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On the Possible Accuracy of TVD Schemes

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Abstract

The paper presents a detailed analysis of the possible accuracy available for TVD schemes in one dimension with emphasis to the semi-discrete 1-D TVD schemes. The analysis shows that the widely accepted statement [1] of degeneration of accuracy at critical points for TVD schemes should be corrected. We have theorem: TVD schemes using flux limiters φ of the form [1], [2] may be second-order accurate at critical points if $\varphi(3) + \varphi(-1) = 2$, but cannot be uniformly second-order accurate in the whole neighborhood of critical point. If $\varphi(1) = 1$, then the TVD schemes are second-order accurate in the region of smooth solutions sufficiently far from the critical points. Two ways are suggested to improve the accuracy. Numerical example is given.

Keywords: Semi-discrete schemes, TVD, flux limiter, degeneration of accuracy.

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We consider numerical approximation methods to the initial value problems for scalar conservation laws

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0 \tag{1}$$

with initial conditions

$$w(x,0) = w_0(x) \in \mathcal{BV}(\mathbb{R}) .$$

First, we consider the five-point stencil method of line:

$$\frac{\partial u_j}{\partial t} + \frac{1}{\Delta x}\Delta_- h_{j+\frac{1}{2}} = 0 \qquad |j| < \infty \tag{2}$$

where

$$\Delta_\pm v_j = \pm(v_{j\pm 1} - v_j) ,$$

and

$$h_{j+\frac{1}{2}} = h\left(u_{j+2}, u_{j+1}, u_j, u_{j-1}\right)$$

is the numerical flux, a Lipschitz continuous function of its arguments, and

$$h(w,w,w,w) = f(w) .$$

Suppose the scheme (2) is of incremental form

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta x}\Delta_- h_{j+\frac{1}{2}} = \left(C_{j+\frac{1}{2}}\Delta_+ u_j - D_{j-\frac{1}{2}}\Delta_- u_j\right)/\Delta x \tag{3}$$

with

$$\begin{aligned} \text{a)}\quad C_{j+\frac{1}{2}} &= C\left(u_{j+2}, u_{j+1}, u_j, u_{j-1}\right) \geq 0 , \\ \text{b)}\quad D_{j-\frac{1}{2}} &= D\left(u_{j+1}, u_j, u_{j-1}, u_{j-2}\right) \geq 0 . \end{aligned}$$

Osher and Chakravarthy showed in [1] that the scheme (3) is TVD, i.e.,

$$\sum_j |\Delta_+ u_j(t_2)| \leq \sum_j |\Delta_+ u_j(t_1)| , \quad t_2 > t_1 \geq 0 .$$

We say that the scheme (1) is P–th order accurate at point x, if for any $u(x) \in C^m$, $m \geq P+1$

$$\frac{1}{\Delta x}\Delta_- h_j\Big(u(x+2\Delta x) , u(x+\Delta) , u(x) , u(x-\Delta x)\Big) = \frac{\partial f\big(u(x)\big)}{\partial x} + \mathcal{O}(\Delta x^P) .$$

Osher et. al gave in [1] a theorem: Approximation (3) is at most first-order accurate at nonsonic critical points of u (a sonic point $\overline{u}$ is one that $f'(\overline{u}) = 0$).

We cite their argument here. Denote

$$C_{j+\frac{1}{2}} = C_{j+\frac{1}{2}}\Big(u(x+2\Delta x)\ ,\ u(x+\Delta x)\ ,\ u(x)\ ,\ u(x-\Delta x)\Big)$$

and

$$C_{j+\frac{1}{2}}(u) = C_{j+\frac{1}{2}}\Big(u(x)\ ,\ u(x)\ ,\ u(x)\ ,\ u(x)\Big)\ .$$

For any $u(x) \in C^2$, we have

$$\begin{aligned}
&-\frac{1}{\Delta x}\Delta_- h_j\Big(u(x+2\Delta x)\ ,\ u(x+\Delta x)\ ,\ u(x)\ ,\ u(x-\Delta x)\Big) \\
&= \Big(C_{j+\frac{1}{2}}\Delta_+ u_j - D_{j-\frac{1}{2}}\Delta_- u_j\Big)/\Delta x \\
&= C_{j+\frac{1}{2}}\left(u_x + \frac{\Delta x}{2}u_{xx} + 0(\Delta x)\right) - D_{j-\frac{1}{2}}\left(u_x - \frac{\Delta x}{2}u_{xx} + 0(\Delta x)\right) \\
&= \Big(C_{j+\frac{1}{2}}(u) - D_{j-\frac{1}{2}}(u)\Big)u_x + \frac{\Delta x}{2}u_{xx}\Big(C_{j+\frac{1}{2}}(u) + D_{j-\frac{1}{2}}(u)\Big) \\
&\quad + \left[\Big(C_{j+\frac{1}{2}} - C_{j+\frac{1}{2}}(u)\Big) - \Big(D_{j-\frac{1}{2}} - D_{j-\frac{1}{2}}(u)\Big)\right]u_x + 0(\Delta x)\ .
\end{aligned}$$

Second-order accuracy at critical points ($u_x = 0$) means

$$C_{j+\frac{1}{2}}(u) + D_{j-\frac{1}{2}}(u) = 0\ . \tag{4}$$

Next, they said that consistency of (3) with (1) implies

$$C_{j+\frac{1}{2}}(u) - D_{j-\frac{1}{2}}(u) = -f'(u) \tag{5}$$

which is not zero at nonsonic points.

From (4), (5), they have

$$\begin{aligned}
C_{j+\frac{1}{2}}(u) &= -\frac{1}{2}f'(u) \\
D_{j-\frac{1}{2}}(u) &= \frac{1}{2}f'(u)\ ,
\end{aligned} \tag{6}$$

which cannot be both positive. This means one of (3a), (3b) must fail at nonsonic u.

The carelessness of these arguments lies in the statemet (5). Actually the consistency of (3) with (1) only means

$$\Big(C_{j+\frac{1}{2}}(u) - D_{j-\frac{1}{2}}(u)\Big)u_x = -f'(u)u_x \;, \tag{7}$$

hence, (5) is certainly true for all noncritical u. At the critical points (even nonsonic u) (7) is satisfied always, and (6) may fail.

Now let us find the condition when the second-order accuracy is available at nonsonic critical points. For the sake of simplicity, we consider the case of a scalar linear equation

$$\frac{\partial w}{\partial t} + a\frac{\partial w}{\partial x} = 0 \;, \quad a = \text{ const } > 0 \;.$$

Consider the TVD scheme with flux limiter φ:

$$\frac{\partial u_j}{\partial t} = -\frac{a}{\Delta x}(u_j - u_{j-1}) - \frac{a}{2\Delta x}\Big[\varphi_j(u_{j+1} - u_j) - \varphi_{j-1}(u_j - u_{j-1})\Big] \tag{8}$$

which can be written as

$$\begin{aligned} \frac{\partial u_j}{\partial t} &= -\frac{a}{\Delta x}\left[1 + \frac{1}{2}\left(\frac{\varphi_j}{r_j} - \varphi_{j-1}\right)\right](u_j - u_{j-1}) \\ &= \frac{1}{\Delta x}\Big(C_{j+\frac{1}{2}}(u_{j+1} - u_j) - D_{j-\frac{1}{2}}(u_j - u_{j-1})\Big) \end{aligned} \tag{9}$$

with $\varphi_j = \varphi(r_j)$, $r_j = (u_j - u_{j-1})/(u_{j+1} - u_j)$. We see that

$$\begin{aligned} C_{j+\frac{1}{2}} &= 0 \\ D_{j-\frac{1}{2}} &= a\left[1 + \frac{1}{2}\left(\frac{\varphi_j}{r_j} - \varphi_{j-1}\right)\right] \quad \forall j \end{aligned}$$

We force $D_{j-\frac{1}{2}} \geq 0$ for (8) to be TVD. We have

$$1 + \frac{1}{2}\left(\frac{\varphi_j}{r_j} - \varphi_{j-1}\right) \geq 0 \qquad \forall j$$

or

$$\varphi_{j-1} \leq 2 + \frac{\varphi_j}{r_j} \qquad \forall j \;.$$

We simply require

$$\frac{\varphi(r)}{r} \geq 0 \qquad \forall r \;.$$

So

$$\varphi(r) \leq 2 \qquad\qquad \forall r\,.$$

Next, we require that $\varphi(r) > -\infty$ and $\varphi(r)$ is Lipschitz continuous in r, hence

$$0 \leq \frac{\varphi(r)}{r} < +\infty \text{ and } \varphi(0) = 0\,.$$

The TVD region of the scheme (8) is depicted in Fig. 1.

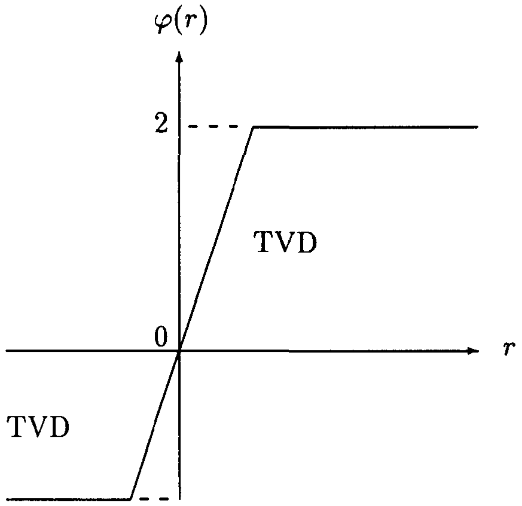

Figure 1

Define

$$r(x, \Delta x) = \frac{u(x) - u(x - \Delta x)}{u(x + \Delta x) - u(x)}\,, \quad r_j = r(x_j, \Delta x)\,.$$

For $u \in C^3$, the R.H.S. of (8) reduces to

$$\begin{aligned}
&-a\left[u_x - \frac{\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2)\right]_j \;-\; \frac{a}{2}\Bigg[\varphi(r_j)\left(u_x + \frac{\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2)\right)_j \\
&\qquad\qquad\qquad\qquad\qquad -\; \varphi(r_{j-1})\left(u_x - \frac{\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2)\right)_j\Bigg] \\
&= -au_x - \frac{a}{2}\big[\varphi(r_j) - \varphi(r_{j-1})\big]_j u_x + \frac{a\Delta x}{2}\left[1 - \frac{1}{2}\big(\varphi(r_j) + \varphi(r_{j-1})\big)\right]_j u_{xx} \\
&\qquad\qquad\qquad\qquad\qquad +\; \mathcal{O}(\Delta x^2)
\end{aligned}$$

The second-order accuracy of the scheme requires

$$\begin{aligned} \Big[\varphi(r_j) - \varphi(r_{j-1})\Big] u_x &= \mathcal{O}(\Delta x^2) \,, \\ \Big[1 - \frac{1}{2}\Big(\varphi(r_j) + \varphi(r_{j-1})\Big)\Big] u_{xx} &= \mathcal{O}(\Delta x) \,. \end{aligned} \tag{10}$$

Consider the noncritical points case when $u_x(x_j) \neq 0$. Then we have

$$\begin{aligned} &r_j = 1 - \Delta x \left(\frac{u_{xx}}{u_x}\right) + \mathcal{O}(\Delta x^2) \\ &r_{j-1} = 1 - \Delta x \left(\frac{u_{xx}}{u_x}\right) + \mathcal{O}(\Delta x^2) \\ &\varphi(r_j) - \varphi(r_{j-1}) = \mathcal{O}(\Delta x^2) \quad (\varphi \text{ is Lipschitz continuous}) \\ &1 - \frac{1}{2}\Big(\varphi(r_j) + \varphi(r_{j-1})\Big) = \Big(1 - \varphi(1)\Big) + \mathcal{O}(\Delta x) \,. \end{aligned}$$

Hence $\varphi(1) = 1$ is necessary for second-order accuracy.

Now suppose $u_x'(x_j) = 0$, but $u_{xx}''(x_j) \neq 0$ (otherwise, (8) is second-order accurate). It is easy to verify that

$$\begin{aligned} r_j &= -1 + \mathcal{O}(\Delta x) \\ r_{j-1} &= \ \ 3 + \mathcal{O}(\Delta x) \,. \end{aligned}$$

The second-order accuracy requirement (10) now reads [3]

$$\varphi(-1) + \varphi(3) = 2 \,. \tag{11}$$

The same condition can be derived for fully discrete TVD schemes of the form [2]

$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n) - \frac{\nu}{2}(1-\nu)\Delta_-\Big[\varphi_j(u_{j+1}^n - u_j^n)\Big] \tag{12}$$

where $\nu = a\Delta t/\Delta x > 0$.

If (11) is satisfied, then the scheme (12) is second-order accurate at critical points. Hence, the widely accepted statement that "second-order TVD schemes degenerate to first-order accurate at smooth maxima and minima" is also incorrect.

Now consider the question of global accuracy of TVD schemes in the nighborhood of critical points. Suppose the solution $u(x,t)$ has a critical point $(x_{\hat{\alpha}}, t)$, $x_{\hat{\alpha}} = x_j + \alpha\Delta x$, α is finite.

The expansion of the R.H.S. of (8) at x_j and $x_{\hat{\alpha}}$ gives the following formulas:

$$
\begin{aligned}
-\frac{a}{\Delta x}(u_j - u_{j-1}) &= -a\frac{\partial u_j}{\partial x} + \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}|_{x=x_{\hat{\alpha}}} + \mathcal{O}(\Delta x^2)\ , \\
r_j &= \frac{u_j - u_{j-1}}{u_{j+1} - u_j} = \frac{2\alpha+1}{2\alpha-1} + \mathcal{O}(\Delta x)\ , \\
r_{j-1} &= \frac{u_{j-1} - u_{j-2}}{u_j - u_{j-1}} = \frac{2\alpha+3}{2\alpha+1} + \mathcal{O}(\Delta x)\ , \\
\frac{\varphi(r_j)}{\Delta x}(u_{j+1} - u_j) &- \frac{\varphi(r_{j-1})}{\Delta x}(u_j - u_{j-1}) \\
&= \frac{\Delta x \partial^2 u}{2\partial x^2}|_{x=x_{\hat{\alpha}}} \cdot \Big[(1-2\alpha)\varphi(r_j) + (1+2\alpha)\varphi(r_{j-1})\Big]\mathcal{O}(\Delta x^2) \\
&= \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}|_{x=x_\alpha} \cdot \left[(1-2\alpha)\varphi\left(\frac{2\alpha+1}{2\alpha-1}\right) + (1+2\alpha)\varphi\left(\frac{2\alpha+3}{2\alpha+1}\right)\right] \\
&\qquad + \mathcal{O}(\Delta x^2)\ .
\end{aligned}
$$

The approximation (8) now reads

$$
\begin{aligned}
\frac{\partial u_j}{\partial t} &= -a\frac{\partial u_j}{\partial x} + \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}|_{x=x_{\hat{\alpha}}} \\
&\quad - \frac{a\Delta x}{4}\frac{\partial^2 u}{\partial x^2}|_{x=x_{\hat{\alpha}}} \cdot \left[(1-2\alpha)\varphi\left(\frac{2\alpha+1}{2\alpha-1}\right) + (1+2\alpha)\varphi\left(\frac{2\alpha+3}{2\alpha+1}\right)\right] \\
&\quad + \mathcal{O}(\Delta x^2)\ .
\end{aligned}
$$

Second-order accuracy of (8) leads to the condition

$$
(1-2\alpha)\varphi\left(\frac{2\alpha+1}{2\alpha-1}\right) + (1+2\alpha)\varphi\left(\frac{2\alpha+3}{2\alpha+1}\right) = 2\ , \tag{13}
$$

which coincides with (11) when $\alpha = 0$, i.e. $x_{\hat{\alpha}} = x_j$ is the critical point. Functional equation (13) with the natural condition $\varphi(0) = 0$ has to be satisfied $\forall|\alpha| < \infty$ for the flux limiter $\varphi(r)$ to reach second-order accuracy. Unfortunately, the only solution of (13) with $\varphi(0) = 0$ is

$$
\varphi(r) = r\ ,
$$

which behaves outside the TVD region (see Fig. 1) of the scheme for r large.

From the above analysis, we conclude that there is no TVD scheme of the form (8) or (12) which has uniform second-order approximation error in the whole neighborhood of critical points.

Now we summarize our results into a theorem.

Theorem. *TVD schemes* (8) *and* (12) *may have second-order accuracy at critical points if* (11) *holds, but cannot be of uniformly second-order accurate in the whole neighborhood of critical points. If* $\varphi(1) = 1$, *then these TVD schemes have second-order accuracy in the region sufficiently far from the critical points of smooth solution.*

We suggest two ways in order to improve the accuracy of the TVD schemes:

- First, we note that in the most practical cases, the set of critical points is of measure zero. Hence, it is worth designing high-order accurate flux limiters φ. The following limiters φ_{W3N} are third-order accurate for smooth solutions in the region sufficiently far away from the critical points

$$\varphi_{\mathrm{W3N}}(r) = \max\Big(0, \min\Big(\frac{2}{1-\nu}, \frac{2r}{\nu}, \big(2-\nu+(1+\nu)r\big)/3\Big)\Big)$$
$$\Big(\text{for fully discrete TVD schemes (12)}\Big)$$

$$\varphi_{\mathrm{W3N}}(r) = \max\Big(0, \min\Big(\frac{2}{1-\nu}, \frac{2r}{\nu}, (2+r)/3\Big)\Big)$$
$$\Big(\text{for semi-discrete TVD schemes (8)}\Big)$$

- Second way is to construct the flux limiter φ_{COM} in the extended TVD region (see Fig.2).

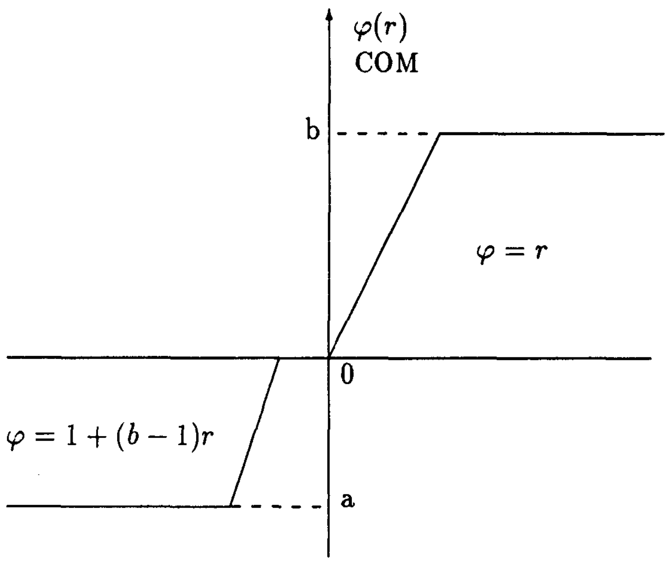

Figure 2

$$\varphi_{\mathrm{COM}}(r) = \begin{cases} \min(r,b) \quad , & \text{if } r \ge 0 \\ \max\Big(\min\big(0, \max(a, 1+(b-1)r)\big)\ ,\ \min(r,b)\Big) & \text{if } r \le 0\ , \end{cases}$$

For schemes (8) $b = 2$, $-\infty < a < 0$; for schemes (12), $b = 2/(1-\nu)$, $a = 1 - \dfrac{2}{\nu}$.

To test the improvement of the accuracy, we consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0\ ,$$
$$u(x,0) = \sin(\pi x)$$
$$u(x+2,t) = u(x,t)\ .$$

The parameters used in computation are $\Delta x = 2/N$, $N = 20,\ 40,\ 80$; $\nu = \Delta t/\Delta x = 0.8,\ 0.5$. Error is estimated in C–norm:

$$\varepsilon_{\Delta x} = \|u - u_{\Delta x}\|_C = \max_j |u(j\Delta x) - u_j| = \mathcal{O}(\Delta x^P)\ ,$$

the order P is taken as

$$P = \log(\varepsilon_{2\Delta x}/\varepsilon_{\Delta x})(\log 2)^{-1}\ .$$

The results are listed in the following two tables:

$\Delta t/\Delta x = 0.8$

Scheme	$\varepsilon_{2/20}$	$\varepsilon_{2/40}$	$\varepsilon_{2/80}$	P 20–40	P 40–80
UNO2	.71–2	.16–2	.40–3	2.14*	2.01*
CO	.618–1	.245–1	.101–1	1.338	1.272
COM	.313–1	.699–2	.240–2	2.161^{Δ}	1.541
W3N	.215–1	.676–2	.215–2	1.667^{Δ}	1.654

Table 1

$$\Delta t/\Delta x = 0.5$$

Scheme	$\varepsilon_{2/20}$	$\varepsilon_{2/40}$	$\varepsilon_{2/80}$	P 20–40	P 40–80
UNO2	.902–2	.114–2	.149–3	2.98*	2.94*
CO	.107–0	.461–1	.220–1	1.213	1.069
COM	.866–1	.207–1	.723–2	2.068^{Δ}	1.514
W3N	.394–1	.104–1	.288–2	1.928^{Δ}	1.844

Table 2

Four schemes are compared. The scheme UNO2 is taken from [4], it is an uniformly second-order non-oscillatory scheme (not TVD). Scheme CO is taken from [2]. It is a TVD scheme, second-order accurate (in the sense of the theorem). Scheme COM is the modification of CO. W3N is the scheme using limiter φ_{W3N}. We can see that both schemes do improve the accuracy of TVD schemes.

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