 \neq \emptyset$ .

Associated with the hyperedge clique  $Q$  is the clique inequality  $\sum_{e \in Q} x_e \leq 1$ .

## 2 Generalizing Odd Set Inequalities

Consider the set packing problem for the hypergraph  $G = (V, E)$ .

In the special case that  $G$  is a graph, the set packing problem becomes an edge packing problem which can be completely described by the following system of

inequalities (Edmonds, 1965):

$$\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \quad (\text{MP i})$$

$$\sum_{e \in E: e \subset V'} x_e \leq \frac{|V'| - 1}{2} \quad \forall V' \subseteq V, |V'| \geq 3, |V'| \text{ odd} \quad (\text{MP ii})$$

$$x_e \geq 0 \quad \forall e \in E \quad (\text{MP iii})$$

The inequalities (MP ii) are called odd set inequalities. Their combinatorial meaning is that for every odd set  $V' \subseteq V$  of  $|V'| = 2k + 1$  vertices there can be at most  $\lfloor \frac{|V'|}{2} \rfloor = \frac{|V'| - 1}{2} = k$  edges connecting pairs of them in a matching. This holds since every edge is incident to two vertices in  $k$ , every vertex can be incident to at most one edge in a matching, and  $k + 1$  edges would need therefore already  $2k + 2 > |V'|$  distinct vertices.

A formal proof of validity for odd set inequalities can be interpreted as a Chvátal-Gomory procedure with coefficient  $\frac{1}{2}$  for all inequalities of type (SSP) (i) for  $v \in V'$  and 0 for all others.

We will generalize these inequalities for the set packing problem, i. e., from graphs to hypergraphs, in three steps. The first one will adapt the odd set inequalities to  $p$ -uniform hypergraphs, i. e., to hypergraphs which have hyperedges all of size  $p$ , where  $p$  can be greater than two. Then, we will tackle hypergraphs with hyperedges of arbitrary size by viewing them as combinations of hyperedges of size  $p$  in the second step. The third step will generalize sets of hyperedges incident to one vertex to hyperedge cliques.

Odd set inequalities can be also written as

$$\sum_{e \in E} \left\lfloor \frac{|\{v \in V' : e \in \delta(v)\}|}{2} \right\rfloor x_e \leq \left\lfloor \frac{|V'| - 1}{2} \right\rfloor,$$

which is a more useful representation for our generalization procedure.

**Step 1.** Let  $G$  be  $p$ -uniform. Applying the idea of odd set inequalities to this situation yields that for every set  $V' \subseteq V$  of  $|V'| = pk + r$ ,  $0 \leq r \leq p - 1$  vertices there can be at most  $\lfloor \frac{|V'|}{p} \rfloor = \frac{|V'| - r}{p} = k$  hyperedges, each connecting  $p$  of them, in a packing. For an example see Figure 1.

This leads to the inequality

$$\sum_{e \in E} \left\lfloor \frac{|\{v \in V' : e \in \delta(v)\}|}{p} \right\rfloor x_e \leq \left\lfloor \frac{|V'|}{p} \right\rfloor \quad \forall V' \subseteq V.$$

The coefficients  $\left\lfloor \frac{|\{v \in V' : e \in \delta(v)\}|}{p} \right\rfloor$  all have value 0 or 1.

The inequality can be also derived using a Chvátal-Gomory procedure with coefficient  $\frac{1}{p}$  for all inequalities of type (SSP) (i) for  $v \in V'$  and 0 for all others.

**Step 2.** Let  $G$  be an arbitrary hypergraph. Choose some  $p \in \mathbb{N}$ ,  $p \geq 2$ . Contrary to the previous case, where all hyperedges had size  $p$ , there now might be hyperedges in the packing that contain more than  $p$  vertices from  $V'$ . The inequality from

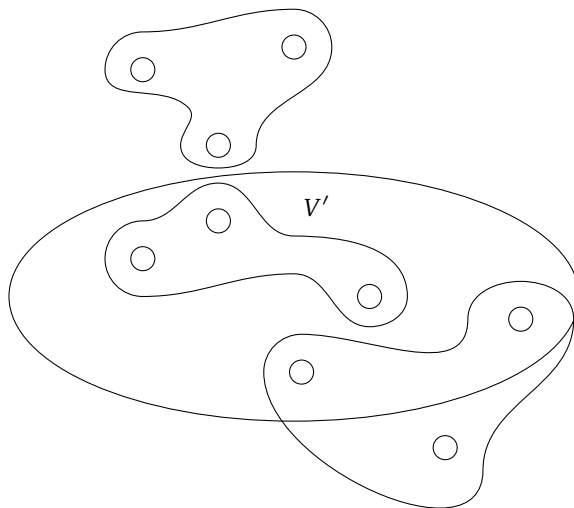


Figure 1: A packing in a 3-uniform hypergraph  $G = (V, E)$  with nine vertices and a vertex subset  $V'$  surrounded by an ellipse. There can be at most  $\lfloor \frac{5}{3} \rfloor = 1$  hyperedge which is a subset of the five vertices in  $V'$ . All other hyperedges in the packing have to have at least one vertex from  $V \setminus V'$ .

Step 1, however, is still true. A hyperedge that contains  $kp + r$  vertices from  $V'$  can be viewed as  $k$  hyperedges of size  $p$  that are contained in  $V'$ . For an example see Figure 2.

This idea leads to the inequality class

$$\sum_{e \in E} \left\lfloor \frac{|\{v \in V' : e \in \delta(v)\}|}{p} \right\rfloor x_e \leq \left\lfloor \frac{|V'|}{p} \right\rfloor \quad \forall V' \subseteq V$$

for arbitrary hypergraphs. The coefficients  $\left\lfloor \frac{|\{v \in V' : e \in \delta(v)\}|}{p} \right\rfloor$  may now have a value greater than 1.

As in the last step, a Chvátal-Gomory procedure with coefficient  $\frac{1}{p}$  for all inequalities of type (SSP) (i) for  $v \in V'$  and 0 for all others yields these inequalities.

**Step 3.** For the third step, observe that for every vertex  $v$  in a graph or hypergraph,  $\delta(v)$  is a hyperedge clique. To get the odd set inequalities or their generalizations in Steps 1 and 2, the Chvátal-Gomory procedure could be applied to the inequalities of type (SSP) (i), which are clique inequalities. In a graph,  $\delta(v)$  is the only type of maximal edge cliques. However, there may be other maximal hyperedge cliques and therefore also other valid clique inequalities for a hypergraph. Applying the previous ideas to also other types of hyperedge cliques for some hyperedge clique set  $\mathcal{Q}' \subseteq \mathcal{Q}$  we get the the inequalities

$$\sum_{e \in E} \left\lfloor \frac{|\{Q \in \mathcal{Q}' : e \in Q\}|}{p} \right\rfloor x_e \leq \left\lfloor \frac{|\mathcal{Q}'|}{p} \right\rfloor \quad \forall \mathcal{Q}' \subseteq \mathcal{Q}, p \in \mathbb{N}, p \geq 2.$$

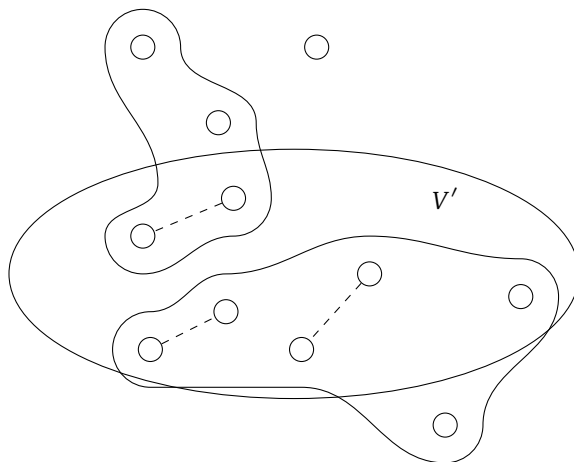


Figure 2: A packing in a hypergraph  $G = (V, E)$  with eleven vertices and a vertex subset  $V'$  surrounded by an ellipse,  $p = 2$ . There can be at most  $\lfloor \frac{7}{2} \rfloor = 3$  edges which are subsets of both the seven vertices in  $V'$  and some hyperedge in the packing. All other possible edges that would connect two vertices that are contained in some hyperedge in the packing have to have at least one vertex from  $V \setminus V'$ .

We remark that for the hypergraph assignment problem (HAP) (Borndörfer and Heismann, 2012), a partitioning problem on bipartite hypergraphs, for which all inequalities valid for the corresponding set packing relaxation are valid, these inequalities can be facet-defining. In the HAP polytope for a certain “complete bipartite hypergraph with three parts in each of the two vertex sets, all parts having size two”, one half of the 30 facet classes (this is modulo symmetry, they contain all together 14049 facets) can be described in this way with  $p = 2$ .

### 3 Comparison with General Clique Family Inequalities

(Pêcher and Wagler, 2006) propose a different generalization of odd set cuts of the set packing problem. These inequalities, “general clique family inequalities”, have a similar structure (division by some  $p \in \mathbb{N}$ , rounding, coefficient for a hyperedge variable depends on the number of hyperedge cliques that contain this hyperedge), however, the resulting inequality is different. Also, to the best of our knowledge no combinatorial interpretation was developed for general clique family inequalities so far.

General clique family inequalities are defined as follows. Let  $\mathcal{Q}' \subseteq \mathcal{Q}$  be a set of at least three edge cliques for the hypergraph  $G = (V, E)$ . Choose an integer  $p$  with  $2 \leq p \leq |\mathcal{Q}'|$ , define  $R := |\mathcal{Q}'| \bmod p$  and choose an integer  $J$  with  $0 \leq J \leq p - R$ . Now define  $E_i := \{e \in E : |\{Q \in \mathcal{Q}' : e \in Q\}| = i\}$  for  $i \in \{1, 2, \dots, |\mathcal{Q}'|\}$  to be the set

	(3)	(4)
$0 \leq i < p - J$	0	0
$p - J \leq i < p$	0	$\frac{i-R}{p-R}$
$p \leq i \leq  \mathcal{Q}' $	$\lfloor \frac{i}{p} \rfloor$	1

Table 1: Coefficient of  $x_e$ ,  $e \in E$  on left hand sides of the inequalities (3) derived in Step 3 and general clique family inequalities (4) depending on the number  $i := |\{Q \in \mathcal{Q}' : e \in Q\}|$  of hyperedge cliques in  $\mathcal{Q}'$  that contain  $e$ .

of hyperedges that are contained in exactly  $i$  hyperedge cliques in  $\mathcal{Q}'$ . The general clique family inequality

$$\sum_{i=p}^{|\mathcal{Q}'|} (p-R) \sum_{e \in E_i} x_e + \sum_{j=1}^J (p-R-j) \sum_{e \in E_{p-j}} x_e \leq b$$

is valid if  $b \geq (p-R) \lfloor \frac{|\mathcal{Q}'|}{p} \rfloor$ .

To compare the general clique family inequalities to our inequalities we rewrite Step 3 as

$$\sum_{i=0}^{|\mathcal{Q}'|} \lfloor \frac{i}{p} \rfloor \sum_{e \in E_i} x_e \leq \lfloor \frac{|\mathcal{Q}'|}{p} \rfloor, \quad (3)$$

and divide both sides of the general clique family inequalities with strongest allowed  $b$  by  $(p-R)$  to get the valid inequality

$$\sum_{i=p}^{|\mathcal{Q}'|} \sum_{e \in E_i} x_e + \sum_{j=1}^J \frac{p-R-j}{p-R} \sum_{e \in E_{p-j}} x_e \leq \lfloor \frac{|\mathcal{Q}'|}{p} \rfloor. \quad (4)$$

Now the right hand sides are equal. The coefficients of  $x_e$ ,  $e \in E$  on the left hand sides are summarized in Table 1 depending on the number  $i := |\{Q \in \mathcal{Q}' : e \in Q\}|$  of edge cliques in  $\mathcal{Q}'$  that contain  $e$ . The table shows that the inequalities concentrate on coefficients for different kinds of hyperedge variables although they employ similar objects. The inequalities derived in this paper have non-zero coefficients only for hyperedges of size  $\geq p$ . These coefficients may differ depending on the hyperedge size and be  $> 1$ , whereas the corresponding coefficients in the general clique family inequalities are all equal to 1. General clique family inequalities, however, have non-zero coefficients for smaller hyperedges.

Thus, the inequality class presented in this paper is different from the general clique family inequalities.



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