



MARTIN BALLERSTEIN[†] AND DENNIS MICHAELS[‡] AND
STEFAN VIGERSKE

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[†] Eidgenössische Technische, Hochschule Zürich, Institut für Operations Research, Rämistrasse 101, 8092 Zurich (Switzerland)

[‡] Technische Universität Dortmund, Fakultät für Mathematik, M/518, Vogelpothsweg 87, 44227 Dortmund (Germany)

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Technical Report

Linear Underestimators for bivariate functions with a fixed convexity behavior*

A documentation for the SCIP constraint handler `cons_bivariate`

Martin Ballerstein[†] Dennis Michaels[‡]
Stefan Vigerske[§]

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[†] Eidgenössische Technische Hochschule Zürich
Institut für Operations Research
Rämistrasse 101, 8092 Zurich (Switzerland)

[‡] Technische Universität Dortmund
Fakultät für Mathematik, M/518
Vogelpothsweg 87, 44227 Dortmund (Germany)

[§] Zuse Institute Berlin
Takustr. 7, 14194 Berlin (Germany)

This is a technical report for the SCIP constraint handler `cons_bivariate`. We describe a cut-generation algorithm for a class of bivariate twice continuously differentiable functions with fixed convexity behavior over a box. Computational results comparing our cut-generation algorithms with state-of-the-art global optimization software on a series of randomly generated test instances are reported and discussed.

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This is a revised version of a Technical Report that has appeared in January 2013. After the original report had been published, it has been recognized that our implementation yield invalid cutting planes for cases where functions are not differentiable on the boundary of the variables domain. In this revised version, computational experiments have been updated, using a corrected version (see Section 4.1.1) of our implementation.

1. Introduction

In this work we derive strong linear underestimators for bivariate functions $f : [l, u] \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$, $(x, y) \mapsto f(x, y)$, satisfying the following assumptions.

(A1) $[l, u] := [l_x, u_x] \times [l_y, u_y] \subseteq \mathbf{R}^2$ is a box with $l_x < u_x$ and $l_y < u_y$,

(A2) f is twice continuously differentiable on $[l, u]$,

(A3) f has a fixed convexity behavior on $[l, u]$, i.e., the signs of the second partial derivatives and the determinant of the Hessian are the same for all points $(x, y) \in [l, u]$.

For a bivariate function $f(x, y)$ with a fixed convexity behavior, we distinguish five convexity patterns of f to derive tight underestimators: (i) convexity on the entire box, (ii) component-wise concavity in each variable, (iii) convexity in x and concavity in y , (iv) concavity in x and convexity in y , and (v) component-wise convexity and indefiniteness in the interior of the box.

Important classes of functions satisfying our assumptions (A1) – (A3) include the family of bivariate quadratic functions and the family of bivariate monomial functions of the form $x^p y^q$ restricted to the nonnegative orthant. Table 1 provides a complete characterization of the convexity behavior for these two families of functions.

Our goal is to provide a cut-generation algorithm for the construction and refinement of a linear relaxation of a possibly nonconvex set $\{(x, y, z) \in \mathbf{R}^3 : f(x, y) \leq cz\}$, where $f(x, y)$ satisfies our assumptions (A1) – (A3) and $c \in \mathbf{R}$. This is achieved by constructing a maximally touching hyperplanes on the graph of the best possible convex underestimator of f on $[l, u]$, the so-called convex envelope which is denoted by $\text{vex}_{[l, u]}[f]$ (cf. [McC76]). The value of $\text{vex}_{[l, u]}[f]$ at $(x_0, y_0) \in [l, u]$ is given by (cf. [Roc70])

$$\text{vex}_{[l, u]}[f](x_0, y_0) = \min \left\{ \sum_{i=1}^3 \lambda_i f(x_i, y_i) \mid x_0 = \sum_{i=1}^3 \lambda_i x_i, y_0 = \sum_{i=1}^3 \lambda_i y_i, \right. \\ \left. 1 = \sum_{i=1}^3 \lambda_i, \lambda_i \geq 0, (x_i, y_i) \in [l, u] \right\}. \quad (1)$$

To determine a maximally touching hyperplane at the graph of $\text{vex}_{[l, u]}[f]$ at a given point $(x_0, y_0) \in [l, u]$, we compute suitable $\alpha, \beta, \delta \in \mathbf{R}$ and $\gamma \in \mathbf{R}_{\geq 0}$ such that $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, for all $(x, y) \in [l, u]$, and equality holds at least at (x_0, y_0) . Substituting $f(x, y)$ by a new variable $z \in \mathbf{R}$ yields the desired hyperplane.

Convexity behavior	$f(x, y) = a_{x,x}x^2 + a_{x,y}xy + a_{y,y}y^2 + b_x x + b_y y + c$	$f(x, y) = x^p y^q, (x, y) \in \mathbf{R}_{\geq 0}^2$
1. f is convex	$a_{x,x} \geq 0, a_{y,y} \geq 0,$ and $a_{x,x}a_{y,y} - a_{x,y}^2 \geq 0$	$p^2 - p \geq 0, q^2 - q \geq 0,$ and $pq(1 - p - q) \geq 0$
2. f is concave in x, y	$a_{x,x} \leq 0$ and $a_{y,y} \leq 0$	$p^2 - p \leq 0$ and $q^2 - q \leq 0$
3. f is non-linear convex in x and concave in y	$a_{x,x} > 0$ and $a_{y,y} \leq 0$	$p^2 - p > 0$ and $q^2 - q \leq 0$
4. f is concave in x and non-linear convex in y	$a_{x,x} \leq 0$ and $a_{y,y} > 0$	$p^2 - p < 0$ and $q^2 - q \leq 0$
5. f is indefinite and non-linear convex in x, y	$a_{x,x} > 0, a_{y,y} > 0,$ and $a_{x,x}a_{y,y} - a_{x,y}^2 < 0$	$p^2 - p \geq 0, q^2 - q \geq 0,$ and $pq(1 - p - q) < 0$

Table 1: Classes of fixed convexity behavior and criteria for bivariate quadratic and bivariate monomial functions. The latter functions are restricted to nonnegative domains $[l, u] \subseteq \mathbf{R}_{\geq 0}^2$.

When f is convex or component-wise concave on the box, an optimal solution of problem (1) can be easily computed which can be used to determine appropriate coefficients $\alpha, \beta, \gamma, \delta$. In the other cases, results from [TS01, JMW08] can be applied to find an optimal solution to the nonconvex problem (1). This is achieved by solving one or two univariate convex subproblems.

In general, these solutions do not lead to explicit formulas for the convex envelope and, hence, for the coefficients of the hyperplanes. However, based on these solutions we can derive the coefficients for the maximally touching hyperplanes using some elementary geometric arguments.

Locatelli and Schoen [LS14] have studied convex envelopes for twice continuously bivariate functions $f : P \rightarrow \mathbf{R}$ over a polytope $P \subseteq \mathbf{R}^2$, where f is indefinite in the interior of P , and either convex or concave over every facet of P . When choosing P to be a box, this class of functions include the indefinite functions studied in this report. Locatelli and Schoen provided a procedure returning, for a given point (x_0, y_0) , the value $\text{vex}_P[f](x_0, y_0)$ and a supporting hyperplane on the graph of $\text{vex}_P[f]$ at (x_0, y_0) . For this, they investigate an alternative representation for the convex envelope reading as

$$\begin{aligned} \text{vex}_{[l,u]}[f](x_0, y_0) = \max \quad & c \\ \text{s. t.} \quad & f(x, y) \geq a(x - x_0) + b(y - y_0) + c, \text{ for all } (x, y) \in P, \\ & a, b, c \in \mathbf{R}, \end{aligned} \quad (2)$$

where a, b, c define the desired supporting hyperplane $z = ax + by + (c - ax_0 - by_0)$. The assumptions on f over P allow the authors to define a family of convex subproblems, where the optimum of problem (2) equals the overall optimum of the convex

subproblems. The number of subproblems is bounded by 3^t , where t represents the number of facets of P over which f is strictly convex. We refer to [LS14], for more details on this analysis, and to [Loc10] for further extensions to multivariate quadratic polynomials and to bivariate polynomials. We also refer to [She97, BST09, TRX13] where connections between the two variants of optimization problems for computing convex envelopes are established for multilinear functions on boxes and for functions with polyhedral envelopes.

For our cut-generation algorithm, we follow [JMW08] and analyze the underlying problems as given in (1). We moreover point out that all the foundations of the presented analysis are already given in the literature. Nevertheless, we provide a detailed step-by-step derivation to provide a self-contained documentation for our code.

The cut-generation algorithm consists of two subroutines based on the evaluation of the convex envelope of f , and on a lifting technique. Let x_0, y_0 , and z_0 be the solution of the current relaxation, where z_0 corresponds to the value of $f(x_0, y_0)$ in the relaxation. If (x_0, y_0) is in the interior of the box $[l, u]$, we use [JMW08, TS01] to solve the optimization problem corresponding to the convex envelope at (x_0, y_0) . The solution of this problem can be used to construct a maximally touching, underestimating hyperplane, i.e., a hyperplane which is not dominated by another underestimating hyperplane. This subroutine is discussed in Section 2. If (x_0, y_0) is in the boundary of the box, the solution of the convex envelope problem may only provide an underestimator which is valid over a facet of the box. In Section 3 we apply a lifting technique to extend this locally valid underestimator to the entire box. Here, we use the results from [GKH⁺06, GHJ⁺08]. The presented formulas are implemented in the constraint handler “cons.bivariate” in the constraint integer programming framework SCIP [Ach07, Ach09]. A computational case study in Section 4 shows the performance of the implemented constraint handler compared to the state-of-the-art solver BARON.

2. Underestimators based on the convex envelope

In this section, we present explicit formulas for maximally touching hyperplanes for the graphs of the convex envelopes of our functions $f(x, y)$ at a given point (x_0, y_0) in the box $[l, u]$, i.e., we determine numbers $\alpha, \beta, \delta \in \mathbf{R}$ and $\gamma \in \mathbf{R}_{\geq 0}$ such that $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, for all $(x, y) \in [l, u]$ with equality at (x_0, y_0) .

2.1. The function $f(x, y)$ is convex in x and y

For a convex function $f : [l, u] \rightarrow \mathbf{R}$, the convex envelope over its domain is given by the function itself. Thus, the best possible linear underestimator of f at (x_0, y_0) is given by the tangent:

$$f(x, y) \geq \nabla f(x_0, y_0)^\top ((x, y) - (x_0, y_0)) + f(x_0, y_0).$$

2.2. The function $f(x, y)$ is concave in x and y

According to [McC76, Tar04, KS13] the convex envelope of a bivariate component-wise concave functions f is given as follows. If $f(l_x, l_y) + f(u_x, u_y) \leq f(l_x, u_y) + f(u_x, l_y)$, the convex envelope is given by

$$\text{vex}_{[l,u]}[f](x, y) = \begin{cases} \alpha_1 x + \beta_1 y - \delta_1, & \text{if } y_0 \leq \frac{u_y - l_y}{u_x - l_x} (x_0 - l_x) + l_y, \\ \alpha_2 x + \beta_2 y - \delta_2, & \text{if } y_0 > \frac{u_y - l_y}{u_x - l_x} (x_0 - l_x) + l_y, \end{cases}$$

where

$$\alpha_1 = \frac{f(u_x, l_y) - f(l_x, l_y)}{u_x - l_x}, \quad \beta_1 = \frac{f(u_x, u_y) - f(u_x, l_y)}{u_y - l_y}, \quad \delta_1 = -\frac{u_x(u_y - l_y)f(l_x, l_y) - (u_x - l_x)l_y f(u_x, u_y) + (u_x l_y - l_x u_y)f(u_x, l_y)}{(u_x - l_x)(u_y - l_y)},$$

$$\alpha_2 = \frac{f(u_x, u_y) - f(l_x, u_y)}{u_x - l_x}, \quad \beta_2 = \frac{f(l_x, u_y) - f(l_x, l_y)}{u_y - l_y}, \quad \delta_2 = -\frac{(u_x - l_x)u_y f(l_x, l_y) - l_x(u_y - l_y)f(u_x, u_y) + (l_x u_y - u_x l_y)f(l_x, u_y)}{(u_x - l_x)(u_y - l_y)}.$$

Otherwise, the convex envelope is given by

$$\text{vex}_{[l,u]}[f](x, y) = \begin{cases} \alpha_1 x + \beta_1 y - \delta_1, & \text{if } y_0 \leq \frac{l_y - u_y}{u_x - l_x} (x_0 - l_x) + u_y, \\ \alpha_2 x + \beta_2 y - \delta_2, & \text{if } y_0 > \frac{l_y - u_y}{u_x - l_x} (x_0 - l_x) + u_y, \end{cases}$$

where

$$\alpha_1 = \frac{f(u_x, l_y) - f(l_x, l_y)}{u_x - l_x}, \quad \beta_1 = \frac{f(l_x, u_y) - f(l_x, l_y)}{u_y - l_y}, \quad \delta_1 = -\frac{(u_x u_y - l_x l_y)f(l_x, l_y) - l_x(u_y - l_y)f(u_x, l_y) - (u_x - l_x)l_y f(l_x, u_y)}{(u_x - l_x)(u_y - l_y)},$$

$$\alpha_2 = \frac{f(u_x, u_y) - f(l_x, u_y)}{u_x - l_x}, \quad \beta_2 = \frac{f(u_x, u_y) - f(u_x, l_y)}{u_y - l_y}, \quad \delta_2 = -\frac{(l_x l_y - u_x u_y)f(u_x, u_y) + u_x(u_y - l_y)f(l_x, u_y) + (u_x - l_x)u_y f(u_x, l_y)}{(u_x - l_x)(u_y - l_y)}.$$

In the following we assume that the given point (x_0, y_0) is in the *interior* of $[l, u]$. The cases where (x_0, y_0) is in the boundary of $[l, u]$ are discussed in Section 3.

2.3. The function f is strictly convex in x and concave in y

We use results from [TS01, JMW08]. It follows that for a given point $(x_0, y_0) \in [l, u]$

$$\begin{aligned} \text{vex}_{[l,u]}[f](x_0, y_0) &= \min v(r, s, t) := t f(r, l_y) + (1 - t) f(s, u_y) \\ \text{s.t.} & \begin{cases} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = t \begin{pmatrix} r \\ l_y \end{pmatrix} + (1 - t) \begin{pmatrix} s \\ u_y \end{pmatrix}, \\ 0 \leq t \leq 1, \quad r, s \in [l_x, u_x]. \end{cases} \end{aligned} \quad (3)$$

Then, Problem (3) can be rewritten to the following univariate convex problem using the identities $t = \frac{y_0 - u_y}{l_y - u_y}$ and $r(s) = \frac{l_y - u_y}{y_0 - u_y} x_0 - \frac{l_y - y_0}{y_0 - u_y} s$ as $l_y < y_0 < u_y$ and $l_x < x_0 < u_x$

$$\min v_{\text{red}}(s) \quad \text{s.t.} \quad \max \left\{ l_x, \frac{y_0 - u_y}{l_y - y_0} \left[\frac{l_y - u_y}{y_0 - u_y} x_0 - u_x \right] \right\} \leq s \leq \min \left\{ \frac{y_0 - u_y}{l_y - y_0} \left[\frac{l_y - u_y}{y_0 - u_y} x_0 - l_x \right], u_x \right\}, \quad (4)$$

where $v_{\text{red}}(s)$ reads as

$$v_{\text{red}}(s) := v\left(r(s), s, \frac{y_0 - u_y}{l_y - u_y}\right) = \frac{y_0 - u_y}{l_y - u_y} f\left(\frac{l_y - u_y}{y_0 - u_y} x_0 - \frac{l_y - y_0}{y_0 - u_y} s, l_y\right) + \frac{l_y - y_0}{l_y - u_y} f(s, u_y).$$

Let s^* denote an optimal solution of the reduced Problem (4). Then, the point (s^*, r^*, t^*) , with $t^* = \frac{y_0 - u_y}{l_y - u_y}$ and $r^* = r(s^*) = \frac{l_y - u_y}{y_0 - u_y} x_0 - \frac{l_y - y_0}{y_0 - u_y} s^*$, is an optimal solution of Problem (3). Moreover, we have that $\text{vex}_{[l, u]}[f](x_0, y_0) = v_{\text{red}}(s^*)$.

By construction, $\text{vex}_{[l_x, u_x] \times [l_y, u_y]}[f](x, y)$ is linear over the segment connecting (r^*, l_y) and (s^*, u_y) which contains the point (x_0, y_0) . Moreover, f is convex in x for a fixed $y \in \{l_y, u_y\}$. Thus, a maximally touching hyperplane on the graph of $\text{vex}_{[l, u]}[f]$ at (x_0, y_0) is defined by the point p_1 and the direction vectors q_1 and q_2 given by

$$p_1 := \begin{pmatrix} x_0 \\ y_0 \\ v_{\text{red}}(s^*) \end{pmatrix}, \quad q_1 := \begin{pmatrix} s^* - r^* \\ u_y - l_y \\ f(s^*, u_y) - f(r^*, l_y) \end{pmatrix}, \quad q_2 := \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(\bar{x}, \bar{y}) \end{pmatrix},$$

where the point $(\bar{x}, \bar{y}) \in \{(r^*, l_y), (s^*, u_y)\}$ has to be chosen as follows. If $l_x < s^* < u_x$, set $(\bar{x}, \bar{y}) = (s^*, u_y)$. If $s^* \in \{l_x, u_x\}$ and $l_x < r^* < u_x$, set $(\bar{x}, \bar{y}) = (r^*, l_y)$. Otherwise, both points $(\bar{x}^1, \bar{y}^1) = (r^*, l_y)$ and $(\bar{x}^2, \bar{y}^2) = (s^*, u_y)$ yield valid inequalities. To understand the choice of (\bar{x}, \bar{y}) we consider the case where $l_x < s^* < u_x$ and $l_x < r^* < u_x$. Then, $\frac{\partial f}{\partial x}(s^*, u_y) = \frac{\partial f}{\partial x}(r^*, l_y)$. If, for instance, $s^* = l_x$ and $l_x < r^* < u_x$, the point (x', u_y) with $\frac{\partial f}{\partial x}(x', u_y) = \frac{\partial f}{\partial x}(r^*, l_y)$ satisfies $x' \leq l_x = s^*$. By convexity of f in x , it follows that $\frac{\partial f}{\partial x}$ is nondecreasing in x and thus $\frac{\partial f}{\partial x}(x', u_y) = \frac{\partial f}{\partial x}(r^*, l_y) \leq \frac{\partial f}{\partial x}(s^*, u_y)$. Then, $f(x, u_y) \geq \frac{\partial f}{\partial x}(s^*, u_y)(x - s^*) + f(s^*, u_y) \geq \frac{\partial f}{\partial x}(r^*, l_y)(x - s^*) + f(s^*, u_y)$ for all $x \in [l, u]$. As the resulting hyperplane underestimates $f(x, y)$ for fixed $y \in \{l_y, u_y\}$, and f is convex in x and concave in y , it follows that the hyperplane is a touching hyperplane on the graph of $\text{vex}_{[l, u]}[f]$ at (x_0, y_0) . This yields the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$ with

$$\begin{aligned} \alpha &= (u_y - l_y) \frac{\partial f}{\partial x}(\bar{x}, \bar{y}), \\ \beta &= f(s^*, u_y) - f(r^*, l_y) - (s^* - r^*) \frac{\partial f}{\partial x}(\bar{x}, \bar{y}), \\ \gamma &= (u_y - l_y), \\ \delta &= \alpha x_0 + \beta y_0 - \gamma v_{\text{red}}(s^*). \end{aligned}$$

2.4. The function f is concave in x and strictly convex in y

Switch the variables and apply Subsection 2.3.

2.5. The function is not convex, but strictly convex in x and y

By Theorem 3.1 in [JMW08], the value of the convex envelope of f on $[l, u]$ at (x_0, y_0) is given by

$$\begin{aligned} \min v(x_1, x_2, y_1, y_2, t) &:= tf(x_1, y_1) + (1-t)f(x_2, y_2) \\ \text{s.t.} \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= t \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \\ 0 \leq t \leq 1, \quad &(x_1, y_1), (x_2, y_2) \in \mathcal{F}, \end{aligned} \quad (5)$$

where \mathcal{F} denotes the boundary of the box $[l, u]$, i.e., either $x_1 \in \{l_x, u_x\}$ or $y_1 \in \{l_y, u_y\}$, and $x_2 \in \{l_x, u_x\}$ or $y_2 \in \{l_y, u_y\}$. This can be used in a case distinction which simplifies Problem (5) by assigning (x_1, y_1) and (x_2, y_2) to different facets of the box $[l, u]$. Thus, we get up to six simplified optimization problems because we have to consider two assignments to parallel facets and four assignments to adjacent facets. The minimum of all six cases yields the value of the convex envelope. According to [JMW08], this case distinction can be avoided by exploiting a geometrical property of indefinite and $(n-1)$ -convex functions: Their concave directions are contained in a pair of orthants of \mathbf{R}^2 . To determine this pair for a given function, we can compute the eigenvector to the negative eigenvalue of the Hessian $\mathcal{H}_f(\bar{x}, \bar{y})$ of f at the midpoint $(\bar{x}, \bar{y}) = (\frac{1}{2}(u_x + l_x), \frac{1}{2}(u_y + l_y))$ of the box. If the eigenvector has entries with different signs, then the concave directions of f at any point in $[l, u]$ are contained in the union $(\mathbf{R}_{\geq 0} \times \mathbf{R}_{\leq 0}) \cup (\mathbf{R}_{\leq 0} \times \mathbf{R}_{\geq 0})$ [**pattern A**], otherwise the concave directions are contained in the union $(\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}) \cup (\mathbf{R}_{\leq 0} \times \mathbf{R}_{\leq 0})$ [**pattern B**]. Each pattern contains still three possible assignments of (x_1, y_1) and (x_2, y_2) to the facets of $[l, u]$.

2.5.1. Pattern A

Pattern A can lead to two possible types of structures for the subdivision of the box w.r.t. the description of the convex envelope. See Figure 1. The figure reflects that all lines in A_i , $i = 1, 2, 3$, connecting the facets of the box are contained in $(\mathbf{R}_{\geq 0} \times \mathbf{R}_{\leq 0}) \cup (\mathbf{R}_{\leq 0} \times \mathbf{R}_{\geq 0})$. The knowledge on the type of subdivision for the box further reduces the number of possible assignments to the endpoints of a minimizing segment.

The type of subdivision can be determined as follows.

Lemma 2.1 (e.g., cf. Ex. 5 in [JMW08]). *If $f(l_x, u_y) + (l_y - u_y) \frac{\partial f}{\partial y}(l_x, u_y) \geq f(u_x, l_y) + (l_x - u_x) \frac{\partial f}{\partial x}(u_x, l_y)$, the structure for the subdivision of the convex envelope corresponds to Figure 1 (a). Otherwise, the structure corresponds to Figure 1 (b).*

Subsequently we discuss the convex envelope with a subdivision as in Figure 1 (a). The formulas for the other case can be derived analogously by interchanging x and y . This gives rise to a situation as in Figure 1 (a).

Note that Lemma 2.1 only provides information about the general shape of the concrete subdivision. To determine a minimizing segment for a given point, we have to solve two auxiliary problems. The minimal value of the two problems is then

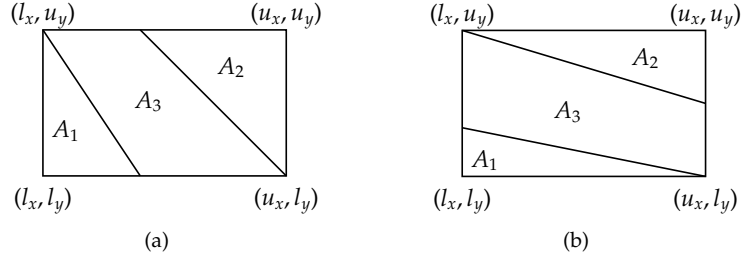


Figure 1: Pattern A: Possible subdivisions of the box w.r.t. the description of the convex envelope.

equivalent to the value of the convex envelope. The first auxiliary problem corresponds to subdomain A_3 , where the endpoints of the possible minimizing segment are contained in the parallel facets given by $y = l_y$ and $y = u_y$. The second auxiliary problem corresponds to the subdomains A_1 and A_2 depending on the position of the point (x_0, y_0) . If the point (x_0, y_0) is below the diagonal of the box connecting (l_x, u_y) and (u_x, l_y) , the endpoints of the possible minimizing segment are contained in the orthogonal facets $x = l_x$ and $y = l_y$ (subdomain A_1). Otherwise, the endpoints of the possible minimizing segment are contained in the orthogonal facets $x = u_x$ and $y = u_y$ (subdomain A_2).

Auxiliary problem 1: Parallel facets In this case the optimization problem reduces to

$$\begin{aligned} \varrho(x_0, y_0) &:= \min t f(r, l_y) + (1-t)f(s, u_y) \\ \text{s.t. } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= t \begin{pmatrix} r \\ l_y \end{pmatrix} + (1-t) \begin{pmatrix} s \\ u_y \end{pmatrix}, \\ 0 &\leq t \leq 1, \quad r, s \in [l_x, u_x]. \end{aligned} \quad (6)$$

This subproblem is identical to the case considered in Subsection 2.3.

Auxiliary problem 2: Orthogonal facets If the point (x_0, y_0) is below the diagonal, i.e., $y_0 \leq \frac{l_y - u_y}{u_x - l_x}(x_0 - l_x) + u_y$, we consider the subproblem

$$\begin{aligned} \omega_1(x_0, y_0) &:= \min t f(l_x, r) + (1-t)f(s, l_y) \\ \text{s.t. } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= t \begin{pmatrix} l_x \\ r \end{pmatrix} + (1-t) \begin{pmatrix} s \\ l_y \end{pmatrix}, \\ 0 &< t < 1, \quad r \in [l_y, u_y], \quad t \in [l_x, u_x]. \end{aligned} \quad (7)$$

Following ([JMW08, Lem. 4.2]), we set $s(t) = (x_0 - l_x t)/(1-t)$ and $r(t) = (y_0 - (1-t)l_y)/t$. Then, Problem (7) can be transformed into the following univariate convex

optimization problem

$$\min v_{\text{red}}(t) := t f\left(l_x, \frac{y_0 - (1-t)l_y}{t}\right) + (1-t)f\left(\frac{x_0 - l_x t}{1-t}, l_y\right), \quad \text{s.t. } t \in \left[\frac{y_0 - l_y}{u_y - l_y}, \frac{u_x - x_0}{u_x - l_x}\right]. \quad (8)$$

An optimal solution $t^* \in (0, 1)$ to Problem (8) yields the point (t^*, s^*, r^*) , with $s^* = s(t^*)$ and $r^* = r(t^*)$, i.e., $\omega_1(x_0, y_0) = v_{\text{red}}(t^*)$.

If the point (x_0, y_0) is above the diagonal, i.e., $y_0 > \frac{l_y - u_y}{u_x - l_x}(x_0 - l_x) + u_y$, we consider the subproblem

$$\begin{aligned} \omega_2(x_0, y_0) &:= \min t f(u_x, r) + (1-t)f(s, u_y) \\ \text{s.t. } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= t \begin{pmatrix} u_x \\ r \end{pmatrix} + (1-t) \begin{pmatrix} s \\ u_y \end{pmatrix}, \\ 0 < t < 1, & r \in [l_y, u_y], s \in [l_x, u_x]. \end{aligned} \quad (9)$$

Using $s(t) = (x_0 - u_x t)/(1-t)$ and $r(t) = (y_0 - (1-t)u_y)/t$, Problem (9) is equivalent to the following univariate convex problem

$$\min v_{\text{red}}(t) := t f\left(u_x, \frac{y_0 - (1-t)u_y}{t}\right) + (1-t)f\left(\frac{x_0 - u_x t}{1-t}, u_y\right), \quad \text{s.t. } t \in \left[\frac{u_y - y_0}{u_y - l_y}, \frac{x_0 - l_x}{u_x - l_x}\right], \quad (10)$$

that can be solved using standard numerical methods. Let $t^* \in (0, 1)$ be an optimal solution of Problem (10). Then, the point (t^*, s^*, r^*) with $s^* = s(t^*)$ and $r^* = r(t^*)$ is an optimal solution for Problem (9).

Thus, the value of the convex envelope is the minimum of the optimal value $\varrho(x_0, y_0)$ of Problem (6) corresponding to the parallel case and of either $\omega_1(x_0, y_0)$ or $\omega_2(x_0, y_0)$ of Problems (7) and (9), respectively, corresponding to the orthogonal case:

$$\text{vex}_{[l,u]}[f](x_0, y_0) = \begin{cases} \min\{\varrho(x_0, y_0), \omega_1(x_0, y_0)\}, & \text{if } y_0 \leq \frac{l_y - u_y}{u_x - l_x}(x_0 - l_x) + u_y, \\ \min\{\varrho(x_0, y_0), \omega_2(x_0, y_0)\}, & \text{if } y_0 > \frac{l_y - u_y}{u_x - l_x}(x_0 - l_x) + u_y. \end{cases}$$

To construct supporting hyperplanes, we thus need to consider three cases:

- (i) $\text{vex}_{[l,u]}[f](x_0, y_0) = \varrho(x_0, y_0)$,
- (ii) $y_0 \leq \frac{l_y - u_y}{u_x - l_x}(x_0 - l_x) + u_y$ and $\text{vex}_{[l,u]}[f](x_0, y_0) = \omega_1(x_0, y_0)$,
- (iii) $y_0 > \frac{l_y - u_y}{u_x - l_x}(x_0 - l_x) + u_y$ and $\text{vex}_{[l,u]}[f](x_0, y_0) = \omega_2(x_0, y_0)$.

Case (i): A minimizing segment is given by the optimal solution of Problem (6), and the formulas for a linear underestimator can be derived as in Subsection 2.3.

Case (ii): A minimizing segment is given by an optimal solution (t^*, s^*, r^*) to Problem (7). The desired hyperplane on the graph of the convex envelope at the point (x_0, y_0) is given by the two points $p_1 = (l_x, r^*, f(l_x, r^*))$ and $p_2 = (s^*, l_y, f(s^*, l_y))$

and a direction vector q . If $s^* \neq u_x$, then $q = (1, 0, \frac{\partial f}{\partial x}(s^*, l_y))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (r^* - l_y) \frac{\partial f}{\partial x}(s^*, l_y), \\ \beta &= (s^* - l_x) \frac{\partial f}{\partial x}(s^*, l_y) + f(l_x, r^*) - f(s^*, l_y), \\ \gamma &= r^* - l_y, \\ \delta &= \alpha l_x + \beta r^* - \gamma f(l_x, r^*).\end{aligned}$$

If $s^* = u_x$ and $r^* \neq u_y$, then $q = (0, 1, \frac{\partial f}{\partial y}(l_x, r^*))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (r^* - l_y) \frac{\partial f}{\partial y}(l_x, r^*) - f(l_x, r^*) + f(s^*, l_y), \\ \beta &= (s^* - l_x) \frac{\partial f}{\partial y}(l_x, r^*), \\ \gamma &= s^* - l_x, \\ \delta &= \alpha l_x + \beta r^* - \gamma f(l_x, r^*).\end{aligned}$$

If $s^* = u_x$ and $r^* = u_y$, then $q = (1, 0, \min\{\frac{\partial f}{\partial x}(s^*, l_y), \frac{\partial f}{\partial x}(l_x, r^*)\})$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (r^* - l_y) \min\{\frac{\partial f}{\partial x}(s^*, l_y), \frac{\partial f}{\partial x}(l_x, r^*)\}, \\ \beta &= (s^* - l_x) \min\{\frac{\partial f}{\partial x}(s^*, l_y), \frac{\partial f}{\partial x}(l_x, r^*)\} + f(l_x, r^*) - f(s^*, l_y), \\ \gamma &= r^* - l_y, \\ \delta &= \alpha l_x + \beta r^* - \gamma f(l_x, r^*).\end{aligned}$$

Case (iii): A minimizing segment is given by an optimal solution (t^*, s^*, r^*) to Problem (9). The desired hyperplane on the graph of the convex envelope at the point (x_0, y_0) is given by the two points $p_1 = (u_x, r^*, f(u_x, r^*))$ and $p_2 = (s^*, u_y, f(s^*, u_y))$ and a direction vector q . If $s^* \neq l_x$, then $q = (1, 0, \frac{\partial f}{\partial x}(s^*, u_y))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (u_y - r^*) \frac{\partial f}{\partial x}(s^*, u_y), \\ \beta &= (u_x - s^*) \frac{\partial f}{\partial x}(s^*, u_y) - f(u_x, r^*) + f(s^*, u_y), \\ \gamma &= u_y - r^*, \\ \delta &= \alpha s^* + \beta u_y - \gamma f(s^*, u_y).\end{aligned}$$

If $s^* = l_x$ and $r^* \neq l_y$, then $q = (0, 1, \frac{\partial f}{\partial y}(u_x, r^*))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (u_y - r^*) \frac{\partial f}{\partial y}(u_x, r^*) + f(u_x, r^*) - f(s^*, u_y), \quad \beta = (u_x - s^*) \frac{\partial f}{\partial y}(u_x, r^*), \quad \gamma = u_x - s^*, \\ \delta &= \alpha s^* + \beta u_y - \gamma f(s^*, u_y).\end{aligned}$$

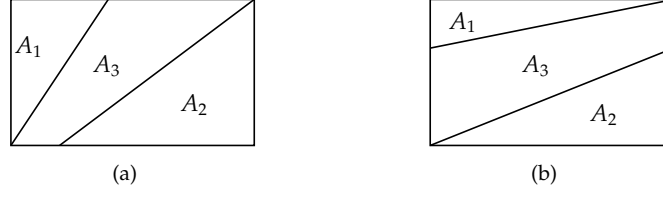


Figure 2: Pattern B: Possible subdivisions of the box w.r.t the description of the convex envelope.

If $s^* = l_x$ and $r^* = l_y$, then $q = (1, 0, \min\{\frac{\partial f}{\partial x}(s^*, u_y), \frac{\partial f}{\partial x}(u_x, r^*)\})$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (u_y - r^*) \min\{\frac{\partial f}{\partial x}(s^*, u_y), \frac{\partial f}{\partial x}(u_x, r^*)\}, \\ \beta &= (u_x - s^*) \min\{\frac{\partial f}{\partial x}(s^*, u_y), \frac{\partial f}{\partial x}(u_x, r^*)\} - f(u_x, r^*) + f(s^*, u_y), \\ \gamma &= u_y - s^*, \\ \delta &= \alpha s^* + \beta u_y - \gamma f(s^*, u_y).\end{aligned}$$

2.5.2. Pattern B

Figure 2 displays the two possible structures for the subdivision of the box w.r.t. the description of the convex envelope for pattern B. If $f(l_x, l_y) + (u_y - l_y)\frac{\partial f}{\partial y}(l_x, l_y) \geq f(u_x, u_y) + (l_x - u_x)\frac{\partial f}{\partial x}(u_x, u_y)$, the structure corresponds to Figure 2 (a). Otherwise, the structure of the subdivision corresponds to Figure 2 (b).

Subsequently, we discuss the convex envelope with a subdivision as in Figure 2 (a). The formulas for the other case can be derived analogously by interchanging x and y leading to a situation as in Figure 2 (a).

Analogously to pattern A, we have to solve two auxiliary problems:

Auxiliary problem 1: Parallel facets In this case the optimization problem reduces to

$$\begin{aligned}\varrho(x_0, y_0) &:= \min \quad t f(r, l_y) + (1 - t) f(s, u_y) \\ \text{s.t.} \quad &\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = t \begin{pmatrix} r \\ l_y \end{pmatrix} + (1 - t) \begin{pmatrix} s \\ u_y \end{pmatrix}, \\ &0 \leq t \leq 1, \quad r, s \in [l_x, u_x].\end{aligned}\tag{11}$$

This subproblem is identical to the case considered in Subsection 2.3.

Auxiliary problem 2: Orthogonal facets If the point (x_0, y_0) is above the diagonal connecting (l_x, l_y) and (u_x, u_y) , i.e., $y_0 \geq \frac{u_y - l_y}{u_x - l_x}(x_0 - l_x) + l_y$, we consider the subproblem

$$\begin{aligned} \omega_1(x_0, y_0) &:= \min t f(r, u_y) + (1-t)f(l_x, s) \\ \text{s.t. } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= t \begin{pmatrix} r \\ u_y \end{pmatrix} + (1-t) \begin{pmatrix} l_x \\ s \end{pmatrix}, \\ 0 < t < 1, & r \in [l_x, u_x], s \in [l_y, u_y]. \end{aligned} \quad (12)$$

Using $r(t) = l_x + (x_0 - l_x)/t$ and $s(t) = (y_0 - tu_y)/(1-t)$, Problem (12) can be transformed into the following univariate convex optimization problem

$$\min v_{\text{red}}(t) := t f\left(l_x + \frac{x_0 - l_x}{t}, u_y\right) + (1-t)f\left(l_x, \frac{y_0 - tu_y}{1-t}\right), \quad \text{s.t. } t \in \left[\frac{x_0 - l_x}{u_x - l_x}, \frac{y_0 - l_y}{u_y - l_y}\right].$$

Let $t^* \in (0, 1)$ be the optimal solution. Then, the point (t^*, s^*, r^*) with $s^* = s(t^*)$ and $r^* = r(t^*)$ is an optimal solution for Problem (12).

If the point (x_0, y_0) is below the diagonal connecting (l_x, l_y) and (u_x, u_y) , i.e., $y_0 < \frac{u_y - l_y}{u_x - l_x}(x_0 - l_x) + l_y$, we consider the subproblem

$$\begin{aligned} \omega_2(x_0, y_0) &:= \min t f(r, l_y) + (1-t)f(u_x, s) \\ \text{s.t. } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= t \begin{pmatrix} r \\ l_y \end{pmatrix} + (1-t) \begin{pmatrix} u_x \\ s \end{pmatrix}, \\ 0 < t < 1, & r \in [l_x, u_x], y \in [l_y, u_y]. \end{aligned} \quad (13)$$

Using $r(t) = u_x + (x_0 - u_x)/t$ and $s(t) = (y_0 - tl_y)/(1-t)$, Problem (13) can be transformed into the following univariate convex optimization problem

$$\min v_{\text{red}}(t) := t f\left(u_x + \frac{x_0 - u_x}{t}, l_y\right) + (1-t)f\left(u_x, \frac{y_0 - tl_y}{1-t}\right), \quad \text{s.t. } t \in \left[\frac{u_x - x_0}{u_x - l_x}, \frac{u_y - y_0}{u_y - l_y}\right].$$

Let $t^* \in (0, 1)$ be the optimal solution. Then, the point (t^*, s^*, r^*) with $s^* = s(t^*)$ and $r^* = r(t^*)$ is an optimal solution for Problem (13).

Thus, the value of the convex envelope is the minimum of the optimal value $\varrho(x_0, y_0)$ of Problem (11) corresponding to the parallel case and of either $\omega_1(x_0, y_0)$ or $\omega_2(x_0, y_0)$ of Problems (12) and (13), respectively, corresponding to the orthogonal case:

$$\text{vex}_{[l,u]}[f](x_0, y_0) = \begin{cases} \min\{\varrho(x_0, y_0), \omega_1(x_0, y_0)\}, & \text{if } y_0 \geq \frac{u_y - l_y}{u_x - l_x}(x_0 - l_x) + l_y, \\ \min\{\varrho(x_0, y_0), \omega_2(x_0, y_0)\}, & \text{if } y_0 < \frac{u_y - l_y}{u_x - l_x}(x_0 - l_x) + l_y. \end{cases}$$

To construct supporting hyperplanes, we thus need to consider three cases:

- (i) $\text{vex}_{[l,u]}[f](x_0, y_0) = \varrho(x_0, y_0)$,
- (ii) $y_0 \geq \frac{u_y - l_y}{u_x - l_x}(x_0 - l_x) + l_y$ and $\text{vex}_{[l,u]}[f](x_0, y_0) = \omega_1(x_0, y_0)$,

(iii) $y_0 < \frac{u_y - l_y}{u_x - l_x}(x_0 - l_x) + l_y$ and $\text{vex}_{[l,u]}[f](x_0, y_0) = \omega_2(x_0, y_0)$.

Case (i): A minimizing segment is given by the optimal solution of Problem (11), and the formulas for a linear underestimators can be derived as in Subsection 2.3.

Case (ii): A minimizing segment is given by the optimal solution (r^*, s^*, r^*) to Problem (12). The desired hyperplane on the graph of the convex envelope at the point (x_0, y_0) is given by the two points $p_1 = (r^*, u_y, f(r^*, u_y))$ and $p_2 = (l_x, s^*, f(l_x, s^*))$ and a direction vector q . If $r^* \neq u_x$, then $q = (1, 0, \frac{\partial f}{\partial x}(r^*, u_y))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (u_y - s^*) \frac{\partial f}{\partial x}(r^*, u_y), \\ \beta &= (l_x - r^*) \frac{\partial f}{\partial x}(r^*, u_y) + f(r^*, u_y) - f(l_x, s^*), \\ \gamma &= u_y - s^*, \\ \delta &= \alpha l_x + \beta s^* - \gamma f(l_x, s^*).\end{aligned}$$

If $r^* = u_x$ and $s^* \neq l_y$, then $q = (0, 1, \frac{\partial f}{\partial y}(l_x, s^*))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (s^* - u_y) \frac{\partial f}{\partial y}(l_x, s^*) + f(r^*, u_y) - f(l_x, s^*), \\ \beta &= (r^* - l_x) \frac{\partial f}{\partial y}(l_x, s^*), \\ \gamma &= r^* - l_x, \\ \delta &= \alpha l_x + \beta s^* - \gamma f(l_x, s^*).\end{aligned}$$

If $r^* = u_x$ and $s^* = l_y$, then $q = (1, 0, \min\{\frac{\partial f}{\partial x}(r^*, u_y), \frac{\partial f}{\partial x}(l_x, s^*)\})$ yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (u_y - s^*) \min\{\frac{\partial f}{\partial x}(r^*, u_y), \frac{\partial f}{\partial x}(l_x, s^*)\}, \\ \beta &= (l_x - r^*) \min\{\frac{\partial f}{\partial x}(r^*, u_y), \frac{\partial f}{\partial x}(l_x, s^*)\} + f(r^*, u_y) - f(l_x, s^*), \\ \gamma &= (u_y - s^*), \\ \delta &= \alpha l_x + \beta s^* - \gamma f(l_x, s^*).\end{aligned}$$

Case (iii): A minimizing segment is given by the optimal solution (t^*, s^*, r^*) to Problem (13). The desired hyperplane on the graph of the convex envelope at the point (x_0, y_0) is now given by the two points $p_1 = (r^*, l_y, f(r^*, l_y))$ and $p_2 = (u_x, s^*, f(u_x, s^*))$ and a direction vector q . If $r^* \neq l_x$, then $q = (1, 0, \frac{\partial f}{\partial x}(r^*, l_y))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (s^* - l_y) \frac{\partial f}{\partial x}(r^*, l_y), \\ \beta &= (r^* - u_x) \frac{\partial f}{\partial x}(r^*, l_y) - f(r^*, l_y) + f(u_x, s^*), \\ \gamma &= s^* - l_y, \\ \delta &= \alpha u_x + \beta s^* - \gamma f(u_x, s^*).\end{aligned}$$

If $r^* = l_x$ and $s^* \neq u_y$, then $q = (0, 1, \frac{\partial f}{\partial y}(u_x, s^*))$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (l_y - s^*) \frac{\partial f}{\partial y}(u_x, s^*) - f(r^*, l_y) + f(u_x, s^*), \\ \beta &= (u_x - r^*) \frac{\partial f}{\partial y}(u_x, s^*), \\ \gamma &= u_x - r^*, \\ \delta &= \alpha u_x + \beta s^* - \gamma f(u_x, s^*).\end{aligned}$$

If $r^* = u_x$ and $s^* = l_y$, then $q = (1, 0, \min\{\frac{\partial f}{\partial x}(r^*, l_y), \frac{\partial f}{\partial x}(u_x, s^*)\})$, yielding the linear underestimator $\gamma f(x, y) \geq \alpha x + \beta y - \delta$, where

$$\begin{aligned}\alpha &= (s^* - l_y) \min\{\frac{\partial f}{\partial x}(r^*, l_y), \frac{\partial f}{\partial x}(u_x, s^*)\} \\ \beta &= (r^* - u_x) \min\{\frac{\partial f}{\partial x}(r^*, l_y), \frac{\partial f}{\partial x}(u_x, s^*)\} - f(r^*, l_y) + f(u_x, s^*), \\ \gamma &= s^* - l_y, \\ \delta &= \alpha u_x + \beta s^* - \gamma f(u_x, s^*).\end{aligned}$$

3. Underestimators on the boundary by lifting

In this section we calculate linear underestimators for f when the given point (x_0, y_0) is in the boundary of the domain $[l, u]$, and f is either convex in one variable and concave in the other, or f is convex in both variables and indefinite. In these cases, an optimal solution of the underlying optimization problems in Equations (3) and (5) provides a lower-dimensional estimator that is, in general, only valid over the corresponding face of the box. In the following, we apply the *lifting technique* to the lower-dimensional estimators to compute estimators that are valid on the entire box.

The concept of the lifting was introduced by Padberg in [Pad75] to compute tight linear inequalities for linear zero-one problems. It has been adopted in [GKH⁺06, GHJ⁺08, BMSMW10] to derive linear and convex underestimators for concrete examples of low-dimensional non-linear functions by exploiting certain analytic and geometric properties. In a general setting, the lifting technique has been studied in [RT10].

Applied to our setting, the key idea of the lifting procedure is the following (see [GHJ⁺08]). Given a bivariate function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$, $(x, y) \mapsto f(x, y)$ on a box $[l, u] := [l_x, u_x] \times [l_y, u_y] \subseteq \mathbf{R}^2$. We first fix one variable to one of its bounds. For the purpose of illustration assume that x is fixed to its lower bound l_x . Assume furthermore that we have a function $g : \mathbf{R} \rightarrow \mathbf{R}$ underestimating $f(l_x, y)$ over $[l_y, u_y]$ at hand. The functions which we consider in this work are either convex or concave when restricted to a facet of the box so that we can underestimate them by a tangent or a secant, respectively. Our aim is to determine a best possible *lifting coefficient* $\mu \in \mathbf{R}$ such that

$$f(x, y) \geq \mu(x - l_x) + g(y) \quad \text{holds for all } (x, y) \in [l, u].$$

This gives rise to the following non-linear optimization problem

$$\mu := \inf \left\{ \frac{f(x,y) - g(y)}{x - l_x} \mid x \in (l_x, u_x], y \in [l_y, u_y] \right\}. \quad (14)$$

If we fix x to its upper bound u_x and assume that $h : \mathbf{R} \rightarrow \mathbf{R}$ is an underestimating function for $f(u_x, y)$ on $[l_y, u_y]$, we determine a best possible number $\tau \in \mathbf{R}$ with

$$f(x, y) \geq \tau(x - u_x) + h(y) \quad \text{for all } (x, y) \in [l, u].$$

Using that $x - u_x \leq 0$ for $x \in [l_x, u_x]$, we arrive at the following optimization task

$$\tau := \sup \left\{ \frac{f(x,y) - h(y)}{x - u_x} \mid x \in (l_x, u_x], y \in [l_y, u_y] \right\}. \quad (15)$$

We remark that the optimization problems can be, in general, extremely difficult to solve. In the following we exploit the specific structure of our bivariate functions to determine appropriate lifting coefficients.

To complete our *cut-generation procedure* from the previous section, we have to investigate the lifting

1. from a facet over which the function is concave into a direction in which the function is convex,
2. from a facet over which the function is convex into a direction in which the function is concave,
3. from a facet over which the function is convex into a direction in which the function is convex.

For this, we use elementary arguments (or slightly different versions of them) that have been already used in [GKH⁺06, GHJ⁺08]. Cases 1 and 2 concern bivariate functions being convex in one variable and concave in the second one. Case 3 concerns 1-convex indefinite functions.

3.1. Lifting from a facet over which the function is concave into a direction in which the function is convex

Let $f : [l, u] \rightarrow \mathbf{R}$ be a bivariate function that is convex in x and concave in y . We assume that the given point (x_0, y_0) is contained in the boundary of the box where $x_0 \in \{l_x, u_x\}$ and $l_y \leq y_0 \leq u_y$. As f is concave in y , the best linear underestimator for $f(x_0, y)$ on $[l_y, u_y]$ is given by the secant $s : \mathbf{R} \rightarrow \mathbf{R}$ on the graph of $f(x_0, y)$ through the points $(l_y, f(x_0, l_y))$ and $(u_y, f(x_0, u_y))$, i.e., s is given by

$$s(y) := \frac{f(x_0, u_y) - f(x_0, l_y)}{u_y - l_y} (y - l_y) + f(x_0, l_y).$$

Next, we extend $s(y)$ to a globally valid underestimator of the form

$$f(x, y) \geq \alpha(x - x_0) + s(y).$$

case (a): $x_0 = l_x$. We will argue that

$$\alpha = \frac{\partial f}{\partial x}(l_x, \bar{y}), \quad \text{where } \bar{y} := \begin{cases} l_y, & \text{if } \frac{\partial f}{\partial x}(l_x, u_y) \geq \frac{\partial f}{\partial x}(l_x, l_y), \\ u_y, & \text{otherwise.} \end{cases}$$

We lift the underestimator into a direction in which the function is convex. As the underestimator and the function coincide at $y \in \{l_y, u_y\}$, it follows that $\alpha \leq \frac{\partial f}{\partial x}(l_x, \bar{y})$. Along the line $y = \bar{y}$, the lifting coefficient $\frac{\partial f}{\partial x}(l_x, \bar{y})$ is the best possible. The resulting linear underestimator is valid for f over $[l, u]$ because (i) it underestimates f along the lines $y = l_y$ and $y = u_y$ and (ii) it underestimates f along each segment from (x, l_y) to (x, u_y) for all $x \in [l_x, u_x]$ as it is linear in y while f is concave in y . Therefore, a *best possible* linear underestimator is given by

$$f(x, y) \geq \frac{\partial f}{\partial x}(l_x, \bar{y}) (x - l_x) + \frac{f(l_x, u_y) - f(l_x, l_y)}{u_y - l_y} (y - l_y) + f(l_x, l_y).$$

case (b): $x_0 = u_x$. Analogously to (a) the best possible lifting coefficient is given by

$$\alpha = \frac{\partial f}{\partial x}(u_x, \bar{y}), \quad \text{where } \bar{y} := \begin{cases} l_y, & \text{if } \frac{\partial f}{\partial x}(u_x, u_y) \leq \frac{\partial f}{\partial x}(u_x, l_y), \\ u_y, & \text{otherwise.} \end{cases}$$

This yields the following linear underestimator

$$f(x, y) \geq \frac{\partial f}{\partial x}(u_x, \bar{y}) (x - u_x) + \frac{f(u_x, u_y) - f(u_x, l_y)}{u_y - l_y} (y - l_y) + f(u_x, l_y).$$

3.2. Lifting from a facet over which the function is convex into a direction in which the function is concave

Let $f : [l, u] \rightarrow \mathbf{R}$ be a bivariate function that is convex in x and concave in y and consider the point (x_0, y_0) , where $l_x < x_0 < u_x$ and $y_0 \in \{l_y, u_y\}$. As f is convex when y is fixed, the best linear underestimator is given by the tangent $t : \mathbf{R} \rightarrow \mathbf{R}$ on the graph of $f(x, y_0)$ at $(x_0, f(x_0, y_0))$ with

$$t(x) := \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + f(x_0, y_0).$$

Next, we extend $t(x)$ to a globally valid underestimator of the form

$$f(x, y) \geq t(x) + \beta(y - y_0).$$

To determine an appropriate β , we use a geometric argument that has been also applied in [GKH⁺06].

case (a): $y_0 = l_y$. For every fixed $x \in [l_x, u_x]$ the segment connecting the points $(x, l_y, t(x))$ and $(x, u_y, f(x, u_y))$ underestimates $f(x, y)$ on $[l_y, u_y]$ as $t(x) \leq f(x, l_y)$ and f is concave for every fixed x . A valid lifting coefficient $\beta \in \mathbf{R}$ is given by the minimal slope $\gamma(x) = \frac{f(x, u_y) - t(x)}{u_y - l_y}$ over $x \in [l_x, u_x]$. Note that $\gamma(x)$ is convex because f is convex. This means that each critical point x satisfying the following first-order condition forms a global minimum of γ :

$$\frac{\partial \gamma}{\partial x}(x) = \frac{1}{u_y - l_y} \left(\frac{\partial f}{\partial x}(x, u_y) - t'(x) \right) \stackrel{!}{=} 0.$$

Therefore, $\beta = \frac{f(\bar{x}, u_y) - t(\bar{x})}{u_y - l_y}$, where \bar{x} is a critical point satisfying the first-order condition, provided such point exists and is contained in the relevant domain $[l_x, u_x]$. If such a point does not exist, set $\bar{x} = l_x$ if $\gamma(l_x) \leq \gamma(u_x)$, and $\bar{x} = u_x$ otherwise.

This leads to the following linear underestimator

$$f(x, y) \geq \frac{\partial f}{\partial x}(x_0, l_y) (x - x_0) + f(x_0, l_y) + \frac{f(\bar{x}, u_y) - t(\bar{x})}{u_y - l_y} (y - l_y).$$

case (b): $y_0 = u_y$. Similar to case (a), for every fixed $x' \in [l_x, u_x]$ the segment connecting the points $(x', l_y, f(x', l_y))$ and $(x', u_y, t(x'))$ underestimates $f(x', y)$ on $[l_y, u_y]$. A valid lifting coefficient $\beta \in \mathbf{R}$ is given by the maximal slope $\gamma(x) = \frac{f(x, l_y) - t(x)}{l_y - u_y}$ over $x \in [l_x, u_x]$. As $l_y - u_y < 0$ and f is convex in x , it follows that γ is concave. Thus, each point satisfying the following first-order condition provides a global maximum for γ :

$$\frac{\partial \gamma}{\partial x}(x) = \frac{1}{l_y - u_y} \left(\frac{\partial f}{\partial x}(x, l_y) - t'(x) \right) \stackrel{!}{=} 0.$$

Let \bar{x} be a critical point satisfying the first-order condition, provided such point exists and is contained in $[l_x, u_x]$. Otherwise we set $\bar{x} = l_x$, if $\gamma(l_x) \geq \gamma(u_x)$, and $\bar{x} = u_x$, else. Then, $\beta = \gamma(\bar{x})$ yields the following linear underestimator

$$f(x, y) \geq \frac{\partial f}{\partial x}(x_0, u_y) (x - x_0) + f(x_0, u_y) + \frac{f(\bar{x}, l_y) - t(\bar{x})}{l_y - u_y} (y - u_y).$$

3.3. Lifting from a facet over which the function is convex into a direction in which the function is convex.

Let $f : [l, u] \rightarrow \mathbf{R}$ be a bivariate function that is strictly convex in both x and y but indefinite. Note that the lifting technique has been already applied in a similar setting in [GH]⁺08] for a class of functions which are convex in each variable but not necessarily indefinite at each point of the box.

Consider a point $(x_0, y_0) \in [l, u]$ which is contained in the boundary of the box, i.e., $x_0 \in \{l_x, u_x\}$ or $y_0 \in \{l_y, u_y\}$. W.l.o.g., we assume $x_0 \in \{l_x, u_x\}$. Otherwise we can interchange the variables x and y and consider the function $f(y, x)$ on $[l_y, u_y] \times [l_x, u_x]$

and the point (y_0, x_0) . As f is convex in y , the best convex underestimator for $f(x_0, y)$ is the function $f(x_0, y)$ itself. A valid lifting coefficient $\alpha \in \mathbf{R}$ is required to satisfy

$$f(x, y) \geq \alpha(x - x_0) + f(x_0, y), \quad \text{for all } (x, y) \in [l, u].$$

For this, we define

$$\mu(x, y) := \frac{f(x, y) - f(x_0, y)}{x - x_0}.$$

The best possible lifting coefficient α is given by the infimum and the supremum of $\mu(x, y)$ over $[l, u]$ for $x_0 = l_x$ and $x_0 = u_x$, respectively.

case (a): $x_0 = l_x$. A valid lifting coefficient α corresponds to the infimum of $\mu(x, y)$. As already mentioned in [GH]⁺08], $\mu(x, y)$ is the differential quotient of f in x , for every fixed $y \in [l_y, u_y]$. By convexity of f in x , it follows that $\mu(x, y) \geq \frac{\partial f}{\partial x}(l_x, y)$ for all $(x, y) \in [l, u]$. We can exploit the assumptions on f to show monotonicity of $\frac{\partial f}{\partial x}(l_x, y)$ in y . Formally the assumptions on f mean

- $\frac{\partial^2 f}{\partial x^2}(x, y) > 0$, $\frac{\partial^2 f}{\partial y^2}(x, y) > 0$ for all (x, y) in the interior of $[l, u]$,
- $\frac{\partial^2 f}{\partial x^2}(x, y) \frac{\partial^2 f}{\partial y^2}(x, y) - [\frac{\partial^2 f}{\partial x \partial y}(x, y)]^2 < 0$ for all (x, y) in the interior of $[l, u]$.

Thus, $[\frac{\partial^2 f}{\partial x \partial y}(x, y)]^2 > \frac{\partial^2 f}{\partial x^2}(x, y) \frac{\partial^2 f}{\partial y^2}(x, y) > 0$ for all (x, y) in the interior of $[l, u]$. As we assume f to be twice continuously differentiable, it follows that $\frac{\partial^2 f}{\partial x \partial y}(x, y)$ is either non-positive or non-negative over $[l, u]$ which implies monotonicity of $\frac{\partial f}{\partial x}(l_x, y)$ in y . Thus, $\mu(x, y) \geq \frac{\partial f}{\partial x}(l_x, y) \geq \frac{\partial f}{\partial x}(l_x, y^*)$ for all $(x, y) \in [l, u]$, where

$$y^* := \begin{cases} l_y, & \text{if } \frac{\partial f}{\partial x}(l_x, l_y) \leq \frac{\partial f}{\partial x}(l_x, u_y), \\ u_y, & \text{otherwise.} \end{cases}$$

A linear underestimator for $f(x, y)$ w.r.t. the given point (x_0, y_0) can be obtained as follows:

$$\begin{aligned} f(x, y) &\geq \frac{\partial f}{\partial x}(l_x, y^*)(x - l_x) + f(l_x, y) \\ &\geq \frac{\partial f}{\partial x}(l_x, y^*)(x - l_x) + \frac{\partial f}{\partial y}(l_x, y_0)(y - y_0) + f(l_x, y_0). \end{aligned}$$

case (b): $x_0 = u_x$. With

$$y^* := \begin{cases} l_y, & \text{if } \frac{\partial f}{\partial x}(u_x, l_y) \geq \frac{\partial f}{\partial x}(u_x, u_y), \\ u_y, & \text{otherwise,} \end{cases}$$

we obtain the following linear underestimator

$$f(x, y) \geq \frac{\partial f}{\partial x}(u_x, y^*)(x - u_x) + \frac{\partial f}{\partial y}(u_x, y_0)(y - y_0) + f(u_x, y_0).$$

We remark that the assumptions on f are crucial to use the derived lifting coefficient. The next example deals with a convex function and shows that the derived lifting coefficients are not valid for this class of functions.

Example 3.1. Consider the bivariate function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ given by $f(x, y) := x^2 + x(y^2 - 2y)$ restricted to the box $[1, 2] \times [0, 2]$ over which f is convex. We fix x to its lower bound 1 and consider the point $(x_0, y_0) = (1, 0.5)$. Assume we want to lift the tangent $t(y) = -y + 0.75$ on the graph of $f(1, y)$ at $y_0 = 0.5$ along the line $x = 1$ to a valid underestimator of the form $f(x, y) = t(y) + \alpha(x - 1)$. Note that the first partial derivative $\frac{\partial f}{\partial x}(1, y) = y^2 - 2y + 2$ attains its global minimum at $y = 1 \in [0, 2]$. Therefore, we can choose α to be $\frac{\partial f}{\partial x}(1, 1) = 1$ yielding $f(x, y) \geq x - y - 0.25$. We remark that for $y \in \{l_y, u_y\}$, $\frac{\partial f}{\partial x}(1, y) = 2$. Setting $\alpha = 2$ gives the following relation $f(x, y) \geq 2x - y - 1.25$ that does not hold for the point $(1.5, 0.5)$, for instance

$$f(1.5, 0.5) = 1.125 \quad \not\geq \quad 1.25 = 2 \cdot 1.5 - 0.5 - 1.25.$$

The following two examples show in detail that $\frac{\partial^2 f}{\partial x \partial y}$ is monotone if f is a bivariate quadratic or monomial function.

Example 3.2. For an arbitrary bivariate quadratic function given by $f(x, y) := a_{x,x}x^2 + a_{x,y}xy + a_{y,y}y^2 + b_x x + b_y y + c$, the second partial derivative $\frac{\partial^2 f}{\partial x \partial y}(l_x, y) = a_{x,y}$ is a constant, that allows us to consider the endpoints of the interval only. Note that if the constant $a_{x,y} = 0$, our bivariate quadratic function f is separable.

Example 3.3. For any monomial function of the form $f(x, y) = x^p y^q$, for some $p, q \neq 0$, restricted to a box $[l, u] \subseteq \mathbf{R}_{\geq}^2$, we have that

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = pqx^{p-1}y^{q-1}$$

is either non-negative for all $(x, y) \in \mathbf{R}_{\geq 0}^2$ or non-positive for all $(x, y) \in \mathbf{R}_{\geq 0}^2$. Thus, critical points can only occur, if one of the lower bounds are equal to zero and one variable is fixed to this lower bound for the lifting step. In such a case, the function f is the zero function over the corresponding facet of the box. It follows that it is again sufficient to consider the endpoints of the interval of the second variable, only.

4. Computations

This section presents a computational study of the linear underestimator technique as discussed in the previous sections.

4.1. Implementation

The underestimator techniques have been implemented as a new constraint handler in the constraint integer programming framework SCIP [Ach07, Ach09, Vig13]. The

new constraint handler `cons_bivariate` has been released the first time with SCIP 2.1.0 in October 2011 as a beta version.

SCIP solves MINLPs by a branch-and-bound algorithm. The problem is recursively split into smaller subproblems, thereby creating a search tree and implicitly enumerating all potential solutions. At each subproblem, domain propagation is performed to exclude further values from the variables' domains, and a linear relaxation is solved to achieve a local lower bound – assuming the problem is to minimize the objective function. The relaxation is strengthened by adding further linear inequalities, which cut off the optimal solution of the relaxation. Primal heuristics are used as supplementary methods to improve the upper bound.

A constraint handler in SCIP defines the semantics and the algorithms to process constraints of a certain class. Each constraint handler has to implement an enforcement method. In enforcement, the handler has to decide whether the optimal solution of the linear relaxation satisfies all of its constraints. If the solution violates one or more constraints, the handler may resolve the infeasibility by adding linear inequalities, performing a domain reduction, or a branching.

Our constraint handler handles bivariate constraints of the form $\ell \leq f(x, y) + cz \leq r$, where $f : [l_x, u_x] \times [l_y, u_y] \rightarrow \mathbf{R}$ is a bivariate function with fixed convexity behavior, $c \in \mathbf{R}$, $\ell \in \mathbf{R} \cup \{-\infty\}$, and $u \in \mathbf{R} \cup \{\infty\}$. The function $f(x, y)$ has to be passed to the constraint handler in form of an expression trees. Additionally, the convexity behavior of the function (recall Table 1) has to be specified. For enforcement and during separation rounds, the constraint handler generates a linear inequality from a linear under- or overestimators of $f(x, y)$ (as described in the previous sections). The univariate convex optimization problems are solved by Newton's method. If the generated inequality does not cut off the optimal solution of the linear relaxation, spatial branching is applied on either x or y . E.g., if $f(x, y)$ is convex in x and concave in y and the current relaxation optimum $(\hat{x}, \hat{y}, \hat{z})$ violates the inequality $f(\hat{x}, \hat{y}) + c\hat{z} \leq r$, then variable y is proposed as branching candidate to SCIP. From all branching candidates that are registered by all constraint handlers, SCIP selects a branching variable and branching point according to a pseudo-costs based variable selection rule, see [BLL⁺09, Vig13] for details. Further, a feasibility-based bound tightening (FBBT) rule is applied to deduce tighter variable bounds for x , y , or z from the constraint and the bounds on these variables, see [Vig13] for details.

During presolve, SCIP reformulates a MINLP into a form where it can construct a linear relaxation. The reformulation mainly consists of introducing new auxiliary variables and nonlinear constraints for subexpressions of nonlinear functions. For example, a general monomial function $x^p y^q$ has so far been reformulated by SCIP into a product $w_1 w_2$ and two new constraints $w_1 = x^p$ and $w_2 = y^q$, because SCIP knows how to compute linear under- and overestimators for these functions. With the new constraint handler for bivariate functions, there is no more need for reformulating monomials $x^p y^q$ with $l_x \geq 0$ and $l_y \geq 0$. Additionally, bivariate quadratic functions $a_x x + a_y y + a_{xy} xy + a_{xx} x^2 + a_{yy} y^2$ are automatically recognized as bivariate terms and handled by the new constraint handler.

4.1.1. Non-Differentiability on Boundary

The separation algorithms of the previous sections make use of assumption (A2), that is, twice continuous differentiability of $f(x, y)$ over the complete domain $[l, u]$. However, the instances of our test set (Section 4.2), can contain monomials of the form $x^p y^q$, $x, y \geq 0$, where p and q may be smaller than one. In such cases, functions $f(x, y)$ would not be differentiable on the boundary ($x = 0$ or $y = 0$).

Our implementation extends to cases where the bivariate function $f(x, y)$ is not differentiable at the boundary by recognizing that such a situation occurs and then abandons the cut generation at first. Next, if the non-differentiability was recognized for the reference point that we attempt to cut off, we move the reference point away from the bounds by 0.1% (relative to the domain width) and try again, even though the cut generated this way may not be the tightest possible. If no cut could be generated, infeasibility of the constraint has to be enforced by other means, that is, branching and bound tightening.

Additionally, in the case of Section 2.3, e.g., $f(x, y)$ being convex in x and concave in y , when both s^* and r^* are on the bounds, two choices for (\bar{x}, \bar{y}) are available. Here, if $f(x, y)$ is not differentiable for the first choice (\bar{x}^1, \bar{y}^1) , the implementation will try (\bar{x}^2, \bar{y}^2) instead.

4.2. Test set

Initially we considered the problem libraries GLOBALLIB [GLO] and MINLPLIB [BDM03]. However, they contain only a few instances with bivariate quadratic terms or monomials, mainly of the form x/y . To investigate the computational benefit of having a convex underestimator for bivariate functions at hand, we created a set of nonlinear optimization problems where bivariate functions occur in form of quadratic functions and monomials, e.g., $3x_1^2 + x_1x_2 - x_2^2 + 2x_1^{0.3}x_2^{1.5} - 4x_2^{1.2}x_3^{2.5}$. The created instances can be found at <http://opus4.kobv.de/opus4-zib/frontdoor/index/index/docId/1764>.

The random generation of problems with constraints can lead to infeasibility. As the proposed constraint handler aims at strong lower bounds on the problem, feasible problems are required in order to compare the quality of the bounds. This is achieved by the following problem class where we vary the number of variables $Nvars$ and constraints $Ncons$, and the maximum degree Deg over all constraints.

$$\begin{aligned} & \min \epsilon \\ \text{s. t. } & \sum_{i=1}^{Deg-1} \sum_{j=1}^{Deg-i} \sum_{k=1}^{Nvars} \sum_{l=k+1}^{Nvars} a_{c,i,j,k,l} x_k^{p_{c,i,j,k}} x_l^{q_{c,i,j,l}} + \sum_{k=1}^{Nvars} b_{c,k} x_k^2 \leq \epsilon, \quad \text{for all } c \in \{1, \dots, Ncons\}, \\ & x \in [l, u], \end{aligned}$$

where

- $p_{c,i,j,k}$: If $i = j = 1$ then $p_{c,i,j,k} = 1$. Otherwise, $p_{c,i,j,k}$ is uniformly random set to a value in $\{(i-1) + 0.2, (i-1) + 0.4, \dots, (i-1) + 1\}$.

- $q_{c,i,j,l}$: If $i = j = 1$ then $q_{c,i,j,l} = 1$. Otherwise, $q_{c,i,j,l}$ is uniformly random set to a value in $\{(j-1) + 0.2, (j-1) + 0.4, \dots, (j-1) + 1\}$.
- $a_{c,i,j,k,l}$: If $i = j = 1, k$ odd, and $l = k + 1$, then $a_{c,i,j,k,l}$ is uniformly random in $\{-4, -3, \dots, 3, 4\}$. If $i > 1$ or $j > 1$, then $a_{c,i,j,k,l}$ is with probability $2/Nvars$ in $\{-4, -3, \dots, 3, 4\}$. Otherwise, we set $a_{c,i,j,k,l} = 0$.
- $b_{c,k}$: The coefficient $b_{c,k}$ is chosen uniformly at random from $\{-4, -3, \dots, 3, 4\}$.
- $[l, u]$: For each $k \in \{1, \dots, Nvars\}$ the lower bound l_k is uniformly at random set to a value in $\{0, 1, 2, 3, 4\}$. The upper bound u_k is the sum of $l_k + 1$ and a value which is chosen uniformly at random from $\{0, 1, 2, 3, 4\}$. To avoid numerical inconsistencies we check that $u_k^{\text{Deg}} \leq 2000$.

The condition $i = j = 1$ deals with the quadratic case. It ensures that integer exponents leading to quadratic terms $x_i x_j$ are generated. The condition $(l = k + 1), l$ odd, leads to bivariate quadratic terms $x_1 x_2, x_3 x_4, \dots$ and thereby the univariate quadratic terms x_k^2 can be associated to a unique bivariate quadratic monomial.

The implemented methods are of particular interest if the optimal or intermediate solutions are attained in the interior of the underlying boxes $[l, u]$. Otherwise, only the lifting methods are executed. Thus, an ellipsoid constraint is optionally added to the problems which cuts off the boundary of the box. It reads

$$\sum_{k=1}^{Nvars} \left(\frac{x_k - \text{midpoint}_k}{\text{intervalllength}_k/2} \right)^2 = \sum_{k=1}^{Nvars} \left(\frac{x_k - (u_k + l_k)/2}{(u_k - l_k)/2} \right)^2 \leq 1 \quad (16)$$

The following settings are considered.

- $Nvars \in \{10, 20, 30\}$.
- $Deg \in \{2, 3, 4, 5\}$.
- $Ncons \in \{1, 2, 3, 5, 10\}$.
- Enable/Disable the ellipsoid constraint (16).

Hence, there are $3 \cdot 5 \cdot 4 \cdot 2 = 120$ different settings. For each setting we generate 10 random instances which leads to 1200 instances in total.

4.3. Experimental Setup

We used GAMS 24.4.1 under Linux 3.13 (64bit) to compare SCIP 3.1 (with the new constraint handler enabled or disabled) with BARON 14.4.0 [TS05]¹. Both solvers use CPLEX 12.6.1.0 for solving LP relaxations. SCIP uses Ipopt for finding local optimal solutions to an NLP, BARON automatically chooses between CONOPT, IPOPT, MINOS, and SNOPT. We used a timelimit of 30 minutes, a feasibility tolerance of 10^{-6} , and an optimality gap tolerance of 0.01%.

¹Originally, we also compared with COUENNE. However, a check for feasibility by GAMS/Examiner of the solution points provided by the solver failed in \approx half of the instances, so we decided to exclude COUENNE.

	SCIP	SCIP(bivar)	BARON
#solved	322	478	724
#fastest	157	364	587
#best dual bound	480	655	1184
time (sh. geom. mean)	748.7	549.7	564.1
nodes (sh. geom. mean)	3650.0	3206.9	11594.5
dual gap (arith. mean)	41%	31%	17%

Table 2: Computational results for 1194 randomly generated polynomial instances.

4.4. Results

Table 2 summarizes the results for SCIP, SCIP(bivar), which stands for SCIP with the new constraint handler enabled, and BARON when run on all 1,200 instances, excluding those 6 instances for which at least one of the 3 algorithms aborted due to numerical difficulties or the solvers reported inconsistent bounds on the optimal value. The detailed results of the single instances can be found in Section A. All experiments for this test set were run on a Dell PowerEdge M620 blade with 64 GB RAM and Intel Xeon E5-2670 v2 CPUs running at 2.50GHz.

First, we report the number of instances which are solved and solved fastest by an algorithm, and for which an algorithm computes the best dual bound. An algorithm is called fastest if it is within one second or within 1% of the minimal solution time for an instance. A dual bound for a solver is called best dual bound, if the bound is within 1% of the best dual bound for that instance. Second, for each solver we calculated mean values of the solution time (in which unsolved instances are accounted for with the time limit), the number of processed nodes, and the dual gap at termination. The mean values are computed according to [Ach07, Section A.3], where the shifted geometric mean, defined as

$$\left(\prod_{i \in [n]} \max(\varepsilon, v_i + s) \right)^{1/n} - s,$$

is calculated with $\varepsilon = 1$ and $s = 10$ for solution times and with $\varepsilon = 1$ and $s = 100$ for node counts. The *dual gap* for a problem with dual bound \underline{v} and best known objective value v^* is defined as²

$$\text{dual gap} := \begin{cases} 0, & \text{if } \underline{v} = v^*, \\ 2, & \text{if } \underline{v} * v^* < 0 \text{ or } |\underline{v}| = \infty \text{ or } |v^*| = \infty, \\ \frac{|v^* - \underline{v}|}{\max(|v^*|, |\underline{v}|)}, & \text{otherwise.} \end{cases} \quad (17)$$

²We use a value of 2 for cases where a bound is not available (i.e., $\pm\infty$) or bounds have different sign to allow for the computation of meaningful arithmetic means.

	SCIP(bivar)	BARON
#fastest	371	348
time (sh. geom. mean)	94.2	120.3
nodes (sh. geom. mean)	10065.5	10342.2

Table 3: Summary of 476 instances solved by SCIP(bivar) as well as BARON.

The results in Table 2 allow an overall ranking of the three algorithms as the ranking for most of the individual performance parameters is the same. BARON clearly outperforms SCIP and SCIP(bivar), BARON solves 246 instances more than SCIP(bivar), which also results in much better average dual bounds and dual gaps. However SCIP(bivar) is slightly faster w.r.t. mean solution time than BARON (unsolved instances were accounted with the timelimit of 1800s for this measure). Further, SCIP and SCIP(bivar) enumerated much less nodes than BARON.

We further restrict our attention to those 476 instances which are solved by both BARON and SCIP(bivar), see Table 3. For these instances, SCIP(bivar) is the fastest algorithm for 5% more instances than BARON. The mean of the computation times shows that BARON is 28% slower than SCIP(bivar). The small solving time of SCIP(bivar) is likely related to the low number of processed branch-and-bound nodes, which indicates the strength of the relaxations used in SCIP(bivar). This claim is supported by the direct comparison of SCIP and SCIP(bivar) restricted to those 318 instances which are solved by both algorithms. SCIP needs a mean of 77 seconds and 13677 nodes while SCIP(bivar) uses only 34 seconds and 2952 nodes. Thus, SCIP(bivar) can utilize the improved relaxations to avoid branching steps and prune nodes earlier, thus accelerating the solving process.

In Figure 3, we refine the analysis of the dual gaps for the 1,200 instances w.r.t. the number of variables N_{Vars} , the number of constraints N_{Cons} , the maximal degree Deg of the polynomials, and the presence of the ellipsoid constraint in Equation (16). For 10 variables per instance, we observe that SCIP(bivar) and BARON have about the same dual gap, which is close to zero. For instances with more variables, the dual gap of BARON increase to more than 30%, while the dual gaps of SCIP and SCIP(bivar) increase to $\approx 60\%$. A similar picture is obtained by an increase of the number of constraints or the maximal degree of the monomials, even though the increase is more modest. Here, SCIP and SCIP(bivar) yield similar mean dual gaps for instances with only one constraint or instances with only quadratic constraints³, while SCIP(bivar) clearly outperforms SCIP for instances with several or non-quadratic constraints. Finally, if the ellipsoid constraint is enabled, the dual gaps of all solvers increase by factors between 5 and 7. The activation of the ellipsoid constraint forces the optimal solution to be attained in the interior of the given domains, which obviously causes some problems for the algorithms due to weaker relaxations.

³Recall, that for degree two, we construct only uni- and bivariate quadratic monomials, but no monomials with fractional exponents like $x^{0.2}y^{1.4}$, which is allowed for larger degrees, see also Section 4.2.

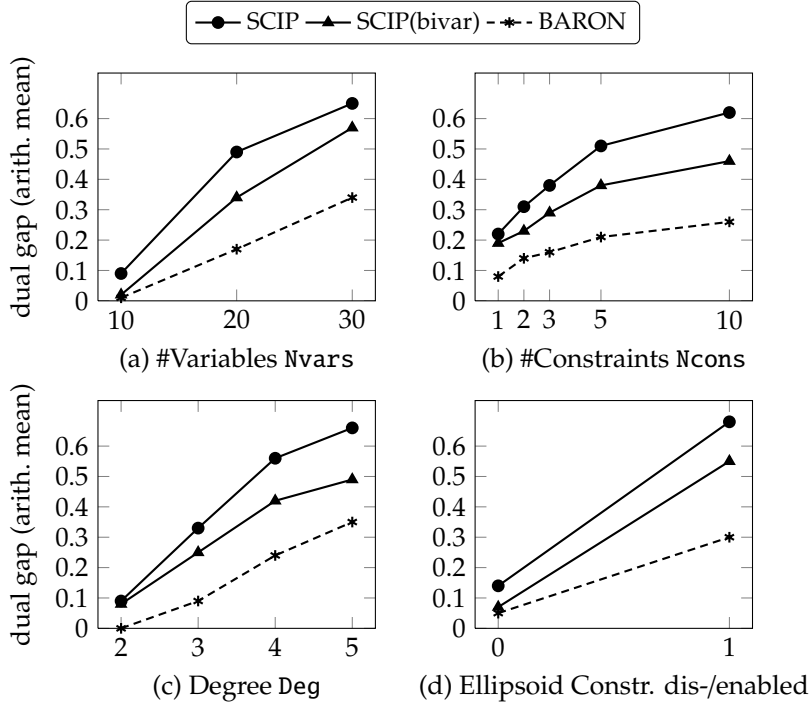


Figure 3: Dual gaps of the solvers w.r.t. to the number of variables N_{vars} , constraints N_{cons} , the maximal degree Deg , and the ellipsoid constraint dis- and enabled.

Integral Exponents

In the tests above, we constructed monomials with fractional exponents like $x^{0.4}y^{1.8}$. A last comparison is devoted to instances with integral exponents only, i.e., we round up the exponents such that we obtain monomials like x^1y^2 . Table 4 compares the computational results of SCIP and SCIP(bivar) applied to a test set of 1,200 instances with integral exponents. All experiments for this test set were run on a Dell PowerEdge M610 blade with 48 GB RAM and Intel Xeon X5672 CPUs running at 3.20GHz.

In contrast to the instances with fractional exponents, see Table 2, SCIP can here solve slightly more instances than SCIP(bivar).

To understand the good performance of SCIP on this test set, consider the monomials $x^{0.4}y^{1.8}$ and $x^{2.6}y^{1.6}$. SCIP introduces new variables $v_{0.4}$, $v_{2.6}$, and $w_{2.6}$, $w_{1.6}$ for the univariate convex or concave monomials $x^{0.4}$, $x^{2.6}$, and $y^{1.8}$, $y^{1.6}$, respectively. Afterwards, it relaxes the univariate monomials and the bilinear product terms $v_{0.4}w_{1.8}$ and $v_{2.6}w_{1.6}$ by their convex and concave envelopes. If only integral exponents are allowed, the monomials change to x^1y^2 and x^3y^2 . Thus, less variables are introduced, namely v_1 , v_3 , and w_2 , and the bilinear terms v_1w_2 and v_3w_2 have a common variable which is helpful in the process of relaxation. Table 5 depicts the average quotient of the num-

	SCIP	SCIP(bivar)
#solved	623	603
#fastest	426	476
#best dual bound	1059	819
time (sh. geom. mean)	415.6	334.5
nodes (sh. geom. mean)	3731.4	1362.9
dual gap (arith. mean)	33%	37%

Table 4: Computational results for 1,200 randomly generated polynomial instances with integral exponents.

Maximal degree		2	3	4	5
Nvars	Fractional exponents	2.76	0.80	0.76	0.81
	Integral exponents	2.76	4.26	2.71	3.93
Ncons	Fractional exponents	9.89	0.78	0.74	0.80
	Integral exponents	9.89	5.77	2.94	4.25

Table 5: Average growth of number of variables and constraints in reformulated problem from SCIP to SCIP(bivar) (only instances without ellipsoid constraint are considered).

ber of variables or constraints in the reformulated problems between SCIP(bivar) and SCIP. It is seen that the number of constraints that have to be processed by SCIP(bivar) is much larger than those that are processed by SCIP, while for the instances with fractional exponents and degree > 2 , SCIP(bivar) has to process even less constraints than SCIP.

A possible explanation for SCIP(bivar)'s bad performance is the absence of monomials like $x^{0.4}y^{1.8}$, which are strictly concave in one variable and convex in the other one. This conjecture is further affirmed by Table 6, which displays the dual gaps for SCIP and SCIP(bivar) with respect to the maximal degree of the programs with fractional and integral exponents. The numbers show a higher dual gap of SCIP(bivar) for integral exponents compared to fractional exponents, while the dual gap of SCIP is smaller for integral exponents compared to fractional exponents.

5. Conclusions

The new constraint handler used in SCIP(bivar) can reduce the solution time and improve the dual bounds of programs containing bivariate functions with a fixed convexity behavior. While the implemented methods of the constraint handler are applicable for general bivariate functions with a fixed convexity behavior, an auto-

		Maximal degree			
		2	3	4	5
SCIP	Fractional exponents	9%	33%	56%	66%
	Integral exponents	9%	29%	44%	50%
SCIP(bivar)	Fractional exponents	8%	25%	42%	49%
	Integral exponents	9%	35%	47%	56%

Table 6: Dual gaps of SCIP and SCIP(bivar) for instances with fractional exponents and instances with integral exponents (arithmetic mean).

matic detection of the convexity behavior has been only implemented for the class of bivariate quadratic functions and for the class of monomial functions restricted to the nonnegative orthant (cf. Table 1). Using the callable library of SCIP, a user can manually provide the convexity type of a function to the constraint handler so that the advantages of this new tool can already be exploited for further functions.

Finally, we remark that for many specific bivariate functions of practical relevance, explicit formulas for the convex envelope are available (e.g., see [TS01, JMW08, KS13, KS12]). An incorporation of such explicit formulas could be helpful for our constraint handler to avoid the numerical solution of the nonlinear subproblems involved in the cut generation.

References

- [Ach07] Tobias Achterberg, *Constraint integer programming*, Ph.D. thesis, TU Berlin, 2007, urn:nbn:de:0297-zib-11129.
- [Ach09] ———, *SCIP: Solving Constraint Integer Programs*, *Mathematical Programming Computation* **1** (2009), no. 1, 1–41, doi:10.1007/s12532-008-0001-1.
- [BDM03] Michael R. Bussieck, Arne Stolbjerg Drud, and Alexander Meeraus, *MINLPLib - a collection of test models for mixed-integer nonlinear programming*, *INFORMS Journal on Computing* **15** (2003), no. 1, 114–119, doi:10.1287/ijoc.15.1.114.15159 .
- [BLL⁺09] Pietro Belotti, Jon Lee, Leo Liberti, François Margot, and Andreas Wächter, *Branching and bounds tightening techniques for non-convex MINLP*, *Optimization Methods and Software* **24** (2009), no. 4-5, 597–634, doi:10.1080/10556780903087124.
- [BMSMW10] Martin Ballerstein, Dennis Michaels, Andreas Seidel-Morgenstern, and Robert Weismantel, *A theoretical study of continuous counter-current chromatography for adsorption isotherms with inflection points*,

- Computers & Chemical Engineering **34** (2010), no. 4, 447 – 459, doi:10.1016/j.compchemeng.2009.10.001.
- [BST09] Xiaowei Bao, Nikolaos V. Sahinidis, and Mohit Tawarmalani, *Multi-term polyhedral relaxations for nonconvex, quadratically constrained quadratic programs*, Optimization Methods Software **24** (2009), no. 4-5, 485–504, doi:10.1080/1055678090288318.
- [GHJ⁺08] Jignesh Gangadwala, Utz-Uwe Haus, Matthias Jach, Achim Kienle, Dennis Michaels, and Robert Weismantel, *Global analysis of combined reaction distillation processes*, Computers & Chemical Engineering **32** (2008), no. 1-2, 343–355, doi:10.1016/j.compchemeng.2007.04.015.
- [GKH⁺06] Jignesh Gangadwala, Achim Kienle, Utz-Uwe Haus, Dennis Michaels, and Robert Weismantel, *Global Bounds on Optimal Solutions for the Production of 2,3-Dimethylbutene-1*, Industrial & Engineering Chemistry Research **45** (2006), no. 7, 2261–2271, doi:10.1021/ie050584j.
- [GLO] GLOBAL Library, <http://www.gamsworld.org/global/globallib.htm>.
- [JMW08] Matthias Jach, Dennis Michaels, and Robert Weismantel, *The convex envelope of $(n-1)$ -convex functions*, SIAM Journal on Optimization **19** (2008), no. 3, 1451–1466, doi:10.1137/07069359X.
- [KS12] Aida Khajavirad and Nikolaos V. Sahinidis, *Convex envelopes of products of convex and component-wise concave functions*, Journal of Global Optimization **52** (2012), no. 3, 391–409, doi:10.1007/s10898-011-9747-5.
- [KS13] ———, *Convex envelopes generated from finitely many compact convex sets*, Mathematical Programming **137** (2013), no. 1-2, 371–408, doi:10.1007/s10107-011-0496-5.
- [Loc10] Marco Locatelli, *Convex envelopes for quadratic and polynomial functions over polytopes*, Manuscript, 11/03/2010, available at http://www.optimization-online.org/DB_FILE/2010/11/2788.pdf, 2010.
- [LS14] Marco Locatelli and Fabio Schoen, *On convex envelopes for bivariate functions over polytopes*, Mathematical Programming **144** (2014), no. 1-2, 65–91, doi:10.1007/s10107-012-0616-x.
- [McC76] Garth P. McCormick, *Computability of global solutions to factorable non-convex programs. I: Convex underestimating problems*, Mathematical Programming **10** (1976), no. 1, 147–175, doi:10.1007/BF01580665.
- [Pad75] Manfred W. Padberg, *A note on 0/1 programming*, Operational Research **23** (1975), no. 4, 833–837, doi:10.1287/opre.23.4.833.

- [Roc70] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics. Princeton, NJ: Princeton University Press, 1970.
- [RT10] Jean-Philippe P. Richard and Mohit Tawarmalani, *Lifting inequalities: a framework for generating strong cuts for nonlinear programs*, *Mathematical Programming* **121** (2010), no. 1, 61–104, doi:10.1007/s10107-008-0226-9.
- [She97] Hanif D. Sherali, *Convex envelopes of multilinear functions over a unit hypercube and over special sets*, *Acta Mathematica Vietnamica* **22** (1997), no. 1, 245–270.
- [Tar04] Fabio Tardella, *On the existence of polyhedral convex envelopes*, *Frontiers in Global Optimization* (C. A. Floudas and P. M. Pardalos, eds.), *Non-convex Optimization and Its Applications*, vol. 74, Kluwer Academic Publisher, 2004, doi:10.1007/978-1-4613-0251-3_30, pp. 563–573.
- [TRX13] Mohit Tawarmalani, Jean-Philippe Richard, and Chuanhui Xiong, *Explicit convex and concave envelopes through polyhedral subdivisions*, *Mathematical Programming* **138** (2013), no. 1-2, 531–577, doi:10.1007/s10107-012-0581-4.
- [TS01] Mohit Tawarmalani and Nikolaos V. Sahinidis, *Semidefinite relaxations of fractional programs via novel convexification techniques*, *Journal of Global Optimization* **20** (2001), no. 2, 133–154, doi:10.1023/A:1011233805045.
- [TS05] ———, *A polyhedral branch-and-cut approach to global optimization*, *Mathematical Programming* **103** (2005), no. 2, 225–249, doi:10.1007/s10107-005-0581-8.
- [Vig13] Stefan Vigerske, *Decomposition of multistage stochastic programs and a constraint integer programming approach to mixed-integer nonlinear programming*, Ph.D. thesis, Humboldt-Universität zu Berlin, 2013, urn:nbn:de:kobv:11-100208240.

A. Detailed computational results

In this section, the results of the solvers on the single instances are reported. We record the number of seconds for solving the problem, or, if a time limit was hit, the lower and upper bound on the optimal value at termination. Additionally, we collect the number of nodes that have been processed.

In the detailed result tables, each entry shows the number of seconds a certain solver needs to solve a problem and the number of branch-and-bound nodes it has processed. If the problem has not been solved within the given time limit and the solver did not abort, then we report either the *dual gap* and *primal gap* in parentheses, or the dual bound and primal bound in brackets. The gaps are reported if the optimal value of the instance is known, otherwise the bounds are given. The *primal gap* is defined analogously to the dual gap, see (17).

A.1. Results for instances with fractional exponents

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	2	1		1	0.06	1	0.04	1	1800.00	20309.66k
10	2	1		2	0.05	1	0.03	1	0.75	1.85k
10	2	1		3	0.07	1	0.04	1	0.04	1
10	2	1		4	0.05	1	0.01	1	0.03	1
10	2	1		5	0.03	1	0.00	1	0.03	1
10	2	1		6	0.08	1	0.05	1	0.05	1
10	2	1		7	0.08	1	0.07	1	0.04	1
10	2	1		8	0.03	1	0.01	1	0.03	1
10	2	1		9	0.05	1	0.01	1	0.03	1
10	2	1		10	0.05	1	0.01	1	0.03	1
10	2	1	✓	1	56.17	14.85k	1804.65	5558.99k	7.17	2.79k
10	2	1	✓	2	49.15	19.60k	1804.05	5191.15k	7.74	1.95k
10	2	1	✓	3	12.10	3.68k	12.17	15.66k	3.16	708
10	2	1	✓	4	82.95	203.71k	1805.33	5797.92k	8.83	3.23k
10	2	1	✓	5	40.59	9.65k	5.91	770	12.62	5.47k
10	2	1	✓	6	54.47	10.24k	24.71	2.68k	1800.00	14556.63k
10	2	1	✓	7	64.64	47.18k	1805.71	3544.38k	11.63	4.40k
10	2	1	✓	8	8.95	2.07k	1804.12	5926.19k	3.52	932
10	2	1	✓	9	87.44	21.03k	1803.48	4399.53k	12.12	4.44k
10	2	1	✓	10	41.32	9.58k	3.63	447	6.19	1.74k
10	2	2		1	0.10	1	0.00	1	0.04	1
10	2	2		2	0.08	1	0.09	8	0.06	1
10	2	2		3	0.40	77	0.37	41	0.07	1
10	2	2		4	0.65	470	0.73	285	0.17	11
10	2	2		5	0.05	1	0.10	1	0.08	5
10	2	2		6	0.08	1	0.03	1	30.24	283.63k
10	2	2		7	0.04	1	0.02	1	0.03	1
10	2	2		8	0.17	4	0.28	5	0.07	7
10	2	2		9	0.13	7	0.68	1.42k	0.08	3
10	2	2		10	0.26	13	0.11	5	0.07	13
10	2	2	✓	1	1800.10	11273.42k	1801.30	5053.08k	7.91	3.34k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	2	2	✓	2	24.25	4.63k	32.22	4.69k	11.99	5.42k
10	2	2	✓	3	1800.08	9959.47k	31.65	5.35k	4.07	755
10	2	2	✓	4	95.00	19.93k	93.72	15.12k	13.60	4.86k
10	2	2	✓	5	1803.26	6390.47k	1803.40	4203.45k	8.02	2.22k
10	2	2	✓	6	1803.19	10441.95k	37.59	21.15k	15.03	7.36k
10	2	2	✓	7	80.07	16.66k	17.42	2.23k	15.76	5.57k
10	2	2	✓	8	1800.03	7952.18k	3.23	503	5.39	1.76k
10	2	2	✓	9	28.42	10.48k	9.85	2.35k	2.35	266
10	2	2	✓	10	32.16	84.36k	12.77	2.78k	5.26	1.75k
10	2	3		1	0.31	64	0.47	89	0.05	1
10	2	3		2	0.37	49	0.47	31	0.11	5
10	2	3		3	0.06	1	0.12	1	0.06	1
10	2	3		4	1800.15	19571.09k	0.99	369	0.12	9
10	2	3		5	0.33	63	0.43	41	1800.00	18509.22k
10	2	3		6	0.38	57	0.16	3	0.05	1
10	2	3		7	0.58	157	0.59	85	0.07	1
10	2	3		8	0.19	25	0.41	37	0.08	3
10	2	3		9	0.07	1	0.05	1	0.04	1
10	2	3		10	1817.81	37111.21k	0.51	81	0.08	3
10	2	3	✓	1	1800.06	8925.75k	20.21	3.48k	5.23	1.30k
10	2	3	✓	2	37.44	12.07k	1802.07	2421.28k	3.42	582
10	2	3	✓	3	15.92	2.57k	34.01	5.13k	13.58	5.75k
10	2	3	✓	4	82.13	26.37k	41.37	9.69k	9.25	2.92k
10	2	3	✓	5	75.73	261.17k	29.52	4.49k	2.66	423
10	2	3	✓	6	30.16	10.21k	6.59	5.04k	3.42	718
10	2	3	✓	7	112.64	27.29k	96.11	14.48k	6.11	1.31k
10	2	3	✓	8	115.00	32.20k	51.82	7.00k	14.30	5.67k
10	2	3	✓	9	1813.18	12876.40k	1804.20	5824.75k	14.15	5.10k
10	2	3	✓	10	49.90	12.56k	12.07	1.83k	11.80	3.36k
10	2	5		1	0.31	29	0.47	57	0.07	1
10	2	5		2	0.35	37	1800.01	10754.36k	0.09	1
10	2	5		3	0.35	41	0.18	1	0.11	9
10	2	5		4	0.45	91	0.55	80	0.13	7
10	2	5		5	0.17	9	1.74	9.20k	0.04	1
10	2	5		6	1800.06	15685.92k	3.99	11.86k	1800.00	10768.41k
10	2	5		7	0.41	47	0.53	53	0.04	1
10	2	5		8	0.22	19	1.07	3.96k	0.04	1
10	2	5		9	0.37	53	0.52	75	0.07	3
10	2	5		10	0.56	209	1.00	220	0.12	5
10	2	5	✓	1	41.19	13.38k	53.00	9.37k	3.62	551
10	2	5	✓	2	25.41	4.79k	49.47	5.40k	17.10	6.50k
10	2	5	✓	3	46.58	17.49k	28.67	5.22k	2.72	302
10	2	5	✓	4	52.26	13.21k	35.73	5.15k	9.01	2.24k
10	2	5	✓	5	13.97	2.75k	9.93	1.65k	1800.00	8098.22k
10	2	5	✓	6	151.78	65.19k	56.94	8.77k	5.90	1.28k
10	2	5	✓	7	33.82	8.72k	27.01	7.01k	1800.00	6408.76k
10	2	5	✓	8	12.55	3.30k	14.92	2.67k	6.16	1.60k
10	2	5	✓	9	44.83	17.55k	3.78	453	2.17	130
10	2	5	✓	10	59.59	15.15k	109.42	13.85k	4.90	970
10	2	10		1	1800.12	15424.62k	1801.12	4850.38k	0.33	43

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	2	10		2	0.39	77	0.34	1	0.06	1
10	2	10		3	1800.17	23070.79k	1809.34	8785.16k	0.08	5
10	2	10		4	0.38	81	0.72	25	1800.00	13520.81k
10	2	10		5	0.37	37	0.77	756	0.04	1
10	2	10		6	0.40	43	0.89	50	0.09	3
10	2	10		7	5.74	73.08k	2.35	8.42k	0.05	1
10	2	10		8	0.35	45	0.32	1	0.09	3
10	2	10		9	0.31	31	0.52	41	0.07	5
10	2	10		10	1800.05	14147.17k	1.19	223	0.32	65
10	2	10	✓	1	100.90	19.45k	648.77	862.30k	7.55	1.90k
10	2	10	✓	2	20.51	4.04k	32.51	5.21k	4.21	929
10	2	10	✓	3	29.88	32.79k	90.77	20.89k	8.52	2.80k
10	2	10	✓	4	1801.35	8040.84k	357.31	74.49k	15.77	4.64k
10	2	10	✓	5	1805.55	8682.87k	1500.99	2140.27k	10.51	2.16k
10	2	10	✓	6	1800.03	7068.40k	151.74	16.51k	5.06	890
10	2	10	✓	7	25.84	7.61k	59.05	8.22k	7.08	1.73k
10	2	10	✓	8	19.66	5.46k	24.96	2.35k	5.49	1.14k
10	2	10	✓	9	87.20	18.01k	1801.97	3937.04k	11.63	3.69k
10	2	10	✓	10	51.21	15.23k	1801.88	3878.78k	4.35	771
10	3	1		1	0.21	1	0.15	1	0.06	1
10	3	1		2	(0.00%)	(∞)	1.17	64	1800.00	9903.72k
10	3	1		3	2.01	18	0.89	17	1800.00	9379.84k
10	3	1		4	3.43	7.46k	1.00	366	1800.00	1711.44k
10	3	1		5	1.04	1	0.74	1	1800.00	5787.04k
10	3	1		6	(0.00%)	(1.26%)	1800.02	44	0.09	1
10	3	1		7	0.20	1	0.14	1	0.06	1
10	3	1		8	1.88	1	0.45	1	0.08	1
10	3	1		9	0.55	65	0.57	29	1800.00	10194.93k
10	3	1		10	0.28	1	0.01	1	0.04	1
10	3	1	✓	1	1800.00	95.00k	86.03	10.65k	1800.00	5426.20k
10	3	1	✓	2	(3.33%)	(0.00%)	1802.73	3456.98k	1800.00	587.94k
10	3	1	✓	3	(0.27%)	(0.00%)	191.31	15.13k	1800.00	898.29k
10	3	1	✓	4	(0.25%)	(0.00%)	191.13	16.27k	1800.00	629.88k
10	3	1	✓	5	(0.19%)	(0.00%)	87.87	9.47k	40.51	7.57k
10	3	1	✓	6	1802.47	374.85k	91.75	11.55k	41.55	6.24k
10	3	1	✓	7	1800.93	1577.96k	230.86	44.28k	1800.00	4206.80k
10	3	1	✓	8	1531.61	423.44k	1052.11	1362.04k	1800.00	798.12k
10	3	1	✓	9	1800.82	607.29k	109.33	14.52k	1800.00	1864.06k
10	3	1	✓	10	1800.00	117.67k	1800.98	1977.76k	1800.00	4753.11k
10	3	2		1	0.57	29	0.70	10	1800.00	7942.80k
10	3	2		2	18.80	57	1.14	31	0.26	1
10	3	2		3	(0.11%)	(99.94%)	1800.03	3943.25k	1800.00	7432.30k
10	3	2		4	1804.15	8847.31k	0.94	260	1800.00	8688.21k
10	3	2		5	(0.00%)	(∞)	1.44	233	0.11	1
10	3	2		6	(0.00%)	(0.26%)	(0.00%)	(0.26%)	1800.00	7859.97k
10	3	2		7	0.89	465	0.68	16	1800.00	7702.79k
10	3	2		8	(6.70%)	(5.68%)	(0.00%)	(99.95%)	1800.00	1247.26k
10	3	2		9	(0.00%)	(∞)	1.14	43	1800.00	11856.78k
10	3	2		10	0.34	2	0.35	6	0.11	1
10	3	2	✓	1	(0.03%)	(0.00%)	1801.37	2633.42k	1800.00	4129.94k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	3	2	✓	2	(0.10%)	(0.00%)	259.08	15.30k	1800.00	2360.38k
10	3	2	✓	3	(0.34%)	(0.00%)	1802.28	3477.48k	83.64	6.69k
10	3	2	✓	4	(1.08%)	(0.00%)	1801.45	1999.21k	363.82	242.88k
10	3	2	✓	5	(0.54%)	(0.00%)	1803.15	4348.81k	1800.00	3850.55k
10	3	2	✓	6	(1.08%)	(0.00%)	1801.38	2009.24k	1800.00	487.10k
10	3	2	✓	7	1801.93	231.37k	142.25	11.54k	1534.46	1302.55k
10	3	2	✓	8	(0.11%)	(0.00%)	1801.94	3023.06k	1800.00	973.50k
10	3	2	✓	9	(0.15%)	(0.00%)	1801.34	2142.05k	1800.00	1216.43k
10	3	2	✓	10	(0.04%)	(0.00%)	1801.71	2365.03k	57.25	3.52k
10	3	3		1	(0.58%)	(0.00%)	(0.08%)	(0.00%)	1800.00	1239.63k
10	3	3		2	(0.29%)	(0.00%)	(0.08%)	(0.00%)	1800.00	1176.44k
10	3	3		3	36.12	115.76k	1.64	67	0.18	1
10	3	3		4	1800.04	5405.15k	1.33	29	1800.00	6631.33k
10	3	3		5	3.77	10.44k	1.04	87	1800.00	1268.71k
10	3	3		6	0.82	207	1802.44	4951.78k	1800.00	6718.16k
10	3	3		7	1.00	184	1.19	81	1800.00	5691.09k
10	3	3		8	0.52	98	0.29	1	0.11	1
10	3	3		9	(0.07%)	(17.56%)	1802.44	2056.86k	1800.00	2451.23k
10	3	3		10	0.85	19	0.61	35	1800.00	5698.40k
10	3	3	✓	1	(2.96%)	(0.00%)	1800.32	670.14k	1800.00	358.79k
10	3	3	✓	2	(0.33%)	(0.00%)	1800.95	1485.83k	1800.00	634.26k
10	3	3	✓	3	(0.50%)	(0.00%)	1801.07	1810.79k	1800.00	447.18k
10	3	3	✓	4	(0.24%)	(0.00%)	1801.07	1574.93k	1800.00	1384.44k
10	3	3	✓	5	(5.03%)	(0.00%)	1800.24	973.32k	1800.00	449.01k
10	3	3	✓	6	(0.03%)	(0.00%)	300.23	23.12k	1800.00	1134.01k
10	3	3	✓	7	(1.98%)	(0.00%)	1800.95	1166.15k	1800.00	858.75k
10	3	3	✓	8	(0.06%)	(0.00%)	319.29	18.66k	1800.00	3477.19k
10	3	3	✓	9	(41.43%)	(0.00%)	(5.20%)	(0.00%)	1800.00	602.92k
10	3	3	✓	10	(0.03%)	(0.00%)	383.39	16.65k	1800.00	1096.02k
10	3	5		1	(2.42%)	(0.00%)	(1.49%)	(0.00%)	(0.11%)	(0.00%)
10	3	5		2	(0.39%)	(∞)	(0.00%)	(0.59%)	1800.00	3640.33k
10	3	5		3	(2.12%)	(0.00%)	(0.08%)	(0.00%)	1.57	1
10	3	5		4	(4.56%)	(0.00%)	(0.00%)	(20.82%)	1800.00	490.17k
10	3	5		5	1.44	1.93k	0.50	2	0.16	1
10	3	5		6	1.11	652	0.83	70	1800.00	4654.23k
10	3	5		7	(10.30%)	(0.00%)	(0.67%)	(0.00%)	(0.25%)	(0.00%)
10	3	5		8	2.27	565	3.57	1.72k	0.38	1
10	3	5		9	(2.82%)	(0.00%)	2.04	185	1800.00	1358.70k
10	3	5		10	0.48	1	0.50	1	0.43	21
10	3	5	✓	1	(1.42%)	(0.00%)	923.78	20.84k	1800.00	435.16k
10	3	5	✓	2	(0.31%)	(0.00%)	1800.88	1442.38k	1800.00	369.10k
10	3	5	✓	3	(0.53%)	(0.00%)	656.60	14.74k	169.95	4.35k
10	3	5	✓	4	(3.67%)	(0.00%)	1800.21	320.96k	1800.00	288.30k
10	3	5	✓	5	(2.56%)	(0.00%)	1800.63	246.58k	1800.00	551.75k
10	3	5	✓	6	(0.95%)	(0.00%)	1116.80	410.51k	1800.00	452.93k
10	3	5	✓	7	(0.86%)	(0.00%)	1800.73	944.21k	1800.00	491.69k
10	3	5	✓	8	(0.58%)	(0.00%)	1800.69	1131.78k	1800.00	593.86k
10	3	5	✓	9	(0.09%)	(0.00%)	1800.61	2233.09k	1800.00	539.04k
10	3	5	✓	10	(0.34%)	(0.00%)	1800.58	820.08k	1800.00	658.04k
10	3	10		1	(2.84%)	(0.00%)	1800.58	1023.17k	1800.00	497.25k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	3	10		2	(0.00%)	(99.87%)	(0.00%)	(0.23%)	1800.00	2259.54k
10	3	10		3	(0.00%)	(0.58%)	(0.00%)	(99.68%)	1800.00	2943.58k
10	3	10		4	(3.39%)	(0.83%)	(0.16%)	(0.19%)	1800.00	580.72k
10	3	10		5	(0.24%)	(99.93%)	(0.25%)	(99.93%)	1800.00	956.57k
10	3	10		6	72.98	158.78k	2.18	95	1800.00	1039.32k
10	3	10		7	0.96	71	1801.32	2147.17k	1800.00	979.80k
10	3	10		8	(0.72%)	(0.00%)	51.77	34.18k	1800.00	692.59k
10	3	10		9	8.41	1.15k	4.45	185	1800.00	2053.62k
10	3	10		10	(4.05%)	(0.00%)	(0.18%)	(0.00%)	1800.00	579.86k
10	3	10	✓	1	(4.15%)	(0.00%)	(0.19%)	(0.00%)	1800.00	208.75k
10	3	10	✓	2	(1.22%)	(0.00%)	1426.57	14.39k	1800.00	216.26k
10	3	10	✓	3	(0.68%)	(0.00%)	1800.38	175.25k	1800.00	272.28k
10	3	10	✓	4	(2.79%)	(0.00%)	(0.19%)	(0.00%)	1800.00	240.07k
10	3	10	✓	5	(6.16%)	(0.00%)	(0.41%)	(0.00%)	1800.00	107.86k
10	3	10	✓	6	(4.13%)	(0.00%)	(0.39%)	(0.00%)	1800.00	92.45k
10	3	10	✓	7	(0.06%)	(0.00%)	1037.91	13.48k	1670.95	336.29k
10	3	10	✓	8	(2.72%)	(0.00%)	(0.22%)	(0.00%)	1800.00	125.81k
10	3	10	✓	9	(3.73%)	(0.00%)	1800.62	59.50k	324.63	1.87k
10	3	10	✓	10	(7.38%)	(0.00%)	(0.86%)	(0.00%)	1800.00	4.87k
10	4	1		1	(0.19%)	(0.00%)	4.10	3.07k	3.38	128
10	4	1		2	1.85	1.75k	0.78	187	1800.00	3599.39k
10	4	1		3	1805.03	10158.83k	1803.20	12321.20k	1800.00	7017.92k
10	4	1		4	(0.00%)	(∞)	1.27	128	1800.00	5406.44k
10	4	1		5	1800.01	6556.41k	1805.72	5824.98k	0.44	3
10	4	1		6	(1.62%)	(0.00%)	1.11	378	1.38	41
10	4	1		7	(2.33%)	(0.00%)	1800.10	8395.86k	1800.00	8053.43k
10	4	1		8	0.30	1	0.19	1	0.07	1
10	4	1		9	1810.38	13156.86k	0.42	5	0.48	19
10	4	1		10	0.31	1	0.03	1	0.05	1
10	4	1	✓	1	(0.61%)	(0.00%)	464.10	31.92k	1800.00	1144.25k
10	4	1	✓	2	(3.47%)	(0.00%)	1800.28	1339.20k	1800.00	496.31k
10	4	1	✓	3	1800.00	134.94k	196.96	18.71k	1800.00	1748.10k
10	4	1	✓	4	(16.92%)	(0.00%)	(0.07%)	(0.00%)	1800.00	315.76k
10	4	1	✓	5	1800.00	45.32k	1801.49	2689.80k	1800.00	1696.28k
10	4	1	✓	6	(16.25%)	(0.00%)	1216.75	96.55k	1800.00	326.94k
10	4	1	✓	7	(0.07%)	(0.00%)	1802.84	4509.61k	1800.00	948.92k
10	4	1	✓	8	(0.09%)	(0.00%)	1801.27	1868.79k	1800.00	1487.78k
10	4	1	✓	9	(0.17%)	(0.00%)	1801.34	2210.78k	1800.00	1904.90k
10	4	1	✓	10	(1.90%)	(0.00%)	1800.40	1367.04k	1800.00	1384.94k
10	4	2		1	(0.39%)	(1.95%)	1804.36	3782.97k	1800.00	3226.86k
10	4	2		2	1801.62	3201.34k	1800.77	2307.22k	4.88	33
10	4	2		3	(0.02%)	(∞)	1802.57	2556.74k	7.44	137
10	4	2		4	1.16	164	0.99	23	1800.00	3164.56k
10	4	2		5	(0.21%)	(∞)	(0.02%)	(19.57%)	1800.00	930.59k
10	4	2		6	1800.01	2983.50k	(0.00%)	(0.59%)	3.72	27
10	4	2		7	12.30	15.95k	1.36	93	3.96	9
10	4	2		8	(15.98%)	(∞)	(0.00%)	(∞)	3.81	83
10	4	2		9	4.69	2.41k	1.56	95	3.80	5
10	4	2		10	56.34	159.94k	0.84	25	1800.00	978.49k
10	4	2	✓	1	(28.46%)	(0.00%)	(0.22%)	(0.00%)	1800.00	254.55k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	4	2	✓	2	(19.12%)	(0.00%)	(0.69%)	(0.00%)	1800.00	116.41k
10	4	2	✓	3	(6.60%)	(0.00%)	(0.27%)	(0.00%)	1800.00	289.03k
10	4	2	✓	4	(20.09%)	(0.00%)	(0.55%)	(0.00%)	1800.00	190.37k
10	4	2	✓	5	(8.65%)	(0.00%)	(0.12%)	(0.00%)	1800.00	280.89k
10	4	2	✓	6	(6.47%)	(0.00%)	(0.13%)	(0.00%)	1800.00	629.79k
10	4	2	✓	7	(7.24%)	(0.00%)	(0.56%)	(0.00%)	1800.00	307.52k
10	4	2	✓	8	(3.70%)	(0.00%)	1024.39	24.80k	1800.00	378.05k
10	4	2	✓	9	(50.67%)	(0.00%)	(6.65%)	(0.00%)	1800.00	75.07k
10	4	2	✓	10	(11.60%)	(0.00%)	1800.25	1372.48k	1800.00	469.93k
10	4	3		1	1800.03	3080.97k	3.29	377	1800.00	582.07k
10	4	3		2	1801.93	4602.46k	1803.65	3297.90k	5.05	79
10	4	3		3	1802.02	3519.50k	4.35	5.07k	5.19	27
10	4	3		4	316.49	1114.75k	1800.05	4185.03k	1800.00	516.76k
10	4	3		5	(0.19%)	(3.40%)	(0.01%)	(0.12%)	1800.00	1124.65k
10	4	3		6	1803.76	6431.05k	1800.02	3655.92k	1800.00	683.63k
10	4	3		7	(18.98%)	(16.20%)	(0.44%)	(10.09%)	(0.10%)	(0.00%)
10	4	3		8	1802.28	3464.59k	2.45	151	7.10	103
10	4	3		9	(6.23%)	(∞)	(0.30%)	(3.95%)	2.46	1
10	4	3		10	3.89	739	1.63	143	4.48	49
10	4	3	✓	1	(69.20%)	(0.00%)	(9.47%)	(0.00%)	(0.12%)	(0.00%)
10	4	3	✓	2	(5.01%)	(0.00%)	(0.44%)	(0.00%)	1800.00	139.97k
10	4	3	✓	3	(3.67%)	(0.00%)	1800.00	68.57k	1800.00	418.53k
10	4	3	✓	4	(12.23%)	(0.00%)	(0.04%)	(0.00%)	1800.00	200.64k
10	4	3	✓	5	(96.89%)	(37.69%)	(73.69%)	(0.00%)	(0.38%)	(0.00%)
10	4	3	✓	6	(0.74%)	(0.00%)	1800.30	706.61k	1800.00	392.93k
10	4	3	✓	7	(81.09%)	(0.00%)	(14.70%)	(0.00%)	(9.49%)	(0.00%)
10	4	3	✓	8	(2.70%)	(0.00%)	(0.07%)	(0.00%)	1800.00	295.96k
10	4	3	✓	9	(20.38%)	(0.00%)	(0.61%)	(0.00%)	1800.00	11.25k
10	4	3	✓	10	(8.12%)	(0.00%)	(0.53%)	(0.00%)	1800.00	244.24k
10	4	5		1	12.52	1.42k	8.67	217	1800.00	2136.69k
10	4	5		2	1800.49	936.68k	12.63	2.29k	17.91	203
10	4	5		3	(43.11%)	(0.00%)	62.49	847	1800.00	389.93k
10	4	5		4	1800.00	747.68k	186.81	93.91k	1800.00	498.69k
10	4	5		5	(3.42%)	(0.00%)	(0.16%)	(0.00%)	(0.05%)	(0.00%)
10	4	5		6	(4.38%)	(99.85%)	(0.35%)	(15.16%)	1800.00	365.33k
10	4	5		7	1800.00	1076.86k	1800.44	747.82k	12.18	89
10	4	5		8	1801.19	1342.50k	1.83	113	1800.00	1221.68k
10	4	5		9	(0.16%)	(0.00%)	40.54	983	60.55	107
10	4	5		10	5.64	1.18k	2.27	188	2.21	3
10	4	5	✓	1	(7.63%)	(0.00%)	(1.52%)	(0.00%)	1800.00	33.62k
10	4	5	✓	2	(5.37%)	(0.00%)	(0.23%)	(0.00%)	1800.00	75.77k
10	4	5	✓	3	(62.71%)	(0.00%)	(11.91%)	(0.00%)	(8.11%)	(0.00%)
10	4	5	✓	4	(2.45%)	(0.00%)	(0.06%)	(0.00%)	1800.00	134.22k
10	4	5	✓	5	(23.91%)	(0.00%)	(3.54%)	(0.00%)	(0.10%)	(0.00%)
10	4	5	✓	6	(13.19%)	(0.00%)	(0.27%)	(0.00%)	1800.00	33.13k
10	4	5	✓	7	(∞)	(0.00%)	(25.59%)	(0.00%)	(3.61%)	(0.00%)
10	4	5	✓	8	(52.42%)	(0.00%)	(5.93%)	(0.00%)	1800.00	126.80k
10	4	5	✓	9	(22.20%)	(0.00%)	(4.88%)	(0.00%)	(13.87%)	(0.00%)
10	4	5	✓	10	(35.75%)	(0.00%)	(5.46%)	(0.00%)	(19.04%)	(0.00%)
10	4	10		1	(0.93%)	(0.98%)	(0.00%)	(0.98%)	1800.00	605.44k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	4	10		2	1800.01	234.18k	111.39	687	1800.00	481.64k
10	4	10		3	(46.54%)	(0.00%)	(10.82%)	(0.00%)	(0.03%)	(0.00%)
10	4	10		4	1800.00	1016.67k	29.58	237	43.44	53
10	4	10		5	(9.89%)	(99.77%)	(0.91%)	(4.23%)	(0.07%)	(0.00%)
10	4	10		6	(1.34%)	(0.00%)	79.45	697	1800.00	98.83k
10	4	10		7	(0.15%)	(0.00%)	214.51	3.10k	324.10	117
10	4	10		8	(0.00%)	(3.10%)	(0.00%)	(1.78%)	30.46	287
10	4	10		9	(32.51%)	(0.00%)	(0.04%)	(0.00%)	1800.00	87.82k
10	4	10		10	(0.19%)	(99.69%)	(0.01%)	(26.49%)	1800.00	250.05k
10	4	10	✓	1	(1.25%)	(0.00%)	(0.28%)	(0.00%)	1800.00	28.34k
10	4	10	✓	2	(22.99%)	(0.00%)	(6.71%)	(0.00%)	(0.05%)	(0.00%)
10	4	10	✓	3	(26.71%)	(0.00%)	(9.80%)	(0.00%)	(0.13%)	(0.00%)
10	4	10	✓	4	(28.91%)	(0.00%)	(5.09%)	(0.00%)	(0.06%)	(0.00%)
10	4	10	✓	5	(26.90%)	(0.00%)	(5.06%)	(0.00%)	1800.00	9.74k
10	4	10	✓	6	(12.27%)	(0.00%)	(4.42%)	(0.00%)	1800.00	4.63k
10	4	10	✓	7	(58.33%)	(0.00%)	(20.36%)	(0.00%)	(40.77%)	(0.00%)
10	4	10	✓	8	(13.23%)	(0.00%)	(4.91%)	(0.00%)	1800.00	11.53k
10	4	10	✓	9	(98.42%)	(0.00%)	(12.16%)	(0.00%)	1800.00	18.01k
10	4	10	✓	10	(19.36%)	(0.00%)	(4.63%)	(0.00%)	(0.61%)	(0.00%)
10	5	1		1	(0.00%)	(4.01%)	1.08	5	0.43	17
10	5	1		2	0.39	1	0.20	1	0.11	1
10	5	1		3	0.47	1	0.35	1	0.13	1
10	5	1		4	11.90	26.82k	1.44	300	3.96	11
10	5	1		5	1.18	133	0.84	103	1.08	5
10	5	1		6	(1.61%)	(18.66%)	(0.30%)	(4.22%)	1800.00	2517.98k
10	5	1		7	(0.90%)	(3.29%)	1800.06	4949.69k	1800.00	3494.44k
10	5	1		8	1800.04	4732.54k	1800.02	4717.05k	1800.00	4001.49k
10	5	1		9	(0.04%)	(21.03%)	11.98	12.39k	1800.00	4389.46k
10	5	1		10	(0.29%)	(3.16%)	1800.06	5449.82k	1800.00	1123.63k
10	5	1	✓	1	(31.26%)	(0.00%)	(1.24%)	(0.00%)	1800.00	732.47k
10	5	1	✓	2	(0.12%)	(0.00%)	1800.00	1153.16k	1800.00	1053.51k
10	5	1	✓	3	(15.95%)	(0.00%)	1800.00	578.59k	1800.00	628.96k
10	5	1	✓	4	(0.05%)	(0.00%)	1800.00	1574.75k	1800.00	855.15k
10	5	1	✓	5	(1.35%)	(0.00%)	1800.00	84.83k	1800.00	824.39k
10	5	1	✓	6	(7.54%)	(0.00%)	(0.04%)	(0.00%)	1800.00	391.55k
10	5	1	✓	7	(5.10%)	(0.00%)	(0.07%)	(0.00%)	1800.00	166.04k
10	5	1	✓	8	(6.46%)	(0.00%)	(0.05%)	(0.00%)	1800.00	427.58k
10	5	1	✓	9	(5.28%)	(0.00%)	(0.09%)	(0.00%)	1800.00	476.58k
10	5	1	✓	10	(0.57%)	(0.00%)	1800.00	891.98k	1800.00	657.57k
10	5	2		1	(2.09%)	(46.17%)	(0.00%)	(40.87%)	12.75	7
10	5	2		2	(0.25%)	(0.00%)	1802.84	2781.43k	1800.00	559.85k
10	5	2		3	239.31	131.36k	1800.64	1027.88k	17.38	441
10	5	2		4	184.80	471.14k	0.72	1	0.57	1
10	5	2		5	4.21	6.55k	1800.95	1874.50k	1.15	1
10	5	2		6	4.88	10.10k	1378.27	2722.01k	1800.00	2704.18k
10	5	2		7	1801.61	2960.67k	3.14	1.37k	3.45	1
10	5	2		8	1804.38	4260.59k	1.02	17	2.51	3
10	5	2		9	328.02	375.56k	1800.13	1129.85k	1800.00	434.03k
10	5	2		10	(0.11%)	(0.00%)	1802.88	1250.27k	1800.00	772.38k
10	5	2	✓	1	(18.32%)	(0.00%)	(0.52%)	(0.00%)	1800.00	117.97k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	5	2	✓	2	(∞)	(0.00%)	(10.80%)	(0.00%)	(11.35%)	(0.00%)
10	5	2	✓	3	(3.51%)	(0.00%)	(0.41%)	(0.00%)	1800.00	37.11k
10	5	2	✓	4	(∞)	(0.00%)	1800.00	501.23k	1800.00	264.70k
10	5	2	✓	5	(2.99%)	(0.00%)	(0.47%)	(0.00%)	1800.00	256.57k
10	5	2	✓	6	(3.02%)	(0.00%)	1800.00	35.75k	1800.00	283.68k
10	5	2	✓	7	(61.87%)	(0.00%)	(4.70%)	(0.00%)	1800.00	64.94k
10	5	2	✓	8	(22.15%)	(0.00%)	(3.15%)	(0.00%)	1800.00	32.14k
10	5	2	✓	9	(44.13%)	(0.00%)	(5.53%)	(0.00%)	(0.21%)	(0.00%)
10	5	2	✓	10	(5.35%)	(0.00%)	(0.47%)	(0.00%)	1800.00	3.57k
10	5	3		1	1802.15	2878.29k	3.46	307	9.71	9
10	5	3		2	(5.73%)	(0.00%)	1800.00	223.23k	1800.00	50.92k
10	5	3		3	(0.48%)	(4.70%)	(0.00%)	(∞)	1800.00	536.17k
10	5	3		5	1800.02	2133.71k	2.05	49	5.31	1
10	5	3		6	1800.00	711.74k	9.00	153	1800.00	283.81k
10	5	3		7	1800.01	1998.50k	1800.33	1660.41k	8.74	7
10	5	3		8	1801.05	876.52k	4.94	275	13.74	19
10	5	3		9	(0.00%)	(0.43%)	(0.00%)	(99.52%)	1.92	1
10	5	3		10	1800.00	382.07k	1800.00	469.87k	23.51	53
10	5	3	✓	1	(13.44%)	(0.00%)	(3.02%)	(0.00%)	(0.03%)	(0.00%)
10	5	3	✓	2	(6.21%)	(0.00%)	(0.34%)	(0.00%)	1800.00	8.82k
10	5	3	✓	3	(12.61%)	(0.00%)	(2.54%)	(0.00%)	(0.11%)	(0.00%)
10	5	3	✓	4	(11.47%)	(0.00%)	(1.14%)	(0.00%)	1800.00	22.33k
10	5	3	✓	5	(36.94%)	(0.00%)	(4.74%)	(0.00%)	(0.03%)	(0.00%)
10	5	3	✓	6	(26.16%)	(0.00%)	(4.19%)	(0.00%)	(0.80%)	(0.00%)
10	5	3	✓	7	(47.34%)	(0.00%)	(2.05%)	(0.00%)	(23.53%)	(0.00%)
10	5	3	✓	8	(5.65%)	(0.00%)	(0.13%)	(0.00%)	1800.00	265.66k
10	5	3	✓	9	(10.80%)	(0.00%)	(1.45%)	(0.00%)	(0.03%)	(0.00%)
10	5	3	✓	10	(64.14%)	(0.00%)	(13.45%)	(0.00%)	(0.45%)	(0.00%)
10	5	5		1	(10.86%)	(0.00%)	1800.69	485.00k	1800.00	186.24k
10	5	5		2	(15.01%)	(0.00%)	(1.44%)	(0.00%)	1800.00	69.37k
10	5	5		3	(31.91%)	(99.27%)	(0.20%)	(0.00%)	(0.04%)	(0.02%)
10	5	5		4	(0.10%)	(0.00%)	153.43	1.47k	1800.00	87.30k
10	5	5		5	1800.00	103.27k	115.27	1.70k	1800.00	111.53k
10	5	5		6	1.47	18	1.38	25	2.08	9
10	5	5		7	(2.11%)	(0.00%)	(0.04%)	(∞)	79.23	5
10	5	5		8	1800.00	1055.91k	10.68	734	(0.31%)	(0.00%)
10	5	5		9	1800.01	576.21k	21.55	243	1800.00	151.53k
10	5	5		10	(∞)	(99.92%)	(3.16%)	(99.92%)	(0.77%)	(0.00%)
10	5	5	✓	1	(5.55%)	(0.00%)	(1.18%)	(0.00%)	1800.00	14.85k
10	5	5	✓	2	(15.67%)	(0.00%)	(1.09%)	(0.00%)	(0.04%)	(0.00%)
10	5	5	✓	3	(26.79%)	(0.00%)	(6.63%)	(0.00%)	(0.26%)	(0.00%)
10	5	5	✓	4	(17.49%)	(0.00%)	(2.80%)	(0.00%)	(11.22%)	(0.00%)
10	5	5	✓	5	(60.76%)	(0.00%)	(22.23%)	(0.00%)	(0.56%)	(0.00%)
10	5	5	✓	6	(1.41%)	(0.00%)	(0.09%)	(0.00%)	1800.00	92.52k
10	5	5	✓	7	(31.47%)	(0.00%)	(2.38%)	(0.00%)	1800.00	10.92k
10	5	5	✓	8	(7.94%)	(0.00%)	(1.34%)	(0.00%)	1800.00	307
10	5	5	✓	9	(17.91%)	(0.00%)	(3.48%)	(0.00%)	(0.46%)	(0.00%)
10	5	5	✓	10	(16.31%)	(0.00%)	(2.24%)	(0.00%)	1800.00	16.79k
10	5	10		1	(3.58%)	(0.00%)	157.28	505	1800.00	86.61k
10	5	10		2	(2.94%)	(0.00%)	9.46	262	18.28	17

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	5	10		3	(6.56%)	(97.77%)	(0.00%)	(0.13%)	13.81	21
10	5	10		4	1800.00	121.05k	119.01	415	275.18	147
10	5	10		5	(1.67%)	(96.09%)	(0.01%)	(96.09%)	1800.00	179.34k
10	5	10		6	(81.67%)	(99.55%)	(8.93%)	(7.94%)	47.10	51
10	5	10		7	(40.14%)	(∞)	(3.27%)	(∞)	1800.00	40.06k
10	5	10		8	1800.00	808.63k	7.02	93	1800.00	149.20k
10	5	10		9	1801.08	1772.19k	3.65	7	1.20	1
10	5	10		10	(27.84%)	(∞)	(1.85%)	(∞)	(0.21%)	(0.00%)
10	5	10	✓	1	(11.03%)	(0.00%)	(5.93%)	(0.00%)	1800.00	3.61k
10	5	10	✓	2	(11.06%)	(0.00%)	(3.30%)	(0.00%)	(0.83%)	(0.00%)
10	5	10	✓	3	(19.72%)	(0.00%)	(5.68%)	(0.00%)	(0.11%)	(0.00%)
10	5	10	✓	4	(6.76%)	(0.00%)	(2.42%)	(0.00%)	(11.64%)	(0.00%)
10	5	10	✓	5	(5.85%)	(0.00%)	(0.85%)	(0.00%)	1800.00	3.00k
10	5	10	✓	6	(15.61%)	(0.00%)	(3.93%)	(0.00%)	(0.37%)	(0.00%)
10	5	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(55.34%)	(0.00%)
10	5	10	✓	8	(12.82%)	(0.00%)	(3.63%)	(0.00%)	(13.87%)	(0.00%)
10	5	10	✓	9	(9.77%)	(0.00%)	(2.85%)	(0.00%)	1800.00	6.94k
10	5	10	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(39.66%)	(0.00%)
20	2	1		1	0.10	1	0.14	1	1800.00	15209.82k
20	2	1		2	0.04	1	0.07	1	0.06	1
20	2	1		3	0.11	1	0.06	1	1800.00	14967.11k
20	2	1		4	0.07	1	0.10	1	1800.00	7581.36k
20	2	1		5	0.06	1	0.06	1	0.05	1
20	2	1		6	0.03	1	0.01	1	0.04	1
20	2	1		7	0.07	1	0.11	2	0.04	1
20	2	1		8	0.06	1	0.07	1	0.04	1
20	2	1		9	0.07	1	0.05	1	0.03	1
20	2	1		10	0.06	1	0.05	1	0.03	1
20	2	1	✓	1	(0.91%)	(0.00%)	737.72	363.44k	1800.00	2326.95k
20	2	1	✓	2	(2.24%)	(0.00%)	(0.99%)	(0.00%)	1800.00	534.93k
20	2	1	✓	3	(7.43%)	(0.00%)	(1.01%)	(0.00%)	1800.00	636.50k
20	2	1	✓	4	(2.52%)	(0.00%)	1800.00	69.75k	1800.00	2522.45k
20	2	1	✓	5	(17.23%)	(0.00%)	(9.38%)	(0.00%)	1800.00	515.69k
20	2	1	✓	6	(6.00%)	(0.00%)	(4.04%)	(0.00%)	1800.00	635.73k
20	2	1	✓	7	(5.01%)	(0.00%)	(0.11%)	(0.00%)	1800.00	2447.54k
20	2	1	✓	8	(0.51%)	(0.00%)	(0.43%)	(0.00%)	1800.00	678.47k
20	2	1	✓	9	(5.65%)	(0.00%)	(2.79%)	(0.00%)	1800.00	661.29k
20	2	1	✓	10	(5.25%)	(0.00%)	(1.99%)	(0.00%)	1800.00	1113.98k
20	2	2		1	393.69	4777.79k	0.15	3	0.08	1
20	2	2		2	0.05	1	0.10	1	0.04	1
20	2	2		3	0.35	49	0.46	59	1800.00	7375.79k
20	2	2		4	0.17	1	0.35	40	0.10	5
20	2	2		5	0.20	15	0.07	1	1.69	5.25k
20	2	2		6	0.23	23	0.13	1	0.07	1
20	2	2		7	1800.15	19166.17k	5.90	46.24k	0.11	3
20	2	2		8	0.58	237	1815.57	10823.07k	0.10	3
20	2	2		9	1804.10	16627.06k	1809.87	9184.90k	1800.00	7279.87k
20	2	2		10	0.10	1	0.11	1	0.06	1
20	2	2	✓	1	(3.49%)	(0.00%)	(3.13%)	(0.00%)	1800.00	481.42k
20	2	2	✓	2	(15.97%)	(0.00%)	(4.47%)	(0.00%)	1800.00	741.68k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	2	2	✓	3	(15.93%)	(0.00%)	(15.98%)	(0.00%)	1800.00	799.05k
20	2	2	✓	4	(3.73%)	(0.00%)	(2.53%)	(0.00%)	1800.00	525.18k
20	2	2	✓	5	(0.96%)	(0.00%)	1800.09	28.43k	1800.00	2798.21k
20	2	2	✓	6	(8.39%)	(0.00%)	(9.47%)	(0.00%)	1800.00	480.89k
20	2	2	✓	7	(35.46%)	(0.00%)	(35.00%)	(0.00%)	1800.00	419.77k
20	2	2	✓	8	(8.01%)	(0.00%)	(5.64%)	(0.00%)	1800.00	457.31k
20	2	2	✓	9	(1.04%)	(0.00%)	(0.39%)	(0.00%)	840.83	252.35k
20	2	2	✓	10	(3.60%)	(0.00%)	(0.13%)	(0.00%)	1800.00	1273.26k
20	2	3		1	0.24	13	0.24	5	0.09	3
20	2	3		2	1811.62	20834.04k	0.68	272	0.23	21
20	2	3		3	0.20	17	1800.01	9978.69k	0.05	1
20	2	3		4	0.32	37	0.51	49	0.05	1
20	2	3		5	0.38	49	0.55	88	1800.00	8357.19k
20	2	3		6	0.30	27	0.91	1.55k	0.07	1
20	2	3		7	0.46	67	0.38	73	0.05	1
20	2	3		8	0.09	1	0.19	1	0.21	1
20	2	3		9	0.51	95	0.59	11	0.11	7
20	2	3		10	0.14	1	0.26	4	0.08	1
20	2	3	✓	1	(13.38%)	(0.00%)	(2.24%)	(0.00%)	1800.00	526.46k
20	2	3	✓	2	(1.92%)	(0.00%)	1800.00	13.23k	1800.00	7270.59k
20	2	3	✓	3	(28.98%)	(0.00%)	(26.07%)	(0.00%)	1800.00	564.05k
20	2	3	✓	4	(0.71%)	(0.00%)	(0.09%)	(0.00%)	1800.00	2572.65k
20	2	3	✓	5	(1.50%)	(0.00%)	1800.00	22.41k	1800.00	2089.54k
20	2	3	✓	6	(2.11%)	(0.00%)	(4.75%)	(0.00%)	1800.00	608.26k
20	2	3	✓	7	(17.62%)	(0.00%)	(18.26%)	(0.00%)	1800.00	398.29k
20	2	3	✓	8	(7.95%)	(0.00%)	(13.52%)	(0.00%)	1800.00	1228.85k
20	2	3	✓	9	(42.34%)	(0.00%)	(42.03%)	(0.00%)	1800.00	450.56k
20	2	3	✓	10	(6.78%)	(0.00%)	(1.16%)	(0.00%)	1800.00	2497.59k
20	2	5		1	0.77	119	0.81	125	0.08	1
20	2	5		2	1800.17	22673.14k	129.27	675.90k	0.12	1
20	2	5		3	0.50	57	0.73	71	0.08	1
20	2	5		4	0.88	267	1.15	661	0.04	1
20	2	5		5	0.61	166	1800.12	10990.70k	0.16	31
20	2	5		6	1800.05	10068.80k	6.97	2.41k	0.19	7
20	2	5		7	0.57	143	0.77	211	0.05	1
20	2	5		8	0.52	128	0.62	7	1800.00	12147.03k
20	2	5		9	1800.07	29730.03k	0.64	103	0.10	3
20	2	5		10	0.85	211	0.99	258	0.13	5
20	2	5	✓	1	(∞)	(0.00%)	(∞)	(0.00%)	1800.00	835.02k
20	2	5	✓	2	(4.03%)	(0.00%)	(10.42%)	(0.00%)	1800.00	415.22k
20	2	5	✓	3	(5.24%)	(0.00%)	(6.41%)	(0.00%)	1800.00	445.36k
20	2	5	✓	4	(3.83%)	(0.00%)	(4.43%)	(0.00%)	1800.00	417.78k
20	2	5	✓	5	(3.34%)	(0.00%)	(0.79%)	(0.00%)	1800.00	4523.90k
20	2	5	✓	6	(30.66%)	(0.00%)	(37.92%)	(0.00%)	1800.00	1055.69k
20	2	5	✓	7	(2.55%)	(0.00%)	(0.24%)	(0.00%)	1232.45	383.26k
20	2	5	✓	8	(1.80%)	(0.00%)	(3.16%)	(0.00%)	1800.00	495.21k
20	2	5	✓	9	(4.42%)	(0.00%)	(2.72%)	(0.00%)	1800.00	1054.60k
20	2	5	✓	10	(17.28%)	(0.00%)	(33.84%)	(0.00%)	1800.00	432.09k
20	2	10		1	1802.50	5938.38k	1800.76	2887.73k	0.43	7
20	2	10		2	4.09	1.94k	7.98	1.84k	0.90	139

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	2	10		3	1800.16	12944.98k	2.54	1.03k	0.07	1
20	2	10		4	1800.08	10534.31k	8.20	1.29k	0.26	1
20	2	10		5	9.12	31.89k	3.24	573	0.17	3
20	2	10		6	1800.03	13253.46k	1801.97	4375.14k	1800.00	6663.35k
20	2	10		7	1800.04	8688.69k	36.62	6.80k	1800.00	5489.10k
20	2	10		8	1.56	629	1800.85	2304.92k	0.08	1
20	2	10		9	1800.00	7992.59k	1802.07	4077.24k	0.55	9
20	2	10		10	1800.06	10992.90k	2.95	969	0.37	29
20	2	10	✓	1	(7.04%)	(0.00%)	(8.22%)	(0.00%)	1800.00	914.08k
20	2	10	✓	2	(0.79%)	(0.00%)	(0.54%)	(0.00%)	1800.00	1670.04k
20	2	10	✓	3	(2.87%)	(0.00%)	(3.79%)	(0.00%)	1800.00	1500.21k
20	2	10	✓	4	(4.82%)	(0.00%)	(8.80%)	(0.00%)	1800.00	286.44k
20	2	10	✓	5	(8.44%)	(0.00%)	(6.44%)	(0.00%)	1800.00	2536.38k
20	2	10	✓	6	(5.84%)	(0.00%)	(7.71%)	(0.00%)	1800.00	385.17k
20	2	10	✓	7	(11.72%)	(0.00%)	(13.23%)	(0.00%)	1800.00	717.97k
20	2	10	✓	8	(3.59%)	(0.00%)	(4.19%)	(0.00%)	1800.00	1662.41k
20	2	10	✓	9	(4.91%)	(0.00%)	(3.40%)	(0.00%)	1800.00	355.42k
20	2	10	✓	10	(3.54%)	(0.00%)	(3.91%)	(0.00%)	1800.00	611.40k
20	3	1		1	0.38	1	0.21	1	0.08	1
20	3	1		2	0.59	32	0.50	15	1800.00	3530.60k
20	3	1		3	(0.58%)	(0.51%)	1801.60	6713.63k	1800.00	4802.54k
20	3	1		4	1.08	1	0.02	1	0.06	1
20	3	1		5	(1.00%)	(0.00%)	(0.00%)	(3.22%)	2.06	175
20	3	1		6	(2.62%)	(1.86%)	(0.08%)	(∞)	1800.00	704.48k
20	3	1		7	(0.00%)	(∞)	0.05	1	0.05	1
20	3	1		8	(0.00%)	(7.34%)	1.10	5	1800.00	2297.84k
20	3	1		9	(0.00%)	(∞)	0.72	3	0.08	1
20	3	1		10	(0.24%)	(0.00%)	1801.96	10289.00k	1800.00	2857.99k
20	3	1	✓	1	(86.51%)	(0.00%)	(48.24%)	(0.00%)	(27.89%)	(0.00%)
20	3	1	✓	2	(82.66%)	(0.00%)	(73.39%)	(0.00%)	1800.00	166.92k
20	3	1	✓	3	(18.49%)	(0.00%)	(22.42%)	(0.00%)	(2.26%)	(0.00%)
20	3	1	✓	4	(5.77%)	(0.00%)	(2.82%)	(0.00%)	1800.00	32.80k
20	3	1	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(52.38%)	(0.00%)
20	3	1	✓	6	(52.36%)	(0.00%)	(36.75%)	(0.00%)	(9.89%)	(0.00%)
20	3	1	✓	7	(8.56%)	(0.00%)	(9.38%)	(0.00%)	1800.00	10.52k
20	3	1	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(85.82%)	(0.00%)
20	3	1	✓	9	(39.46%)	(0.00%)	(35.33%)	(0.00%)	(5.18%)	(0.00%)
20	3	1	✓	10	(45.10%)	(0.00%)	(28.55%)	(0.00%)	(13.26%)	(0.00%)
20	3	2		1	(13.89%)	(0.00%)	1804.90	3735.97k	1800.00	1199.53k
20	3	2		2	(0.00%)	(0.34%)	(0.00%)	(∞)	0.83	1
20	3	2		3	(0.07%)	(0.02%)	(0.02%)	(99.98%)	1800.00	1435.72k
20	3	2		4	(1.14%)	(0.00%)	1802.86	4539.10k	1800.00	735.86k
20	3	2		5	(0.73%)	(0.00%)	1800.02	2576.45k	1.54	7
20	3	2		6	(0.99%)	(0.00%)	1802.80	3082.05k	1800.00	403.19k
20	3	2		7	(0.11%)	(0.00%)	1800.05	4171.12k	1800.00	859.07k
20	3	2		8	(0.22%)	(4.50%)	1800.01	1975.74k	1800.00	1449.68k
20	3	2		9	1802.67	4238.06k	3.27	579	1800.00	793.22k
20	3	2		10	1802.90	6832.61k	1.80	178	1.02	25
20	3	2	✓	1	(45.77%)	(0.00%)	(19.53%)	(0.00%)	(9.72%)	(0.00%)
20	3	2	✓	2	(16.45%)	(0.00%)	(9.57%)	(0.00%)	(2.80%)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	3	2	✓	3	(25.27%)	(0.00%)	(16.81%)	(0.00%)	(7.56%)	(0.00%)
20	3	2	✓	4	(38.34%)	(0.00%)	(27.12%)	(0.00%)	(16.90%)	(0.00%)
20	3	2	✓	5	(58.06%)	(0.00%)	(48.23%)	(0.00%)	(19.99%)	(0.00%)
20	3	2	✓	6	(93.88%)	(0.00%)	(90.99%)	(0.00%)	(75.14%)	(0.00%)
20	3	2	✓	7	(19.88%)	(0.00%)	(8.37%)	(0.00%)	(1.90%)	(0.00%)
20	3	2	✓	8	(58.79%)	(0.00%)	(28.23%)	(0.00%)	(8.35%)	(0.00%)
20	3	2	✓	9	(11.83%)	(0.00%)	(7.28%)	(0.00%)	(1.60%)	(0.00%)
20	3	2	✓	10	(15.51%)	(0.00%)	(13.89%)	(0.00%)	(0.03%)	(0.00%)
20	3	3		1	(0.01%)	(14.50%)	1800.03	2012.95k	4.97	19
20	3	3		2	(0.00%)	(∞)	1800.00	888.33k	5.47	9
20	3	3		3	(1.25%)	(0.03%)	1800.00	224.97k	1800.00	563.98k
20	3	3		4	(1.31%)	(0.00%)	(0.21%)	(0.00%)	(0.08%)	(0.00%)
20	3	3		5	(0.04%)	(0.00%)	1800.01	1460.23k	9.48	287
20	3	3		6	(0.18%)	(0.00%)	1803.89	2266.76k	1800.00	230.36k
20	3	3		7	1.54	158	0.96	51	1800.00	1660.28k
20	3	3		8	(1.50%)	(0.00%)	(0.08%)	(0.00%)	1800.00	404.08k
20	3	3		9	(0.00%)	(∞)	(0.00%)	(∞)	0.83	1
20	3	3		10	(2.92%)	(0.02%)	(1.36%)	(0.00%)	(0.54%)	(0.00%)
20	3	3	✓	1	(∞)	(4.44%)	(∞)	(4.44%)	562.27	13.42k
20	3	3	✓	2	(72.67%)	(0.00%)	(63.62%)	(0.00%)	(27.45%)	(0.00%)
20	3	3	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	3	3	✓	4	(16.38%)	(0.00%)	(14.50%)	(0.00%)	(3.17%)	(0.00%)
20	3	3	✓	5	(18.86%)	(0.00%)	(15.62%)	(0.00%)	(2.44%)	(0.00%)
20	3	3	✓	6	(32.50%)	(0.00%)	(30.44%)	(0.00%)	(9.96%)	(0.00%)
20	3	3	✓	7	(29.18%)	(0.00%)	(23.52%)	(0.00%)	(11.14%)	(0.00%)
20	3	3	✓	8	(49.04%)	(0.00%)	(21.27%)	(0.00%)	(9.50%)	(0.00%)
20	3	3	✓	9	(17.18%)	(0.00%)	(12.98%)	(0.00%)	(4.43%)	(0.00%)
20	3	3	✓	10	(90.14%)	(0.00%)	(39.38%)	(0.00%)	(25.24%)	(0.00%)
20	3	5		1	(0.67%)	(0.00%)	1800.72	1141.88k	1800.00	137.58k
20	3	5		2	(0.41%)	(0.00%)	193.05	18.52k	1800.00	492.81k
20	3	5		3	(0.00%)	(99.90%)	(0.00%)	(99.90%)	1800.00	421.41k
20	3	5		4	(0.99%)	(0.00%)	1801.03	1214.89k	1800.00	199.65k
20	3	5		6	232.05	167.23k	10.10	690	18.03	45
20	3	5		7	1800.58	1098.69k	1801.37	2018.22k	1800.00	264.67k
20	3	5		8	(0.15%)	(0.00%)	1800.00	792.29k	1800.00	240.34k
20	3	5		9	(0.38%)	(0.40%)	1800.00	448.81k	1800.00	231.38k
20	3	5		10	(0.13%)	(0.00%)	(0.08%)	(0.00%)	(0.03%)	(0.03%)
20	3	5	✓	1	(59.51%)	(0.00%)	(49.14%)	(0.00%)	(25.18%)	(0.00%)
20	3	5	✓	2	(∞)	(0.00%)	(53.96%)	(0.00%)	(14.05%)	(0.00%)
20	3	5	✓	3	(21.97%)	(0.00%)	(9.52%)	(0.00%)	(1.58%)	(0.00%)
20	3	5	✓	4	(70.59%)	(0.00%)	(53.88%)	(0.00%)	(21.31%)	(0.00%)
20	3	5	✓	5	(85.11%)	(0.00%)	(62.50%)	(0.00%)	(17.83%)	(0.00%)
20	3	5	✓	6	(59.32%)	(0.00%)	(36.52%)	(0.00%)	(13.64%)	(0.00%)
20	3	5	✓	7	(40.79%)	(0.00%)	(26.81%)	(0.00%)	(5.17%)	(0.00%)
20	3	5	✓	8	(48.41%)	(0.00%)	(33.00%)	(0.00%)	(8.88%)	(0.00%)
20	3	5	✓	9	(65.95%)	(0.00%)	(51.35%)	(0.00%)	(21.16%)	(0.00%)
20	3	5	✓	10	(12.39%)	(0.00%)	(5.34%)	(0.00%)	(2.41%)	(0.00%)
20	3	10		1	(39.21%)	(0.00%)	(4.45%)	(0.00%)	1800.00	63.28k
20	3	10		2	(8.58%)	(∞)	(2.31%)	(∞)	1800.00	139.48k
20	3	10		3	(6.97%)	(0.00%)	1800.28	61.19k	1800.00	55.10k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	3	10		4	(0.67%)	(0.00%)	(0.14%)	(0.00%)	(0.03%)	(0.00%)
20	3	10		5	(74.77%)	(0.00%)	(59.70%)	(0.00%)	(49.18%)	(0.00%)
20	3	10		6	1800.49	816.80k	1800.86	647.67k	1800.00	139.20k
20	3	10		8	(59.52%)	(∞)	(1.14%)	(0.81%)	(0.02%)	(0.00%)
20	3	10		9	(0.27%)	(0.00%)	(0.06%)	(66.90%)	1800.00	298.83k
20	3	10		10	(0.58%)	(0.00%)	(0.00%)	(85.57%)	37.76	25
20	3	10	✓	1	(51.83%)	(0.00%)	(35.41%)	(0.00%)	(7.39%)	(0.00%)
20	3	10	✓	2	(67.13%)	(0.00%)	(40.44%)	(0.00%)	(6.30%)	(0.00%)
20	3	10	✓	3	(31.85%)	(0.00%)	(16.81%)	(0.00%)	(2.21%)	(0.00%)
20	3	10	✓	4	(60.88%)	(0.00%)	(36.59%)	(0.00%)	(4.42%)	(0.00%)
20	3	10	✓	5	(52.17%)	(0.00%)	(33.34%)	(0.00%)	(5.09%)	(0.00%)
20	3	10	✓	6	(37.87%)	(0.00%)	(26.50%)	(0.00%)	(6.76%)	(0.00%)
20	3	10	✓	7	(∞)	(0.00%)	(61.31%)	(0.00%)	(14.39%)	(0.00%)
20	3	10	✓	8	(55.09%)	(0.00%)	(39.60%)	(0.00%)	(10.50%)	(0.00%)
20	3	10	✓	9	(25.16%)	(0.00%)	(13.76%)	(0.00%)	(8.04%)	(0.00%)
20	3	10	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	4	1		1	(7.71%)	(0.00%)	1.09	27	1.94	9
20	4	1		2	(0.00%)	(∞)	1.61	5	1.01	7
20	4	1		3	(0.10%)	(17.51%)	1800.00	1542.08k	1800.00	618.04k
20	4	1		4	1800.01	2134.64k	(0.00%)	(0.02%)	6.88	23
20	4	1		5	(0.01%)	(∞)	1803.32	4498.48k	14.14	77
20	4	1		6	(0.11%)	(∞)	3.34	1.52k	3.19	43
20	4	1		7	(0.00%)	(∞)	(0.00%)	(∞)	0.08	1
20	4	1		8	4.16	9.42k	0.37	1	1800.00	651.07k
20	4	1		9	(0.47%)	(0.00%)	65.50	127.86k	1800.00	599.12k
20	4	1		10	(4.07%)	(∞)	(0.09%)	(0.59%)	1800.00	1022.03k
20	4	1	✓	1	(66.74%)	(0.00%)	(56.19%)	(0.00%)	(29.23%)	(0.00%)
20	4	1	✓	2	(28.10%)	(0.00%)	(15.30%)	(0.00%)	(2.17%)	(0.00%)
20	4	1	✓	3	(61.81%)	(0.00%)	(49.05%)	(0.00%)	(21.49%)	(0.00%)
20	4	1	✓	4	(22.32%)	(0.00%)	(15.24%)	(0.00%)	(1.10%)	(0.00%)
20	4	1	✓	5	(24.70%)	(0.00%)	(17.08%)	(0.00%)	(5.95%)	(0.00%)
20	4	1	✓	6	(61.82%)	(0.00%)	(44.41%)	(0.00%)	(24.17%)	(0.00%)
20	4	1	✓	7	(16.94%)	(0.00%)	(12.07%)	(0.00%)	(4.46%)	(0.00%)
20	4	1	✓	8	(∞)	(0.00%)	(59.36%)	(0.00%)	(25.42%)	(0.00%)
20	4	1	✓	9	(∞)	(0.00%)	(∞)	(0.00%)	(48.47%)	(0.00%)
20	4	1	✓	10	(44.96%)	(0.00%)	(24.97%)	(0.00%)	(4.52%)	(0.00%)
20	4	2		1	1800.73	1999.03k	5.57	1.02k	15.51	69
20	4	2		2	(0.55%)	(0.00%)	5.82	353	4.94	5
20	4	2		3	(1.58%)	(0.00%)	1800.00	1043.38k	1800.00	260.55k
20	4	2		4	(1.38%)	(27.52%)	(0.00%)	(∞)	8.61	45
20	4	2		5	(0.20%)	(0.63%)	(0.00%)	(∞)	1800.00	330.00k
20	4	2		6	(0.52%)	(0.00%)	1800.98	1756.97k	32.55	637
20	4	2		7	(0.40%)	(∞)	(0.06%)	(∞)	1800.00	255.12k
20	4	2		8	(0.04%)	(0.30%)	(0.00%)	(0.67%)	1800.00	255.96k
20	4	2		9	(0.06%)	(0.00%)	1800.00	904.24k	12.57	63
20	4	2		10	(0.75%)	(0.00%)	(0.28%)	(0.00%)	(0.08%)	(0.00%)
20	4	2	✓	1	(56.43%)	(0.00%)	(40.00%)	(0.00%)	(18.75%)	(0.00%)
20	4	2	✓	2	(42.97%)	(0.00%)	(31.13%)	(0.00%)	(12.68%)	(0.00%)
20	4	2	✓	3	(75.60%)	(0.00%)	(58.43%)	(0.00%)	(24.77%)	(0.00%)
20	4	2	✓	4	(∞)	(0.00%)	(67.51%)	(0.00%)	(19.11%)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	4	2	✓	5	(91.13%)	(0.00%)	(85.10%)	(0.00%)	(68.83%)	(0.00%)
20	4	2	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(46.93%)	(0.00%)
20	4	2	✓	7	(56.33%)	(0.00%)	(41.81%)	(0.00%)	(29.03%)	(0.00%)
20	4	2	✓	8	(94.02%)	(0.00%)	(89.35%)	(0.00%)	(75.13%)	(0.00%)
20	4	2	✓	9	(∞)	(0.00%)	(98.79%)	(0.00%)	(29.30%)	(0.00%)
20	4	2	✓	10	(74.15%)	(0.00%)	(40.64%)	(0.00%)	(14.53%)	(0.00%)
20	4	3		1	(2.89%)	(0.06%)	1802.12	1155.69k	1800.00	230.78k
20	4	3		2	(1.20%)	(∞)	(0.06%)	(0.00%)	37.62	169
20	4	3		3	1800.00	348.78k	1800.98	1243.98k	29.92	7
20	4	3		4	(5.69%)	(0.00%)	1800.92	1209.10k	1800.00	88.70k
20	4	3		5	(49.47%)	(∞)	(14.32%)	(∞)	(0.46%)	(0.00%)
20	4	3		6	(0.01%)	(∞)	(0.00%)	(0.08%)	9.25	117
20	4	3		7	(0.44%)	(3.16%)	1800.00	120.03k	1800.00	110.95k
20	4	3		8	1800.26	840.01k	1801.14	1899.83k	1800.00	294.14k
20	4	3		9	(0.63%)	(0.01%)	(0.16%)	(1.52%)	1800.00	90.42k
20	4	3		10	(5.69%)	(2.86%)	(0.22%)	(1.69%)	1800.00	330.79k
20	4	3	✓	1	(55.49%)	(0.00%)	(44.84%)	(0.00%)	(10.64%)	(0.00%)
20	4	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(27.64%)	(0.00%)
20	4	3	✓	3	(∞)	(0.00%)	(71.23%)	(0.00%)	(23.70%)	(0.00%)
20	4	3	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	4	3	✓	5	(76.87%)	(0.00%)	(40.73%)	(0.00%)	(13.74%)	(0.00%)
20	4	3	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(97.50%)	(0.00%)
20	4	3	✓	7	(87.33%)	(0.00%)	(78.12%)	(0.00%)	(46.20%)	(0.00%)
20	4	3	✓	8	(∞)	(0.00%)	(90.63%)	(0.00%)	(27.30%)	(0.00%)
20	4	3	✓	9	(∞)	(0.00%)	(73.03%)	(0.00%)	(25.40%)	(0.00%)
20	4	3	✓	10	(84.33%)	(0.00%)	(40.95%)	(0.00%)	(15.30%)	(0.00%)
20	4	5		1	(6.34%)	(0.00%)	(0.26%)	(0.00%)	(0.40%)	(0.00%)
20	4	5		2	(∞)	(99.80%)	(∞)	(99.80%)	(52.40%)	(0.00%)
20	4	5		3	(∞)	(99.71%)	(47.61%)	(99.71%)	(5.63%)	(0.00%)
20	4	5		4	(7.30%)	(∞)	(0.21%)	(0.00%)	(0.03%)	(0.00%)
20	4	5		5	(9.57%)	(6.86%)	(2.83%)	(99.58%)	1800.00	537.05k
20	4	5		6	(44.22%)	(0.00%)	1796.13	11.20k	1800.00	17.02k
20	4	5		8	(23.92%)	(1.15%)	(11.93%)	(∞)	(3.94%)	(0.00%)
20	4	5		9	(12.86%)	(∞)	(5.56%)	(∞)	(0.07%)	(0.00%)
20	4	5		10	(24.82%)	(∞)	(7.09%)	(∞)	(3.85%)	(0.00%)
20	4	5	✓	1	(73.86%)	(0.00%)	(60.47%)	(0.00%)	(31.09%)	(0.00%)
20	4	5	✓	2	(∞)	(0.00%)	(83.02%)	(0.00%)	(39.71%)	(0.00%)
20	4	5	✓	3	(69.88%)	(0.00%)	(43.20%)	(0.00%)	(15.54%)	(0.00%)
20	4	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(82.13%)	(0.00%)
20	4	5	✓	5	(41.34%)	(0.00%)	(20.57%)	(0.00%)	(3.26%)	(0.00%)
20	4	5	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(76.03%)	(0.00%)
20	4	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(77.79%)	(0.00%)
20	4	5	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	4	5	✓	9	(∞)	(0.00%)	(86.90%)	(0.00%)	(33.72%)	(0.00%)
20	4	5	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10		1	(54.65%)	(∞)	(13.91%)	(10.10%)	(4.95%)	(0.00%)
20	4	10		2	(63.60%)	(∞)	(26.33%)	(0.00%)	(11.12%)	(0.21%)
20	4	10		3	(46.81%)	(0.00%)	(11.18%)	(0.00%)	(4.30%)	(0.00%)
20	4	10		4	(46.94%)	(0.00%)	(27.34%)	(0.00%)	(27.35%)	(0.92%)
20	4	10		5	(∞)	(11.60%)	(∞)	(11.60%)	(∞)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	4	10		6	(30.99%)	(47.10%)	(5.94%)	(99.49%)	(0.06%)	(0.00%)
20	4	10		7	(13.90%)	(∞)	(8.92%)	(∞)	(8.66%)	(0.00%)
20	4	10		8	(87.41%)	(0.00%)	(3.08%)	(0.00%)	(0.19%)	(0.00%)
20	4	10		9	(21.87%)	(1.17%)	(7.72%)	(0.00%)	(3.00%)	(0.00%)
20	4	10		10	(∞)	(100.00%)	(∞)	(100.00%)	(∞)	(0.00%)
20	4	10	✓	1	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	2	(∞)	(0.00%)	(80.42%)	(0.00%)	(19.31%)	(0.00%)
20	4	10	✓	3	(∞)	(0.00%)	(73.66%)	(0.00%)	(40.83%)	(0.00%)
20	4	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(34.05%)	(0.00%)
20	4	10	✓	5	(∞)	(0.00%)	(63.58%)	(0.00%)	(16.10%)	(0.00%)
20	4	10	✓	6	(69.81%)	(0.00%)	(45.17%)	(0.00%)	(15.95%)	(0.00%)
20	4	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	8	(75.29%)	(0.00%)	(38.99%)	(0.00%)	(9.49%)	(0.00%)
20	4	10	✓	9	(73.12%)	(0.00%)	(71.82%)	(0.00%)	(54.18%)	(0.00%)
20	4	10	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(51.40%)	(0.00%)
20	5	1		1	(1.44%)	(0.00%)	(0.11%)	(0.00%)	1800.00	265.66k
20	5	1		2	(0.21%)	(0.00%)	1802.89	4578.70k	1800.00	414.62k
20	5	1		3	2.51	960	114.61	317.50k	2.52	5
20	5	1		5	581.93	508.50k	5.08	968	10.94	149
20	5	1		6	4.87	3.93k	1.22	69	1.09	1
20	5	1		7	(0.19%)	(∞)	(0.02%)	(∞)	2.10	3
20	5	1		8	(2.23%)	(∞)	1800.00	1316.26k	1800.00	481.07k
20	5	1		9	(0.47%)	(0.00%)	1806.12	2723.69k	1800.00	340.26k
20	5	1		10	1800.01	1947.07k	(0.00%)	(0.15%)	1800.00	2067.20k
20	5	1	✓	1	(65.59%)	(0.00%)	(43.00%)	(0.00%)	(11.77%)	(0.00%)
20	5	1	✓	2	(54.65%)	(0.00%)	(22.93%)	(0.00%)	(6.42%)	(0.00%)
20	5	1	✓	3	(52.93%)	(0.00%)	(31.60%)	(0.00%)	(17.58%)	(0.00%)
20	5	1	✓	4	(59.31%)	(0.00%)	(30.88%)	(0.00%)	(17.84%)	(0.00%)
20	5	1	✓	5	(62.48%)	(0.00%)	(37.11%)	(0.00%)	(19.84%)	(0.00%)
20	5	1	✓	6	(52.56%)	(0.00%)	(26.16%)	(0.00%)	(5.72%)	(0.00%)
20	5	1	✓	7	(39.77%)	(0.00%)	(25.85%)	(0.00%)	(10.44%)	(0.00%)
20	5	1	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(29.27%)	(0.00%)
20	5	1	✓	9	(74.84%)	(0.00%)	(55.94%)	(0.00%)	(27.33%)	(0.00%)
20	5	1	✓	10	(97.45%)	(0.00%)	(95.33%)	(0.00%)	(86.91%)	(0.00%)
20	5	2		1	(14.02%)	(∞)	(0.05%)	(∞)	1800.00	131.52k
20	5	2		2	(4.92%)	(9.28%)	(7.56%)	(99.41%)	(0.06%)	(0.00%)
20	5	2		3	(1.14%)	(0.00%)	49.91	827	1800.00	141.54k
20	5	2		4	(0.36%)	(∞)	(0.03%)	(∞)	1800.00	167.89k
20	5	2		5	(9.26%)	(1.31%)	(0.42%)	(0.31%)	(0.02%)	(0.00%)
20	5	2		6	(29.34%)	(17.11%)	(2.45%)	(∞)	402.10	2.23k
20	5	2		7	(3.44%)	(0.00%)	425.28	7.57k	1800.00	72.84k
20	5	2		8	(3.85%)	(0.00%)	(0.11%)	(0.00%)	1800.00	89.39k
20	5	2		9	(4.74%)	(∞)	1800.00	141.41k	1800.00	61.94k
20	5	2		10	(0.02%)	(∞)	1802.31	131.06k	1800.00	77.65k
20	5	2	✓	1	(69.77%)	(0.00%)	(57.13%)	(0.00%)	(34.66%)	(0.00%)
20	5	2	✓	2	(53.06%)	(0.00%)	(28.58%)	(0.00%)	(8.99%)	(0.00%)
20	5	2	✓	3	(56.39%)	(0.00%)	(43.29%)	(0.00%)	(22.06%)	(0.00%)
20	5	2	✓	4	(59.92%)	(0.00%)	(38.52%)	(0.00%)	(18.56%)	(0.00%)
20	5	2	✓	5	(52.16%)	(0.00%)	(31.73%)	(0.00%)	(8.47%)	(0.00%)
20	5	2	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	5	2	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	5	2	✓	8	(78.18%)	(0.00%)	(73.59%)	(0.00%)	(48.79%)	(0.00%)
20	5	2	✓	9	(63.45%)	(0.00%)	(43.59%)	(0.00%)	(26.80%)	(0.00%)
20	5	2	✓	10	(73.38%)	(0.00%)	(47.71%)	(0.00%)	(34.00%)	(0.00%)
20	5	3		1	(29.82%)	(∞)	(1.56%)	(4.09%)	(0.11%)	(0.00%)
20	5	3		2	3.54	7	1.98	11	2.75	29
20	5	3		3	(0.02%)	(96.89%)	1800.00	306.90k	1800.00	88.81k
20	5	3		4	(5.07%)	(0.00%)	(0.26%)	(0.00%)	(0.04%)	(0.00%)
20	5	3		5	(9.27%)	(0.26%)	(2.02%)	(∞)	(0.20%)	(0.00%)
20	5	3		6	(11.97%)	(∞)	(1.31%)	(∞)	1800.00	13.20k
20	5	3		7	1800.00	491.92k	48.05	903	153.11	647
20	5	3		8	(1.80%)	(0.06%)	(0.74%)	(∞)	(0.02%)	(0.00%)
20	5	3		9	(61.22%)	(98.51%)	(3.99%)	(98.51%)	1800.00	56.43k
20	5	3		10	(21.56%)	(0.00%)	(3.24%)	(0.00%)	(2.26%)	(0.00%)
20	5	3	✓	1	(∞)	(0.00%)	(∞)	(0.00%)	(66.33%)	(0.00%)
20	5	3	✓	2	(24.77%)	(0.00%)	(6.97%)	(0.00%)	(2.58%)	(0.00%)
20	5	3	✓	3	(53.71%)	(0.00%)	(23.64%)	(0.00%)	(9.83%)	(0.00%)
20	5	3	✓	4	(83.36%)	(0.00%)	(70.88%)	(0.00%)	(52.52%)	(0.00%)
20	5	3	✓	5	(62.52%)	(0.00%)	(30.90%)	(0.00%)	(13.60%)	(0.00%)
20	5	3	✓	6	(71.01%)	(0.00%)	(60.66%)	(0.00%)	(40.53%)	(0.00%)
20	5	3	✓	7	(99.82%)	(0.00%)	(59.66%)	(0.00%)	(25.57%)	(0.00%)
20	5	3	✓	8	(41.67%)	(0.00%)	(29.86%)	(0.00%)	(12.62%)	(0.00%)
20	5	3	✓	9	(64.29%)	(0.00%)	(45.02%)	(0.00%)	(20.37%)	(0.00%)
20	5	3	✓	10	(∞)	(0.00%)	(59.41%)	(0.00%)	(26.72%)	(0.00%)
20	5	5		1	(58.24%)	(0.00%)	(33.63%)	(0.00%)	(31.12%)	(0.00%)
20	5	5		2	(34.84%)	(99.47%)	(5.10%)	(99.47%)	(0.11%)	(0.00%)
20	5	5		3	(31.91%)	(7.42%)	1800.00	192.48k	1800.00	41.23k
20	5	5		4	(82.33%)	(∞)	(42.92%)	(99.01%)	(35.41%)	(0.00%)
20	5	5		5	(0.13%)	(97.64%)	(0.02%)	(97.64%)	(0.02%)	(0.00%)
20	5	5		6	(36.73%)	(∞)	(1.85%)	(0.06%)	(0.80%)	(0.00%)
20	5	5		7	(16.07%)	(∞)	(5.00%)	(∞)	1800.00	8.82k
20	5	5		8	(9.88%)	(∞)	305.74	877	1800.00	7.26k
20	5	5		9	(∞)	(98.45%)	(48.22%)	(98.45%)	(28.12%)	(0.00%)
20	5	5		10	(∞)	(99.48%)	(∞)	(99.48%)	(∞)	(0.00%)
20	5	5	✓	1	(94.95%)	(0.00%)	(91.51%)	(0.00%)	(87.44%)	(0.00%)
20	5	5	✓	2	(81.23%)	(0.00%)	(48.80%)	(0.00%)	(23.25%)	(0.00%)
20	5	5	✓	3	(80.32%)	(0.00%)	(36.68%)	(0.00%)	(16.90%)	(0.00%)
20	5	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(71.24%)	(0.00%)
20	5	5	✓	5	(60.12%)	(0.00%)	(25.82%)	(0.00%)	(12.63%)	(0.00%)
20	5	5	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(62.52%)	(0.00%)
20	5	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(54.09%)	(0.00%)
20	5	5	✓	8	(78.45%)	(0.00%)	(66.46%)	(0.00%)	(39.59%)	(0.00%)
20	5	5	✓	9	(∞)	(0.00%)	(59.07%)	(0.00%)	(32.06%)	(0.00%)
20	5	5	✓	10	(∞)	(0.00%)	(86.84%)	(0.00%)	(47.33%)	(0.00%)
20	5	10		1	(25.07%)	(0.00%)	(12.78%)	(0.00%)	(5.90%)	(0.55%)
20	5	10		2	(28.77%)	(0.00%)	(16.37%)	(0.00%)	(7.82%)	(0.00%)
20	5	10		3	(75.05%)	(∞)	(37.01%)	(∞)	(8.01%)	(0.00%)
20	5	10		4	(∞)	(57.61%)	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10		5	(77.51%)	(0.00%)	(61.88%)	(0.00%)	(57.97%)	(0.01%)
20	5	10		6	(45.84%)	(0.00%)	(17.31%)	(0.00%)	(16.64%)	(0.04%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	5	10		7	(62.85%)	(0.00%)	(30.53%)	(0.00%)	(20.15%)	(0.27%)
20	5	10		8	(∞)	(97.86%)	(51.79%)	(97.86%)	(28.84%)	(0.00%)
20	5	10		9	(80.09%)	(∞)	(66.83%)	(∞)	(59.80%)	(0.00%)
20	5	10		10	(∞)	(99.55%)	(16.29%)	(99.55%)	(0.08%)	(0.00%)
20	5	10	✓	1	(84.50%)	(0.00%)	(54.33%)	(0.00%)	(21.90%)	(0.00%)
20	5	10	✓	2	(60.81%)	(0.00%)	(42.62%)	(0.00%)	(11.83%)	(0.00%)
20	5	10	✓	3	(∞)	(0.27%)	(67.17%)	(0.00%)	(29.22%)	(0.00%)
20	5	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(59.01%)	(0.00%)
20	5	10	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	6	(87.01%)	(0.00%)	(48.87%)	(0.00%)	(19.90%)	(0.00%)
20	5	10	✓	7	(80.83%)	(0.00%)	(54.65%)	(0.00%)	(18.66%)	(0.00%)
20	5	10	✓	8	(∞)	(0.00%)	(63.07%)	(0.00%)	(29.28%)	(0.00%)
20	5	10	✓	9	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	10	(∞)	(0.00%)	(65.23%)	(0.00%)	(37.11%)	(0.00%)
30	2	1		1	0.08	1	0.01	1	0.04	1
30	2	1		2	0.12	1	0.07	1	1800.00	12249.13k
30	2	1		3	0.05	1	0.10	1	0.06	1
30	2	1		4	1800.05	21326.47k	0.19	1	1800.00	7397.48k
30	2	1		5	0.07	1	0.07	1	0.04	1
30	2	1		6	0.11	1	0.07	1	0.10	1
30	2	1		7	0.08	1	0.08	1	0.05	1
30	2	1		8	0.08	1	0.11	1	0.09	1
30	2	1		9	0.37	17	0.07	1	0.04	1
30	2	1		10	0.11	1	0.07	1	1800.00	6296.47k
30	2	1	✓	1	(16.93%)	(0.00%)	(10.17%)	(0.00%)	1800.00	195.74k
30	2	1	✓	2	(23.96%)	(0.00%)	(20.10%)	(0.00%)	(8.92%)	(0.00%)
30	2	1	✓	3	(25.17%)	(0.00%)	(11.74%)	(0.00%)	1800.00	390.50k
30	2	1	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(0.08%)	(0.00%)
30	2	1	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	1800.00	382.03k
30	2	1	✓	6	(9.88%)	(0.00%)	(4.69%)	(0.00%)	1800.00	324.46k
30	2	1	✓	7	(36.68%)	(0.00%)	(17.30%)	(0.00%)	1800.00	224.61k
30	2	1	✓	8	(20.29%)	(0.00%)	(12.29%)	(0.00%)	1800.00	299.31k
30	2	1	✓	9	(11.45%)	(0.00%)	(6.83%)	(0.00%)	1800.00	338.15k
30	2	1	✓	10	(10.67%)	(0.00%)	(1.52%)	(0.00%)	1800.00	337.45k
30	2	2		1	0.38	113	0.50	19	1800.00	5368.85k
30	2	2		2	1805.06	16609.83k	0.94	484	1800.00	5559.32k
30	2	2		3	0.21	11	0.05	1	0.07	1
30	2	2		4	0.11	1	0.05	1	0.11	1
30	2	2		5	0.35	37	0.52	11	1800.00	6456.81k
30	2	2		6	2.41	4.83k	0.99	228	0.19	7
30	2	2		7	0.94	163	1800.03	9176.28k	0.16	4
30	2	2		8	0.10	1	1.29	5.27k	0.10	1
30	2	2		9	0.47	49	0.58	55	0.22	19
30	2	2		10	0.69	241	0.84	156	0.16	1
30	2	2	✓	1	(∞)	(0.00%)	(96.62%)	(0.00%)	(0.03%)	(0.00%)
30	2	2	✓	2	(26.90%)	(0.00%)	(21.30%)	(0.00%)	1800.00	323.25k
30	2	2	✓	3	(8.17%)	(0.00%)	(5.68%)	(0.00%)	1800.00	296.47k
30	2	2	✓	4	(35.42%)	(0.00%)	(35.74%)	(0.00%)	(0.03%)	(0.00%)
30	2	2	✓	5	(9.20%)	(0.00%)	(7.50%)	(0.00%)	1800.00	217.61k
30	2	2	✓	6	(28.50%)	(0.00%)	(21.38%)	(0.00%)	1800.00	99.08k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	2	2	✓	7	(35.66%)	(0.00%)	(27.93%)	(0.00%)	(3.55%)	(0.00%)
30	2	2	✓	8	(12.82%)	(0.00%)	(7.34%)	(0.00%)	1800.00	326.89k
30	2	2	✓	9	(26.84%)	(0.00%)	(29.14%)	(0.00%)	(11.61%)	(0.00%)
30	2	2	✓	10	(8.27%)	(0.00%)	(5.03%)	(0.00%)	1800.00	282.09k
30	2	3		1	0.53	111	0.70	61	0.18	7
30	2	3		2	1800.03	20087.39k	1805.88	8573.53k	0.17	7
30	2	3		3	1.33	594	1808.21	8128.69k	1800.00	4461.48k
30	2	3		4	1801.97	11069.00k	0.90	255	1800.00	4748.16k
30	2	3		5	1.13	442	2.09	1.89k	0.07	1
30	2	3		6	1.26	370	1.47	451	0.09	1
30	2	3		7	0.59	201	0.32	223	1800.00	5833.60k
30	2	3		8	1.21	1.61k	1.00	167	0.06	1
30	2	3		9	1800.05	22776.98k	0.40	110	0.13	1
30	2	3		10	0.47	80	0.17	3	0.09	1
30	2	3	✓	1	(14.44%)	(0.00%)	(14.14%)	(0.00%)	1800.00	212.41k
30	2	3	✓	2	(30.57%)	(0.00%)	(34.58%)	(0.00%)	1800.00	236.13k
30	2	3	✓	3	(59.95%)	(0.00%)	(55.63%)	(0.00%)	(0.05%)	(0.00%)
30	2	3	✓	4	(6.60%)	(0.00%)	(2.06%)	(0.00%)	1800.00	368.71k
30	2	3	✓	5	(29.05%)	(0.00%)	(26.92%)	(0.00%)	1800.00	233.03k
30	2	3	✓	6	(74.14%)	(0.00%)	(76.45%)	(0.00%)	(0.06%)	(0.00%)
30	2	3	✓	7	(6.39%)	(0.00%)	(4.63%)	(0.00%)	1800.00	241.54k
30	2	3	✓	8	(22.26%)	(0.00%)	(20.22%)	(0.00%)	1800.00	188.67k
30	2	3	✓	9	(20.05%)	(0.00%)	(19.08%)	(0.00%)	1800.00	101.61k
30	2	3	✓	10	(36.83%)	(0.00%)	(31.79%)	(0.00%)	1800.00	271.99k
30	2	5		1	1800.06	9570.90k	24.38	67.02k	1800.00	5434.94k
30	2	5		2	1800.05	5131.55k	18.88	2.91k	1800.00	4169.59k
30	2	5		3	1800.05	13378.63k	0.74	143	0.11	5
30	2	5		4	1800.02	3108.27k	77.69	27.96k	1800.00	3978.53k
30	2	5		5	1800.03	5273.22k	1803.17	4673.03k	0.43	29
30	2	5		6	1800.04	9215.55k	5.17	708	0.23	3
30	2	5		7	3.31	713	3.32	721	0.10	1
30	2	5		8	1800.01	2791.90k	1801.03	1738.59k	1800.00	2678.42k
30	2	5		9	1.22	289	892.30	2095.61k	0.14	1
30	2	5		10	1801.96	9326.53k	42.95	143.91k	1800.00	5221.27k
30	2	5	✓	1	(36.11%)	(0.00%)	(37.35%)	(0.00%)	1800.00	257.83k
30	2	5	✓	2	(55.75%)	(0.00%)	(44.81%)	(0.00%)	(3.17%)	(0.00%)
30	2	5	✓	3	(33.58%)	(0.00%)	(29.89%)	(0.00%)	1800.00	215.99k
30	2	5	✓	4	(38.95%)	(0.00%)	(42.23%)	(0.00%)	(1.74%)	(0.00%)
30	2	5	✓	5	(18.71%)	(0.00%)	(17.56%)	(0.00%)	1800.00	315.37k
30	2	5	✓	6	(60.64%)	(0.00%)	(57.12%)	(0.00%)	1800.00	230.43k
30	2	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(0.05%)	(0.00%)
30	2	5	✓	8	(83.45%)	(0.00%)	(85.65%)	(0.00%)	(28.00%)	(0.00%)
30	2	5	✓	9	(13.24%)	(0.00%)	(14.25%)	(0.00%)	1800.00	284.40k
30	2	5	✓	10	(26.22%)	(0.00%)	(20.89%)	(0.00%)	1800.00	258.29k
30	2	10		1	1800.06	7437.59k	1800.96	2315.22k	1800.00	4249.12k
30	2	10		2	1800.04	6020.82k	1800.60	1508.79k	1800.00	3084.66k
30	2	10		3	1800.03	4256.76k	1800.82	2165.34k	1800.00	5112.28k
30	2	10		4	1800.03	3630.57k	125.97	5.74k	0.41	3
30	2	10		5	1800.04	5166.44k	1800.74	1448.64k	0.37	5
30	2	10		6	17.31	25.00k	12.08	1.89k	0.16	1

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	2	10		7	1802.02	7471.73k	11.50	3.25k	1800.00	6430.47k
30	2	10		8	1801.66	4027.98k	1800.69	1348.08k	0.49	3
30	2	10		9	1800.05	8809.23k	7.24	2.15k	1800.00	9258.85k
30	2	10		10	1800.04	3687.27k	1801.40	2262.45k	1800.00	4141.98k
30	2	10	✓	1	(11.10%)	(0.00%)	(16.53%)	(0.00%)	1800.00	210.17k
30	2	10	✓	2	(43.24%)	(0.00%)	(48.70%)	(0.00%)	1800.00	235.15k
30	2	10	✓	3	(31.31%)	(0.00%)	(31.61%)	(0.00%)	1800.00	190.03k
30	2	10	✓	4	(21.16%)	(0.00%)	(26.15%)	(0.00%)	1800.00	312.57k
30	2	10	✓	5	(56.42%)	(0.00%)	(67.92%)	(0.00%)	(7.13%)	(0.00%)
30	2	10	✓	6	(19.38%)	(0.00%)	(19.81%)	(0.00%)	1800.00	38.24k
30	2	10	✓	7	(7.68%)	(0.00%)	(7.64%)	(0.00%)	1800.00	232.91k
30	2	10	✓	8	(17.97%)	(0.00%)	(21.51%)	(0.00%)	1800.00	227.46k
30	2	10	✓	9	(18.76%)	(0.00%)	(22.44%)	(0.00%)	(0.04%)	(0.00%)
30	2	10	✓	10	(61.81%)	(0.00%)	(73.76%)	(0.00%)	1800.00	212.94k
30	3	1		1	(0.00%)	(1.17%)	1.73	9	1.07	19
30	3	1		2	(0.09%)	(0.00%)	(0.00%)	(1.94%)	2.52	43
30	3	1		3	(0.00%)	(∞)	1.73	1	0.20	7
30	3	1		4	(0.38%)	(∞)	19.16	34.17k	1800.00	1229.66k
30	3	1		5	(0.41%)	(∞)	(0.43%)	(0.00%)	1800.00	619.29k
30	3	1		6	713.92	3523.37k	0.81	1	0.17	1
30	3	1		7	(0.00%)	(∞)	1.49	170	1.53	29
30	3	1		8	1.46	59	0.44	1	0.10	1
30	3	1		9	(0.00%)	(∞)	1.16	22	0.20	1
30	3	1		10	35.57	33.21k	1800.03	6057.41k	1800.00	1084.08k
30	3	1	✓	1	(83.81%)	(0.00%)	(80.15%)	(0.00%)	(55.23%)	(0.00%)
30	3	1	✓	2	(35.73%)	(0.00%)	(32.21%)	(0.00%)	(12.18%)	(0.00%)
30	3	1	✓	3	(28.31%)	(0.00%)	(22.97%)	(0.00%)	(16.12%)	(0.00%)
30	3	1	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(52.24%)	(0.00%)
30	3	1	✓	5	(42.06%)	(0.00%)	(39.98%)	(0.00%)	(15.62%)	(0.00%)
30	3	1	✓	6	(15.92%)	(0.00%)	(13.48%)	(0.00%)	(3.95%)	(0.00%)
30	3	1	✓	7	(58.23%)	(0.00%)	(56.65%)	(0.00%)	(26.73%)	(0.00%)
30	3	1	✓	8	(71.38%)	(0.00%)	(52.05%)	(0.00%)	(20.67%)	(0.00%)
30	3	1	✓	9	(28.65%)	(0.00%)	(24.91%)	(0.00%)	(11.29%)	(0.00%)
30	3	1	✓	10	(15.78%)	(0.00%)	(14.95%)	(0.00%)	(5.48%)	(0.00%)
30	3	2		1	(0.20%)	(0.00%)	1802.73	1544.23k	1800.00	486.85k
30	3	2		2	(0.00%)	(5.74%)	17.26	555	2.91	17
30	3	2		3	(5.72%)	(∞)	(0.94%)	(0.13%)	(0.30%)	(0.00%)
30	3	2		4	(2.05%)	(∞)	(2.06%)	(∞)	1800.00	598.39k
30	3	2		5	1800.02	3155.18k	1800.03	4085.31k	1800.00	551.72k
30	3	2		6	(0.42%)	(0.00%)	(0.38%)	(0.00%)	1800.00	576.73k
30	3	2		7	1802.81	5960.28k	2.72	167	1800.00	691.72k
30	3	2		8	(0.10%)	(∞)	1800.01	1516.30k	1800.00	371.38k
30	3	2		9	1800.02	2162.37k	(0.00%)	(0.03%)	14.47	563
30	3	2		10	(0.13%)	(0.00%)	1803.79	1962.69k	1800.00	235.35k
30	3	2	✓	1	(66.89%)	(0.00%)	(53.49%)	(0.00%)	(19.56%)	(0.00%)
30	3	2	✓	2	(88.79%)	(0.00%)	(86.56%)	(0.00%)	(64.88%)	(0.00%)
30	3	2	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(27.73%)	(0.00%)
30	3	2	✓	4	(43.57%)	(0.00%)	(35.66%)	(0.00%)	(11.21%)	(0.00%)
30	3	2	✓	5	(89.51%)	(0.00%)	(88.58%)	(0.00%)	(69.48%)	(0.00%)
30	3	2	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	3	2	✓	7	(83.84%)	(0.00%)	(57.64%)	(0.00%)	(25.69%)	(0.00%)
30	3	2	✓	8	(63.27%)	(0.00%)	(56.44%)	(0.00%)	(24.81%)	(0.00%)
30	3	2	✓	9	(37.16%)	(0.00%)	(32.34%)	(0.00%)	(12.01%)	(0.00%)
30	3	2	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(67.10%)	(0.00%)
30	3	3		1	(0.06%)	(0.00%)	1801.84	954.20k	1800.00	378.25k
30	3	3		2	(1.36%)	(∞)	1800.00	493.67k	1800.00	333.66k
30	3	3		3	(1.21%)	(0.75%)	(0.06%)	(0.00%)	1800.00	143.94k
30	3	3		4	1800.02	2697.53k	1801.77	2864.64k	1800.00	903.67k
30	3	3		5	1801.05	1911.46k	11.10	1.51k	19.63	95
30	3	3		6	(0.11%)	(0.15%)	(0.00%)	(0.05%)	1800.00	303.68k
30	3	3		7	(0.40%)	(0.33%)	(0.04%)	(∞)	1800.00	165.71k
30	3	3		8	103.42	166.43k	2.18	352	3.80	19
30	3	3		9	(0.63%)	(∞)	(0.00%)	(∞)	1800.00	219.76k
30	3	3		10	30.03	11.95k	1801.60	2339.23k	1800.00	272.72k
30	3	3	✓	1	(93.06%)	(0.00%)	(64.65%)	(0.00%)	(20.59%)	(0.00%)
30	3	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(44.74%)	(0.00%)
30	3	3	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(41.18%)	(0.00%)
30	3	3	✓	4	(70.63%)	(0.00%)	(68.47%)	(0.00%)	(41.13%)	(0.00%)
30	3	3	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(29.34%)	(0.00%)
30	3	3	✓	6	(61.05%)	(0.00%)	(60.69%)	(0.00%)	(28.72%)	(0.00%)
30	3	3	✓	7	(62.24%)	(0.00%)	(55.33%)	(0.00%)	(34.83%)	(0.00%)
30	3	3	✓	8	(∞)	(0.00%)	(86.02%)	(0.00%)	(35.39%)	(0.00%)
30	3	3	✓	9	(65.95%)	(0.00%)	(62.63%)	(0.00%)	(35.35%)	(0.00%)
30	3	3	✓	10	(29.20%)	(0.00%)	(26.51%)	(0.00%)	(9.72%)	(0.00%)
30	3	5		1	(0.00%)	(99.96%)	(0.00%)	(22.83%)	4.60	7
30	3	5		2	(0.18%)	(0.70%)	(0.00%)	(∞)	1800.00	365.95k
30	3	5		3	(1.18%)	(4.42%)	(0.01%)	(0.79%)	1800.00	235.62k
30	3	5		4	(2.88%)	(0.00%)	(0.67%)	(0.00%)	(0.26%)	(0.00%)
30	3	5		5	(4.68%)	(1.22%)	(2.81%)	(∞)	(0.82%)	(0.00%)
30	3	5		6	(9.05%)	(∞)	(0.33%)	(0.00%)	1800.00	20.36k
30	3	5		7	(17.32%)	(∞)	(2.56%)	(∞)	1800.00	55.00k
30	3	5		8	(5.64%)	(0.00%)	251.04	3.13k	1800.00	434.46k
30	3	5		9	(4.11%)	(0.01%)	(0.16%)	(0.00%)	1800.00	123.02k
30	3	5		10	(2.12%)	(2.31%)	(0.83%)	(∞)	1800.00	100.16k
30	3	5	✓	1	(39.38%)	(0.00%)	(30.23%)	(0.00%)	(7.85%)	(0.00%)
30	3	5	✓	2	(72.33%)	(0.00%)	(65.83%)	(0.00%)	(16.45%)	(0.00%)
30	3	5	✓	3	(92.66%)	(0.00%)	(67.77%)	(0.00%)	(22.26%)	(0.00%)
30	3	5	✓	4	(43.89%)	(0.00%)	(39.01%)	(0.00%)	(17.15%)	(0.00%)
30	3	5	✓	5	(50.85%)	(0.00%)	(43.59%)	(0.00%)	(8.39%)	(0.00%)
30	3	5	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(93.78%)	(0.00%)
30	3	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(47.10%)	(0.00%)
30	3	5	✓	8	(∞)	(0.00%)	(96.90%)	(0.00%)	(18.84%)	(0.00%)
30	3	5	✓	9	(53.83%)	(0.00%)	(43.19%)	(0.00%)	(14.36%)	(0.00%)
30	3	5	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(42.56%)	(0.00%)
30	3	10		1	(40.31%)	(3.79%)	(32.46%)	(∞)	(2.06%)	(0.00%)
30	3	10		2	(1.22%)	(14.00%)	(0.05%)	(0.01%)	1800.00	24.21k
30	3	10		3	(11.87%)	(∞)	(8.02%)	(∞)	(1.68%)	(0.00%)
30	3	10		4	(55.11%)	(∞)	(41.89%)	(∞)	(18.58%)	(0.00%)
30	3	10		5	(95.54%)	(99.94%)	(37.80%)	(99.94%)	1800.00	8.44k
30	3	10		6	(23.00%)	(8.16%)	(12.78%)	(∞)	1800.00	25.10k

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	3	10		7	(21.27%)	(∞)	(6.12%)	(∞)	(0.07%)	(0.00%)
30	3	10		8	(∞)	(99.94%)	(40.11%)	(99.94%)	(3.49%)	(0.00%)
30	3	10		9	(42.09%)	(24.84%)	(29.58%)	(∞)	(2.88%)	(0.00%)
30	3	10		10	(49.13%)	(99.97%)	(2.44%)	(38.90%)	1800.00	80.47k
30	3	10	✓	1	(∞)	(0.00%)	(76.98%)	(0.00%)	(29.64%)	(0.00%)
30	3	10	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(49.91%)	(0.00%)
30	3	10	✓	3	(∞)	(0.00%)	(85.74%)	(0.00%)	(27.67%)	(0.00%)
30	3	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(42.16%)	(0.00%)
30	3	10	✓	5	(72.62%)	(0.00%)	(57.54%)	(0.00%)	(19.90%)	(0.00%)
30	3	10	✓	6	(∞)	(0.00%)	(86.32%)	(0.00%)	(29.67%)	(0.00%)
30	3	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(69.80%)	(0.00%)
30	3	10	✓	8	(79.14%)	(0.00%)	(61.25%)	(0.00%)	(23.53%)	(0.00%)
30	3	10	✓	9	(95.04%)	(0.00%)	(77.43%)	(0.00%)	(27.19%)	(0.00%)
30	3	10	✓	10	(72.21%)	(0.00%)	(52.82%)	(0.00%)	(15.53%)	(0.00%)
30	4	1		1	(0.10%)	(0.29%)	1800.02	2480.33k	1800.00	384.06k
30	4	1		2	(0.70%)	(0.24%)	11.58	6.25k	1800.00	387.55k
30	4	1		3	1185.35	555.01k	1803.95	4204.33k	1800.00	340.53k
30	4	1		4	(1.08%)	(0.24%)	(0.06%)	(0.00%)	1800.00	808.74k
30	4	1		5	100.83	17.77k	5.30	1.10k	1800.00	497.60k
30	4	1		6	(2.21%)	(∞)	(0.05%)	(0.38%)	1800.00	439.39k
30	4	1		8	(0.04%)	(∞)	1805.82	5854.94k	1800.00	543.02k
30	4	1		9	(0.13%)	(∞)	(0.01%)	(0.13%)	1800.00	498.05k
30	4	1		10	(0.00%)	(∞)	(0.00%)	(0.10%)	1.57	11
30	4	1	✓	1	(33.74%)	(0.00%)	(28.51%)	(0.00%)	(8.18%)	(0.00%)
30	4	1	✓	2	(93.29%)	(0.00%)	(91.78%)	(0.00%)	(77.24%)	(0.00%)
30	4	1	✓	3	(58.04%)	(0.00%)	(52.93%)	(0.00%)	(21.39%)	(0.00%)
30	4	1	✓	4	(49.63%)	(0.00%)	(52.19%)	(0.00%)	(21.12%)	(0.00%)
30	4	1	✓	5	(85.23%)	(0.00%)	(80.46%)	(0.00%)	(56.21%)	(0.00%)
30	4	1	✓	6	(81.09%)	(0.00%)	(66.78%)	(0.00%)	(44.64%)	(0.00%)
30	4	1	✓	7	(43.07%)	(0.00%)	(39.52%)	(0.00%)	(14.22%)	(0.00%)
30	4	1	✓	8	(34.86%)	(0.00%)	(27.93%)	(0.00%)	(7.82%)	(0.00%)
30	4	1	✓	9	(78.51%)	(0.00%)	(71.22%)	(0.00%)	(40.75%)	(0.00%)
30	4	1	✓	10	(37.87%)	(0.00%)	(33.40%)	(0.00%)	(13.38%)	(0.00%)
30	4	2		1	(0.02%)	(∞)	(0.00%)	(∞)	1800.00	98.30k
30	4	2		2	(0.18%)	(1.46%)	(0.00%)	(∞)	1800.00	167.47k
30	4	2		3	(0.14%)	(0.65%)	(0.05%)	(0.31%)	(0.02%)	(0.00%)
30	4	2		4	(1.87%)	(∞)	(0.30%)	(0.78%)	1800.00	106.41k
30	4	2		5	(0.04%)	(0.59%)	11.45	1.96k	1800.00	160.77k
30	4	2		6	(0.85%)	(∞)	146.21	9.49k	35.64	457
30	4	2		7	(3.82%)	(∞)	(0.12%)	(∞)	1800.00	130.93k
30	4	2		8	(0.38%)	(0.17%)	(0.33%)	(∞)	(0.04%)	(0.00%)
30	4	2		9	(0.17%)	(0.31%)	(0.01%)	(0.31%)	23.37	137
30	4	2		10	(63.18%)	(32.44%)	1800.76	964.43k	1800.00	60.19k
30	4	2	✓	1	(68.56%)	(0.00%)	(57.38%)	(0.00%)	(26.13%)	(0.00%)
30	4	2	✓	2	(65.12%)	(0.00%)	(56.17%)	(0.00%)	(26.69%)	(0.00%)
30	4	2	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(27.98%)	(0.00%)
30	4	2	✓	4	(51.31%)	(0.00%)	(43.99%)	(0.00%)	(23.11%)	(0.00%)
30	4	2	✓	5	(41.52%)	(0.00%)	(36.78%)	(0.00%)	(19.20%)	(0.00%)
30	4	2	✓	6	(64.50%)	(0.00%)	(50.92%)	(0.00%)	(22.21%)	(0.00%)
30	4	2	✓	7	(64.83%)	(0.00%)	(61.91%)	(0.00%)	(32.10%)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	4	2	✓	8	(68.59%)	(0.00%)	(63.70%)	(0.00%)	(39.26%)	(0.00%)
30	4	2	✓	9	(59.33%)	(0.00%)	(52.03%)	(0.00%)	(34.54%)	(0.00%)
30	4	2	✓	10	(83.41%)	(0.00%)	(64.42%)	(0.00%)	(21.36%)	(0.00%)
30	4	3		1	(1.85%)	(0.99%)	(0.77%)	(∞)	(0.03%)	(0.00%)
30	4	3		2	(12.19%)	(∞)	1800.00	93.74k	(1.68%)	(0.00%)
30	4	3		3	(8.07%)	(∞)	(0.57%)	(∞)	1800.00	37.96k
30	4	3		4	(0.23%)	(0.00%)	1800.80	834.02k	1800.00	88.64k
30	4	3		5	(0.01%)	(∞)	1801.40	362.84k	21.24	77
30	4	3		6	(3.84%)	(∞)	(0.46%)	(∞)	1800.00	19.89k
30	4	3		7	(7.81%)	(∞)	(1.18%)	(∞)	(0.07%)	(0.00%)
30	4	3		8	(3.81%)	(1.67%)	(0.69%)	(∞)	1800.00	23.36k
30	4	3		9	(6.52%)	(2.52%)	(2.88%)	(∞)	(0.58%)	(0.00%)
30	4	3		10	(28.67%)	(0.46%)	(8.67%)	(∞)	(0.07%)	(0.00%)
30	4	3	✓	1	(77.02%)	(0.00%)	(73.30%)	(0.00%)	(44.10%)	(0.00%)
30	4	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	3	✓	3	(73.97%)	(0.00%)	(67.62%)	(0.00%)	(49.50%)	(0.00%)
30	4	3	✓	4	(57.05%)	(0.00%)	(49.11%)	(0.00%)	(22.59%)	(0.00%)
30	4	3	✓	5	(∞)	(0.00%)	(92.72%)	(0.00%)	(23.85%)	(0.00%)
30	4	3	✓	6	(68.27%)	(0.00%)	(60.59%)	(0.00%)	(32.86%)	(0.00%)
30	4	3	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(46.51%)	(0.00%)
30	4	3	✓	8	(75.02%)	(0.00%)	(66.95%)	(0.00%)	(48.51%)	(0.00%)
30	4	3	✓	9	(80.17%)	(0.00%)	(76.72%)	(0.00%)	(54.19%)	(0.00%)
30	4	3	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5		1	(7.88%)	(0.00%)	(2.01%)	(0.00%)	(0.61%)	(0.03%)
30	4	5		2	(8.25%)	(0.00%)	(1.88%)	(0.67%)	(1.57%)	(0.00%)
30	4	5		3	(15.39%)	(∞)	(4.82%)	(∞)	(1.56%)	(0.00%)
30	4	5		4	(61.10%)	(∞)	(32.35%)	(∞)	(20.92%)	(0.00%)
30	4	5		5	(25.50%)	(∞)	(2.40%)	(∞)	1800.00	15.51k
30	4	5		6	(22.75%)	(0.88%)	(7.42%)	(0.85%)	(4.06%)	(0.00%)
30	4	5		7	(16.17%)	(∞)	(0.84%)	(5.99%)	1800.00	14.78k
30	4	5		8	(28.40%)	(1.34%)	(10.08%)	(1.34%)	(8.06%)	(0.00%)
30	4	5		9	(27.49%)	(∞)	(10.81%)	(∞)	(5.98%)	(0.00%)
30	4	5		10	(18.05%)	(0.00%)	(8.25%)	(0.00%)	(5.23%)	(0.00%)
30	4	5	✓	1	(78.19%)	(0.00%)	(74.60%)	(0.00%)	(55.21%)	(0.00%)
30	4	5	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	3	(97.25%)	(0.00%)	(95.59%)	(0.00%)	(87.70%)	(0.00%)
30	4	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(66.75%)	(0.00%)
30	4	5	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(34.87%)	(0.00%)
30	4	5	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	9	(∞)	(0.00%)	(∞)	(0.00%)	(93.37%)	(0.00%)
30	4	5	✓	10	(91.78%)	(0.00%)	(87.67%)	(0.00%)	(71.90%)	(0.00%)
30	4	10		1	(44.30%)	(∞)	(35.10%)	(∞)	(25.80%)	(0.00%)
30	4	10		2	(60.13%)	(∞)	(40.51%)	(∞)	(4.05%)	(0.00%)
30	4	10		3	(69.35%)	(∞)	(61.56%)	(∞)	(45.39%)	(0.00%)
30	4	10		4	(65.17%)	(∞)	(45.61%)	(∞)	(38.39%)	(0.00%)
30	4	10		5	(99.17%)	(0.00%)	(97.98%)	(0.00%)	(97.28%)	(0.00%)
30	4	10		6	(∞)	(99.84%)	(∞)	(99.84%)	(58.04%)	(0.00%)
30	4	10		7	(79.30%)	(6.51%)	(66.15%)	(2.44%)	(61.89%)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	4	10		8	(22.98%)	(0.00%)	(12.43%)	(0.00%)	(10.66%)	(0.00%)
30	4	10		9	(52.20%)	(∞)	(35.88%)	(∞)	(22.00%)	(0.00%)
30	4	10		10	(∞)	(31.59%)	(66.00%)	(0.00%)	(25.56%)	(0.00%)
30	4	10	✓	1	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(30.84%)	(0.00%)
30	4	10	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(49.97%)	(0.00%)
30	4	10	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(39.63%)	(0.00%)
30	4	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	9	(∞)	(0.00%)	(∞)	(0.00%)	(81.27%)	(0.00%)
30	4	10	✓	10	(98.60%)	(0.00%)	(81.52%)	(0.00%)	(27.29%)	(0.00%)
30	5	1		1	(1.13%)	(∞)	1800.00	1054.42k	1800.00	180.88k
30	5	1		2	1800.59	345.42k	74.87	12.71k	1800.00	190.40k
30	5	1		3	(1.29%)	(∞)	(0.00%)	(∞)	1800.00	106.31k
30	5	1		4	(2.49%)	(0.00%)	43.66	5.48k	79.06	839
30	5	1		5	(0.89%)	(0.08%)	(0.01%)	(0.32%)	1800.00	257.98k
30	5	1		6	(3.81%)	(∞)	(0.13%)	(∞)	(0.02%)	(0.00%)
30	5	1		7	(0.03%)	(0.06%)	(0.00%)	(∞)	1800.00	217.06k
30	5	1		8	1803.01	530.74k	1801.64	2191.27k	36.89	361
30	5	1		9	(1.51%)	(0.83%)	(0.22%)	(∞)	(0.07%)	(0.00%)
30	5	1		10	(0.12%)	(0.30%)	1800.01	1626.74k	3.46	19
30	5	1	✓	1	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(74.13%)	(0.00%)
30	5	1	✓	4	(71.70%)	(0.00%)	(63.08%)	(0.00%)	(39.69%)	(0.00%)
30	5	1	✓	5	(69.20%)	(0.00%)	(58.41%)	(0.00%)	(35.21%)	(0.00%)
30	5	1	✓	6	(61.16%)	(0.00%)	(53.91%)	(0.00%)	(23.45%)	(0.00%)
30	5	1	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1	✓	8	(66.86%)	(0.00%)	(61.58%)	(0.00%)	(43.57%)	(0.00%)
30	5	1	✓	9	(54.78%)	(0.00%)	(46.89%)	(0.00%)	(22.15%)	(0.00%)
30	5	1	✓	10	(39.17%)	(0.00%)	(29.89%)	(0.00%)	(15.22%)	(0.00%)
30	5	2		1	(41.88%)	(5.13%)	(5.09%)	(5.13%)	(9.99%)	(0.00%)
30	5	2		2	(3.39%)	(0.27%)	(1.57%)	(∞)	1800.00	33.28k
30	5	2		3	(7.56%)	(1.98%)	(0.98%)	(∞)	615.52	1.83k
30	5	2		4	(16.29%)	(∞)	(0.16%)	(3.72%)	1800.00	20.90k
30	5	2		5	(20.02%)	(0.00%)	(2.24%)	(0.00%)	(2.38%)	(0.02%)
30	5	2		6	(17.70%)	(4.05%)	1800.00	358.61k	1800.00	90.97k
30	5	2		7	(0.16%)	(0.03%)	(0.02%)	(0.00%)	1800.00	52.37k
30	5	2		8	(18.12%)	(0.59%)	(2.47%)	(0.80%)	(2.15%)	(0.00%)
30	5	2		9	(35.14%)	(∞)	(5.54%)	(∞)	(3.98%)	(0.00%)
30	5	2		10	(11.81%)	(1.15%)	(0.99%)	(∞)	1800.00	34.58k
30	5	2	✓	1	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	2	✓	2	(92.56%)	(0.00%)	(87.49%)	(0.00%)	(68.94%)	(0.00%)
30	5	2	✓	3	(62.51%)	(0.00%)	(53.52%)	(0.00%)	(33.88%)	(0.00%)
30	5	2	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	2	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(98.37%)	(0.00%)
30	5	2	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	2	✓	7	(98.70%)	(0.00%)	(98.19%)	(0.00%)	(96.69%)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	5	2	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	2	✓	9	(84.49%)	(0.00%)	(75.73%)	(0.00%)	(59.72%)	(0.00%)
30	5	2	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3		1	(5.02%)	(∞)	(0.19%)	(∞)	1800.00	10.54k
30	5	3		2	(10.38%)	(8.19%)	(3.36%)	(∞)	(2.18%)	(0.00%)
30	5	3		3	(21.70%)	(∞)	(2.30%)	(0.17%)	(1.46%)	(0.00%)
30	5	3		4	(58.95%)	(1.03%)	(20.94%)	(0.00%)	(19.39%)	(0.03%)
30	5	3		5	(21.38%)	(∞)	(1.23%)	(∞)	1800.00	4.84k
30	5	3		6	(48.02%)	(∞)	(17.41%)	(∞)	(9.76%)	(0.00%)
30	5	3		7	(11.14%)	(0.00%)	(0.31%)	(0.00%)	(2.54%)	(0.00%)
30	5	3		8	(30.28%)	(5.20%)	(2.71%)	(1.13%)	(1.48%)	(0.00%)
30	5	3		9	(40.80%)	(∞)	(9.01%)	(∞)	(0.08%)	(0.00%)
30	5	3		10	(47.95%)	(∞)	(13.43%)	(0.00%)	(13.59%)	(0.34%)
30	5	3	✓	1	(97.08%)	(0.00%)	(96.60%)	(0.00%)	(90.47%)	(0.00%)
30	5	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	4	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	9	(∞)	(0.00%)	(∞)	(0.00%)	(61.24%)	(0.00%)
30	5	3	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5		1	(36.96%)	(∞)	(19.38%)	(∞)	(14.22%)	(0.00%)
30	5	5		2	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5		3	(57.18%)	(∞)	(35.11%)	(0.00%)	(33.03%)	(0.00%)
30	5	5		4	(95.92%)	(∞)	(90.88%)	(∞)	(88.28%)	(0.00%)
30	5	5		5	(35.46%)	(∞)	(10.74%)	(0.68%)	(8.38%)	(0.00%)
30	5	5		6	(31.37%)	(∞)	(17.45%)	(0.10%)	(15.06%)	(0.00%)
30	5	5		7	(22.98%)	(0.00%)	(10.84%)	(0.00%)	(6.58%)	(0.00%)
30	5	5		8	(86.94%)	(0.00%)	(64.61%)	(0.24%)	(61.57%)	(0.00%)
30	5	5		9	(44.90%)	(∞)	(26.38%)	(∞)	(21.84%)	(0.00%)
30	5	5		10	(40.27%)	(∞)	(14.01%)	(0.00%)	(11.42%)	(0.02%)
30	5	5	✓	1	(99.16%)	(0.00%)	(98.89%)	(0.00%)	(98.13%)	(0.00%)
30	5	5	✓	2	(90.68%)	(0.00%)	(63.01%)	(0.00%)	(24.19%)	(0.00%)
30	5	5	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	4	(∞)	(91.01%)	(91.76%)	(0.00%)	(41.34%)	(0.00%)
30	5	5	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(58.71%)	(0.00%)
30	5	5	✓	9	(∞)	(0.00%)	(∞)	(0.00%)	(58.88%)	(0.00%)
30	5	5	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10		1	(72.14%)	(95.02%)	(29.24%)	(95.02%)	(17.77%)	(0.00%)
30	5	10		2	(79.21%)	(0.00%)	(64.85%)	(0.00%)	(59.21%)	(0.29%)
30	5	10		3	(79.63%)	(7.67%)	(72.63%)	(0.00%)	(68.39%)	(0.48%)
30	5	10		4	(∞)	(0.00%)	(72.09%)	(0.00%)	(48.91%)	(0.00%)
30	5	10		5	(82.18%)	(0.00%)	(74.67%)	(0.00%)	(69.76%)	(0.32%)
30	5	10		6	(73.90%)	(0.00%)	(63.74%)	(∞)	(55.90%)	(0.12%)
30	5	10		7	(∞)	(0.53%)	(∞)	(99.70%)	(∞)	(0.00%)

instance					SCIP		SCIP(bivar)		BARON	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	5	10		8	(31.16%)	(0.00%)	(19.49%)	(∞)	(19.77%)	(1.58%)
30	5	10		9	(61.48%)	(0.64%)	(46.50%)	(∞)	(40.57%)	(0.00%)
30	5	10		10	(83.54%)	(∞)	(54.62%)	(∞)	(45.12%)	(0.00%)
30	5	10	✓	1	(90.46%)	(0.00%)	(70.42%)	(0.00%)	(36.64%)	(0.00%)
30	5	10	✓	2	(∞)	(0.00%)	(∞)	(0.00%)	(56.21%)	(0.00%)
30	5	10	✓	3	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	4	(99.70%)	(0.00%)	(90.01%)	(0.00%)	(39.65%)	(0.00%)
30	5	10	✓	5	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	6	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)	(57.75%)	(0.00%)
30	5	10	✓	8	(∞)	(0.00%)	(∞)	(0.00%)	(46.91%)	(0.00%)
30	5	10	✓	9	(∞)	(0.00%)	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	10	(∞)	(0.00%)	(∞)	(0.00%)	(59.14%)	(0.00%)

A.2. Results for instances with integral exponents

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	2	1		1	0.05	1	0.02	1
10	2	1		2	0.04	1	0.01	1
10	2	1		3	0.05	1	0.05	1
10	2	1		4	0.03	1	0.01	1
10	2	1		5	0.02	1	0.01	1
10	2	1		6	0.05	1	0.05	1
10	2	1		7	0.04	1	0.05	1
10	2	1		8	0.02	1	0.00	1
10	2	1		9	0.03	1	0.00	1
10	2	1		10	0.04	1	0.01	1
10	2	1	✓	1	1800.04	6937.40k	1804.04	5251.76k
10	2	1	✓	2	56.87	19.00k	1801.75	4520.41k
10	2	1	✓	3	13.30	3.68k	12.64	13.58k
10	2	1	✓	4	49.44	12.47k	1800.13	4560.91k
10	2	1	✓	5	1802.43	7955.43k	6.14	796
10	2	1	✓	6	59.23	9.98k	27.00	2.68k
10	2	1	✓	7	66.25	12.47k	1804.66	3064.13k
10	2	1	✓	8	9.98	2.06k	1802.58	7146.65k
10	2	1	✓	9	98.53	22.02k	1802.90	4002.24k
10	2	1	✓	10	1801.91	8033.24k	3.93	520
10	2	2		1	0.05	1	0.01	1
10	2	2		2	0.06	1	0.07	8
10	2	2		3	0.32	77	0.28	41
10	2	2		4	0.84	470	0.60	285
10	2	2		5	0.04	1	0.10	1
10	2	2		6	0.05	1	0.03	1
10	2	2		7	0.03	1	0.01	1
10	2	2		8	0.12	4	0.20	5
10	2	2		9	0.10	7	0.59	1.42k
10	2	2		10	0.18	13	0.09	5
10	2	2	✓	1	42.58	12.01k	1801.20	4660.71k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	2	2	✓	2	27.07	4.92k	33.13	4.66k
10	2	2	✓	3	758.66	2414.47k	33.55	5.34k
10	2	2	✓	4	153.20	240.17k	80.31	10.83k
10	2	2	✓	5	63.88	32.25k	1803.07	4111.42k
10	2	2	✓	6	57.72	12.47k	37.11	6.57k
10	2	2	✓	7	129.78	192.09k	1771.20	4926.10k
10	2	2	✓	8	1800.04	7158.25k	3.37	458
10	2	2	✓	9	49.72	20.70k	10.67	3.91k
10	2	2	✓	10	16.35	5.82k	13.69	2.54k
10	2	3		1	0.24	64	0.40	89
10	2	3		2	0.30	49	0.30	31
10	2	3		3	0.06	1	0.10	1
10	2	3		4	1800.15	18372.54k	0.99	369
10	2	3		5	0.22	63	0.31	41
10	2	3		6	0.29	57	0.13	3
10	2	3		7	0.50	157	0.43	85
10	2	3		8	0.14	25	0.30	37
10	2	3		9	0.06	1	0.04	1
10	2	3		10	1815.48	32572.81k	0.38	81
10	2	3	✓	1	39.57	9.17k	15.43	2.41k
10	2	3	✓	2	43.28	13.38k	1801.96	2187.25k
10	2	3	✓	3	17.77	2.48k	33.77	4.06k
10	2	3	✓	4	94.12	27.77k	45.52	6.39k
10	2	3	✓	5	82.72	261.17k	32.36	3.91k
10	2	3	✓	6	1800.04	6779.34k	6.36	721
10	2	3	✓	7	117.38	24.47k	104.92	13.29k
10	2	3	✓	8	123.72	29.93k	56.73	6.75k
10	2	3	✓	9	1807.61	11649.61k	175.13	81.39k
10	2	3	✓	10	1804.26	11093.36k	13.49	1.50k
10	2	5		1	0.26	29	0.41	57
10	2	5		2	0.24	37	1800.01	10232.65k
10	2	5		3	0.28	41	0.15	1
10	2	5		4	0.35	91	0.44	72
10	2	5		5	0.12	9	6.41	44.55k
10	2	5		6	3.23	6.27k	1801.31	4610.92k
10	2	5		7	0.29	47	0.39	53
10	2	5		8	0.17	19	1.13	3.96k
10	2	5		9	0.28	53	1804.98	8576.33k
10	2	5		10	0.63	209	0.91	220
10	2	5	✓	1	45.35	14.28k	60.13	10.36k
10	2	5	✓	2	27.60	4.52k	55.82	5.61k
10	2	5	✓	3	49.11	16.14k	31.08	5.22k
10	2	5	✓	4	58.56	13.27k	1803.02	4325.52k
10	2	5	✓	5	14.76	2.43k	10.41	1.68k
10	2	5	✓	6	167.81	65.19k	1800.06	7282.54k
10	2	5	✓	7	33.39	7.55k	27.46	4.35k
10	2	5	✓	8	14.31	3.15k	16.20	2.63k
10	2	5	✓	9	44.21	12.69k	5.54	3.98k
10	2	5	✓	10	214.34	771.62k	120.38	13.23k
10	2	10		1	1800.11	14004.38k	4.22	1.67k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	2	10		2	0.30	73	0.26	1
10	2	10		3	1800.13	20603.37k	1651.94	7491.24k
10	2	10		4	0.34	81	0.69	25
10	2	10		5	0.27	37	0.74	756
10	2	10		6	0.28	43	0.76	50
10	2	10		7	6.13	73.08k	2.18	7.78k
10	2	10		8	0.31	45	0.44	26
10	2	10		9	0.26	31	0.45	43
10	2	10		10	1800.05	13518.29k	1.00	234
10	2	10	✓	1	114.01	20.46k	376.06	24.81k
10	2	10	✓	2	25.87	5.04k	33.06	2.87k
10	2	10	✓	3	35.35	33.94k	97.82	8.87k
10	2	10	✓	4	268.57	480.13k	1801.19	2934.19k
10	2	10	✓	5	70.19	13.73k	153.80	76.07k
10	2	10	✓	6	51.01	14.54k	171.28	16.19k
10	2	10	✓	7	1802.07	6726.55k	75.14	8.44k
10	2	10	✓	8	22.70	6.13k	27.86	2.73k
10	2	10	✓	9	99.91	19.30k	193.68	17.92k
10	2	10	✓	10	56.21	17.28k	189.61	19.63k
10	3	1		1	0.12	1	0.06	1
10	3	1		2	0.16	1	0.15	1
10	3	1		3	0.11	1	0.01	1
10	3	1		4	0.17	1	0.41	30
10	3	1		5	0.18	1	0.35	27
10	3	1		6	0.11	1	0.02	1
10	3	1		7	0.16	1	0.14	1
10	3	1		8	0.10	1	0.07	1
10	3	1		9	0.24	9	0.20	1
10	3	1		10	0.11	1	0.01	1
10	3	1	✓	1	477.33	311.43k	1800.68	2499.14k
10	3	1	✓	2	1583.65	400.10k	1801.59	459.08k
10	3	1	✓	3	1801.44	3366.84k	1801.10	1998.38k
10	3	1	✓	4	1800.75	1879.86k	1802.39	3269.52k
10	3	1	✓	5	254.38	36.99k	1802.43	3251.36k
10	3	1	✓	6	435.01	115.25k	1800.00	631.95k
10	3	1	✓	7	288.73	29.52k	1801.08	1225.28k
10	3	1	✓	8	618.50	61.99k	1800.18	1652.31k
10	3	1	✓	9	1800.73	544.80k	(0.05%)	(0.00%)
10	3	1	✓	10	1800.96	2364.48k	1800.00	468.07k
10	3	2		1	0.44	44	894.36	2818.78k
10	3	2		2	0.16	1	0.29	1
10	3	2		3	0.35	2	0.45	22
10	3	2		4	0.15	1	316.84	1445.68k
10	3	2		5	13.64	99.82k	0.94	189
10	3	2		6	0.18	1	0.28	1
10	3	2		7	1807.32	12256.72k	1386.09	4224.84k
10	3	2		8	1.25	395	1.84	1.15k
10	3	2		9	0.24	7	0.16	1
10	3	2		10	0.44	8	0.61	34
10	3	2	✓	1	1800.76	472.45k	1801.31	2045.84k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	3	2	✓	2	1801.00	2536.29k	415.46	28.64k
10	3	2	✓	3	1440.82	464.25k	1801.30	1767.64k
10	3	2	✓	4	1009.13	180.76k	1420.51	719.60k
10	3	2	✓	5	1129.44	155.87k	1149.36	627.69k
10	3	2	✓	6	959.30	43.72k	1800.58	972.32k
10	3	2	✓	7	1372.65	228.98k	1800.40	1418.52k
10	3	2	✓	8	1800.72	569.50k	1801.51	1714.84k
10	3	2	✓	9	302.73	39.00k	1801.83	2789.21k
10	3	2	✓	10	790.68	80.48k	1800.50	1342.24k
10	3	3		1	0.61	133	1.25	232
10	3	3		2	0.36	35	0.83	19
10	3	3		3	0.20	4	0.44	4
10	3	3		4	0.54	85	0.96	408
10	3	3		5	1809.70	12559.25k	0.79	105
10	3	3		6	0.12	1	0.03	1
10	3	3		7	0.54	31	1.06	128
10	3	3		8	0.17	1	0.25	1
10	3	3		9	0.78	109	1.84	144
10	3	3		10	0.14	1	1.04	287
10	3	3	✓	1	1800.00	227.88k	(0.07%)	(0.00%)
10	3	3	✓	2	318.00	126.66k	1800.00	671.66k
10	3	3	✓	3	1802.15	3732.76k	1800.72	1150.89k
10	3	3	✓	4	1801.44	1717.20k	1800.26	564.49k
10	3	3	✓	5	1801.01	695.14k	1800.39	308.12k
10	3	3	✓	6	1519.01	384.95k	1603.29	1587.77k
10	3	3	✓	7	1200.25	335.12k	(0.99%)	(0.00%)
10	3	3	✓	8	695.48	139.16k	473.74	19.55k
10	3	3	✓	9	931.63	44.19k	(0.18%)	(0.00%)
10	3	3	✓	10	822.98	455.77k	1800.27	1005.13k
10	3	5		1	0.61	99	1.40	93
10	3	5		2	0.76	117	1.76	168
10	3	5		3	0.24	1	0.99	37
10	3	5		4	0.48	31	0.99	41
10	3	5		5	(0.14%)	(0.00%)	(0.14%)	(0.00%)
10	3	5		6	1800.02	5456.95k	1.66	131
10	3	5		7	0.28	1	0.57	68
10	3	5		8	0.41	19	0.79	33
10	3	5		9	0.62	177	1.72	309
10	3	5		10	0.20	1	0.76	11
10	3	5	✓	1	(0.05%)	(0.00%)	(1.03%)	(0.00%)
10	3	5	✓	2	974.64	53.31k	(0.15%)	(0.00%)
10	3	5	✓	3	191.55	44.38k	1800.24	512.31k
10	3	5	✓	4	1724.08	260.66k	1800.13	485.47k
10	3	5	✓	5	645.07	47.21k	1800.34	174.28k
10	3	5	✓	6	1800.62	538.93k	(0.79%)	(0.00%)
10	3	5	✓	7	417.35	31.20k	1800.44	424.00k
10	3	5	✓	8	378.43	27.00k	1800.35	763.81k
10	3	5	✓	9	1801.78	2642.63k	1800.56	508.41k
10	3	5	✓	10	1800.70	1850.19k	1800.49	196.34k
10	3	10		1	1803.21	4887.70k	1800.64	1040.58k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	3	10		2	0.66	1	2.20	67
10	3	10		3	1.29	275	7.93	351
10	3	10		4	1803.02	3755.52k	1800.53	656.51k
10	3	10		5	1800.59	3218.52k	16.54	707
10	3	10		6	1801.86	2867.99k	1801.06	1054.84k
10	3	10		7	0.64	103	1801.37	1840.78k
10	3	10		8	0.74	101	1800.00	1091.96k
10	3	10		9	1.46	349	11.11	487
10	3	10		10	6.64	15.53k	86.73	40.54k
10	3	10	✓	1	1801.02	1128.91k	(0.66%)	(0.00%)
10	3	10	✓	2	1800.89	887.88k	(1.53%)	(0.00%)
10	3	10	✓	3	1800.56	738.15k	(0.37%)	(0.00%)
10	3	10	✓	4	1801.57	189.84k	(0.15%)	(0.00%)
10	3	10	✓	5	1800.18	673.37k	(1.17%)	(0.00%)
10	3	10	✓	6	1800.00	60.08k	(1.89%)	(0.00%)
10	3	10	✓	7	428.80	56.84k	1800.00	12.37k
10	3	10	✓	8	1803.27	184.70k	(0.19%)	(0.00%)
10	3	10	✓	9	1800.60	808.38k	(0.22%)	(0.00%)
10	3	10	✓	10	(0.03%)	(0.00%)	(1.24%)	(0.00%)
10	4	1		1	0.40	1	0.59	3
10	4	1		2	1.10	73	0.87	57
10	4	1		3	0.32	3	0.27	1
10	4	1		4	1803.09	8335.55k	0.95	563
10	4	1		5	0.45	41	0.32	1
10	4	1		6	2.49	5.92k	1.12	171
10	4	1		7	0.60	29	0.48	15
10	4	1		8	0.81	1.55k	1800.09	9238.72k
10	4	1		9	0.62	93	0.60	153
10	4	1		10	0.76	291	0.32	1
10	4	1	✓	1	1800.80	1073.35k	1800.72	2282.03k
10	4	1	✓	2	(0.64%)	(0.00%)	(0.05%)	(0.00%)
10	4	1	✓	3	1800.00	153.07k	1800.71	986.82k
10	4	1	✓	4	(1.72%)	(0.00%)	1800.00	76.92k
10	4	1	✓	5	1800.58	641.86k	1800.64	1190.48k
10	4	1	✓	6	(1.49%)	(0.00%)	(0.15%)	(0.00%)
10	4	1	✓	7	1801.59	1106.48k	1800.01	2452.12k
10	4	1	✓	8	1801.05	501.51k	1800.21	1784.77k
10	4	1	✓	9	1800.00	61.41k	1800.37	1837.58k
10	4	1	✓	10	(3.57%)	(0.00%)	(0.38%)	(0.00%)
10	4	2		1	0.78	582	0.87	1
10	4	2		2	1.86	497	1.16	127
10	4	2		3	60.41	259.55k	1802.17	3328.80k
10	4	2		4	1805.03	6165.54k	1.20	185
10	4	2		5	1801.78	5415.07k	3.37	3.82k
10	4	2		6	1.89	570	1.88	307
10	4	2		7	0.86	119	1.22	75
10	4	2		8	2.39	235	3.95	3.42k
10	4	2		9	1818.09	11965.18k	1801.61	3615.70k
10	4	2		10	0.82	567	1805.42	7111.08k
10	4	2	✓	1	(0.98%)	(0.00%)	(0.69%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	4	2	✓	2	(15.83%)	(0.00%)	(6.80%)	(0.00%)
10	4	2	✓	3	(0.03%)	(0.00%)	1800.00	58.65k
10	4	2	✓	4	(4.83%)	(0.00%)	(9.86%)	(0.00%)
10	4	2	✓	5	1801.65	112.03k	1800.00	32.10k
10	4	2	✓	6	(0.61%)	(0.00%)	(0.76%)	(0.00%)
10	4	2	✓	7	(0.77%)	(0.00%)	(1.68%)	(0.00%)
10	4	2	✓	8	(0.96%)	(0.00%)	(0.55%)	(0.00%)
10	4	2	✓	9	(1.08%)	(0.00%)	(0.18%)	(0.00%)
10	4	2	✓	10	1800.14	201.18k	1800.26	905.54k
10	4	3		1	1801.79	2804.07k	10.05	609
10	4	3		2	1800.85	3043.25k	6.24	257
10	4	3		3	4.20	685	5.10	301
10	4	3		4	1803.38	5663.44k	1.49	87
10	4	3		5	1800.45	1273.08k	28.47	2.45k
10	4	3		6	0.29	1	1.04	27
10	4	3		7	7.70	3.60k	6.69	481
10	4	3		8	25.53	86.32k	5.75	342
10	4	3		9	0.66	19	1.07	15
10	4	3		10	0.93	244	1.22	95
10	4	3	✓	1	1800.23	80.92k	(0.02%)	(0.00%)
10	4	3	✓	2	1802.98	223.41k	1800.00	43.17k
10	4	3	✓	3	1800.00	133.74k	1800.00	21.05k
10	4	3	✓	4	(0.40%)	(0.00%)	(0.30%)	(0.00%)
10	4	3	✓	5	(24.85%)	(0.00%)	(26.22%)	(0.00%)
10	4	3	✓	6	1800.71	1207.05k	1442.06	16.55k
10	4	3	✓	7	(77.46%)	(0.00%)	(70.34%)	(0.00%)
10	4	3	✓	8	(0.02%)	(0.00%)	(0.20%)	(0.00%)
10	4	3	✓	9	(5.31%)	(0.00%)	(6.12%)	(0.00%)
10	4	3	✓	10	(4.31%)	(0.00%)	(11.27%)	(0.00%)
10	4	5		1	1.36	395	2.98	139
10	4	5		2	1802.70	3899.34k	5.55	179
10	4	5		3	1802.77	4546.06k	24.19	423
10	4	5		4	6.49	885	566.29	33.23k
10	4	5		5	1802.17	3619.79k	11.06	739
10	4	5		6	1804.28	4445.94k	5.28	149
10	4	5		7	1802.15	3683.70k	10.69	677
10	4	5		8	1803.30	5018.77k	8.53	643
10	4	5		9	1800.02	2858.35k	132.27	1.34k
10	4	5		10	1803.94	5311.74k	3.79	129
10	4	5	✓	1	(0.86%)	(0.00%)	(0.63%)	(0.00%)
10	4	5	✓	2	(2.59%)	(0.00%)	(4.24%)	(0.00%)
10	4	5	✓	3	(34.51%)	(0.00%)	(51.69%)	(0.00%)
10	4	5	✓	4	(0.05%)	(0.00%)	(0.29%)	(0.00%)
10	4	5	✓	5	(4.49%)	(0.00%)	(5.12%)	(0.00%)
10	4	5	✓	6	(0.13%)	(0.00%)	(0.59%)	(0.00%)
10	4	5	✓	7	(0.19%)	(0.00%)	(4.17%)	(0.00%)
10	4	5	✓	8	(5.94%)	(0.00%)	(12.60%)	(0.00%)
10	4	5	✓	9	(14.39%)	(0.00%)	(27.84%)	(0.00%)
10	4	5	✓	10	(6.23%)	(0.00%)	(21.94%)	(0.00%)
10	4	10		1	2.68	1.35k	106.68	58.03k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	4	10		2	1800.01	1533.76k	251.32	1.17k
10	4	10		3	1800.00	1300.68k	166.43	1.27k
10	4	10		4	44.84	3.44k	206.98	1.01k
10	4	10		5	1800.82	1416.19k	826.15	5.32k
10	4	10		6	1801.32	2259.51k	64.81	801
10	4	10		7	1800.02	2160.87k	114.37	577
10	4	10		8	1801.06	1471.97k	205.33	1.53k
10	4	10		9	1801.02	1757.64k	73.15	443
10	4	10		10	104.17	46.89k	151.38	893
10	4	10	✓	1	(0.03%)	(0.00%)	(0.47%)	(0.00%)
10	4	10	✓	2	(7.18%)	(0.00%)	(19.19%)	(0.00%)
10	4	10	✓	3	(6.05%)	(0.00%)	(17.08%)	(0.00%)
10	4	10	✓	4	(4.68%)	(0.00%)	(19.48%)	(0.00%)
10	4	10	✓	5	(1.28%)	(0.00%)	(7.92%)	(0.00%)
10	4	10	✓	6	(0.37%)	(0.00%)	(7.58%)	(0.00%)
10	4	10	✓	7	(29.49%)	(0.00%)	(84.27%)	(0.00%)
10	4	10	✓	8	(4.61%)	(0.00%)	(8.35%)	(0.00%)
10	4	10	✓	9	(2.03%)	(0.00%)	(25.69%)	(0.00%)
10	4	10	✓	10	(1.64%)	(0.00%)	(17.03%)	(0.00%)
10	5	1		1	0.77	79	1.05	1
10	5	1		2	0.16	1	0.19	1
10	5	1		3	0.77	243	0.91	65
10	5	1		4	0.93	1.03k	1.15	211
10	5	1		5	0.78	384	0.78	73
10	5	1		6	0.75	506	0.83	89
10	5	1		7	1.05	1.28k	0.76	71
10	5	1		8	1.39	542	0.96	87
10	5	1		9	(0.00%)	(0.30%)	1800.63	5148.93k
10	5	1		10	0.40	141	0.70	112
10	5	1	✓	1	(4.81%)	(0.00%)	(1.72%)	(0.00%)
10	5	1	✓	2	1800.35	543.21k	1364.77	69.76k
10	5	1	✓	3	(0.78%)	(0.00%)	1800.00	174.51k
10	5	1	✓	4	803.46	400.52k	658.07	27.62k
10	5	1	✓	5	1800.00	38.60k	1800.00	66.67k
10	5	1	✓	6	(0.75%)	(0.00%)	(0.07%)	(0.00%)
10	5	1	✓	7	(0.64%)	(0.00%)	(0.06%)	(0.00%)
10	5	1	✓	8	(5.68%)	(0.00%)	(2.05%)	(0.00%)
10	5	1	✓	9	(0.09%)	(0.00%)	(0.07%)	(0.00%)
10	5	1	✓	10	1800.00	44.49k	1800.00	45.52k
10	5	2		1	5.60	131	3.33	103
10	5	2		2	14.23	1.64k	9.13	217
10	5	2		3	1.93	3.06k	2.73	311
10	5	2		4	0.45	5	1.02	16
10	5	2		5	15.45	65.74k	1.44	83
10	5	2		6	0.19	1	0.87	7
10	5	2		7	1807.53	7786.70k	6.60	161
10	5	2		8	3.96	1.58k	1.77	39
10	5	2		9	0.91	202	1.69	105
10	5	2		10	1.29	13	1.48	49
10	5	2	✓	1	(19.19%)	(0.00%)	(21.45%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	5	2	✓	2	(14.30%)	(0.00%)	(8.68%)	(0.00%)
10	5	2	✓	3	1800.37	84.23k	(0.75%)	(0.00%)
10	5	2	✓	4	(0.07%)	(0.00%)	(0.05%)	(0.00%)
10	5	2	✓	5	(0.35%)	(0.00%)	(0.58%)	(0.00%)
10	5	2	✓	6	(0.05%)	(0.00%)	(0.26%)	(0.00%)
10	5	2	✓	7	1800.36	504.62k	(0.45%)	(0.00%)
10	5	2	✓	8	(6.30%)	(0.00%)	(4.28%)	(0.00%)
10	5	2	✓	9	(4.08%)	(0.00%)	(2.24%)	(0.00%)
10	5	2	✓	10	(0.19%)	(0.00%)	(0.36%)	(0.00%)
10	5	3		1	1802.61	4314.84k	35.98	859
10	5	3		2	723.83	1031.89k	11.90	295
10	5	3		3	0.96	235	6.35	710
10	5	3		4	1.40	75	92.81	15.71k
10	5	3		5	3.93	210	3.83	67
10	5	3		6	1802.78	3990.33k	51.54	819
10	5	3		7	1803.03	4727.64k	21.77	1.52k
10	5	3		8	1804.57	6063.30k	2.73	157
10	5	3		9	0.53	1	1.40	7
10	5	3		10	1803.64	5071.31k	17.62	365
10	5	3	✓	1	(14.43%)	(0.00%)	(27.56%)	(0.00%)
10	5	3	✓	2	(1.32%)	(0.00%)	(2.67%)	(0.00%)
10	5	3	✓	3	(0.92%)	(0.00%)	(2.01%)	(0.00%)
10	5	3	✓	4	(1.89%)	(0.00%)	(3.42%)	(0.00%)
10	5	3	✓	5	(1.56%)	(0.00%)	(7.84%)	(0.00%)
10	5	3	✓	6	(1.70%)	(0.00%)	(5.25%)	(0.00%)
10	5	3	✓	7	(25.75%)	(0.00%)	(47.83%)	(0.00%)
10	5	3	✓	8	(0.44%)	(0.00%)	(1.10%)	(0.00%)
10	5	3	✓	9	1800.00	19.83k	(0.21%)	(0.00%)
10	5	3	✓	10	(∞)	(0.00%)	(∞)	(1.52%)
10	5	5		1	1804.50	6133.59k	4.38	129
10	5	5		2	2.57	1.08k	9.50	141
10	5	5		3	88.66	5.63k	470.01	2.31k
10	5	5		4	10.19	842	28.02	227
10	5	5		5	1800.03	4601.02k	24.07	103
10	5	5		6	0.42	1	1.03	5
10	5	5		7	377.14	639.90k	57.36	624
10	5	5		8	(0.00%)	(2.47%)	96.23	5.01k
10	5	5		9	1801.86	2805.51k	1800.09	149.58k
10	5	5		10	1802.73	3488.96k	28.40	497
10	5	5	✓	1	(0.05%)	(0.00%)	(3.43%)	(0.00%)
10	5	5	✓	2	(0.18%)	(0.00%)	(4.27%)	(0.00%)
10	5	5	✓	3	(3.47%)	(0.00%)	(15.07%)	(0.00%)
10	5	5	✓	4	(1.84%)	(0.00%)	(4.88%)	(0.00%)
10	5	5	✓	5	(13.42%)	(0.00%)	(32.22%)	(0.00%)
10	5	5	✓	6	1800.00	26.13k	(0.21%)	(0.00%)
10	5	5	✓	7	(0.74%)	(0.00%)	(7.42%)	(0.00%)
10	5	5	✓	8	(0.30%)	(0.00%)	(7.80%)	(0.00%)
10	5	5	✓	9	(9.71%)	(0.00%)	(22.60%)	(0.00%)
10	5	5	✓	10	(6.58%)	(0.00%)	(9.73%)	(0.00%)
10	5	10		1	77.78	73.64k	388.19	589

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
10	5	10		2	86.69	126.97k	82.71	1.12k
10	5	10		3	1801.05	1515.16k	172.09	1.14k
10	5	10		4	232.63	267.88k	117.90	335
10	5	10		5	1802.44	2248.27k	49.58	381
10	5	10		6	1801.84	1667.51k	308.22	863
10	5	10		7	223.16	5.36k	(7.38%)	(0.10%)
10	5	10		8	1800.01	1684.44k	101.97	411
10	5	10		9	1.16	121	7.40	47
10	5	10		10	1800.96	1022.64k	1771.89	3.21k
10	5	10	✓	1	(22.54%)	(0.00%)	(51.82%)	(0.00%)
10	5	10	✓	2	(0.19%)	(0.00%)	(2.94%)	(0.00%)
10	5	10	✓	3	(3.45%)	(0.00%)	(14.15%)	(0.00%)
10	5	10	✓	4	(0.18%)	(0.00%)	(10.01%)	(0.00%)
10	5	10	✓	5	(0.79%)	(0.00%)	(3.93%)	(0.00%)
10	5	10	✓	6	(2.85%)	(0.00%)	(21.45%)	(0.00%)
10	5	10	✓	7	(7.55%)	(0.00%)	(56.37%)	(0.00%)
10	5	10	✓	8	(0.65%)	(0.00%)	(10.36%)	(0.00%)
10	5	10	✓	9	(0.37%)	(0.00%)	(2.39%)	(0.00%)
10	5	10	✓	10	(15.25%)	(0.00%)	(∞)	(0.00%)
20	2	1		1	0.04	1	0.12	1
20	2	1		2	0.03	1	0.06	1
20	2	1		3	0.07	1	0.04	1
20	2	1		4	0.05	1	0.07	1
20	2	1		5	0.06	1	0.06	1
20	2	1		6	0.02	1	0.00	1
20	2	1		7	0.06	1	0.06	2
20	2	1		8	0.04	1	0.04	1
20	2	1		9	0.05	1	0.03	1
20	2	1		10	0.04	1	0.04	1
20	2	1	✓	1	(1.03%)	(0.00%)	326.49	3.05k
20	2	1	✓	2	(2.32%)	(0.00%)	(1.10%)	(0.00%)
20	2	1	✓	3	(8.01%)	(0.00%)	(2.86%)	(0.00%)
20	2	1	✓	4	(2.61%)	(0.00%)	1800.00	56.90k
20	2	1	✓	5	(17.74%)	(0.00%)	(10.47%)	(0.00%)
20	2	1	✓	6	(6.44%)	(0.00%)	(4.30%)	(0.00%)
20	2	1	✓	7	(5.63%)	(0.00%)	(0.10%)	(0.00%)
20	2	1	✓	8	(0.52%)	(0.00%)	(0.44%)	(0.00%)
20	2	1	✓	9	(5.50%)	(0.00%)	(5.24%)	(0.00%)
20	2	1	✓	10	(5.53%)	(0.00%)	(2.14%)	(0.00%)
20	2	2		1	437.13	4777.79k	0.11	3
20	2	2		2	0.04	1	0.08	1
20	2	2		3	0.26	49	0.41	59
20	2	2		4	0.13	1	0.10	1
20	2	2		5	0.13	15	0.06	1
20	2	2		6	0.16	23	0.10	1
20	2	2		7	1800.13	17812.55k	0.57	323
20	2	2		8	0.43	237	1814.17	10315.84k
20	2	2		9	1800.04	11598.12k	6.82	7.45k
20	2	2		10	0.08	1	0.09	1
20	2	2	✓	1	(3.77%)	(0.00%)	(3.37%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	2	2	✓	2	(18.36%)	(0.00%)	(5.31%)	(0.00%)
20	2	2	✓	3	(15.92%)	(0.00%)	(16.88%)	(0.00%)
20	2	2	✓	4	(3.92%)	(0.00%)	(2.70%)	(0.00%)
20	2	2	✓	5	(1.16%)	(0.00%)	(0.04%)	(0.00%)
20	2	2	✓	6	(9.24%)	(0.00%)	(12.76%)	(0.00%)
20	2	2	✓	7	(38.26%)	(0.00%)	(35.26%)	(0.00%)
20	2	2	✓	8	(8.28%)	(0.00%)	(6.00%)	(0.00%)
20	2	2	✓	9	(1.18%)	(0.00%)	(0.38%)	(0.00%)
20	2	2	✓	10	(4.24%)	(0.00%)	(0.11%)	(0.00%)
20	2	3		1	0.14	13	0.20	5
20	2	3		2	1809.21	18782.52k	0.62	272
20	2	3		3	0.14	17	1800.01	9415.93k
20	2	3		4	0.26	37	0.41	49
20	2	3		5	0.31	49	0.45	88
20	2	3		6	0.22	27	0.75	1.55k
20	2	3		7	0.36	67	0.50	73
20	2	3		8	0.05	1	0.14	1
20	2	3		9	0.37	95	0.41	11
20	2	3		10	0.10	1	0.22	4
20	2	3	✓	1	(14.16%)	(0.00%)	(2.22%)	(0.00%)
20	2	3	✓	2	(2.14%)	(0.00%)	(0.42%)	(0.00%)
20	2	3	✓	3	(29.40%)	(0.00%)	(28.80%)	(0.00%)
20	2	3	✓	4	(0.81%)	(0.00%)	(0.11%)	(0.00%)
20	2	3	✓	5	(1.61%)	(0.00%)	(0.04%)	(0.00%)
20	2	3	✓	6	(2.12%)	(0.00%)	(5.13%)	(0.00%)
20	2	3	✓	7	(20.21%)	(0.00%)	(18.50%)	(0.00%)
20	2	3	✓	8	(10.50%)	(0.00%)	(14.16%)	(0.00%)
20	2	3	✓	9	(43.27%)	(0.00%)	(41.11%)	(0.00%)
20	2	3	✓	10	(7.55%)	(0.00%)	(6.69%)	(0.00%)
20	2	5		1	0.63	103	0.81	125
20	2	5		2	1800.12	20371.60k	0.31	1
20	2	5		3	0.38	57	0.57	71
20	2	5		4	0.82	267	1.10	661
20	2	5		5	0.39	166	1800.10	10250.39k
20	2	5		6	1802.96	9039.06k	7.61	2.05k
20	2	5		7	0.58	143	0.79	183
20	2	5		8	0.43	128	0.66	7
20	2	5		9	1800.07	26305.26k	0.70	103
20	2	5		10	0.86	211	1.07	256
20	2	5	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
20	2	5	✓	2	(4.47%)	(0.00%)	(9.81%)	(0.00%)
20	2	5	✓	3	(5.46%)	(0.00%)	(6.82%)	(0.00%)
20	2	5	✓	4	(3.88%)	(0.00%)	(4.65%)	(0.00%)
20	2	5	✓	5	(3.46%)	(0.00%)	(1.45%)	(0.00%)
20	2	5	✓	6	(32.24%)	(0.00%)	(38.58%)	(0.00%)
20	2	5	✓	7	(3.84%)	(0.00%)	(0.32%)	(0.00%)
20	2	5	✓	8	(1.83%)	(0.00%)	(3.49%)	(0.00%)
20	2	5	✓	9	(4.69%)	(0.00%)	(3.45%)	(0.00%)
20	2	5	✓	10	(18.36%)	(0.00%)	(28.84%)	(0.00%)
20	2	10		1	1800.02	4125.00k	1801.50	2718.76k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	2	10		2	4.46	1.99k	470.69	1044.80k
20	2	10		3	1800.12	10573.43k	25.52	72.33k
20	2	10		4	1800.05	9134.35k	8.90	1.29k
20	2	10		5	9.76	31.89k	3.59	615
20	2	10		6	1800.05	11283.96k	1801.57	4321.45k
20	2	10		7	1800.04	7650.64k	35.07	2.98k
20	2	10		8	1.72	629	1800.78	2250.24k
20	2	10		9	1800.06	11458.29k	1802.17	4088.15k
20	2	10		10	1800.07	9459.89k	2.94	965
20	2	10	✓	1	(7.32%)	(0.00%)	(8.34%)	(0.00%)
20	2	10	✓	2	(0.79%)	(0.00%)	(0.56%)	(0.00%)
20	2	10	✓	3	(3.03%)	(0.00%)	(4.05%)	(0.00%)
20	2	10	✓	4	(5.56%)	(0.00%)	(9.06%)	(0.00%)
20	2	10	✓	5	(8.33%)	(0.00%)	(7.50%)	(0.00%)
20	2	10	✓	6	(6.57%)	(0.00%)	(8.02%)	(0.00%)
20	2	10	✓	7	(12.77%)	(0.00%)	(13.44%)	(0.00%)
20	2	10	✓	8	(3.93%)	(0.00%)	(4.62%)	(0.00%)
20	2	10	✓	9	(5.26%)	(0.00%)	(3.49%)	(0.00%)
20	2	10	✓	10	(3.61%)	(0.00%)	(3.91%)	(0.00%)
20	3	1		1	0.24	1	0.03	1
20	3	1		2	0.12	1	0.12	1
20	3	1		3	0.46	55	0.54	5
20	3	1		4	0.26	1	0.51	21
20	3	1		5	0.23	1	0.16	1
20	3	1		6	0.20	3	0.28	1
20	3	1		7	0.31	1	0.60	1
20	3	1		8	0.48	197	0.56	41
20	3	1		9	0.44	23	0.46	7
20	3	1		10	0.17	91	0.10	1
20	3	1	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
20	3	1	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
20	3	1	✓	3	(5.89%)	(0.00%)	(11.27%)	(0.00%)
20	3	1	✓	4	(4.89%)	(0.00%)	(4.36%)	(0.00%)
20	3	1	✓	5	(85.23%)	(0.00%)	(90.46%)	(0.00%)
20	3	1	✓	6	(67.09%)	(0.00%)	(∞)	(0.00%)
20	3	1	✓	7	(24.91%)	(0.00%)	(26.84%)	(0.00%)
20	3	1	✓	8	(14.17%)	(0.00%)	(18.69%)	(0.00%)
20	3	1	✓	9	(50.82%)	(0.00%)	(47.91%)	(0.00%)
20	3	1	✓	10	(15.68%)	(0.00%)	(15.89%)	(0.00%)
20	3	2		1	0.53	51	1.25	151
20	3	2		2	0.25	19	1.38	85
20	3	2		3	0.49	71	1.12	107
20	3	2		4	1804.44	6094.93k	1.94	197
20	3	2		5	1.52	4.72k	1.20	133
20	3	2		6	1800.69	6538.45k	1.90	259
20	3	2		7	0.67	23	1.70	65
20	3	2		8	0.34	99	1.08	19
20	3	2		9	1.93	1.48k	1800.74	2508.74k
20	3	2		10	0.50	141	1804.09	5779.00k
20	3	2	✓	1	(8.84%)	(0.00%)	(8.53%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	3	2	✓	2	(33.90%)	(0.00%)	(42.19%)	(0.00%)
20	3	2	✓	3	(37.95%)	(0.00%)	(53.01%)	(0.00%)
20	3	2	✓	4	(69.24%)	(0.00%)	(70.92%)	(0.00%)
20	3	2	✓	5	(43.84%)	(0.00%)	(50.47%)	(0.00%)
20	3	2	✓	6	(22.66%)	(0.00%)	(30.28%)	(0.00%)
20	3	2	✓	7	(13.87%)	(0.00%)	(19.15%)	(0.00%)
20	3	2	✓	8	(40.09%)	(0.00%)	(45.17%)	(0.00%)
20	3	2	✓	9	(24.04%)	(0.00%)	(32.48%)	(0.00%)
20	3	2	✓	10	(15.63%)	(0.00%)	(21.19%)	(0.00%)
20	3	3		1	64.21	205.92k	1802.87	3171.32k
20	3	3		2	(1.85%)	(0.00%)	(1.85%)	(0.00%)
20	3	3		3	4.23	151	6.42	213
20	3	3		4	1804.15	3973.07k	1800.25	1206.14k
20	3	3		5	0.86	293	2.38	305
20	3	3		6	1812.64	12761.31k	2.25	203
20	3	3		7	0.38	1	1800.04	4478.61k
20	3	3		8	1.15	411	5.00	1.59k
20	3	3		9	0.56	77	1.76	91
20	3	3		10	0.96	780	11.49	17.31k
20	3	3	✓	1	(66.02%)	(0.00%)	(73.60%)	(0.00%)
20	3	3	✓	2	(52.15%)	(0.00%)	(60.18%)	(0.00%)
20	3	3	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
20	3	3	✓	4	(18.39%)	(0.00%)	(25.31%)	(0.00%)
20	3	3	✓	5	(9.57%)	(0.00%)	(14.71%)	(0.00%)
20	3	3	✓	6	(46.00%)	(0.00%)	(64.10%)	(0.00%)
20	3	3	✓	7	(21.33%)	(0.00%)	(27.64%)	(0.00%)
20	3	3	✓	8	(22.36%)	(0.00%)	(38.64%)	(0.00%)
20	3	3	✓	9	(30.02%)	(0.00%)	(37.13%)	(0.00%)
20	3	3	✓	10	(92.50%)	(0.00%)	(93.73%)	(0.00%)
20	3	5		1	1802.73	3876.04k	5.62	327
20	3	5		2	1801.60	3286.32k	1800.37	700.95k
20	3	5		3	1801.66	2364.74k	1800.72	860.00k
20	3	5		4	1.12	193	9.74	547
20	3	5		5	1800.07	2490.22k	40.80	11.61k
20	3	5		6	3.44	449	(1.72%)	(0.00%)
20	3	5		7	1802.24	3617.88k	22.90	1.29k
20	3	5		8	0.89	338	1800.70	1639.19k
20	3	5		9	1804.39	4990.81k	5.23	227
20	3	5		10	0.55	33	1.21	37
20	3	5	✓	1	(75.26%)	(0.00%)	(79.41%)	(0.00%)
20	3	5	✓	2	(43.36%)	(0.00%)	(53.56%)	(0.00%)
20	3	5	✓	3	(26.07%)	(0.00%)	(42.34%)	(0.00%)
20	3	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	3	5	✓	5	(76.41%)	(0.00%)	(∞)	(0.00%)
20	3	5	✓	6	(41.91%)	(0.00%)	(60.23%)	(0.00%)
20	3	5	✓	7	(79.12%)	(0.00%)	(84.77%)	(0.00%)
20	3	5	✓	8	(82.74%)	(0.00%)	(∞)	(0.00%)
20	3	5	✓	9	(34.08%)	(0.00%)	(45.45%)	(0.00%)
20	3	5	✓	10	(12.63%)	(0.00%)	(19.48%)	(0.00%)
20	3	10		1	1802.10	2598.30k	1800.38	479.29k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	3	10		2	1800.84	984.65k	(0.55%)	(0.00%)
20	3	10		3	1802.01	3655.85k	175.73	1.42k
20	3	10		4	1801.34	2370.84k	1800.33	358.71k
20	3	10		5	1801.51	1254.07k	(30.63%)	(6.73%)
20	3	10		6	0.86	64	7.02	215
20	3	10		7	121.81	1.87k	1468.60	5.01k
20	3	10		8	1801.51	1924.68k	704.64	3.24k
20	3	10		9	9.71	448	178.68	1.16k
20	3	10		10	13.47	1.63k	163.76	2.55k
20	3	10	✓	1	(16.86%)	(0.00%)	(21.10%)	(0.00%)
20	3	10	✓	2	(27.14%)	(0.00%)	(37.84%)	(0.00%)
20	3	10	✓	3	(14.49%)	(0.00%)	(21.50%)	(0.00%)
20	3	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	3	10	✓	5	(56.40%)	(0.00%)	(∞)	(0.00%)
20	3	10	✓	6	(37.18%)	(0.00%)	(52.30%)	(0.00%)
20	3	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
20	3	10	✓	8	(53.85%)	(0.00%)	(74.55%)	(0.00%)
20	3	10	✓	9	(21.86%)	(0.00%)	(40.91%)	(0.00%)
20	3	10	✓	10	(86.65%)	(0.00%)	(∞)	(0.00%)
20	4	1		1	4.26	6.78k	1.51	411
20	4	1		2	1802.75	5153.20k	2.48	1.22k
20	4	1		3	85.54	1.49k	1803.04	4792.78k
20	4	1		4	1805.25	7525.15k	1.62	696
20	4	1		5	2.70	3.46k	1800.04	6367.69k
20	4	1		6	0.54	69	1.14	3
20	4	1		7	3.98	8.80k	1.40	463
20	4	1		8	4.82	4.73k	1803.28	6063.41k
20	4	1		9	8.75	631	1.83	403
20	4	1		10	1.36	1.03k	0.99	161
20	4	1	✓	1	(81.42%)	(0.00%)	(88.07%)	(0.00%)
20	4	1	✓	2	(33.24%)	(0.00%)	(24.57%)	(0.00%)
20	4	1	✓	3	(75.14%)	(0.00%)	(73.35%)	(0.00%)
20	4	1	✓	4	(15.13%)	(0.00%)	(19.97%)	(0.00%)
20	4	1	✓	5	(26.02%)	(0.00%)	(21.12%)	(0.00%)
20	4	1	✓	6	(68.82%)	(0.00%)	(59.67%)	(0.00%)
20	4	1	✓	7	(19.30%)	(0.00%)	(16.08%)	(0.00%)
20	4	1	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
20	4	1	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
20	4	1	✓	10	(94.00%)	(0.00%)	(80.05%)	(0.00%)
20	4	2		1	48.32	13.02k	18.68	1.56k
20	4	2		2	17.83	4.44k	1801.54	2009.72k
20	4	2		3	23.65	22.47k	2.33	373
20	4	2		4	1802.14	2627.81k	1801.57	2133.61k
20	4	2		5	1800.71	558.60k	1800.02	1691.20k
20	4	2		6	68.30	2.39k	4.80	277
20	4	2		7	19.27	7.36k	3.47	139
20	4	2		8	98.46	52.41k	1800.01	1442.96k
20	4	2		9	1.31	342	1.28	46
20	4	2		10	475.14	34.46k	49.16	3.88k
20	4	2	✓	1	(62.27%)	(0.00%)	(61.65%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	4	2	✓	2	(38.15%)	(0.00%)	(34.33%)	(0.00%)
20	4	2	✓	3	(66.99%)	(0.00%)	(63.77%)	(0.00%)
20	4	2	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	4	2	✓	5	(24.60%)	(0.00%)	(22.33%)	(0.00%)
20	4	2	✓	6	(47.06%)	(0.00%)	(47.65%)	(0.00%)
20	4	2	✓	7	(72.73%)	(0.00%)	(73.35%)	(0.00%)
20	4	2	✓	8	(79.33%)	(0.00%)	(78.42%)	(0.00%)
20	4	2	✓	9	(65.05%)	(0.00%)	(70.82%)	(0.00%)
20	4	2	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
20	4	3		1	6.62	6.92k	17.65	15.84k
20	4	3		2	1800.04	3467.45k	29.57	1.66k
20	4	3		3	1801.20	1675.71k	74.46	1.21k
20	4	3		4	1802.58	4244.92k	6.56	361
20	4	3		5	(0.17%)	(0.00%)	224.76	4.25k
20	4	3		6	2.51	85	2.32	67
20	4	3		7	63.74	10.44k	24.88	1.14k
20	4	3		8	1802.01	3521.04k	13.90	545
20	4	3		9	1803.58	4539.46k	4.48	265
20	4	3		10	(2.44%)	(1.07%)	353.77	4.61k
20	4	3	✓	1	(47.86%)	(0.00%)	(32.57%)	(0.00%)
20	4	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
20	4	3	✓	3	(63.40%)	(0.00%)	(68.51%)	(0.00%)
20	4	3	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	4	3	✓	5	(58.69%)	(0.00%)	(44.91%)	(0.00%)
20	4	3	✓	6	(63.91%)	(0.00%)	(82.98%)	(0.00%)
20	4	3	✓	7	(96.43%)	(0.00%)	(96.52%)	(0.00%)
20	4	3	✓	8	(99.24%)	(0.00%)	(∞)	(0.00%)
20	4	3	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
20	4	3	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
20	4	5		1	1800.00	1301.44k	333.99	1.42k
20	4	5		2	1800.01	1057.48k	1113.29	3.25k
20	4	5		3	(79.69%)	(0.00%)	(79.27%)	(0.00%)
20	4	5		4	1800.00	92.54k	1198.35	7.20k
20	4	5		5	(40.05%)	(0.00%)	(61.93%)	(0.00%)
20	4	5		6	1800.01	944.50k	983.74	8.74k
20	4	5		7	1800.01	1431.18k	109.05	1.39k
20	4	5		8	(5.27%)	(0.00%)	(7.87%)	(0.00%)
20	4	5		9	1800.02	2262.68k	70.04	675
20	4	5		10	647.33	7.97k	234.28	3.08k
20	4	5	✓	1	(70.75%)	(0.00%)	(75.06%)	(0.00%)
20	4	5	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
20	4	5	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
20	4	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	4	5	✓	5	(46.30%)	(0.00%)	(41.16%)	(0.00%)
20	4	5	✓	6	(77.86%)	(0.00%)	(78.83%)	(0.00%)
20	4	5	✓	7	(70.35%)	(0.00%)	(73.83%)	(0.00%)
20	4	5	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
20	4	5	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
20	4	5	✓	10	(37.43%)	(0.00%)	(51.73%)	(0.00%)
20	4	10		1	(14.83%)	(0.00%)	(14.02%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	4	10		2	(48.89%)	(0.00%)	(56.82%)	(0.00%)
20	4	10		3	(1.85%)	(0.00%)	(2.22%)	(0.00%)
20	4	10		4	(27.46%)	(0.00%)	(37.14%)	(0.00%)
20	4	10		5	1800.00	608.75k	577.29	809
20	4	10		6	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10		7	(16.03%)	(0.00%)	(19.58%)	(9.00%)
20	4	10		8	(42.45%)	(0.00%)	(56.89%)	(0.00%)
20	4	10		9	(29.28%)	(0.00%)	(35.37%)	(0.03%)
20	4	10		10	(49.48%)	(0.00%)	(64.18%)	(0.00%)
20	4	10	✓	1	(95.73%)	(0.00%)	(96.87%)	(0.00%)
20	4	10	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	3	(58.74%)	(0.00%)	(83.48%)	(0.00%)
20	4	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	6	(60.97%)	(0.00%)	(85.62%)	(0.00%)
20	4	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
20	4	10	✓	10	(74.92%)	(0.00%)	(∞)	(0.00%)
20	5	1		1	4.25	1.75k	2.26	137
20	5	1		2	15.27	1.62k	1802.01	3240.84k
20	5	1		3	1.72	274	3.62	265
20	5	1		4	6.10	3.49k	4.73	249
20	5	1		5	38.03	2.59k	1802.56	4113.77k
20	5	1		6	1.20	1	2.99	77
20	5	1		7	(0.00%)	(0.73%)	1800.02	2536.93k
20	5	1		8	3.99	2.47k	1.55	53
20	5	1		9	123.06	1.77k	1801.08	2054.49k
20	5	1		10	3.95	816	1800.43	3720.02k
20	5	1	✓	1	(51.14%)	(0.00%)	(42.61%)	(0.00%)
20	5	1	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
20	5	1	✓	3	(55.89%)	(0.00%)	(55.36%)	(0.00%)
20	5	1	✓	4	(73.60%)	(0.00%)	(51.72%)	(0.00%)
20	5	1	✓	5	(61.68%)	(0.00%)	(55.07%)	(0.00%)
20	5	1	✓	6	(68.31%)	(0.00%)	(65.10%)	(0.00%)
20	5	1	✓	7	(37.93%)	(0.00%)	(38.09%)	(0.00%)
20	5	1	✓	8	(52.77%)	(0.00%)	(55.20%)	(0.00%)
20	5	1	✓	9	(70.16%)	(0.00%)	(71.75%)	(0.00%)
20	5	1	✓	10	(48.73%)	(0.00%)	(52.53%)	(0.00%)
20	5	2		1	1800.04	3656.47k	7.45	471
20	5	2		2	24.70	13.04k	10.77	459
20	5	2		3	516.77	209.21k	1800.00	790.99k
20	5	2		4	9.94	3.04k	22.79	1.27k
20	5	2		5	19.09	1.52k	1800.72	1026.56k
20	5	2		6	(0.06%)	(0.00%)	588.87	3.69k
20	5	2		7	103.24	2.33k	93.62	1.24k
20	5	2		8	55.54	6.99k	100.39	917
20	5	2		9	1800.79	1246.01k	179.45	3.11k
20	5	2		10	1800.61	1604.58k	81.28	7.45k
20	5	2	✓	1	(56.79%)	(0.00%)	(59.85%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	5	2	✓	2	(39.99%)	(0.00%)	(39.00%)	(0.00%)
20	5	2	✓	3	(69.74%)	(0.00%)	(67.14%)	(0.00%)
20	5	2	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	5	2	✓	5	(36.25%)	(0.00%)	(49.74%)	(0.00%)
20	5	2	✓	6	(71.33%)	(0.00%)	(75.91%)	(0.00%)
20	5	2	✓	7	(77.97%)	(0.00%)	(81.64%)	(0.00%)
20	5	2	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
20	5	2	✓	9	(50.10%)	(0.00%)	(51.73%)	(0.00%)
20	5	2	✓	10	(81.13%)	(0.00%)	(61.41%)	(0.00%)
20	5	3		1	1801.03	1122.65k	836.80	5.35k
20	5	3		2	1.17	172	2.79	29
20	5	3		3	24.37	17.21k	16.06	1.21k
20	5	3		4	(16.40%)	(0.00%)	(17.17%)	(0.00%)
20	5	3		5	20.40	2.18k	33.95	523
20	5	3		6	(1.60%)	(0.69%)	(4.41%)	(0.00%)
20	5	3		7	1800.03	2664.81k	150.13	3.42k
20	5	3		8	1800.54	964.38k	820.03	12.82k
20	5	3		9	993.16	133.47k	423.67	2.48k
20	5	3		10	875.73	348.44k	352.15	2.07k
20	5	3	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
20	5	3	✓	2	(18.88%)	(0.00%)	(11.07%)	(0.00%)
20	5	3	✓	3	(23.88%)	(0.00%)	(26.35%)	(0.00%)
20	5	3	✓	4	(70.80%)	(0.00%)	(75.97%)	(0.00%)
20	5	3	✓	5	(55.54%)	(0.00%)	(67.81%)	(0.00%)
20	5	3	✓	6	(81.08%)	(0.00%)	(86.95%)	(0.00%)
20	5	3	✓	7	(77.62%)	(0.00%)	(76.19%)	(0.00%)
20	5	3	✓	8	(30.05%)	(0.00%)	(36.57%)	(0.00%)
20	5	3	✓	9	(49.47%)	(0.00%)	(63.62%)	(0.00%)
20	5	3	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
20	5	5		1	(34.28%)	(0.00%)	(28.88%)	(0.00%)
20	5	5		2	1800.84	1348.65k	191.85	1.09k
20	5	5		3	1800.00	428.60k	(9.56%)	(1.22%)
20	5	5		4	1800.69	396.96k	(3.01%)	(0.00%)
20	5	5		5	6.23	355	16.79	119
20	5	5		6	1800.96	779.27k	907.50	845
20	5	5		7	1800.77	872.34k	(1.92%)	(0.00%)
20	5	5		8	(0.87%)	(0.00%)	(3.61%)	(0.00%)
20	5	5		9	(∞)	(0.00%)	(∞)	(0.00%)
20	5	5		10	(79.17%)	(0.00%)	(83.41%)	(35.05%)
20	5	5	✓	1	(90.83%)	(0.00%)	(92.17%)	(0.00%)
20	5	5	✓	2	(36.74%)	(0.00%)	(43.92%)	(0.00%)
20	5	5	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
20	5	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
20	5	5	✓	5	(41.73%)	(0.00%)	(46.10%)	(0.00%)
20	5	5	✓	6	(∞)	(0.00%)	(∞)	(0.00%)
20	5	5	✓	7	(90.65%)	(0.00%)	(∞)	(0.00%)
20	5	5	✓	8	(44.05%)	(0.00%)	(55.65%)	(0.00%)
20	5	5	✓	9	(95.70%)	(0.00%)	(∞)	(0.00%)
20	5	5	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10		1	1800.83	341.10k	(10.81%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
20	5	10		2	(0.97%)	(0.00%)	(33.60%)	(0.00%)
20	5	10		3	(29.04%)	(0.00%)	(43.15%)	(0.00%)
20	5	10		4	(18.12%)	(0.00%)	(65.22%)	(0.69%)
20	5	10		5	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10		6	(0.04%)	(0.00%)	(24.67%)	(0.00%)
20	5	10		7	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10		8	1800.00	740.14k	(∞)	(0.00%)
20	5	10		9	(63.63%)	(0.00%)	(72.17%)	(6.94%)
20	5	10		10	1800.00	840.45k	(9.69%)	(0.00%)
20	5	10	✓	1	(38.67%)	(0.00%)	(64.80%)	(0.00%)
20	5	10	✓	2	(43.94%)	(0.00%)	(78.60%)	(0.00%)
20	5	10	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	4	(58.20%)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	6	(42.88%)	(0.00%)	(64.45%)	(0.00%)
20	5	10	✓	7	(86.15%)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	8	(87.25%)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
20	5	10	✓	10	(82.14%)	(0.00%)	(∞)	(0.00%)
30	2	1		1	0.06	1	0.01	1
30	2	1		2	0.08	1	0.06	1
30	2	1		3	0.02	1	0.04	1
30	2	1		4	1800.02	18802.15k	0.15	1
30	2	1		5	0.06	1	0.07	1
30	2	1		6	0.09	1	0.06	1
30	2	1		7	0.06	1	0.06	1
30	2	1		8	0.07	1	0.10	1
30	2	1		9	0.30	17	0.05	1
30	2	1		10	0.11	1	0.07	1
30	2	1	✓	1	(17.55%)	(0.00%)	(11.78%)	(0.00%)
30	2	1	✓	2	(23.66%)	(0.00%)	(20.09%)	(0.00%)
30	2	1	✓	3	(23.58%)	(0.00%)	(11.68%)	(0.00%)
30	2	1	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	2	1	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
30	2	1	✓	6	(10.36%)	(0.00%)	(4.55%)	(0.00%)
30	2	1	✓	7	(36.68%)	(0.00%)	(17.65%)	(0.00%)
30	2	1	✓	8	(20.38%)	(0.00%)	(11.96%)	(0.00%)
30	2	1	✓	9	(11.45%)	(0.00%)	(7.73%)	(0.00%)
30	2	1	✓	10	(11.47%)	(0.00%)	(2.70%)	(0.00%)
30	2	2		1	0.33	113	0.50	19
30	2	2		2	1804.50	15159.71k	1.13	484
30	2	2		3	0.16	11	0.10	1
30	2	2		4	0.08	1	0.10	1
30	2	2		5	0.25	37	0.39	11
30	2	2		6	2.04	635	0.93	228
30	2	2		7	1.01	263	1800.03	8686.87k
30	2	2		8	0.04	1	1.31	5.27k
30	2	2		9	0.39	49	0.44	55
30	2	2		10	0.75	309	0.64	156
30	2	2	✓	1	(∞)	(0.00%)	(∞)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	2	2	✓	2	(29.61%)	(0.00%)	(20.54%)	(0.00%)
30	2	2	✓	3	(8.05%)	(0.00%)	(6.11%)	(0.00%)
30	2	2	✓	4	(36.74%)	(0.00%)	(35.93%)	(0.00%)
30	2	2	✓	5	(9.86%)	(0.00%)	(8.68%)	(0.00%)
30	2	2	✓	6	(32.51%)	(0.00%)	(25.23%)	(0.00%)
30	2	2	✓	7	(37.08%)	(0.00%)	(28.93%)	(0.00%)
30	2	2	✓	8	(14.43%)	(0.00%)	(7.44%)	(0.00%)
30	2	2	✓	9	(26.87%)	(0.00%)	(28.90%)	(0.00%)
30	2	2	✓	10	(8.37%)	(0.00%)	(6.42%)	(0.00%)
30	2	3		1	0.39	111	0.57	60
30	2	3		2	1800.02	17765.97k	1805.55	8177.68k
30	2	3		3	1.45	594	1801.00	5491.76k
30	2	3		4	1802.98	11842.67k	0.92	300
30	2	3		5	1.00	442	1.99	1.88k
30	2	3		6	1.29	370	1.58	451
30	2	3		7	0.58	173	0.69	229
30	2	3		8	0.78	183	0.98	167
30	2	3		9	1800.04	19529.73k	0.80	110
30	2	3		10	0.24	80	0.36	3
30	2	3	✓	1	(15.59%)	(0.00%)	(15.70%)	(0.00%)
30	2	3	✓	2	(31.42%)	(0.00%)	(34.68%)	(0.00%)
30	2	3	✓	3	(60.22%)	(0.00%)	(55.20%)	(0.00%)
30	2	3	✓	4	(6.70%)	(0.00%)	(3.80%)	(0.00%)
30	2	3	✓	5	(30.07%)	(0.00%)	(30.23%)	(0.00%)
30	2	3	✓	6	(74.51%)	(0.00%)	(76.45%)	(0.00%)
30	2	3	✓	7	(6.76%)	(0.00%)	(4.16%)	(0.00%)
30	2	3	✓	8	(22.13%)	(0.00%)	(20.24%)	(0.00%)
30	2	3	✓	9	(20.19%)	(0.00%)	(19.08%)	(0.00%)
30	2	3	✓	10	(37.24%)	(0.00%)	(31.79%)	(0.00%)
30	2	5		1	1800.05	8484.40k	25.30	67.02k
30	2	5		2	1800.05	6695.06k	21.27	6.80k
30	2	5		3	0.77	61	0.80	225
30	2	5		4	1800.03	3783.28k	82.05	27.96k
30	2	5		5	1800.00	3979.46k	1802.09	3587.92k
30	2	5		6	1800.03	8168.85k	5.61	738
30	2	5		7	3.72	713	3.65	721
30	2	5		8	1800.01	2200.69k	1800.86	1612.90k
30	2	5		9	0.96	213	911.15	2095.61k
30	2	5		10	1801.85	8152.33k	45.90	143.91k
30	2	5	✓	1	(36.57%)	(0.00%)	(37.30%)	(0.00%)
30	2	5	✓	2	(56.42%)	(0.00%)	(53.23%)	(0.00%)
30	2	5	✓	3	(34.52%)	(0.00%)	(30.21%)	(0.00%)
30	2	5	✓	4	(39.08%)	(0.00%)	(43.09%)	(0.00%)
30	2	5	✓	5	(19.03%)	(0.00%)	(17.63%)	(0.00%)
30	2	5	✓	6	(61.23%)	(0.00%)	(53.40%)	(0.00%)
30	2	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	2	5	✓	8	(83.74%)	(0.00%)	(85.84%)	(0.00%)
30	2	5	✓	9	(14.20%)	(0.00%)	(13.78%)	(0.00%)
30	2	5	✓	10	(26.34%)	(0.00%)	(20.47%)	(0.00%)
30	2	10		1	1800.03	4790.40k	1802.84	2954.72k

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	2	10		2	1800.03	5017.14k	1800.58	1217.82k
30	2	10		3	1800.01	3799.24k	1801.08	2141.67k
30	2	10		4	1800.02	3164.39k	141.03	5.74k
30	2	10		5	1800.04	4638.77k	1800.01	1537.33k
30	2	10		6	1803.41	5968.91k	18.95	3.74k
30	2	10		7	1801.77	6645.22k	12.22	2.92k
30	2	10		8	1801.52	3560.19k	1800.57	1128.06k
30	2	10		9	1800.06	7762.47k	7.04	1.88k
30	2	10		10	1800.02	3242.48k	1801.20	1958.45k
30	2	10	✓	1	(11.82%)	(0.00%)	(16.53%)	(0.00%)
30	2	10	✓	2	(44.43%)	(0.00%)	(50.78%)	(0.00%)
30	2	10	✓	3	(31.54%)	(0.00%)	(31.16%)	(0.00%)
30	2	10	✓	4	(22.16%)	(0.00%)	(26.19%)	(0.00%)
30	2	10	✓	5	(57.49%)	(0.00%)	(73.27%)	(0.00%)
30	2	10	✓	6	(20.06%)	(0.00%)	(19.91%)	(0.00%)
30	2	10	✓	7	(9.28%)	(0.00%)	(7.64%)	(0.00%)
30	2	10	✓	8	(18.55%)	(0.00%)	(21.51%)	(0.00%)
30	2	10	✓	9	(18.92%)	(0.00%)	(21.72%)	(0.00%)
30	2	10	✓	10	(64.07%)	(0.00%)	(73.90%)	(0.00%)
30	3	1		1	0.54	45	1.02	1
30	3	1		2	13.52	84.55k	0.97	87
30	3	1		3	0.55	63	0.43	3
30	3	1		4	0.56	27	0.87	49
30	3	1		5	0.48	25	0.81	7
30	3	1		6	0.27	1	0.72	1
30	3	1		7	0.49	13	0.81	124
30	3	1		8	0.26	1	0.25	1
30	3	1		9	0.51	41	0.57	3
30	3	1		10	0.58	67	1.11	356
30	3	1	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
30	3	1	✓	2	(35.20%)	(0.00%)	(39.37%)	(0.00%)
30	3	1	✓	3	(31.97%)	(0.00%)	(33.08%)	(0.00%)
30	3	1	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	3	1	✓	5	(47.85%)	(0.00%)	(53.03%)	(0.00%)
30	3	1	✓	6	(21.23%)	(0.00%)	(24.46%)	(0.00%)
30	3	1	✓	7	(71.17%)	(0.00%)	(75.30%)	(0.00%)
30	3	1	✓	8	(69.27%)	(0.00%)	(70.01%)	(0.00%)
30	3	1	✓	9	(52.82%)	(0.00%)	(62.37%)	(0.00%)
30	3	1	✓	10	(24.54%)	(0.00%)	(29.31%)	(0.00%)
30	3	2		1	0.57	25	0.98	55
30	3	2		2	1807.75	7685.16k	3.10	786
30	3	2		3	1810.55	8856.95k	2.64	839
30	3	2		4	0.67	120	1.55	107
30	3	2		5	0.58	41	0.95	5
30	3	2		6	0.33	105	1.80	236
30	3	2		7	1800.05	9688.96k	1.08	47
30	3	2		8	1.32	529	7.20	2.98k
30	3	2		9	1808.85	8063.23k	1801.30	2971.53k
30	3	2		10	0.92	560	2.50	1.20k
30	3	2	✓	1	(32.91%)	(0.00%)	(37.70%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	3	2	✓	2	(71.16%)	(0.00%)	(71.73%)	(0.00%)
30	3	2	✓	3	(87.89%)	(0.00%)	(98.24%)	(0.00%)
30	3	2	✓	4	(78.90%)	(0.00%)	(97.10%)	(0.00%)
30	3	2	✓	5	(74.40%)	(0.00%)	(78.82%)	(0.00%)
30	3	2	✓	6	(64.70%)	(0.00%)	(96.65%)	(0.00%)
30	3	2	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	3	2	✓	8	(47.54%)	(0.00%)	(50.11%)	(0.00%)
30	3	2	✓	9	(34.99%)	(0.00%)	(43.41%)	(0.00%)
30	3	2	✓	10	(55.15%)	(0.00%)	(59.93%)	(0.00%)
30	3	3		1	1.55	2.71k	4.25	3.11k
30	3	3		2	1806.65	7476.36k	5.61	435
30	3	3		3	5.58	663	25.99	1.13k
30	3	3		4	(0.16%)	(0.00%)	(0.16%)	(0.00%)
30	3	3		5	0.98	73	1.98	109
30	3	3		6	852.77	3241.41k	45.83	73.95k
30	3	3		7	0.64	10	1.65	93
30	3	3		8	1805.54	8095.16k	7.20	1.51k
30	3	3		9	1802.98	5050.30k	1801.25	1698.09k
30	3	3		10	1800.06	5865.06k	1801.51	2074.47k
30	3	3	✓	1	(94.82%)	(0.00%)	(98.92%)	(0.00%)
30	3	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	3	3	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	3	3	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	3	3	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
30	3	3	✓	6	(∞)	(0.00%)	(∞)	(0.00%)
30	3	3	✓	7	(67.18%)	(0.00%)	(68.75%)	(0.00%)
30	3	3	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
30	3	3	✓	9	(31.72%)	(0.00%)	(36.94%)	(0.00%)
30	3	3	✓	10	(50.71%)	(0.00%)	(54.96%)	(0.00%)
30	3	5		1	7.41	229	40.98	687
30	3	5		2	1801.24	2292.18k	1148.47	73.32k
30	3	5		3	1801.26	2045.06k	(3.31%)	(3.04%)
30	3	5		4	1801.83	2686.39k	1800.45	679.01k
30	3	5		5	1801.11	2058.47k	144.43	1.74k
30	3	5		6	333.78	401.68k	(0.08%)	(0.00%)
30	3	5		7	1800.13	944.62k	67.89	1.57k
30	3	5		8	640.48	331.72k	(0.03%)	(0.00%)
30	3	5		9	13.08	4.16k	1801.00	1052.02k
30	3	5		10	1.58	539	14.77	2.70k
30	3	5	✓	1	(63.28%)	(0.00%)	(86.36%)	(0.00%)
30	3	5	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	3	5	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	3	5	✓	4	(62.71%)	(0.00%)	(68.15%)	(0.00%)
30	3	5	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
30	3	5	✓	6	(51.39%)	(0.00%)	(60.19%)	(0.00%)
30	3	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	3	5	✓	8	(71.75%)	(0.00%)	(86.61%)	(0.00%)
30	3	5	✓	9	(92.14%)	(0.00%)	(∞)	(0.00%)
30	3	5	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
30	3	10		1	1801.23	1136.65k	(7.50%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	3	10		2	7.36	527	110.09	1.09k
30	3	10		3	978.98	16.45k	(10.17%)	(0.00%)
30	3	10		4	(51.43%)	(0.00%)	(61.74%)	(0.00%)
30	3	10		5	1800.85	709.11k	(56.55%)	(0.00%)
30	3	10		6	(0.22%)	(0.00%)	(8.23%)	(0.29%)
30	3	10		7	1800.01	1336.86k	1800.00	186.74k
30	3	10		8	1802.21	1645.87k	(32.21%)	(0.00%)
30	3	10		9	1800.97	1411.13k	(0.02%)	(0.00%)
30	3	10		10	1801.61	1394.88k	(0.49%)	(0.00%)
30	3	10	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
30	3	10	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	3	10	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	3	10	✓	4	(∞)	(0.00%)	(∞)	(1.64%)
30	3	10	✓	5	(56.80%)	(0.00%)	(80.40%)	(0.00%)
30	3	10	✓	6	(∞)	(0.00%)	(∞)	(0.00%)
30	3	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	3	10	✓	8	(75.01%)	(0.00%)	(87.16%)	(0.00%)
30	3	10	✓	9	(77.49%)	(0.00%)	(∞)	(0.00%)
30	3	10	✓	10	(74.68%)	(0.00%)	(∞)	(0.00%)
30	4	1		1	0.80	225	1.68	1
30	4	1		2	199.16	273.06k	3.08	1.58k
30	4	1		3	0.60	1	0.72	5
30	4	1		4	1.20	429	1.32	98
30	4	1		5	1801.01	962.92k	1802.23	3466.98k
30	4	1		6	1802.67	2934.00k	1802.70	4675.33k
30	4	1		7	1142.93	4198.76k	0.94	1
30	4	1		8	5.50	1.79k	1.21	94
30	4	1		9	10.67	925	2.86	265
30	4	1		10	0.93	786	2.80	3.12k
30	4	1	✓	1	(35.57%)	(0.00%)	(34.79%)	(0.00%)
30	4	1	✓	2	(90.67%)	(0.00%)	(91.64%)	(0.00%)
30	4	1	✓	3	(61.39%)	(0.00%)	(65.43%)	(0.00%)
30	4	1	✓	4	(74.81%)	(0.00%)	(75.67%)	(0.00%)
30	4	1	✓	5	(∞)	(0.00%)	(∞)	(0.09%)
30	4	1	✓	6	(90.39%)	(0.00%)	(89.52%)	(0.00%)
30	4	1	✓	7	(49.13%)	(0.00%)	(50.38%)	(0.00%)
30	4	1	✓	8	(47.72%)	(0.00%)	(44.30%)	(0.00%)
30	4	1	✓	9	(83.21%)	(0.00%)	(85.53%)	(0.00%)
30	4	1	✓	10	(42.41%)	(0.00%)	(42.07%)	(0.00%)
30	4	2		1	1800.62	1100.09k	61.38	1.89k
30	4	2		2	1662.85	205.86k	35.53	17.72k
30	4	2		3	26.75	1.17k	10.65	345
30	4	2		4	1803.29	615.68k	1801.40	1710.52k
30	4	2		5	4.55	3.62k	3.57	127
30	4	2		6	450.16	49.61k	28.84	1.29k
30	4	2		7	1800.01	1297.80k	86.07	1.88k
30	4	2		8	1800.80	1379.01k	48.01	6.29k
30	4	2		9	1800.53	128.49k	82.62	5.95k
30	4	2		10	1800.01	1133.33k	97.11	2.06k
30	4	2	✓	1	(58.91%)	(0.00%)	(62.71%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	4	2	✓	2	(61.90%)	(0.00%)	(63.18%)	(0.00%)
30	4	2	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	4	2	✓	4	(53.97%)	(0.00%)	(61.36%)	(0.00%)
30	4	2	✓	5	(40.57%)	(0.00%)	(40.58%)	(0.00%)
30	4	2	✓	6	(60.90%)	(0.00%)	(60.81%)	(0.00%)
30	4	2	✓	7	(57.38%)	(0.00%)	(61.82%)	(0.00%)
30	4	2	✓	8	(60.12%)	(0.00%)	(60.88%)	(0.00%)
30	4	2	✓	9	(79.84%)	(0.00%)	(73.17%)	(0.00%)
30	4	2	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
30	4	3		1	26.97	3.76k	13.93	225
30	4	3		2	(0.05%)	(0.00%)	713.07	4.37k
30	4	3		3	(4.17%)	(0.00%)	349.00	2.02k
30	4	3		4	(0.13%)	(0.00%)	147.39	3.30k
30	4	3		5	(0.04%)	(0.00%)	375.93	8.64k
30	4	3		6	1800.00	8.99k	536.02	4.76k
30	4	3		7	1800.01	431.48k	1800.01	1165.93k
30	4	3		8	1800.01	1872.74k	106.27	3.11k
30	4	3		9	(9.77%)	(0.00%)	(3.22%)	(0.00%)
30	4	3		10	(11.46%)	(0.00%)	703.77	3.22k
30	4	3	✓	1	(66.52%)	(0.00%)	(68.09%)	(0.00%)
30	4	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	4	3	✓	3	(74.44%)	(0.00%)	(74.74%)	(0.00%)
30	4	3	✓	4	(50.52%)	(0.00%)	(50.38%)	(0.00%)
30	4	3	✓	5	(88.67%)	(0.00%)	(86.82%)	(0.00%)
30	4	3	✓	6	(73.69%)	(0.00%)	(74.64%)	(0.00%)
30	4	3	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	4	3	✓	8	(85.56%)	(0.00%)	(85.92%)	(0.00%)
30	4	3	✓	9	(74.68%)	(0.00%)	(79.25%)	(0.00%)
30	4	3	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5		1	(8.34%)	(0.00%)	(9.92%)	(0.00%)
30	4	5		2	(4.03%)	(0.00%)	(1.32%)	(0.00%)
30	4	5		3	(2.03%)	(0.00%)	679.29	2.42k
30	4	5		4	(8.58%)	(0.00%)	(17.21%)	(0.00%)
30	4	5		5	(26.82%)	(0.00%)	(14.58%)	(0.00%)
30	4	5		6	(12.53%)	(0.00%)	(10.00%)	(3.89%)
30	4	5		7	(14.29%)	(0.00%)	(5.68%)	(0.00%)
30	4	5		8	(18.44%)	(0.00%)	(12.83%)	(0.00%)
30	4	5		9	(4.76%)	(0.00%)	(1.80%)	(0.00%)
30	4	5		10	1800.00	44.54k	1169.86	14.56k
30	4	5	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	3	(88.60%)	(0.00%)	(85.95%)	(0.00%)
30	4	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	6	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
30	4	5	✓	10	(95.89%)	(0.00%)	(96.28%)	(0.00%)
30	4	10		1	(46.87%)	(2.95%)	(48.74%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	4	10		2	(37.00%)	(0.00%)	(39.36%)	(1.18%)
30	4	10		3	(32.37%)	(0.00%)	(42.54%)	(0.00%)
30	4	10		4	(34.43%)	(0.00%)	(32.59%)	(0.00%)
30	4	10		5	(57.56%)	(0.00%)	(63.35%)	(0.00%)
30	4	10		6	(25.98%)	(0.00%)	(46.07%)	(0.00%)
30	4	10		7	(45.81%)	(0.00%)	(48.12%)	(0.00%)
30	4	10		8	(10.99%)	(0.00%)	(13.96%)	(0.00%)
30	4	10		9	(28.84%)	(0.00%)	(36.02%)	(2.79%)
30	4	10		10	(58.38%)	(0.00%)	(60.02%)	(0.00%)
30	4	10	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	6	(91.42%)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
30	4	10	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1		1	(0.07%)	(0.00%)	309.66	23.91k
30	5	1		2	278.97	56.82k	20.14	2.46k
30	5	1		3	(2.00%)	(0.00%)	1800.02	2164.85k
30	5	1		4	(1.93%)	(0.00%)	111.44	3.87k
30	5	1		5	3.87	445	4.38	256
30	5	1		6	1802.50	567.69k	1801.30	1982.37k
30	5	1		7	140.96	54.88k	1801.85	2666.38k
30	5	1		8	1800.21	400.08k	71.57	3.67k
30	5	1		9	4.36	1.60k	2.45	93
30	5	1		10	1.49	235	2.96	23
30	5	1	✓	1	(70.81%)	(0.00%)	(70.75%)	(0.00%)
30	5	1	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1	✓	5	(59.05%)	(0.00%)	(57.55%)	(0.00%)
30	5	1	✓	6	(57.06%)	(0.00%)	(61.16%)	(0.00%)
30	5	1	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	5	1	✓	8	(52.01%)	(0.00%)	(52.57%)	(0.00%)
30	5	1	✓	9	(47.65%)	(0.00%)	(48.85%)	(0.00%)
30	5	1	✓	10	(41.31%)	(0.00%)	(38.96%)	(0.00%)
30	5	2		1	(33.93%)	(0.00%)	(13.31%)	(0.00%)
30	5	2		2	1800.00	39.02k	(3.34%)	(0.00%)
30	5	2		3	(4.67%)	(0.00%)	(2.38%)	(0.00%)
30	5	2		4	786.27	17.37k	285.87	3.61k
30	5	2		5	(4.86%)	(0.00%)	(1.94%)	(0.00%)
30	5	2		6	(2.76%)	(0.00%)	(3.33%)	(0.00%)
30	5	2		7	257.93	31.49k	155.66	1.27k
30	5	2		8	(16.68%)	(1.62%)	(8.74%)	(0.00%)
30	5	2		9	1800.64	863.86k	484.29	4.38k
30	5	2		10	1800.00	76.69k	1463.84	15.85k
30	5	2	✓	1	(∞)	(0.00%)	(∞)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	5	2	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	5	2	✓	3	(56.86%)	(0.00%)	(57.28%)	(0.00%)
30	5	2	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	5	2	✓	5	(83.45%)	(0.00%)	(85.22%)	(0.00%)
30	5	2	✓	6	(98.90%)	(0.00%)	(99.07%)	(0.00%)
30	5	2	✓	7	(68.79%)	(0.00%)	(68.01%)	(0.00%)
30	5	2	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
30	5	2	✓	9	(67.74%)	(0.00%)	(71.74%)	(0.00%)
30	5	2	✓	10	(95.01%)	(0.00%)	(95.37%)	(0.00%)
30	5	3		1	1800.00	204.16k	1183.16	9.18k
30	5	3		2	1800.85	597.11k	833.75	7.82k
30	5	3		3	(0.20%)	(0.00%)	(2.14%)	(0.00%)
30	5	3		4	(36.73%)	(0.00%)	(23.67%)	(2.68%)
30	5	3		5	(9.79%)	(0.00%)	(11.64%)	(0.00%)
30	5	3		6	(48.71%)	(0.00%)	(39.88%)	(0.00%)
30	5	3		7	(0.89%)	(0.00%)	(2.56%)	(0.00%)
30	5	3		8	(10.90%)	(0.00%)	(6.24%)	(0.00%)
30	5	3		9	1800.77	457.01k	(1.70%)	(0.00%)
30	5	3		10	(15.72%)	(0.00%)	(8.30%)	(0.00%)
30	5	3	✓	1	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	5	(87.11%)	(0.00%)	(89.13%)	(0.00%)
30	5	3	✓	6	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
30	5	3	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5		1	(20.62%)	(0.00%)	(23.83%)	(0.00%)
30	5	5		2	1801.19	405.35k	1527.88	2.77k
30	5	5		3	(59.72%)	(0.00%)	(54.84%)	(19.01%)
30	5	5		4	(40.57%)	(0.00%)	(44.97%)	(0.00%)
30	5	5		5	(17.33%)	(0.00%)	(25.55%)	(0.00%)
30	5	5		6	(11.41%)	(0.00%)	(21.28%)	(0.00%)
30	5	5		7	(18.52%)	(0.00%)	(21.11%)	(0.00%)
30	5	5		8	(27.11%)	(2.59%)	(46.46%)	(0.00%)
30	5	5		9	(16.70%)	(0.00%)	(33.16%)	(0.00%)
30	5	5		10	(19.25%)	(0.00%)	(22.08%)	(0.00%)
30	5	5	✓	1	(79.34%)	(0.00%)	(82.24%)	(0.00%)
30	5	5	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	5	(90.81%)	(0.00%)	(94.14%)	(0.00%)
30	5	5	✓	6	(89.65%)	(0.00%)	(91.73%)	(0.00%)
30	5	5	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
30	5	5	✓	10	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10		1	(7.89%)	(0.00%)	(46.99%)	(0.00%)

instance					SCIP		SCIP(bivar)	
Nvars	Deg	Ncons	(16)	#	time (dgap)	nodes (pgap)	time (dgap)	nodes (pgap)
30	5	10		2	(∞)	(0.00%)	(∞)	(1.62%)
30	5	10		3	(77.43%)	(0.00%)	(81.55%)	(57.86%)
30	5	10		4	(98.31%)	(0.00%)	(∞)	(0.00%)
30	5	10		5	(71.52%)	(0.00%)	(72.74%)	(5.29%)
30	5	10		6	(92.46%)	(0.00%)	(94.24%)	(0.00%)
30	5	10		7	(77.38%)	(0.00%)	(82.94%)	(0.00%)
30	5	10		8	(15.31%)	(0.00%)	(29.85%)	(0.00%)
30	5	10		9	(49.46%)	(0.00%)	(48.91%)	(0.00%)
30	5	10		10	(∞)	(0.00%)	(∞)	(3.00%)
30	5	10	✓	1	(66.22%)	(0.00%)	(88.36%)	(0.00%)
30	5	10	✓	2	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	3	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	4	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	5	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	6	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	7	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	8	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	9	(∞)	(0.00%)	(∞)	(0.00%)
30	5	10	✓	10	(∞)	(0.00%)	(∞)	(0.00%)