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Metric Inequalities for Routings on Direct Connections*

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Abstract

We consider multi-commodity flow problems in which capacities are installed on paths. In this setting, it is often important to distinguish between flows on direct connection routes, using single paths, and flows that include path switching. We show that the concept of metric inequalities to characterize capacities that support a multi-commodity flow can be generalized to deal with direct connections.

1 Introduction

Network design problems deal with installing capacities on arcs to support a multi-commodity flow routing of a given demand. A key component of such models are the *metric inequalities* by Iri [3] and Kakusho & Onaga [5] that characterize the feasible edge capacities, see [6, 1] for surveys. We consider a generalization in which capacities are installed on paths. This has applications, e.g., in public transport, where the paths correspond to lines [2, 4]. It is then important whether a commodity is routed on direct connections on single paths or whether one has to switch paths, in which case a *switching penalty* arises. Because of these penalties, direct connection routes are preferred, unless routes with path-switching are forced by a lack of capacity. The task is to design a system of paths with associated capacities such that a weighted sum of path and routing costs, including switching penalties, is minimized. This paper presents a tractable model to handle such problems. It is based on a novel concept of generalized *metric inequalities for direct connections*.

We use the following notation. Let $G = (V, E)$ be an undirected graph, \mathcal{P} a set of explicitly given elementary *paths* in G , and $d \in \mathbb{Q}_{\geq 0}^{V \times V}$ a *demand* for each pair of *origin-destination nodes* (OD-nodes) $s, t \in V$. We denote by $D = \{(s, t) \in V \times V : d_{st} > 0\}$

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the set of all OD-pairs with positive demand. On each path $p \in \mathcal{P}$ we can install a *capacity* $k \in K$ out of a finite set $K \subseteq \mathbb{N}$ at a *cost* $c_{p,k}$.

The demand is routed along a directed *routing graph* (V, A) that arises from the graph $G = (V, E)$ by replacing each edge $e \in E$ by two antiparallel arcs $a(e)$ and $\bar{a}(e)$; let conversely $e(a)$ be the undirected edge corresponding to such an arc $a \in A$. We also denote the routing graph by G . Each arc in the routing graph is associated with a *cost* τ_a . We say that a path $p \in \mathcal{P}$ that covers an edge e in the undirected graph covers the two antiparallel arcs $a(e)$ and $\bar{a}(e)$ in the directed routing graph. Let $\mathcal{P}(a) = \{p \in \mathcal{P} : e(a) \in p\}$ be the set of all paths that covers arc a . We denote by \mathcal{R}_{st} the set of all elementary directed *st-routes* (i. e., paths) from s to t in G and by $\mathcal{R} = \bigcup_{(s,t) \in D} \mathcal{R}_{st}$ the set of all *routes*; they will be generated.

A *direct connection st-route* (*st-dcroute*) is an *st-route* $r = (s, a_0, v_1, \dots, v_r, a_j, t)$, where $e(a_i) \in p$, $i = 0, \dots, j$, for one $p \in \mathcal{P}$, i. e., the demand can be routed along one path from origin s directly to destination t without path-switching. Let \mathcal{R}_{st}^0 be the set of all *st-dcroutes*, $\mathcal{R}_{st}^0(a) = \{r \in \mathcal{R}_{st}^0 : a \in r\}$ the set of *st-dcroutes* that pass over arc a , and $\mathcal{R}^0 = \bigcup_{(s,t) \in D} \mathcal{R}_{st}^0$, $\mathcal{R}^0(a) = \bigcup_{(s,t) \in D} \mathcal{R}_{st}^0(a)$ their unions. We denote by $\mathcal{P}_r = \{p \in \mathcal{P} : r \subset p\}$ the set of all paths that support the dcroute r . We set the *cost* of a dcroute $r \in \mathcal{R}^0$ to the sum of the arc costs $\tau_{r,0} = \sum_{a \in r} \tau_a$. For all routes $r \in \mathcal{R}$ that involve a switching between paths we add a *switching penalty* and arrive at a *cost* of $\tau_{r,1} = \sigma + \sum_{a \in r} \tau_a$. The *direct connection network design problem* that we consider is to find a set of paths and associated capacities that (i) supports a routing of the demand and (ii) minimizes a weighted sum of path plus routing costs, including switching penalties.

The paper discusses a family of network design models that integrate direct connection routings. It is structured as follows. In section 2, we start with an (explicit) *direct path connection model*, that associates direct connection routes with the corresponding paths. Section 3 derives the *metric inequalities for a direct connection routing*, that characterize the feasible path capacities. In section 4, we use these constraints to construct an efficiently solvable *complete direct connection model*. Finally, we present an approximative *basic direct connection model* of polynomial size in section 5 that involves a combinatorial subset of the metric inequalities for direct connection routings.

2 Direct Path Connection Model

We will first introduce a model that accounts for the demand on direct connections in an explicit way. To this purpose, we introduce flow variables $z_{r,0}^p$ and $y_{r,1}$ for the demand routed on dcroute r on path p and on route r with at least one path-switch, respectively. Introducing further variables $x_{p,k} \in \{0, 1\}$ for installing capacity k on path

p , we can state a *direct path connection model* as follows:

$$\begin{aligned}
\text{(DPC)} \quad \min \quad & \lambda \sum_{p \in \mathcal{P}} \sum_{k \in K} c_{p,k} x_{p,k} + (1 - \lambda) \left(\sum_{r \in \mathcal{R}^0} \sum_{p \in \mathcal{P}_r} \tau_{r,0} z_{r,0}^p + \sum_{r \in \mathcal{R}} \tau_{r,1} y_{r,1} \right) \\
& \sum_{r \in \mathcal{R}_{st}^0} \sum_{p \in \mathcal{P}_r} z_{r,0}^p + \sum_{r \in \mathcal{R}_{st}} y_{r,1} = d_{st} \quad \forall (s, t) \in D \quad (1)
\end{aligned}$$

$$\sum_{r \in \mathcal{R}^0(a)} \sum_{p \in \mathcal{P}_r} z_{r,0}^p + \sum_{r \in \mathcal{R}(a)} y_{r,1} \leq \sum_{p \in \mathcal{P}(a)} \sum_{k \in K} k x_{p,k} \quad \forall a \in A \quad (2)$$

$$\sum_{r \in \mathcal{R}^0(a): p \in \mathcal{P}_r} z_{r,0}^p \leq \sum_{k \in K} k x_{p,k} \quad \forall a \in A, \forall p \in \mathcal{P}(a) \quad (3)$$

$$\sum_{k \in K} x_{p,k} \leq 1 \quad \forall p \in \mathcal{P} \quad (4)$$

$$x_{p,k} \in \{0, 1\} \quad \forall p \in \mathcal{P}, \forall k \in K \quad (5)$$

$$z_{r,0}^p \geq 0 \quad \forall r \in \mathcal{R}^0, \forall p \in \mathcal{P}_r \quad (6)$$

$$y_{r,1} \geq 0 \quad \forall r \in \mathcal{R}. \quad (7)$$

Model (DPC) minimizes a weighted sum of path and routing costs; the weighing parameter is $\lambda \in [0, 1]$. Equations (1) enforce the demand flow. Inequalities (2) guarantee sufficient total path capacity on each arc. Constraints (3), the *direct path connection constraints*, ensure sufficient capacity for direct connection routes on each arc of each path. Inequalities (4) ensure that at most one capacity is installed on each path.

Model (DPC) includes a variable $z_{r,0}^p$ for the assignment of each direct connection route r to a direct connection path p . This makes the model fairly large. Indeed, a path of length l is usually a direct connection path for $\mathcal{O}(l^2)$ OD-pairs, such that the number of variables is much larger than the number of paths. Moreover, choices between several possible direct connection paths for every dcroute produce lots of degeneracy. We will show next how to overcome these problems and reduce the number of variables by relaxing the explicit assignment of dcroutes to direct connection paths. We will end up with a direct connection route variable $y_{r,0}$ for each route $r \in \mathcal{R}^0$.

3 A Generalization of Metric Inequalities

We eliminate the assignment of direct routes to particular paths in model (DPC) by aggregating the dcroute variables as

$$y_{r,0} = \sum_{p \in \mathcal{P}_r} z_{r,0}^p, \quad (8)$$

i. e., we introduce *path-independent* dcroute variables $y_{r,0}$ for the demand routed directly on r . Such a substitution can be easily done in the objective of model (DPC) and in the constraints (1) and (2). Skipping the direct path connection constraints (3) for the

moment, we arrive at what we will call a “*skeleton*” *direct connection model*

$$\begin{aligned}
\text{(DC-skeleton)} \quad \min \lambda \sum_{p \in \mathcal{P}} \sum_{k \in K} c_{p,k} x_{p,k} + (1 - \lambda) \left(\sum_{r \in \mathcal{R}^0} \tau_{r,0} y_{r,0} + \sum_{r \in \mathcal{R}} \tau_{r,1} y_{r,1} \right) \\
\sum_{r \in \mathcal{R}_{st}^0} y_{r,0} + \sum_{r \in \mathcal{R}_{st}} y_{r,1} = d_{st} \quad \forall (s, t) \in D \quad (9)
\end{aligned}$$

$$\sum_{r \in \mathcal{R}^0(a)} y_{r,0} + \sum_{r \in \mathcal{R}(a)} y_{r,1} \leq \sum_{p \in \mathcal{P}(a)} \sum_{k \in K} k x_{p,k} \quad \forall a \in A \quad (10)$$

$$\sum_{k \in K} x_{p,k} \leq 1 \quad \forall p \in \mathcal{P} \quad (11)$$

$$x_{p,k} \in \{0, 1\} \quad \forall p \in \mathcal{P}, \forall k \in K \quad (12)$$

$$y_{r,0} \geq 0 \quad \forall r \in \mathcal{R}^0 \quad (13)$$

$$y_{r,1} \geq 0 \quad \forall r \in \mathcal{R}. \quad (14)$$

To replace the direct path connection constraints (3) in a way that is compatible with the aggregated dcroute variables, assume we are given a solution (x^*, y^*) of model (DC-skeleton). Such a direct connection routing is possible if and only if the following set of inequalities (C) is satisfied

$$\begin{aligned}
\text{(C)} \quad \sum_{r \in \mathcal{R}^0(a): p \in \mathcal{P}_r} z_{r,0}^p \leq c^p (:= \sum_{k \in K} k x_{p,k}^*) \quad \forall a \in A, \forall p \in \mathcal{P} (a \in p) \quad (\mu_a^p) \\
\sum_{p \in \mathcal{P}_r} z_{r,0}^p = y_{r,0}^* \quad \forall r \in \mathcal{R}^0 \quad (\omega_r) \\
z_{r,0}^p \geq 0 \quad \forall r \in \mathcal{R}^0, \forall p \in \mathcal{P}_r.
\end{aligned}$$

Using the Farkas Lemma either inequality set (C) has a solution or inequality set (\bar{C})

$$\begin{aligned}
\text{(\bar{C})} \quad \sum_{p \in \mathcal{P}} c^p \sum_{a \in p} \mu_a^p + \sum_{r \in \mathcal{R}^0} \omega_r y_{r,0}^* < 0 \\
\sum_{a \in r} \mu_a^p + \omega_r \geq 0 \quad \forall r \in \mathcal{R}^0, \forall p \in \mathcal{P}_r \\
\mu_a^p \geq 0 \quad \forall p \in \mathcal{P}, \forall a \in A.
\end{aligned}$$

Note that we can restrict $\mu_a^p \leq 1$ for all $p \in \mathcal{P}$, $a \in A$. Consider some ω_r with $r \in \mathcal{R}^0$. Then

$$\begin{aligned}
-\omega_r &\leq \sum_{a \in r} \mu_a^p \quad \forall p \in \mathcal{P}_r \\
\Leftrightarrow -\omega_r &= \min_{p \in \mathcal{P}_r} \left\{ \sum_{a \in r} \mu_a^p \right\} \quad (=: \text{dist}_{\mu}^{\mathcal{P}}(r)).
\end{aligned}$$

We then get that (\bar{C}) has a solution if and only if there exists $\mu \in [0, 1]^{\mathcal{P} \times A}$ with

$$\sum_{p \in \mathcal{P}} c^p \sum_{a \in A} \mu_a^p < - \sum_{r \in \mathcal{R}^0} \omega_r y_{r,0}^* = \sum_{r \in \mathcal{R}^0} \text{dist}_\mu^{\mathcal{P}}(r) y_{r,0}^*.$$

This gives a necessary and sufficient feasibility condition for aggregated multi-commodity direct connection flows:

Theorem 3.1. *A capacity vector $c \in \mathbb{R}_+^{\mathcal{P}}$ supports a direct connection routing $y_{r,0}^*$ if and only if*

$$\sum_{p \in \mathcal{P}} c^p \sum_{a \in A} \mu_a^p \geq \sum_{r \in \mathcal{R}^0} \text{dist}_\mu^{\mathcal{P}}(r) y_{r,0}^* \quad \forall \mu \in [0, 1]^{\mathcal{P} \times A}. \quad (15)$$

This result generalizes the one of Iri [3], Kakusho and Onaga [5] for the feasibility of edge capacities to support a multi-commodity flow for a given demand to path capacities and direct connection routings. Hence, the inequalities (15) can be interpreted as *metric inequalities for a direct connection routing*.

4 Complete Direct Connection Model

Inserting the metric inequalities for a direct connection routing into model (DC-skeleton) produces the following *complete direct connection model*

$$\begin{aligned} \text{(DC-complete)} \quad \min \quad & \lambda \sum_{p \in \mathcal{P}} \sum_{k \in K} c_{p,k} x_{p,k} + (1 - \lambda) \left(\sum_{r \in \mathcal{R}^0} \tau_{r,0} y_{r,0} + \sum_{r \in \mathcal{R}} \tau_{r,1} y_{r,1} \right) \\ & \sum_{r \in \mathcal{R}_{st}^0} y_{r,0} + \sum_{r \in \mathcal{R}_{st}} y_{r,1} = d_{st} \quad \forall (s, t) \in D \end{aligned} \quad (16)$$

$$\sum_{r \in \mathcal{R}^0(a)} y_{r,0} + \sum_{r \in \mathcal{R}(a)} y_{r,1} \leq \sum_{p \in \mathcal{P}(a)} \sum_{k \in K} k x_{p,k} \quad \forall a \in A \quad (17)$$

$$\sum_{p \in \mathcal{P}} \sum_{a \in A} \mu_a^p \sum_{k \in K} k x_{p,k} \geq \sum_{(s,t) \in D} \sum_{r \in \mathcal{R}_{st}^0} \text{dist}_\mu^{\mathcal{P}}(r) y_{r,0} \quad \forall \mu \in [0, 1]^{\mathcal{P} \times A} \quad (18)$$

$$\sum_{k \in K} x_{p,k} \leq 1 \quad \forall p \in \mathcal{P} \quad (19)$$

$$x_{p,k} \in \{0, 1\} \quad \forall p \in \mathcal{P}, \forall k \in K \quad (20)$$

$$y_{r,0} \geq 0 \quad \forall r \in \mathcal{R}^0 \quad (21)$$

$$y_{r,1} \geq 0 \quad \forall r \in \mathcal{R}. \quad (22)$$

Proposition 4.1. *Models (DPC) and (DC-complete) are equivalent. More precisely, each solution of model (DPC) can be transformed into a solution of model (DC-complete) and vice versa.*

We argue now that model (DC-complete) is algorithmically tractable, even though it contains a large number of metric inequalities.

Consider the skeleton model (DC-skeleton) plus a (possibly empty) polynomially sized subset of the metric inequalities for a direct connection routing (18). We denote this starting model by (DC-complete^{*}). Let (x^*, y^*) be an (optimal) LP solution for (DC-complete^{*}). If the objective value of the linear program (S)

$$\begin{aligned}
(S) \quad & \min \sum_{p \in \mathcal{P}} \sum_{a \in p} \mu_a^p \sum_{k \in K} k x_{p,k}^* - \sum_{r \in \mathcal{R}^0} \omega_r y_{r,0}^* \\
\text{s.t.} \quad & \sum_{a \in r} \mu_a^p - \omega_r \geq 0 && \forall r \in \mathcal{R}^0, \forall p \in \mathcal{P}_r \\
& \mu_a^p \geq 0 && \forall p \in \mathcal{P}, \forall a \in A \\
& 1 \geq \omega_r \geq 0 && \forall r \in \mathcal{R}^0.
\end{aligned}$$

is negative, then the objective function of (S) gives rise to a violated metric inequality for a direct connection routing.

The feasible region of the separation program is a cone that does not depend on the current solution (x^*, y^*) of (DC-complete^{*}). It therefore suffices to consider a finite number of metric inequalities that correspond to the extremal rays of the feasible region. These arguments prove:

Proposition 4.2. *The metric inequalities for a direct connection routing can be separated in polynomial time.*

Proposition 4.3. *The LP-relaxation of the complete direct connection model (DC-complete) can be solved in polynomial time.*

We remark that one can also compute valid primal LP and IP solutions for model (DC-complete) from a given solution (x^*, y^*) of (DC-complete^{*}) *throughout the cutting plane algorithm* by some path-switching on connections that are currently mistaken as being direct; this is very convenient in practice. More precisely, the linear program

$$\begin{aligned}
(H) \quad & \max \sum_{r \in \mathcal{R}^0} \sum_{p \in \mathcal{P}_r} z_{r,0}^p \\
\text{s.t.} \quad & \sum_{r \in \mathcal{R}^0(a): p \in \mathcal{P}_r} z_{r,0}^p \leq \sum_{k \in K} k x_{p,k}^* && \forall p \in \mathcal{P}, \forall a \in p \\
& \sum_{p \in \mathcal{P}_r} z_{r,0}^p \leq y_{r,0}^* && \forall r \in \mathcal{R}^0 \\
& z_{r,0}^p \geq 0 && \forall r \in \mathcal{R}^0, \forall p \in \mathcal{P}_r.
\end{aligned}$$

gives rise to the maximal demand that can be routed directly according to the path capacities defined by x^* and the estimated amount of direct demand $y_{r,0}^*$, $r \in \mathcal{R}^0$. In fact, $y_{r,0} := \sum_{p \in \mathcal{P}_r} z_{r,0}^p$,

$$y_{r,1} := \begin{cases} y_{r,1}^* + (y_{r,0}^* - \sum_{p \in \mathcal{P}_r} z_{r,0}^p) & \text{for } r \in \mathcal{R}^0 \\ y_{r,1}^* & \text{for } r \notin \mathcal{R}^0 \end{cases}$$

and $x := x^*$ is a valid LP/IP solution for (DC-complete). Of course, this routing is best possible for the capacities defined by x^* if $y_{r,0}^*$, $r \in \mathcal{R}^0$, is estimated correctly, i.e., the better (DC-complete*) approximates (DC-complete) the better are the associated primal solutions.

The duality of the programs (S) and (H) can be seen as follows. We add the constant term $-\sum_{r \in \mathcal{R}^0} y_{r,0}^*$ to the objective of (H) and associate dual variables μ_a^p and ν_r with the set of inequalities. The dual of (H) is then

$$\begin{aligned} \min \quad & \sum_{p \in \mathcal{P}} \sum_{a \in p} \sum_{k \in K} \mu_a^p k x_{p,k}^* + \sum_{r \in \mathcal{R}^0} y_{r,0}^* (\nu_r - 1) \\ \text{s.t.} \quad & \sum_{a \in r} \mu_a^p + \nu_r \geq 1 && \forall r \in \mathcal{R}^0, \forall p \in \mathcal{P}_r \\ & \mu_a^p \geq 0 && \forall p \in \mathcal{P}, \forall a \in p \\ & \nu_r \geq 0 && \forall r \in \mathcal{R}^0. \end{aligned}$$

It is easy to see that the above linear program has always an optimal solution with $\nu_r \in [0, 1]$, $r \in \mathcal{R}^0$. If we substitute $\omega_r := (1 - \nu_r)$, we have $1 \geq \omega_r \geq 0$, $r \in \mathcal{R}^0$, and get the linear program (S).

This relation also gives an interpretation of the objective value of (S). Namely, the optimal objective value of (S) amounts to the demand that cannot be routed directly with the current path capacities.

5 Basic Direct Connection Model

We finally show that a combinatorially motivated compact approximation of the complete direct connection model (DC-complete) provides a provable quality. More precisely, we will show that the difference between the directly routed demand estimated by this approximation and the directly routed demand computed by model (DC-complete) can be bounded by the maximal length of a path.

We define a polynomial subset of metric inequalities (15) as follows. For each route $\tilde{r} \in \mathcal{R}^0$ and each arc $\tilde{a} \in \tilde{r}$ we set

$$\mu_a^p := \begin{cases} 1 & \text{if } a = \tilde{a} \text{ and } p \in \mathcal{P}_r \\ 0 & \text{otherwise.} \end{cases}$$

Then $c^p = \sum_{k \in K} k x_{p,k}$ and inequality (15) reads

$$\sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k} \geq \sum_{\substack{\tilde{r} \in \mathcal{R}^0(a) \\ \mathcal{P}_{\tilde{r}} \subseteq \mathcal{P}_r}} y_{\tilde{r},0}. \quad (23)$$

Such an inequality can be defined for each combination of route $\tilde{r} \in \mathcal{R}^0$ and arc $\tilde{a} \in \tilde{r}$.

Substituting inequalities (23) for the metric inequalities for a direct connection routing, we obtain the following *basic direct connection model*:

$$\begin{aligned} \text{(DC-basic)} \quad \min \quad & \lambda \sum_{p \in \mathcal{P}} \sum_{k \in K} c_{p,k} x_{p,k} + (1 - \lambda) \left(\sum_{r \in \mathcal{R}^0} \tau_{r,0} y_{r,0} + \sum_{r \in \mathcal{R}} \tau_{r,1} y_{r,1} \right) \\ & \sum_{r \in \mathcal{R}_{st}^0} y_{r,0} + \sum_{r \in \mathcal{R}_{st}} y_{r,1} = d_{st} \quad \forall (s, t) \in D \end{aligned} \quad (24)$$

$$\sum_{r \in \mathcal{R}^0(a)} y_{r,0} + \sum_{r \in \mathcal{R}(a)} y_{r,1} \leq \sum_{p \in \mathcal{P}(a)} \sum_{k \in K} k x_{p,k} \quad \forall a \in A \quad (25)$$

$$\sum_{\substack{\tilde{r} \in \mathcal{R}^0(a) \\ \mathcal{P}_{\tilde{r}} \subseteq \mathcal{P}_r}} y_{\tilde{r},0} \leq \sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k} \quad \forall r \in \mathcal{R}^0, \forall a \in A \quad (26)$$

$$\sum_{k \in K} x_{p,k} \leq 1 \quad \forall p \in \mathcal{P} \quad (27)$$

$$x_{p,k} \in \{0, 1\} \quad \forall p \in \mathcal{P}, \forall k \in K \quad (28)$$

$$y_{r,0} \geq 0 \quad \forall r \in \mathcal{R}^0 \quad (29)$$

$$y_{r,1} \geq 0 \quad \forall r \in \mathcal{R}. \quad (30)$$

Model (DC-basic) minimizes, as model (DPC) or (DC-complete), a weighted sum of path and routing costs. Equations (24) enforce the demand flow. Inequalities (25) guarantee sufficient capacity on each arc. The direct connection constraints (26) approximate the sufficiency of capacities for direct connection routes on each arc.

Let $r_1, r_2 \in \mathcal{R}^0$. We say that routes r_1 and r_2 are *comparable* if $\mathcal{P}_{r_1} \subseteq \mathcal{P}_{r_2}$ or $\mathcal{P}_{r_2} \subseteq \mathcal{P}_{r_1}$. A set of routes $\mathcal{R}' \subseteq \mathcal{R}^0$ is a *comparable set of routes* if each two elements $r_1, r_2 \in \mathcal{R}'$ are comparable.

Proposition 5.1. *Model (DC-basic) overestimates the directly routable demand (with respect to model (DC-complete)) by a factor of at most*

$$\max_{p \in \mathcal{P}} \left\lfloor \frac{|\{V \cap p\}|}{2} \right\rfloor.$$

Proof. Consider a solution (x^*, y^*) of model (DC-basic). For every $r \in \mathcal{R}^0$ and $p \in \mathcal{P}_r$ define

$$z_{r,0}^p := y_{r,0}^* \cdot \frac{\sum_{k \in K} k x_{p,k}^*}{\sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k}^*}.$$

Then it holds

$$\text{(i)} \quad \sum_{p \in \mathcal{P}_r} z_{r,0}^p = y_{r,0}^* \quad \forall r \in \mathcal{R}^0$$

$$\text{(ii)} \quad \sum_{\substack{\tilde{r} \in \mathcal{R}^0(a) \\ \mathcal{P}_{\tilde{r}} \subseteq \mathcal{P}_r}} \sum_{p \in \mathcal{P}_r} z_{\tilde{r},0}^p \leq \sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k}^* \quad \forall r \in \mathcal{R}^0, \forall a \in A$$

$$\text{(iii)} \quad z_{r,0}^p \leq \sum_{k \in K} k x_{p,k}^* \quad \forall p \in \mathcal{P}_r.$$

This can be seen as follows:

$$\begin{aligned}
\text{(i)} \quad \sum_{p \in \mathcal{P}_r} z_{r,0}^p &= \sum_{p \in \mathcal{P}_r} y_{r,0}^* \cdot \frac{\sum_{k \in K} k x_{p,k}^*}{\sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k}^*} = y_{r,0}^* & \forall r \in \mathcal{R}^0 \\
\text{(ii)} \quad \sum_{\substack{\tilde{r} \in \mathcal{R}^0(a) \\ \mathcal{P}_{\tilde{r}} \subseteq \mathcal{P}_r}} \sum_{p \in \mathcal{P}_r} z_{\tilde{r},0}^p &\stackrel{(i)}{=} \sum_{\substack{\tilde{r} \in \mathcal{R}^0(a) \\ \mathcal{P}_{\tilde{r}} \subseteq \mathcal{P}_r}} y_{\tilde{r},0}^* \stackrel{(26)}{\leq} \sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k}^* & \forall r \in \mathcal{R}^0, \forall a \in A \\
\text{(iii)} \quad \text{inequality (26) yields } y_{r,0}^* &\leq \sum_{\substack{\tilde{r} \in \mathcal{R}^0(a) \\ \mathcal{P}_{\tilde{r}} \subseteq \mathcal{P}_r}} y_{\tilde{r},0}^* \leq \sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k}^* & \forall p \in \mathcal{P}_r \\
&\Rightarrow z_{r,0}^p = y_{r,0}^* \cdot \frac{\sum_{k \in K} k x_{p,k}^*}{\sum_{p \in \mathcal{P}_r} \sum_{k \in K} k x_{p,k}^*} \leq \sum_{k \in K} k x_{p,k}^* & \forall p \in \mathcal{P}_r.
\end{aligned}$$

Let \tilde{z} be defined as

$$\tilde{z}_{r,0}^p := \frac{1}{\lfloor \frac{|V \cap p|}{2} \rfloor} z_{r,0}^p.$$

We show that \tilde{z} is a solution of (H). Consider a $p \in \mathcal{P}$ and $a \in p$. Let $p = (v_1, \dots, v_m)$ be the nodes the path traverse in this order and let $a = (v_l, v_{l+1})$, $l \in \{1, \dots, m-1\}$. Further denote by $\mathcal{R}^0(a, p) = \{r \in \mathcal{R}^0(a) : p \in \mathcal{P}_r\}$ the set of routes that contain arc a and that are supported by p . Define the following sets of routes

$$\mathcal{R}_i = \begin{cases} \{r \in \mathcal{R}^0(a, p) : r \text{ starts in } v_i\} & i \in \{1, \dots, l\} \\ \{r \in \mathcal{R}^0(a, p) : r \text{ ends in } v_i\} & i \in \{l+1, \dots, m\} \end{cases}$$

Since all routes and the path p are elementary, the sets \mathcal{R}_i , $i = 1, \dots, m$, are comparable sets of routes, and $\mathcal{R}_1 \dot{\cup} \dots \dot{\cup} \mathcal{R}_l$ and $\mathcal{R}_{l+1} \dot{\cup} \dots \dot{\cup} \mathcal{R}_m$ are partitions of the set $\mathcal{R}^0(a, p)$, respectively. Either $l \leq \lfloor \frac{|V \cap p|}{2} \rfloor$ or $m-l \leq \lfloor \frac{|V \cap p|}{2} \rfloor$. Assume w.l.o.g. $l \leq \lfloor \frac{|V \cap p|}{2} \rfloor$. We then get

$$\begin{aligned}
\sum_{r \in \mathcal{R}^0(a, p)} \tilde{z}_{r,0}^p &= \sum_{i=1}^l \sum_{r \in \mathcal{R}_i} \tilde{z}_{r,0}^p = \sum_{i=1}^l \sum_{r \in \mathcal{R}_i} \frac{1}{\lfloor \frac{|V \cap p|}{2} \rfloor} z_{r,0}^p \\
&\stackrel{(ii), (iii)}{\leq} \sum_{i=1}^l \frac{1}{\lfloor \frac{|V \cap p|}{2} \rfloor} \sum_{k \in K} k x_{p,k}^* \leq \sum_{k \in K} k x_{p,k}^*,
\end{aligned}$$

i. e., inequality

$$\sum_{r \in \mathcal{R}^0(a) : p \in \mathcal{P}_r} \tilde{z}_{r,0}^p \leq \sum_{k \in K} k x_{p,k}^* \quad \forall p \in \mathcal{P}, \forall a \in p$$

is satisfied. The inequality

$$\sum_{p \in \mathcal{P}_r} \tilde{z}_{r,0}^p \leq y_{r,0}^* \quad \forall r \in \mathcal{R}^0$$



Figure 1: *Left:* Worst case example for Proposition 5.1 and approximation factor 2. *Right:* General worst case example for Proposition 5.1.

follows immediately with (i).

Hence,

$$\begin{aligned}
 y_{r,0} &:= \sum_{p \in \mathcal{P}_r} \tilde{z}_{r,0}^p, \\
 y_{r,1} &:= \begin{cases} y_{r,1}^* + (y_{r,0}^* - \sum_{p \in \mathcal{P}_r} \tilde{z}_{r,0}^p) & \text{for } r \in \mathcal{R}^0 \\ y_{r,1}^* & \text{for } r \notin \mathcal{R}^0 \end{cases} \\
 x &:= x^*
 \end{aligned}$$

is a solution for (DC-complete). Overall, we get

$$\begin{aligned}
 \sum_{r \in \mathcal{R}^0} y_{r,0} &= \sum_{r \in \mathcal{R}^0} \sum_{p \in \mathcal{P}_r} \tilde{z}_{r,0}^p = \sum_{r \in \mathcal{R}^0} \sum_{p \in \mathcal{P}_r} \frac{1}{\lfloor \frac{|V \cap p|}{2} \rfloor} z_{r,0}^p \\
 &\geq \frac{1}{\max_{p \in \mathcal{P}} \lfloor \frac{|V \cap p|}{2} \rfloor} \sum_{r \in \mathcal{R}^0} y_{r,0}^*,
 \end{aligned}$$

i. e., the number of direct travelers is overestimated by at most

$$\max_{p \in \mathcal{P}} \left\lfloor \frac{|V \cap p|}{2} \right\rfloor.$$

□

The bound is tight for the example illustrated in the left of Figure 1. We have four paths and two routes for two OD pairs with demand 100 each. The routes are incomparable on arc (2, 3), i. e., operating path p_1 and p_4 with capacity 100, respectively, implies a solution for the basic direct connection model (DC-basic) where all demand can be routed directly. But in fact either the demand from 1 to 3 or the one from 2 to 4 cannot be routed directly. This example can be extended to longer paths as illustrated in the right of Figure 1. Here, we have a (red) path with k nodes and $\lfloor \frac{k}{2} \rfloor$ routes. Each route is covered by an individual path. Furthermore, there are paths for each edge $\{j, j+1\}$, $j = 2, \dots, k-2$. Choosing the red path and the edge-paths with enough capacity yields a solution for the basic direct connection model (DC-basic) where all demand can be routed directly. But the capacity of the red path is not necessary sufficient to yield a direct connection for more than one route.

Corollary 5.2. *If each path contains at most 3 nodes then the metric inequalities for a direct connection routing (15) are implied by inequalities (23), i. e., in this case models (DC-basic) and (DC-complete) are equivalent.*

6 Summary

Let us denote by $v_R(M)$ the optimal objective value of relaxation R of an integer programming model M . Considering the IP values and the LP relaxation values of all models of this paper, we, finally, get the following picture:

Proposition 6.1.

$$\left. \begin{array}{l} v_{IP}(\text{DPC}) \\ \parallel \\ v_{IP}(\text{DC-complete}) \end{array} \right\} \geq v_{IP}(\text{DC-basic}) \geq v_{IP}(\text{DC-skeleton}),$$

$$\left. \begin{array}{l} v_{LP}(\text{DPC}) \\ \parallel \\ v_{LP}(\text{DC-complete}) \end{array} \right\} \geq v_{LP}(\text{DC-basic}) \geq v_{LP}(\text{DC-skeleton}).$$

All LPs can be solved in polynomial time.

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