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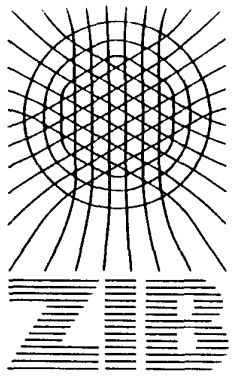

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Multivariate Rational Interpolation

Reconstruction of Rational Functions

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Abstract

In this paper we consider the problem of reconstructing a multivariate rational function, when only its values at sufficiently many points are known. We use for the reconstruction of bivariate rational functions a bivariate rational interpolation operator investigated by Siemaszko [7] and a new one, compare both by examples in a Computer Algebra system, and present their multivariate generalizations.

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Contents

1. Introduction

Rimey [5] asked for all solutions (x, y, z) of the system of equations

$$\begin{aligned} \lambda x + \varepsilon yz - x(x^2 + \alpha y^2 + \beta z^2) &= 0, \\ \lambda y + \varepsilon zx - y(y^2 + \alpha z^2 + \beta x^2) &= 0, \\ \lambda z + \varepsilon xy - z(z^2 + \alpha x^2 + \beta y^2) &= 0. \end{aligned} \tag{1.1}$$

Since the solutions depend on the parameters $\alpha, \beta, \lambda, \varepsilon$, no numerical method can be used for showing the dependency of the solutions on the parameters. Rimey derived by nonstandard methods the 27 solutions explicitly and remarked that he could not find them using packages from Computer Algebra Systems (CAS). The components of the solutions can be described by rational functions in the four parameters. And during the performance of the algorithm in the CAS polynomials occur with coefficients which are complicated rational functions in the parameters. This lead in all considered CAS to a break down by storage problems.

Therefore, Melenk [4] proposed to specify the parameters to fixed (integer) values and solve then the system (1.1). When sufficiently many values of the solutions are known, the rational functions in $\alpha, \beta, \lambda, \varepsilon$ on which the 27 solutions depend can hopefully be reconstructed.

This rises the more general problem on how to reconstruct a multivariate rational function, which is only known to exist and for which at arbitrary points the value can be computed. No a-priori informations on numerator and denominator degrees are given. Obviously, in order to solve this problem, good methods for computing interpolating rational functions must be available.

In the univariate case, the linearized rational interpolation problem is as follows:

$$\begin{aligned} &\text{Find polynomials } p, q \in \mathbb{K}[x], \mathbb{K} \text{ a field, with } \deg(p) \le r,\ \deg(q) \le s, \\ &\quad \text{such that } p(t_i) - y_i q(t_i) = 0, i = 1, \ldots, r+s+1 \text{ holds,} \\ &\quad \text{where } t_1, \ldots, t_{r+s+1} \in \mathbb{K} \text{ are given distinct points and} \\ &\quad y_1, \ldots, y_{r+s+1} \in \mathbb{K} \text{ are given data.} \end{aligned} \tag{1.2}$$

This problem is solvable by a polynomial pair p, q, where not the pair but the rational function p/q is uniquely determined. Therefore, we also say in abuse of notation that p/q solves (1.2), if it is solved by the pair p, q. For a survey and for construction methods see for instance [6].

If the data $y_1, \ldots, y_{r+s+1}$ are known to be the values of a rational function P/Q at distinct points $t_1, \ldots, t_{r+s+1}$, then the solution of (1.2) with $y_i := P(t_i)/Q(t_i)$ reconstructs the rational function in question, provided the degrees of P and Q are not too great, in other words, r and s are sufficiently great numbers.

The situation is more difficult in the multivariate case. Here (1.2) reads as follows:

$$\begin{aligned}&\text{Find polynomials } p, q \in \mathbb{K}[x_1, \ldots, x_n], \text{ with } \deg(p) \leq r, \deg(q) \leq s,\\ &\quad \text{such that } p(t_i) - y_i q(t_i) = 0, i = 1, \ldots, M, \text{ holds,}\\ &\quad \text{where } t_1, \ldots, t_M \in \mathbb{K}^n \text{ are given distinct points and}\\ &\quad y_1, \ldots, y_M \in \mathbb{K} \text{ are given data.}\end{aligned} \tag{1.3}$$

Again, we call p/q a solution, if the pair p, q solves (1.3). The dependence of M on r and s is not as canonical as in the univariate case, and for the notion of degree we may choose between the total degree (sum of exponents) or the degree by components, i.e. $\deg(f) \leq r$ means $r = (r_1, \ldots, r_n)$ and $\deg(f, x_i) \leq r_i$, $i = 1, \ldots, n$. Some methods are known for the construction of a p/q solving (1.3), see [2, 3]. Due to the lack of uniqueness, most of the methods do not reproduce the rational function P/Q as solution of (1.3), when $y_i = P(t_i)/Q(t_i), i = 1, \ldots, M$, even if (the components of) r and s are sufficiently great. When such method is applied to solve problems with increasing degree bounds, in most cases only rational functions with increasing numerator and denominator degrees are produced.

The multivariate reconstruction of a rational function P/Q consists ideally in the construction of a series of rational functions p_M/q_M solving (1.3) where r and s increase with M, such that $p_M/q_M = P/Q$ for all $M \geq M^*, M^*$ a fixed constant. For practical reasons, only a finite number of p_M/q_M's can be computed and P/Q is not known. The verification that the guess made for P/Q is in fact equal to P/Q can not be done with the methods presented here. This check can be eventually made in the area, where the reconstruction problem came from.

Therefore, we must be content with the following procedure. First of all, let us assume that we are always able to find the value of a rational function P/Q for all points in question. We select an isotone sequence of finite sets $T_1 \subset T_2 \subset \ldots \subset \mathbb{K}^n$. Let $T_k := (t_{1,k}, \ldots, t_{m_k,k})$. Then we select for each $k = 1, 2, \ldots$ a rational interpolation operator $\mathcal{P}_k$, which maps every $(y_1, \ldots, y_{m_k}) \in \mathbb{K}^{m_k}$ to a rational function p_k/q_k, such that p and q satisfy

$$p_k(t_{i,k}) - y_i q_k(t_{i,k}) = 0 \quad \text{for} \quad i = 1, \ldots, m_k.$$

(If $\mathcal{P}_k$ is the restriction of $\mathcal{P}_{k+1}$ to T_k, then T_k and $\mathcal{P}_k$ are redundant. We exclude in the following such situations.) Now let P/Q be a rational function and let p_k/q_k be the result of $\mathcal{P}_k$ applied on

$$\big(P(t_{1,k})/Q(t_{1,k}), \ldots, P(t_{m_k,k})/Q(t_{m_k,k})\big).$$

If there are an M and an $M^* \geq 1$, such that $p_M/q_M = \ldots = p_{M+M^*}/q_{M+M^*}$, then we accept p_M/q_M as guess for P/Q and terminate the procedure.

Definition 1. We say the described procedure given by the point sets $T_1 \subset T_2 \subset \ldots$, and the corresponding interpolation operators $\mathcal{P}_k, k = 1, 2, \ldots$ has the reproducing property, if it terminates for every rational function P/Q with a guess for P/Q.

In §3, the author was able to show, that a procedure using bivariate rational interpolation operators investigated by Siemaszko [7] has the reproducing property. A new bivariate method based on univariate rational and polynomial interpolation is proposed. It has no reproducing property, but only in degenerate cases, it does not reproduce the rational function in question. This procedure requires the solution of a lesser number of rational interpolation subproblems by employing polynomial interpolation. Hence, as the examples in §3 confirm, it is faster than Siemaszko's already for moderate sized functions and recovers the function when Siemaszko's fails because of storage problems. The multivariate analogy of the two methods are presented in §4.

2. Univariate Rational Interpolation

In the univariate case, rational interpolation methods with reproducing property are easily found. Take $T_m := \{t_1, \ldots, t_m\}$ with distinct points t_i. And take as rational interpolation operator $\mathcal{P}_m$ the operator which maps $m-$tuples $y_1, \ldots, y_m$ to the rational function p_m/q_m uniquely determined by the interpolation conditions

$$\begin{aligned} p_m(t_i) - y_i q_m(t_i) = 0 \ \text{ for } \ i = 1, \ldots, m, \\ \text{with } \deg(p_m) \le r_m \ \text{ and } \ \deg(q_m) \le s_m, \ r_m + s_m + 1 = m. \end{aligned} \tag{2.1}$$

Let P/Q be a rational function with $y_i = P(t_i)/Q(t_i)$ for all $i \ge 1$. If both r_m and s_m tend to infinity, when $m \to \infty$, then for sufficiently great m both degree bounds $\deg(P) \le r_m$ and $\deg(Q) \le s_m$ are satisfied, i.e. for sufficiently great m all rational functions p_m/q_m are identical with P/Q. In order to guarantee that both r_m and s_m tend to infinity for $m \to \infty$, we make the standard assumption

$$r_m := [m/2], \ s_m := [(m-1)/2], \tag{2.2}$$

where $[a]$ means the integer part of a.

Practical algorithms which calculate the p_m/q_m satisfying (2.1) and (2.2) take advantage from the fact that the rational functions p_m/q_m can be defined recursively. So many quantities computed in previous calculations can be used for abbreviating the calculations for p_m/q_m. The method of Stoer, cf. [2], calculates rational functions p_{ik}/q_{ik} interpolating at $t_i, \ldots, t_{i+k}$ and satisfying the degree bounds corresponding to (2.2),

$$\deg(p_{ik}) \le \left[(k+1-i)/2\right], \ \deg(q_{ik}) \le \left[(k-i)/2\right].$$

It initializes with $p_{ii}/q_{ii} = P(t_i)/Q(t_i)$ and calculates then iteratively

$$\begin{aligned} p_{ik}(x) &:= \lambda_{ik} \cdot (x - t_i) \cdot p_{i+1,k}(x) + \mu_{ik} \cdot (t_k - x) \cdot p_{i,k-1}(x), \\ q_{ik}(x) &:= \lambda_{ik} \cdot (x - t_i) \cdot q_{i+1,k}(x) + \mu_{ik} \cdot (t_k - x) \cdot q_{i,k-1}(x), \end{aligned} \tag{2.3}$$

where the constants λ_{ik} and μ_{ik} are chosen such that the degree restrictions also hold for p_{ik} and q_{ik}. Then $p_m/q_m := p_{1m}/q_{1m}$ solves (2.1) and (2.2).

Stoer's method requires for the computation of $p_{m+1}/q_{m+1} = p_{1,m+1}/q_{1,m+1}$ the additional computation of the m rational functions $p_{i,m+1}/q_{i,m+1}$ for $i = m+1$, $m, \ldots, 2$. An alternative method which avoids the computation of polynomials as long as the final guess is not found can be derived from a method which is attributed to Thiele, cf. [2].

Theorem 1. *Let inverted divided differences of length* 1 *be given by*

$$\Phi[t_i] := P(t_i)/Q(t_i), i = 1, \ldots, m,$$

and inverted divided differences of length $k+1$ *by those of length* k*,*

$$\Phi[t_1, \ldots, t_k, t_i] := \frac{t_i - t_k}{\Phi[t_1, \ldots, t_{k-1}, t_i] - \Phi[t_1, \ldots, t_k]}, k < i \leq m. \tag{2.4}$$

Then the continued fraction

$$\frac{p_m}{q_m} = \frac{P(t_1)}{Q(t_1)} + \frac{x - t_1}{\Phi[t_1, t_2]} + \frac{x - t_2}{\Phi[t_1, t_2, t_3]} + \cdots + \frac{x - t_{m-1}}{\Phi[t_1, \ldots, t_m]} \tag{2.5}$$

is a rational function satisfying (2.1) *and* (2.2)*, provided the denominator of no* $\Phi[t_1, \ldots, t_k]$ *vanishes.*

If the denominators of $\Phi[t_1, \ldots, t_i], i = 1, \ldots, m-1$ don't vanish, but $\Phi[t_1, \ldots, t_m]$ has a vanishing denominator, then by the definition of continued fractions, $p_{m-1}/q_{m-1} = p_m/q_m$ and p_{m-1}/q_{m-1} interpolates also in t_m. It could mean that we did already reconstruct P/Q by this p_{m-1}/q_{m-1}, but it could mean a degeneration as well. In order to overcome such degeneration, we may replace t_m by a better suited $t_k, k > m$, i.e. we look for a $\Phi[t_1, \ldots, t_{m-1}, t_k]$ with nonvanishing denominator and then replace t_k by t_m and vice versa. If no such t_k exists, we have by theorem 1 independent of the selection of t_m always p_{m-1}/q_{m-1} as rational function which interpolates at $t_1, \ldots, t_{m-1}$ and at all possible forthcoming $t_k, k \geq m$. In this case p_{m-1}/q_{m-1} is the guess for P/Q. Using this modification for avoiding degenerate situations, the author together with H. Melenk and W. Neun (both from Konrad-Zuse-Zentrum für Informationstechnik Berlin) implemented Thiele's algorithm in the Computer Algebra system REDUCE.

The following complexity analysis explains why we implemented Thiele's and not Stoer's method. These two methods require exact arithmetic. Therefore, we apply the results of [1] for the comparison. In most applications we have in mind (cf. the example in the beginning) the points can be assumed to be integers of moderate length, and the polynomials have integer coefficients. Then, denoting by $L(a)$ the length of an integer a, and assuming that integer multiplication is performed by Karatsuba's method, we get from [1] for integers a, b

Operation	Complexity
$a + b$	$\min\{L(a), L(b)\}$
$a \cdot b$	$\max\{L(a), L(b)\}^{log3}$
$quot(a, b)$	$L(b)\big(L(a) - L(b) - 1\big)$ $\quad \big(\text{for } L(a) \geq L(b)\big)$
$gcd(a, b)$	$L(a)L(b)$

And for univariate polynomials p, q with integer coefficients using $L(\Sigma a_i x^i) := \max L(a_i)$ simple calculations give for great $L(p), L(q)$, and $L(a)$

Operation	Complexity
$p+q$	$\Big[1+\min\{\deg(p),\deg(q)\}\Big]\,\min\Big\{L(p),L(q)\Big\}$
$a\cdot p$	$\Big[1+\deg(p)\Big]\,\max\Big\{L(a),L(p)\Big\}^{log3}$
$(x-a)p$	$\Big[1+\deg(p)\Big]\Big[\max\{L(a),L(p)\}^{log3}+L(a)+L(p)\Big]$
$a(x-x_i)p$	$\Big[1+\deg(p)\Big]\Big[\max\{L(a),L(p)\}^{log3}+L(p)^{log3}\Big]$ $\Big(\text{if } L(x_i)\ll L(p),L(a)\Big)$

If we denote by L_0 the maximum of all numbers $L\Big(P(x_i)\Big), L\Big(Q(x_i)\Big)$, then we get in Stoer's method that $L(p_{ik}), L(q_{ik})$ are both approximately $2^{k-i}L_0$. And the complexity for computing p_m/q_m by Stoer's method is then approximatey $\frac{4}{9}m4^mL_0^2+m3^mL_0^{log3}$. The first summand came from gcd computations for reducing previous p_i/q_i, whereas the second summand could be replaced by a lesser number, if we knew that some of the gcd's were non-trivial.

Similarly, in Thiele's method we have that numerator and denominator of any inverted divided difference of length k is approximately $2^{k-1}L_0$. And the complexity for computing all inverted divided differences in question and the combination to obtain the guess p_m/q_m is approximately $\frac{16}{3}4^mL_0^2+\frac{29}{2}3^mL_0^{log3}$. This is approximately by a factor m less than the computational amount in Stoer's method.

3. Bivariate Rational Interpolation

Some bivariate rational interpolation operators are presented in [2, 3]. With respect to the better performance of Thiele's method in comparison with Stoer's, we concentrate on interpolation methods which use branched continued fractions, a generalization of continued fractions. However, we got by several examples the impression that most of these methods are not suited for being used in procedures with the reproducing property, because even for rational functions P/Q with low $\deg(P)$ and $\deg(Q)$ the branched continued fraction representation of P/Q seemed not to be finite. And finiteness is an essential tool for proving the reproducing property, as we will see in the following.

Theorem 2. *Let P/Q be a rational function with*

$$\begin{aligned}&\deg(P,x) \le [m_1/2], \deg(P,y) \le [m_2/2],\\ &\deg(Q,x) \le \big[(m_1-1)/2\big], \deg(Q,y) \le \big[(m_2-1)/2\big],\end{aligned}$$

and let $y_1,\dots,y_{m_2}$ be different points in $\mathbb{K}$. Then there are an $s \le m_2$ and polynomials $P_k, Q_k \in \mathbb{K}[x], k = 0,\dots,s$, such that $\deg(P_k) \le 2^{k-1}m_1, \deg(Q_k) \le 2^{k-1}m_1$, and

$$\frac{P(x,y)}{Q(x,y)} = \frac{P_0(x)}{Q_0(x)} + \cfrac{y-y_1}{\Big|\ P_1(x)/Q_1(x)} + \dots + \cfrac{y-y_s}{\Big|\ P_s(x)/Q_s(x)}. \tag{3.1}$$

If $y_1,\dots,y_{m_2}$ are fixed, then s and the P_k/Q_k are uniquely determined.

Remark. The theorem is stated and proved in [7] without the degree bounds for the P_k and Q_k.

Proof. Only the degree bounds for P_k, Q_k must be shown. Let $\rho_s := P_s/Q_s$ and then recursively for $k := s-1,\dots,0$ $\rho_k := P_k/Q_k + (y-y_{k+1})/\rho_{k+1}$. It follows that $\rho_0 = P/Q$. Denoting numerator and denominator of ρ_k by A_k and B_k resp., then by definition of ρ_k

$$\begin{aligned}&\deg(P_k) \le \deg\big(A_k(.,y_{k+1})\big) \le \deg(A_k,x)\\ &\deg(Q_k) \le \deg\big(B_k(.,y_{k+1})\big) \le \deg(B_k,x).\end{aligned}$$

Beginning with the estimates

$$\begin{aligned}&\deg(P_0) \le \deg(A_0,x) \le [m_1/2] \le m_1/2,\\ &\deg(Q_0) \le \deg(B_0,x) \le \big[(m_1-1)/2\big] \le m_1/2,\end{aligned}$$

the recursion

$$A_{k+1}/B_{k+1} = 1/(y - y_{k+1}) \cdot B_k Q_k/(A_k Q_k - B_k P_k)$$

gives by induction

$$\deg(P_k) \leq \deg(A_k, x) \leq 2^{k-1} m_1 \text{ and } \deg(Q_k) \leq \deg(B_k, x) \leq 2^{k-1} m_1.$$

■

The bounds for $\deg(P_k)$ and $\deg(Q_k)$ are in general too pessimistic, since no cancellation of common factors has been assumed. So we have for instance if $y_i := i - 1$ for $i = 1, 2, \ldots$

P/Q	s	$(\deg(P_k), \deg(Q_k))$ $k = 0, \ldots, s$
$\frac{3x^2 + xy - 7y}{y^2 + x + 3}$	4	$(2,1), (2,2), (4,4), (6,6), (4,3)$
$\frac{(x-y)^2 - 2xy}{(x-y)^2 + 1}$	4	$(2,2), (4,3), (6,6), (8,7), (4,4)$
$\frac{(x-y)^4 - 2xy^3}{(x-y)^4 + 1}$	8	$(4,4), (8,5), (10,13), (18,15), (20,23), (28,25),$ $(30,30), (32,31), (16,16)$

The rational interpolation operator described in [7] uses a rectangular grid of interpolation points $T_{(k1,k2)} := \{(x_i, y_j) \mid 1 \leq i \leq k1, 1 \leq j \leq k2\}$. If d_{ij} is the interpolation date at (x_i, y_j), then for every fixed x_i the inverted divided differences $\Phi_i[y_1, \ldots, y_k]$ for the continued fraction of the univariate rational function p_i/q_i in y are calculated, which interpolates d_{ij} at $y_j, j = 1, \ldots, k2$. This gives continued fraction representations

$$\frac{p_i}{q_i} = \Phi_i[y_1] + \frac{y - y_1 \,|}{|\, \Phi_i[y_1, y_2]} + \ldots + \frac{y - y_{s_i - 1} \,|}{|\, \Phi_i[y_1, \ldots, y_{s_i}]}. \tag{3.2}$$

Using for fixed k the quantities $\Phi_i[y_1, \ldots, y_k]$ as interpolation data for the points $x_i, i = 1, \ldots, k1$, then Thiele's method calculates rational functions u_k/v_k in x. This gives then the bivariate rational function

$$\frac{u_1}{v_1} + \frac{y - y_1 \,|}{|\, u_2/v_2} + \ldots + \frac{y - y_{k2-1} \,|}{|\, u_{k2}/v_{k2}}, \tag{3.3}$$

which interpolates d_{ij} at (x_i, y_j) for all $(x_i, y_j) \in T_{(k1,k2)}$.

Now let P/Q be a given bivariate rational function and let d_{ij} be the value of P/Q at (x_i, y_j). If $k2 \geq 2 \cdot \max\{\deg(P,y), \deg(Q,y)\}$, then by the uniqueness of the univariate rational interpolation $p_i(y)/q_i(y) = P(x_i,y)/Q(x_i,y)$. Comparing with the continued fraction of P/Q given by theorem 2, we state $\Phi_i[y_1,\ldots,y_k] = P_k(x_i)/Q_k(x_i)$. If $k1$ is at least two times greater than the maximal degree of all $\deg(P_k)$, $\deg(Q_k)$, then $u_k/v_k = P_k/Q_k$ again by uniqueness of the univariate problem. Hence for sufficiently great $k1$, $k2$, P/Q is reproduced, i.e. the procedure of using this bivariate rational interpolation operator on grids increasing in both directions has the reproducing property.

An alternative to this procedure is the following. We interpolate again on a rectangular grid $T_{(k1,k2)} = \{(x_i, y_j) \mid 1 \leq i \leq k1, 1 \leq j \leq k2\}$ given data d_{ij}. First, we construct for every y_j by Thiele's method the rational function p_j/q_j solving

$$p_j(x_i) - d_{ij}q_j(x_i) = 0, i = 1,\ldots,k1.$$

Let $p_j(x) = \Sigma_{i=0}^{m} a_{ij}x^i$, and $q_j(x) = \Sigma_{i=0}^{n} b_{ij}x^i$, where $m \leq [k1/2]$ is the maximal degree of all p_j and $n \leq [(k1-1)/2]$ that one of all q_j. Cancel all p_j/q_j with $(a_{m,j}, b_{n,j}) = (0,0)$ and renumber the remaining p_l/q_l and renumber the points in $T_{(k1,k2)}$ after removing those with second component $= y_j$ for such j. For simplicity, let us assume $(a_{mj}, b_{nj}) \neq (0,0)$ for $j = 1,\ldots,k2$. Then compute by (an easy modification of) Thiele's method a rational function a_m/b_n in y solving

$$b_{nj}a_m(y_j) - a_{mj}b_n(y_j) = 0, j = 1,\ldots,k_2.$$

Let $c_j := a_m(y_j)/a_{mj}$ if $a_{mj} \neq 0$ and $c_j := b_n(y_j)/b_{nj}$ if $a_{mj} = 0$. Then find by polynomial interpolation for every $i \in \{1,\ldots,k1\}$ the lowest degree polynomial a_i satisfying

$$a_i(y_j) = c_j a_{ij}, j = 1,\ldots,k2$$

and the lowest degree polynomial b_i satisfying

$$b_i(y_j) = c_j b_{ij}, j = 1,\ldots,k2.$$

Then the rational function P^*/Q^* defined by

$$P^*(x,y) := \Sigma_{l=0}^{m} a_l(y)x^l,\ Q^*(x,y) := \Sigma_{l=0}^{n} b_l(y)x^l$$

satisfies

$$\begin{aligned} P^*(x_i,y_j) - d_{ij}Q^*(x_i,y_j) &= \Sigma_l a_l(y_j)x_i^l - d_{ij}\Sigma_l b_l(y_j)x_i^l \\ = c_j\Sigma_l a_{lj}x_i^l - c_j d_{ij}\Sigma_l b_{lj}x_i^l &= c_j\left[p_j(x_i) - d_{ij}q_j(x_i)\right] = 0 \end{aligned} \tag{3.4}$$

If the data d_{ij} are the values of a rational function P/Q evaluated at (x_i, y_j) for $i = 1,\ldots,k1, j = 1,\ldots,k2$, then $P(x,y_j)/Q(x,y_j) = p_j(x)/q_j(x)$ for sufficiently great $k1$ and $j = 1,\ldots,k2$ by construction p_j/q_j and P/Q are reduced. Therefore, only for degenerate cases $P(.,y_j)/Q(.,y_j)$ is not reduced. Hence only in these

cases $P(x,y_j) = c_j p_j(x)$ and $Q(x,y_j) = c_j q_j(x)$ holds, where c_j is not a constant polynomial. Apart of these degenerations, we have by comparing the coefficients of the powers of x,

$$a_i(y_j) = c_j a_{ij}, \quad b_i(y_j) = c_j b_{ij}, \quad j = 1, \ldots, k2, \tag{3.5}$$

when $P(x,y) = \Sigma a_i(y)x^i, Q(x,y) = \Sigma b_i(y)x^i$ and $p_j(x) = \Sigma a_{ij}x^i, q_j(x) = \Sigma b_{ij}x^i$. By (3.5) the univariate polynomials a_i and b_i are uniquely determined if there are sufficiently many interpolation points y_j, i.e. if $k2$ is sufficiently great, and if c_j is known in advance. But this can be reconstructed by a univariate rational interpolation as described above.

Therefore, this new method reconstructs a rational function P/Q, if only for a restricted number of second components y_j of points in all $T_{(k1,k2)}$ the rational function $P(.,y_j)/Q(.,y_j)$ is reducible or both $\deg\big(P(.,y_j)\big)$ and $\deg\big(Q(.,y_j)\big)$ are simultaneously less than $\deg(P,x)$ and $\deg(Q,x)$ resp.

H. Melenk and the author compared the method described by Siemaszko and the new one by some examples. We run the procedures on a CRAY X-MP using REDUCE 3.3 and obtained the following, where x means, that the result was different from the function to be reconstructed.

P/Q	Siemaszko 10×10 points	Siemaszko 20×20 points	New method 10×10 points	New method 20×20 points
$\frac{xy+1}{x+y+2}$	1.2 sec	1.1 sec	0.8 sec	3.2 sec
$\frac{xy-1}{x+y+2}$	0.8 sec	1.2 sec	0.9 sec	3.2 sec
$\frac{3x^2+xy-7y}{y^2+x+3}$	x	6.0 sec	1.2 sec	5.0 sec
$\frac{(x-y)^2-2xy}{(x-y)^2+1}$	x	7.9 sec	1.5 sec	5.5 sec
$\frac{(x-y)^4-4xy^3}{(x-y)^4+1}$	x	x	2.2 sec	11.0 sec
$\frac{(x-y)^6-6xy^5}{(x-y)^6+1}$	x	x	x	20.1 sec

The examples show that the method investigated by Siemaszko is only for rational functions of very low numerator and denominator degree (0 or 1) comparable or better than the new method. For moderate degrees the new method gives better results. The reason is that the functions P_k/Q_k depending on P/Q as in (3.1) have unproportionally high (numerator and denominator) degrees. Hence, their reconstruction by univariate rational interpolation is very time consuming. And we observed in these P_k/Q_k coefficients of remarkable length (more than 100 digits). So there are also storage problems. On the other hand, the new method interpolates only rational functions or polynomials, where the degrees never surpass the degrees of the rational function P/Q.

4. n–variate Rational Interpolation

The method described by Siemaszko can be generalized to the n–variate case in the following way. Consider the grid

$$T_{(k1,\ldots,kn)} := \{(x_{1,i1},\ldots,x_{n,in}) \mid 1 \le i1 \le k1,\ldots,1 \le in \le kn\}.$$

For every fixed $(i2,\ldots,in)$ calculate by Thiele's method the univariate rational function in x_1

$$\frac{P_{i2,\ldots,in}}{Q_{i2,\ldots,in}} = \frac{P^{(0)}_{i2,\ldots,in}}{Q^{(0}_{i2,\ldots,in}} + \frac{x_1 - x_{11}}{\mid P^{(1)}_{i2,\ldots,in}/Q^{(1)}_{i2,\ldots,in}} + \ldots + \frac{x_1 - x_{1s}}{\mid P^{(s)}_{i2,\ldots,in}/Q^{(s)}_{i2,\ldots,in}} \tag{4.1}$$

interpolating $d_{i1,\ldots,in}$ at $x_{1,i1}, 1 \le i1 \le k1$. s can be assumed to be a global variable, because continued fractions like (4.1) can be extended by adding arbitrary terms of similar type provided the first of them has a denominator $P^{(s+1)}_{i2,\ldots,in}/Q^{(s+1)}_{i2,\ldots,in}$ with $Q^{(s+1)}_{i2,\ldots,in} = 0$. The rational function P/Q which we will reconstruct has a continued fraction

$$\frac{P}{Q} = \frac{P^{(0)}}{Q^{(0)}} + \frac{x_1 - x_{11}}{\mid P^{(1)}/Q^{(1)}} + \ldots + \frac{x_1 - x_{1s}}{\mid P^{(s)}/Q^{(s)}}, \tag{4.2}$$

where $P^{(k)}/Q^{(k)}$ depends on $x_2,\ldots,x_n$. Then the $(n-1)$–variate variant of the method calculates $P^{(k)}/Q^{(k)}$ from the value $P^{(k)}_{i2,\ldots,in}/Q^{(k)}_{i2,\ldots,in}$ at point $(x_{2,i2},\ldots,x_{n,in})$.

The degrees of (numerator and denominator) of the rational functions in the intermediate steps of this method will certainly surpass the degrees of the rational function P/Q, as we already detected in the bivariate case. In addition, one n–variate reconstruction requires roughly $\deg(P,x_1)+\deg(Q,x_1)+1$ $(n-1)$–variate reconstructions of rational functions with higher degrees. Therefore, we expect an efficient reconstruction by this method only for rational functions with extremely low numerator and denominator degree. On the other hand, it is a robust method. It always reconstructs the function, if enough time and enough storage is available.

The n–variate generalization of our new method also interpolates on a grid $T_{(k1,\ldots,kn)}$. For every fixed $(i2,\ldots,in)$ calculate the univariate rational function in x_1 $P_{i2,\ldots,in}/Q_{i2,\ldots,in}$ which interpolates $d_{i1,\ldots,in}$ at $x_{1,i1}, 1 \le i1 \le k1$. Let m resp. m' be the maximal degree of all $P_{i2,\ldots,in}$ resp. $Q_{i2,\ldots,in}$ and $a^{(m)}_{i2,\ldots,in}$ resp. $b^{(m')}_{i2,\ldots,in}$ the coefficient of x_1^m in $P_{i2,\ldots,in}$ resp. of $x_1^{m'}$ in $Q_{i2,\ldots,in}$. Reduce now the grid $T_{(k1,\ldots,kn)}$ to a new grid by cancelling prints, such that in the new grid always $\left(a^{(m)}_{i2,\ldots,in}, b^{(m')}_{i2,\ldots,in}\right) \neq (0,0)$ holds. Let w.l.o.g. $T_{(k1,\ldots,kn)}$ be this grid again. Then solve the $(n-1)$–variate rational interpolation problem

$$\begin{gathered} P_m(x_{2,i2},\ldots,x_{n,in}) - \frac{a^{(m)}_{i2,\ldots,in}}{b^{(m')}_{i2,\ldots,in}} Q_{m'}(x_{2,i2},\ldots,x_{n,in}) = 0, \\ 1 \le i2 \le k2,\ldots,1 \le in \le kn. \end{gathered} \tag{4.3}$$

Let

$$\begin{aligned} P_{i2,\ldots,in}(x_1) &= \Sigma_{i=0}^{m} a_{i2,\ldots,in}^{(i)} x_1^i, \\ Q_{i2,\ldots,in}(x_1) &= \Sigma_{i=0}^{m'} b_{i2,\ldots,in}^{(i)} x_1^i \end{aligned}$$

and

$$c_{i2,\ldots,in} := \begin{cases} P_m(x_{2,i2},\ldots,x_{n,in})/a_{i2,\ldots,in}^{(m)} & \text{if } a_{i2,\ldots,in}^{(m)} \neq 0 \\ Q_{m'}(x_{2,i2},\ldots,x_{n,in})/b_{i2,\ldots,in}^{(m')} & \text{if } a_{i2,\ldots,in}^{(m)} = 0. \end{cases}$$

Then the rational function P/Q which we want to reconstruct is given by

$$\begin{aligned} P &= \Sigma_0^m P_i(x_2,\ldots,x_n) x_1^i, \\ Q &= \Sigma_0^{m'} Q_i(x_2,\ldots,x_n) x_1^i, \end{aligned} \tag{4.4}$$

where P_i and Q_i are determined (in case the grid contains enough points) by the interpolation conditions

$$\begin{aligned} P_i(x_{2,i2},\ldots,x_{n,in}) &= c_{i2,\ldots,in} \cdot a_{i2,\ldots,in}^{(i)}, \\ Q_i(x_{2,i2},\ldots,x_{n,in}) &= c_{i2,\ldots,in} \cdot b_{i2,\ldots,in}^{(i)}. \end{aligned}$$

The advantage of this method is that no intermediate rational function has unnecessary great degrees, and that only one $(n-1)$–variate variant of this method is needed for the n–variate method. On the other hand, there are degenerate cases, where the rational function reconstruction fails.

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