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Building Large Compressed PDBs for the Sliding Tile Puzzle

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Building Large Compressed PDBs for the Sliding Tile Puzzle

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Abstract. The performance of heuristic search algorithms depends crucially on the effectiveness of the heuristic. A pattern database (PDB) is a powerful heuristic in the form of a pre-computed lookup table. Larger PDBs provide better bounds and thus allow more cut-offs in the search process. Today, the largest PDB for the 24-puzzle is a 6-6-6-6 PDB with a size of 486 MB.

We created 8-8-8, 9-8-7 and 9-9-6 PDBs that are three orders of magnitude larger (up to 1.4 TB) than the 6-6-6-6 PDB. We show how to compute such large PDBs and we present statistical and empirical data on their efficiency. The largest single PDB gives on average an 8-fold improvement over the 6-6-6-6 PDB. Combining several large PDBs gives on average an 12-fold improvement.

1 Introduction

Heuristic search algorithms are widely used to solve combinatorial optimization problems. While traversing the problem space, the search process is guided by a heuristic function that provides a lower bound on the cost to a goal state. This allows to prune large parts of the search space and thus reduces the overall search effort. The more accurate the heuristic is, the more states can be pruned during the search.

Pattern Databases (PDBs) are powerful heuristic functions in form of a lookup table. They store the exact solution of a relaxed version of the problem. The less the original problem is relaxed the larger is the size of the PDB and thereby the tighter are its bounds.

In this paper we present for the first time very large complete PDBs for the 24-puzzle: a 8-8-8 PDB with 122 GB, a 9-8-7 PDB with 733 GB, and a 9-9-6 PDB with 1381 GB. The largest one gave node savings by up to a factor of 37 compared to the 6-6-6-6 PDB.

We present in Sec. 4 a parallel algorithm that allows to compute very large PDBs on cluster systems with a modest amount of memory. The application of such large PDBs in heuristic search, however, requires a computer with a much higher memory capacity of more than 1.4 TB for the 9-9-6 PDB, for example. While systems with more than one TB of memory are not (yet) commonly used as standard computers, we believe that our work will help in studying the pruning-power of large PDBs.

The rest of the paper is structured as follows. Sec. 2 sets the context of our work by reviewing relevant literature. Thereafter PDBs are introduced in Sec. 3 and the algorithms and compressed data structures for generating large PDBs are presented in Sec. 4. In Sec. 5 we provide a statistical and empirical analysis and we summarize our work in Sec. 6.

2 Background

PDBs were first mentioned by [2]. [5] presented additive PDBs in which the heuristic estimate is computed as the sum of the values of several smaller PDBs. The same authors also proposed a method for compressing a PDB by disregarding the blank and computing the minimum of all blank positions.

PDBs are built with a backward breadth-first search over the complete state space. Large breadth-first searches have been used by [8] to expand the complete graph of the 15-puzzle for the first time. This could only be achieved by keeping the search front on disks and hiding the disk latency with multiple threads.

[3] use *instance dependent PDBs* to utilize large PDBs without completely creating them. They build on the observation of [9] that only the nodes generated by the best-first search algorithm A* are needed in the pattern space to solve an individual instance. For each pattern of the given instance, they perform an A* search from the goal pattern towards the start pattern until the available memory is exhausted. This database is then used for the forward search. When h -values are missing, several smaller PDBs are used instead.

[6] suggest to take the maximum h -value from several smaller PDBs instead of a large one. They show that the accuracy of small h -values is especially important for reducing the number of expanded nodes.

3 Pattern Databases

In this paper we are concerned with sliding tile puzzles. An instance of the $(n - 1)$ -puzzle can be described by n state variables, one for each tile. Each *state variable* describes the position of one specific tile in the tray. A *pattern* considers only a subset of the state variables; the remaining state variables are ignored. Hence, patterns abstract from the original problem by mapping several states to the same point in the pattern space. The number of ignored state variables can be used to control the information loss.

In the $(n - 1)$ -puzzle, a pattern is defined by a subset of the tiles. The position of the pattern tiles, the *pattern tile configuration*, and the blank defines a node in the pattern space. Move operations in the original problem can be analogously applied to nodes in the pattern space by moving either a pattern tile or a non-pattern tile, i.e. a *don't care tile*. Although we count the moves of don't cares, they are indistinguishable from each other. The size of the pattern space for a pattern with k tiles for the $(N - 1)$ -puzzle is $\frac{N!}{(N-k-1)!}$.

The number of moves needed to reach the goal in the pattern space can be used as an admissible heuristic for the move number in the original space.

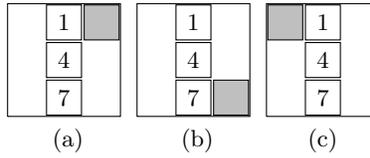


Fig. 1. Patterns with different blank positions (8-puzzle).

Because of the don't care tiles, a path in the original search space can only be longer than the corresponding path in the pattern space and hence the heuristic is admissible, i.e. non-overestimating.

To compute a PDB, we perform a backward breadth-first search from the goal to the start node and record for each visited node the distance from the goal.

3.1 Additive Pattern Databases

Because of space limitations, only small PDBs can be built. To get better heuristic estimates, several PDBs must be combined. However, with the above method, which also counts the movements of don't care tiles, we cannot simply add the h values of PDBs, even when the patterns are disjoint, because the same move would be counted several times. For additive pattern databases [4] we only count the moves of pattern tiles.

The search space is mapped to the pattern space in the following way. Two states of the original space map to the same state in the pattern space, if the pattern tiles are in the same position and the two blank positions can be reached from each other by moving only don't care tiles. There is an edge between two nodes a and b in the pattern space if and only if there are two nodes c and d in the puzzle space where c maps to a and d maps to b and there is an edge between c and d .

Fig. 1 shows an example for the 8-puzzle. Positions (a) and (b) map to the same state in the pattern space, because the blank positions are reachable from each other without moving pattern tiles. Positions (a) and (c), in contrast, do not map to the same state in the pattern space, because at least one pattern tile must be moved to shift the blank to the same position.

To further reduce the memory consumption, we compress the databases by the blank position as described in [4]. This is done by storing for any pattern-tile configuration, independent of the different blank positions, only the minimal distance from the goal node. For the three examples shown in Fig. 1 we only store one (the smallest) distance g in the PDB.

4 Building Compressed PDBs

When building large PDBs we ran into two limits: space and time. Not only do we need to keep the PDB itself in memory, but also the Open and Closed lists

Algorithm 1 BFS in compressed, indexed PDB space.

```
1: PDBArray A
2: initialize array
3: expandedNodes = -1;
4: g = 1;
5: while expandedNodes ≠ 0 do
6:   expandedNodes = 0;
7:   for i = 0 → A.size - 1 do
8:     if A[i].open_list = ∅ then
9:       continue;
10:    end if
11:    expandedNodes++;
12:    pattern = unindex(i);
13:    blanks = unpackBlanks(A[i].open_list);
14:    succs = genSuccs(pattern, blanks);
15:    for j = 0 → succs.size - 1 do
16:      sIndex = index(succs[j]);
17:      rBlanks = reachableBlanks(succs[j]);
18:      pBlanks = packBlanks(rBlanks);
19:      pBlanks -= A[sIndex].closed_list;
20:      A[sIndex].open_list += pBlanks;
21:      A[sIndex].g = min(A[sIndex].g, g);
22:    end for
23:    A[i].closed_list += A[i].open_list;
24:    A[i].open_list = ∅;
25:  end for
26:  g++;
27: end while
```

must be stored. In Sec. 4.1 we describe a sequential algorithm and a compressed data structure for computing large PDBs. In Sec. 4.2 we describe a parallel implementation that uses the combined memory and compute capacity of a cluster as a single resource.

4.1 Sequential Algorithm

Our algorithm for building PDBs builds on ideas of [1]. To store the k -tile PDB, we use an array of $\frac{N!}{(N-k)!}$ elements, one entry for each state of the compressed pattern space. For our 9 tile PDB this results in $\frac{25!}{16!} = 741 \cdot 10^9$ entries. We use a perfect hash function to map a configuration of the pattern to this array. The hash function is reversible so that we can map an array index back to its pattern tile configuration. Each entry in the array is made up of three values: *g*, *open_list* and *closed_list*.

```
struct {
  byte g;
  byte open_list;
```

```

    byte closed_list;
} array_entry;

```

The variable `g` in Alg. 1 stores for each entry the minimum g in which we found that state. Additionally, we need to store for each tuple of a pattern tile configuration and blank position whether it is in the Open or in the Closed list. This could be done by simply storing two bit strings of length $N - k$ in each PDB entry and setting the responsible bit whenever a new blank position is visited.

However, this simple approach can be improved to achieve a further data compression. A *blank partition* is a set of blank positions with a common pattern tile configuration where all blank positions are reachable from each other by only moving don't care tiles. This is shown in Fig. 1: (a) and (b) belong to the same blank partition, while (a) and (c) do not. For patterns with 9 tiles, the pattern tile configurations have no more than 8 blank partitions. We can simply enumerate the blank partitions and only store one bit for each partition in the `open_list` or `closed_list`. In the backward breadth-first search we used pre-computed lookup tables to map the blank positions to blank partitions. To build a PDB with up to 9 tiles, this scheme requires 3 bytes per state, one for `g`, `open_list`, and `closed_list`, respectively.

The breadth-first search over the pattern space is performed as follows (Alg. 1): All `open_lists` and `closed_lists` are initialized with zeroes. The `g` for each state is set to the maximum value. For the initial state, the blank partition of the initial position is set in the `open_list`.

Then the array is scanned repeatedly (line 4). For each entry, we check if the Open list is empty (line 7). If not, we create the pattern tile configuration (line 11), extract all blank positions from the Open list (line 12) and finally generate the successors (line 13). For each successor, we calculate the index in the PDB (line 15), compress the blank positions (line 16-17) and update the successor's entry in the PDB (line 18-20). Note that backward steps are eliminated with the update. Finally, we update the `open_list` and `closed_list` of the current position. This is repeated until the complete pattern space has been visited.

4.2 Parallel Algorithm

For the parallel algorithm, we distribute the array (in disjoint partitions) over all compute nodes. To avoid imbalances in the work load, we do not assign contiguous parts to the nodes but use a hash function for assigning rows of the array to the compute nodes. The parallel algorithm has the same structure as the sequential algorithm (see Fig. 2) but it needs additional communication to move the results to remote compute nodes.

For each g , first each node scans its part of the array and generates the successors as described in Alg. 1. But instead of directly updating the PDB, each node collects the successors locally. In the shuffle phase (Fig. 2), these successors are sent to the nodes storing the corresponding rows in the PDB. Finally, the PDB is updated locally.

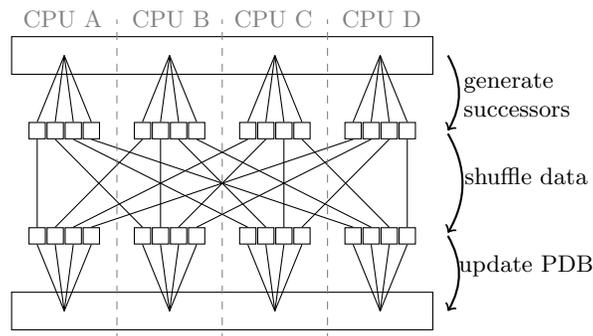
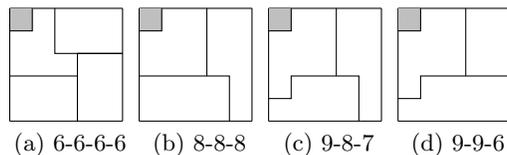


Fig. 2. Workflow of the parallel implementation.

5 Evaluation

We used the presented parallel algorithm to build three large PDBs, 8-8-8, 9-8-7, and 9-9-6, with sizes of 122 GB, 733 GB and 1381 GB, respectively. For comparison, the previously largest 6-6-6-6 PDB has a size of only 488 MB.



In our cluster, each compute node has 2 quad-core Intel Xeon X5570 with 48 GB of main memory. It took about 6 hours to build a single 9 tile PDB on 255 nodes. For the empirical analysis we used an SGI UV 1000, a large shared-memory machine with 64 octo-core Intel Xeon X7560 and 2 TB of main memory.

In the following, we first present a statistical analysis of the performance of our PDBs on a large number of randomly generated positions. Thereafter we show the performance on Korf's set of random 24-puzzle instances. In both cases, we used mirroring [2] to improve the accuracy of the heuristics.

5.1 Statistical Evaluation

We created 100,000,000 random instances of the 24-puzzle and recorded the h -values obtained with the 6-6-6-6, 8-8-8, 9-8-7, and 9-9-6 PDB. Fig. 3 shows the cumulative distribution, i.e. the probability $P(X \leq h)$, that the heuristic value for a random state is less or equal to h . The higher the h -value, the better the pruning power of the heuristic. This is because all heuristics are admissible, i.e. they never overestimate the goal distance. Higher h -values represent therefore tighter bounds on the true value. As can be seen in Fig. 3, all graphs lie close

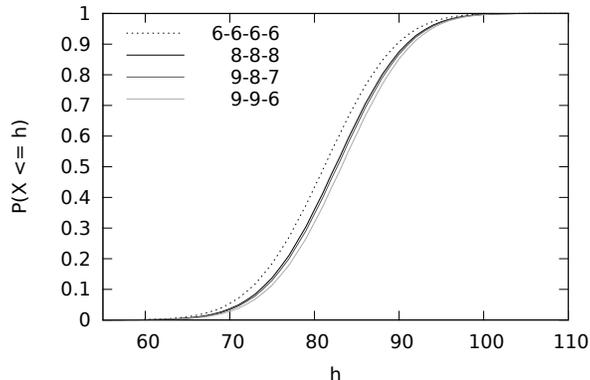


Fig. 3. Cumulative distribution of h -values of 100,000,000 random samples.

together and their order corresponds to the size and pruning power of the PDBs. Interestingly, the new PDBs are distinctively better than the 6-6-6-6 PDB (see the dashed line).

Note that the increased number of small h -values is especially important for the performance of the heuristic [6]. Fig. 4 shows a magnification of the lower left corner of the data in Fig. 3. It can be seen that all curves are clearly distinct and that the large PDBs provide a considerable improvement over the 6-6-6-6 PDB.

| PDB | size [GB] | avg. h | min. h | max. h |
|---------|-----------|----------|----------|----------|
| 6-6-6-6 | 0.488 | 81.85 | 40 | 115 |
| 8-8-8 | 122 | 82.84 | 40 | 116 |
| 9-8-7 | 733 | 83.10 | 43 | 116 |
| 9-9-6 | 1381 | 84.56 | 44 | 116 |

Table 1. Average, minimum and maximum h -values of 100,000,000 random instances.

Table 1 lists the average, minimum, and maximum values. In accordance with Fig. 3, larger PDBs return on average a higher h -value. Checking the extreme values reveals an interesting fact: While the minimum value of the 9-9-6 PDB is 4 moves higher than the lowest value of the 6-6-6-6, its maximum value is only increased by one. Thus, the large PDBs return fewer small values but they do not provide a significantly higher maximum.

5.2 Empirical Evaluation

For the second set of experiments, we used Korf’s fifty random instances [8] and solved them optimally. We present data on the breadth-first iterative deepening

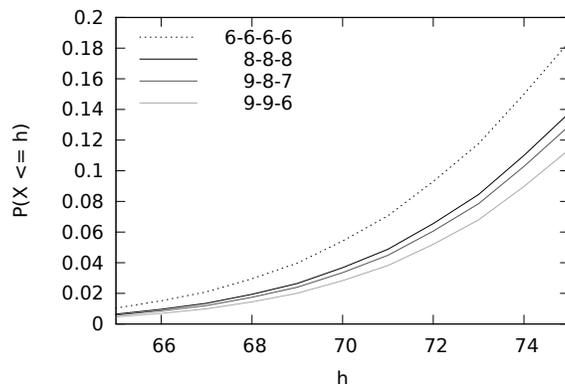


Fig. 4. Magnification of the lower left corner of Fig. 3.

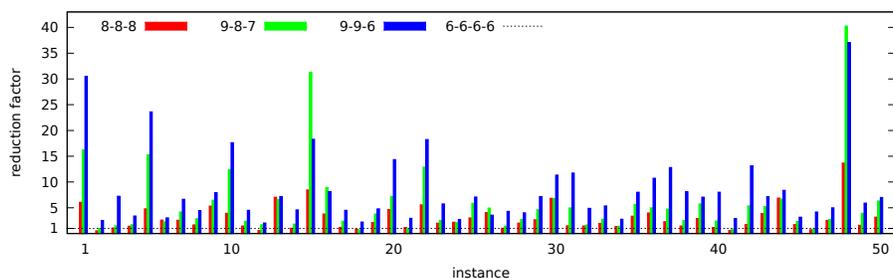


Fig. 5. Reduction factor to 6-6-6-6 PDB on Korf's random set (ordered by IDA* nodes) using BF-IDA*.

A* algorithm (BF-IDA*) [10], a breadth-first variant of IDA* [7]. We chose BF-IDA* over IDA* because its performance does not depend on the node ordering and it therefore allows to better assess the performance of the heuristic. We sorted the 50 instances by the number of expanded nodes with BF-IDA* and solved all instances optimally.

Fig. 5 shows the reduction of node expansions in comparison to the 6-6-6-6 PDB. For each bar we divided the nodes expanded by the 6-6-6-6 PDB by that of the other PDBs. In general, larger PDBs tend to perform better than smaller ones and the gain seems to be independent of the hardness of the problem. However, there are a number of outliers in both directions.

Fig. 6 summarizes Fig. 5 and groups the reduction factors by PDB. Additionally, we added max-of which takes the maximum of the 6-6-6-6, 8-8-8, 9-8-7 and 9-9-6 PDBs. The consumed memory is only marginally larger because of the overlapping partitions. The four PDBs reduce the number of expanded nodes by a median factor of 2.16, 3.86, 6.81 and 9.36. However, there are some out-

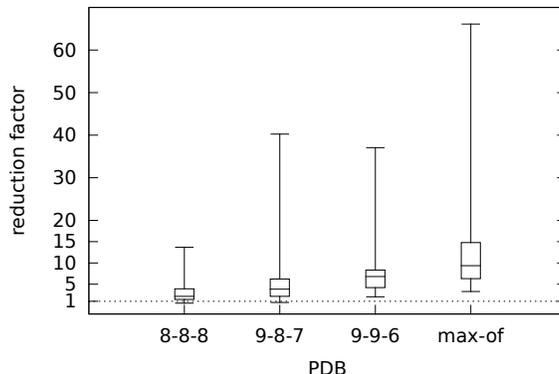


Fig. 6. Reduction factors compared to 6-6-6-6 PDB.

liers towards both ends of the scale. For some instances the number of expanded nodes was higher compared to the 6-6-6-6 PDB. On the other hand, it could be reduced by a factor of up to 10 with the 8-8-8 PDB and up to 31 with the 9-8-7 and 9-9-6 PDBs. The standard deviation seems to slightly increase with the size of the heuristic.

Table 2 shows the full details on each instance and the number of expanded nodes for each PDB. The first column gives the Id used in [8] and the second column states the length d of the shortest path. The number of expanded nodes with the individual PDBs are listed in columns three to seven. Columns eight, nine, ten and eleven give the reduction factor of the 8-8-8, 9-8-7, 9-9-6 and the max-of PDB relative to the 6-6-6-6 PDB.

6 Conclusions

We presented an efficient parallel algorithm and a compact data structure that allowed us to compute for the first time very large compressed PDBs. The parallel algorithm utilizes the aggregated memory of multiple parallel computers to compute and stores the PDB in the main memory.

We computed three additive PDBs for the 24-puzzle, an 8-8-8, 9-8-7 and 9-9-6 PDB. To the best of our knowledge, these are the largest PDBs reported for this domain.

The 9-9-6 PDB gives on average an 8-fold node reduction compared to a 6-6-6-6 PDB on the first 45 of Korf’s random instances of the 24-puzzle. We observed a high variance on the reduction rate, which ranges from 2x to 37x savings (Tab. 2). Hence, it is advisable to use the maximum over several additive PDBs in a practical application. This is feasible, because multiple additive PDBs do not proportionally increase the memory consumption. This is because the same

PDB can be utilized by multiple additive PDBs. As an example, the same 9 PDB can be used in both of our 9-9-6 PDB and the 9-8-7 PDB.

Acknowledgments

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| Id | d | 6-6-6-6 | 8-8-8 | 9-8-7 | 9-9-6 | max-of | r ₁ | r ₂ | r ₃ | r ₄ |
|---------|-----|-------------------|-------------------|-------------------|-------------------|------------------|----------------|----------------|----------------|----------------|
| 40 | 82 | 26,320,497 | 49,291,000 | 26,655,910 | 10,486,000 | 7,166,383 | 0.53 | 0.99 | 2.51 | 3.67 |
| 38 | 96 | 58,097,633 | 9,577,883 | 3,573,949 | 1,906,127 | 1,638,334 | 6.07 | 16.26 | 30.48 | 35.46 |
| 25 | 81 | 127,949,696 | 118,780,897 | 85,141,009 | 17,658,986 | 15,217,162 | 1.08 | 1.50 | 7.25 | 8.41 |
| 44 | 93 | 181,555,996 | 37,853,812 | 11,869,090 | 7,686,937 | 5,547,600 | 4.80 | 15.30 | 23.62 | 32.73 |
| 32 | 97 | 399,045,498 | 281,515,091 | 232,222,028 | 117,317,314 | 67,570,393 | 1.42 | 1.72 | 3.40 | 5.91 |
| 28 | 98 | 450,493,295 | 114,571,662 | 36,263,727 | 25,552,985 | 19,743,793 | 3.93 | 12.42 | 17.63 | 22.82 |
| 22 | 95 | 581,539,254 | 82,503,279 | 88,652,504 | 81,038,427 | 37,858,513 | 7.05 | 6.56 | 7.18 | 15.36 |
| 36 | 90 | 603,580,192 | 408,261,989 | 252,309,866 | 133,482,919 | 95,563,302 | 1.48 | 2.39 | 4.52 | 6.32 |
| 30 | 92 | 661,835,606 | 256,431,250 | 158,409,200 | 99,557,684 | 52,338,447 | 2.58 | 4.18 | 6.65 | 12.65 |
| 1 | 95 | 1,059,622,872 | 199,198,406 | 163,950,295 | 133,060,463 | 63,948,759 | 5.32 | 6.46 | 7.96 | 16.57 |
| 29 | 88 | 1,090,385,785 | 128,886,129 | 34,814,333 | 59,609,938 | 21,223,415 | 8.46 | 31.32 | 18.29 | 51.38 |
| 37 | 100 | 1,646,715,005 | 628,890,120 | 725,323,664 | 542,573,720 | 331,223,844 | 2.62 | 2.27 | 3.04 | 4.97 |
| 16 | 96 | 1,783,144,872 | 1,729,554,795 | 966,783,772 | 387,360,939 | 296,519,726 | 1.03 | 1.84 | 4.60 | 6.01 |
| 5 | 100 | 1,859,102,197 | 3,125,977,623 | 1,078,990,063 | 905,861,248 | 565,263,022 | 0.59 | 1.72 | 2.05 | 3.27 |
| 13 | 101 | 1,979,587,555 | 1,181,771,575 | 690,327,991 | 444,476,728 | 268,475,464 | 1.68 | 2.87 | 4.45 | 7.37 |
| 47 | 92 | 4,385,270,986 | 3,825,636,827 | 4,520,442,316 | 1,479,759,728 | 960,463,883 | 1.15 | 0.97 | 2.96 | 4.57 |
| 3 | 97 | 4,805,007,493 | 5,699,072,723 | 6,731,407,433 | 2,146,564,697 | 1,113,194,453 | 0.84 | 0.71 | 2.24 | 4.32 |
| 4 | 98 | 5,154,861,019 | 1,361,290,863 | 581,368,420 | 632,299,449 | 370,467,747 | 3.79 | 8.87 | 8.15 | 13.91 |
| 26 | 105 | 6,039,700,647 | 4,993,857,550 | 2,525,926,189 | 1,337,993,889 | 955,364,988 | 1.21 | 2.39 | 4.51 | 6.32 |
| 31 | 99 | 7,785,405,374 | 3,653,831,114 | 2,058,364,161 | 1,622,465,469 | 992,726,542 | 2.13 | 3.78 | 4.80 | 7.84 |
| 27 | 99 | 7,884,559,441 | 1,415,859,414 | 611,960,188 | 432,345,846 | 337,466,232 | 5.57 | 12.88 | 18.24 | 23.23 |
| 41 | 106 | 8,064,453,928 | 1,737,010,534 | 1,123,917,776 | 561,944,277 | 455,028,148 | 4.64 | 7.18 | 14.35 | 17.72 |
| 43 | 104 | 8,816,151,498 | 4,378,714,353 | 3,498,876,258 | 1,532,474,999 | 1,090,696,435 | 2.01 | 2.52 | 5.75 | 8.08 |
| 6 | 101 | 9,810,208,759 | 2,397,434,227 | 1,982,606,973 | 2,739,184,006 | 1,053,141,115 | 4.09 | 4.95 | 3.58 | 9.32 |
| 49 | 100 | 11,220,738,849 | 5,526,627,744 | 4,160,235,910 | 2,792,736,271 | 1,587,674,537 | 2.03 | 2.70 | 4.02 | 7.07 |
| 45 | 101 | 17,068,061,084 | 5,614,562,048 | 2,909,124,921 | 2,408,543,192 | 1,339,279,458 | 3.04 | 5.87 | 7.09 | 12.74 |
| 20 | 92 | 20,689,215,063 | 9,014,702,404 | 4,354,383,611 | 1,615,310,063 | 1,378,812,797 | 2.30 | 4.75 | 12.81 | 15.01 |
| 46 | 100 | 21,674,806,323 | 9,872,851,915 | 10,304,210,129 | 8,017,940,089 | 3,402,288,275 | 2.20 | 2.10 | 2.70 | 6.37 |
| 19 | 106 | 22,761,173,348 | 6,759,987,121 | 4,019,764,127 | 2,836,304,399 | 2,125,081,076 | 3.37 | 5.66 | 8.02 | 10.71 |
| 35 | 98 | 23,049,423,391 | 8,584,994,059 | 4,998,934,055 | 3,208,321,325 | 2,369,834,229 | 2.68 | 4.61 | 7.18 | 9.73 |
| 7 | 104 | 27,686,193,468 | 26,781,188,637 | 19,232,502,973 | 6,429,879,587 | 4,395,653,789 | 1.03 | 1.44 | 4.31 | 6.30 |
| 8 | 108 | 29,575,219,906 | 4,318,849,565 | 4,366,429,730 | 2,609,051,057 | 1,727,994,805 | 6.85 | 6.77 | 11.34 | 17.12 |
| 39 | 104 | 34,198,605,172 | 22,810,919,845 | 6,881,101,921 | 2,912,577,301 | 2,428,595,642 | 1.50 | 4.97 | 11.74 | 14.08 |
| 42 | 108 | 37,492,323,962 | 9,339,335,844 | 7,508,532,598 | 3,490,897,448 | 2,697,310,294 | 4.01 | 4.99 | 10.74 | 13.09 |
| 24 | 107 | 38,272,741,957 | 25,802,863,114 | 15,170,752,402 | 4,724,091,699 | 3,837,236,834 | 1.48 | 2.52 | 8.10 | 9.97 |
| 2 | 96 | 40,161,477,151 | 29,318,072,174 | 28,011,360,591 | 14,446,211,551 | 8,963,348,921 | 1.37 | 1.43 | 2.78 | 4.48 |
| 15 | 103 | 52,178,879,610 | 26,951,022,561 | 18,771,225,751 | 9,741,418,794 | 8,075,823,446 | 1.94 | 2.78 | 5.36 | 6.46 |
| 23 | 104 | 54,281,904,788 | 36,611,741,317 | 32,729,241,923 | 11,103,574,065 | 8,930,804,356 | 1.48 | 1.66 | 4.89 | 6.08 |
| 48 | 107 | 58,365,224,981 | 99,614,525,233 | 68,013,167,519 | 19,890,964,633 | 12,563,246,704 | 0.59 | 0.86 | 2.93 | 4.65 |
| 34 | 102 | 59,225,710,222 | 49,923,377,951 | 24,336,781,035 | 7,384,409,074 | 5,346,161,078 | 1.19 | 2.43 | 8.02 | 11.08 |
| 12 | 109 | 76,476,143,041 | 43,132,155,298 | 14,260,876,794 | 5,820,163,959 | 4,265,458,902 | 1.77 | 5.36 | 13.14 | 17.93 |
| 21 | 103 | 98,083,647,769 | 25,411,173,479 | 18,746,227,139 | 13,731,206,789 | 8,402,416,300 | 3.86 | 5.23 | 7.14 | 11.67 |
| 18 | 110 | 126,470,260,027 | 18,375,847,744 | 18,999,810,842 | 15,070,620,942 | 7,809,249,544 | 6.88 | 6.66 | 8.39 | 16.19 |
| 9 | 113 | 132,599,245,368 | 82,839,919,151 | 33,749,539,711 | 22,489,080,304 | 16,927,179,096 | 1.60 | 3.93 | 5.90 | 7.83 |
| 33 | 106 | 134,103,676,989 | 77,163,409,262 | 57,402,766,270 | 42,219,474,099 | 25,271,466,707 | 1.74 | 2.34 | 3.18 | 5.31 |
| 17 | 109 | 143,972,316,747 | 49,516,974,145 | 25,000,824,805 | 20,405,484,237 | 15,304,298,302 | 2.91 | 5.76 | 7.06 | 9.41 |
| 11 | 106 | 309,253,017,124 | 22,602,670,676 | 7,683,989,291 | 8,343,197,181 | 4,678,739,173 | 13.68 | 40.25 | 37.07 | 66.10 |
| 14 | 111 | 312,885,453,572 | 419,699,251,120 | 360,169,788,945 | 74,779,904,961 | 63,056,188,490 | 0.75 | 0.87 | 4.18 | 4.96 |
| 10 | 114 | 525,907,193,133 | 207,752,246,775 | 192,243,603,386 | 105,311,763,457 | 63,629,118,230 | 2.53 | 2.74 | 4.99 | 8.27 |
| 50 | 113 | 1,067,321,687,213 | 334,283,260,227 | 168,384,195,109 | 152,720,707,871 | 100,026,128,248 | 3.19 | 6.34 | 6.99 | 10.67 |
| average | | 71,004,578,707.12 | 33,908,766,050.50 | 23,611,990,572.06 | 11,599,129,942.46 | 7,794,424,738.66 | 3.00 | 5.74 | 8.37 | 12.85 |
| median | | 14,144,399,966.50 | 5,570,594,896.00 | 4,257,309,760.50 | 2,508,797,124.50 | 1,359,046,127.50 | 2.16 | 3.86 | 6.81 | 9.36 |

Table 2. Expanded nodes of all 50 random instances (r_1 : 6-6-6-6/8-8-8, r_2 : 6-6-6-6/9-8-7, r_3 : 6-6-6-6/9-9-6, r_4 : 6-6-6-6/max-of).

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