

JÁCINT SZABÓ<sup>1</sup>

**The set of solutions to nomination  
validation in passive gas transportation  
networks with a generalized flow  
formula**

Herausgegeben vom  
Konrad-Zuse-Zentrum für Informationstechnik Berlin  
Takustraße 7  
D-14195 Berlin-Dahlem

Telefon: 030-84185-0  
Telefax: 030-84185-125

e-mail: [bibliothek@zib.de](mailto:bibliothek@zib.de)  
URL: <http://www.zib.de>

ZIB-Report (Print) ISSN 1438-0064  
ZIB-Report (Internet) ISSN 2192-7782

# The set of solutions to nomination validation in passive gas transportation networks with a generalized flow formula

Jácint Szabó

May 14, 2012

## Abstract

In this paper we give an analytical description to the structure of solutions in the gas nomination validation problem in gas transportation networks. These networks are assumed to contain no active devices, only certain types of pipelines, where the flow of gas is modeled by a generalized version of the quadratic Weymouth's equation. The purpose of considering generalized flow formulas is to be able to adapt our results to various gas network optimization problems involving gas flow formulas beyond Weymouth's equation. Such formulas can appear in certain nodes of Branch&Bound trees, or they can stem from discretization and linearization carried out at active devices. We call a balanced supply – demand vector a nomination, and the passive nomination validation problem is to decide whether there exist pressures at the nodes generating a given nomination. We prove that in our setup the pressure square vectors generating a given nomination form a one-dimensional connected and continuous curve in the pressure square space, and this curve is a line for the classical Weymouth's equation. We also present a visual approach for the easy comprehension of how this solution curve arises; we give a short investigation of the set of feasible nominations; and finally we give a proof that the nomination validation problem in gas networks with active devices is  $\mathcal{NP}$ -complete.

**Keywords** gas transportation network, gas nomination, Weymouth's equation

## 1 Introduction

The efficient transmission of natural gas from producing areas to consumption areas requires an extensive and elaborate gas transportation system. A gas transportation network consists of several components, namely **nodes**, like junctions, entry, and exit nodes; **pipelines**, connecting long distances, where the gas is driven by the pressure difference at the end nodes; **compressor**

---

\*Konrad-Zuse-Zentrum für Informationstechnik Berlin, Takustrasse 7, D-14195 Berlin and IBM Research Lab, Zürich, Säumerstrasse 4, CH-8803 Rüschlikon. E-mail: [jsz@zurich.ibm.com](mailto:jsz@zurich.ibm.com). Research supported by Open Grid Europe GmbH, Kallenbergstr. 5, D-45141 Essen.

**stations**, which increase the pressure in the network; **resistors** and **control valves** which decrease and regulate the pressure; and **valves** which can open or close a pipeline. Pipelines and resistors are called **passive**, while all other devices (compressors, control valves, and valves) are called **active**.

There are lots of optimization problems arising in the operation, maintenance, planning, expansion, and construction of the gas transportation infrastructure. One of them is the nomination validation problem, also called gas transmission or nomination problem, that is to decide if a given nomination, which is by definition a balanced supply and demand vector at the nodes, can be delivered in the network or not. If it can, one has to find appropriate pressures at the nodes and configurations of the active devices generating the given nomination, which minimizes the gas fuel driving the compressor stations (gas fuel minimization). Deciding whether a set of supplies and demands specified at the nodes has the property that every nomination below it is feasible is called the booking validation problem. Finally, if the capacity of the gas network is not enough, then to decide which possible network expansion to carry out, like where to route a new pipeline between two nodes of the network, is called the topology planning problem. More details on these and other optimization problems can be found in [4, 6, 9, 16, 17].

One the most important of these optimization problems is nomination validation. In the current industry practice the network operator concludes gas transportation contracts with its clients, giving them the right to feed in up to a certain amount of gas at a specified entry and to feed out the same amount at a specified exit. The collection of these bounds is called a booking. This is the so-called RAC-type contract (restrictively allocable capacity). To start an actual gas transmission, a pair of clients, a gas supplier and a gas consumer, have to nominate to the network operator the exact amount of gas they want to feed in at one node of the network and out of another, obeying the bounds specified in the RAC-type contract. The network operator has no knowledge of what nominations will be received, but has to guarantee delivery. The nomination validation problem is to decide whether there exist pressures at the nodes and configurations of the active devices without violating physical or operational constraints, such that the specified nomination is delivered. For more on gas transmission contracts we refer to [9].

In practice the nomination validation problem is solved by first making the discrete decisions at the active devices, usually manually, based on some heuristic and the experience of the operator; and then applying simulation techniques [15], dynamic programming [3], or various nonlinear programming (NLP) tools [7, 18] to solve the resulting NLP. A more precise and computationally more challenging approach, which is not widely used yet, is to directly formulate the whole problem as a mixed integer nonlinear problem (MINLP). To attack these MINLP's, advanced Branch&Bound solvers are usually used. At Zuse Institut, Berlin, one such specially tailored solver was developed for solving gas optimization problems, based on the open source Branch&Bound solver SCIP, that combines techniques of mixed integer linear programming, nonlinear optimization and constraint programming in a unique way. For more details on this solver see [1, 10, 19].

In this paper we consider the nomination validation problem in gas transportation networks containing no active devices, only certain types of pipelines where the formula determining the flow of gas is allowed to come from a wide

class of formulas to be defined in Equation (5). This class includes the widely adapted Weymouth’s equation and the Panhandle formulas. Our main result in Theorem 2.3 is to prove that in this framework, for every nomination  $d$ , the pressure *square* vectors generating  $d$  form a 1-dimensional connected and continuous curve in the pressure square space. Moreover, this curve is a straight line for important special cases, like Weymouth’s equation. Theorem 2.3 may find applications in gas network optimization problems which involve gas flow formulas fitting into Equation (5), which is indeed the case in modeling certain active devices, see Examples 2.12–2.14, or after adapting certain discretization and linearization techniques to gas networks (Example 2.11). Furthermore, most of the optimization problems arising in gas transportation networks lead to MINLP’s, which are usually solved by Branch&Bound solvers. Nodes of a Branch&Bound tree where the discrete decisions at the active devices are already made, in many cases involve gas flow formulas which fit into our framework.

Theorem 2.3 generalizes a result of Maugis [13], who proved the special case of Weymouth’s equation. The proof given in the present paper is based on an alternative proof by DeWolf and Smeers [5] to Maugis’ theorem, which however contained an error. Thus in the present paper an improved version, applied to the generalized problem is given.

In Section 3 we give a physical interpretation and visualization of the nomination validation problem in passive gas transportation networks by means of rubber bands. With the help of the rubber band view we show an example to the so-called more pipeline – less throughput phenomenon, and we determine the maximum possible throughput decrease. In Section 4 we consider convexity and topological questions on the set of feasible nominations and bookings, and finally in Section 5 we prove that nomination validation in gas networks containing active devices is  $\mathcal{NP}$ -complete.

There are other physical networks where certain energy heads on the nodes drive the flow over the edges, like pressure drives gas in a gas transportation network. In water supply pipe networks this energy head is the water head, and the Hazen–Williams rule gives the flow on an edge as a function of the head loss. For other liquids or gases the Darcy–Weisbach formula can be applied. For electrical networks the role of Weymouth’s equation is played by Ohm’s law. The methods described in this paper can be also adapted to some extent to these other networks with similar characteristics.

## 2 The set of solutions in the pressure square space to the Generalized Passive Gas Nomination Validation Problem

In this section we consider the gas nomination validation problem in generalized passive gas transportation networks. These networks contain no active devices, only certain types of passive pipelines, where the formula determining the flow of gas is allowed to come from a wide class of formulas to be defined in Equation (5), including the widely adapted Weymouth’s equation, and the Panhandle formulas. Such formulas can arise in modelings of active devices to be described in Examples 2.12–2.14, and in the discretization of compressor characteristic diagrams in Example 2.11. Branch&Bound trees occur frequently in solution

methods for various gas optimization problems. Nodes in these trees where the discrete decisions at active devices are already made may also involve gas flow formulations fitting into our framework. Our goal with this generalized setup is the potential to adapt our results to these settings.

## 2.1 The Passive Gas Nomination Validation Problem

First we introduce the classical PASSIVE GAS NOMINATION VALIDATION PROBLEM (NVP). We model a gas transportation network as a directed network  $G = (V, A)$ , where  $V$  is the set of nodes, that is junctions, entry, and exit nodes of the network, and  $A$  is the set of pipelines connecting them. We introduce gas flow variables  $q_a \in \mathbb{R}$  for each pipeline  $a \in A$ . Although the network is directed, this is only an auxiliary orientation, as negative flow  $q$  on a pipeline  $(u, v)$  connecting  $u$  to  $v$  simply indicates gas physically flowing from  $v$  to  $u$  in the pipeline.  $G$  contains no active devices, only pipelines.

We also introduce pressure variables  $p_v \in \mathbb{R}$  for all nodes  $v \in V$ . These are bounded by minimum and maximum node pressures:

$$\underline{p}_v \leq p_v \leq \bar{p}_v \quad \forall v \in V. \quad (1)$$

Upper bounds usually come from physical limitations, to avoid damages of the network, while lower bounds are either 0 or some positive value determined by contracts with customers. We point out that in Weymouth's equation (3) and in many other formulas determining the amount of gas flowing along a pipeline, only the squares of the pressures, and not the pressures themselves appear. So in this paper we use the notation

$$\pi_v = p_v^2 \quad \forall v \in V$$

throughout. Accordingly, instead of inequality (1), we will refer to the equivalent formulation

$$\underline{\pi}_v \leq \pi_v \leq \bar{\pi}_v \quad \forall v \in V. \quad (2)$$

Although  $\pi$  is introduced as a square of the pressure, implicitly implying non-negativity, later we consider generalizations where this assumption is relaxed. Thus from now on  $\pi$  is also allowed to take negative values.

The most widely adapted model for the amount of gas flowing along a pipeline is **Weymouth's equation** [20]

$$\text{sgn}(q_a) \cdot q_a^2 = \alpha_a \cdot (\pi_u - \pi_v) \quad \forall (u, v) = a \in A, \quad (3)$$

where  $\alpha_a$  is a positive constant for each  $a \in A$ , depending on the temperature, the pressure, and the Reynolds number of the gas; and on the length, diameter, and roughness of the pipeline. There are several other more detailed physical models, most notably the Panhandle formulas, but Weymouth's equation is widely used because it is a good estimation of the physics and is simple. For a detailed overview of other gas flow formulas see [14].

In the NVP we are also given a **nomination** vector  $d : V \rightarrow \mathbb{R}$  with  $\sum_{v \in V} d_v = 0$ , which prescribes the node-wise net gas outflow, and which is nonpositive at entry nodes, nonnegative at exit nodes, and 0 at inner junctions. A nomination specifies the amount of gas that the network operator must deliver. If this is possible, the nomination is said to be feasible, otherwise it is

infeasible. In other words, a nomination  $d$  is **feasible** in the NVP if and only if there exists flow and pressure square vectors  $q \in \mathbb{R}^A$  and  $\pi \in \mathbb{R}^V$  such that (2) and (3) hold, together with the flow conservation constraints

$$\sum_{a \in A, a=(v,u)} q_a - \sum_{a \in A, a=(u,v)} q_a = d_v \quad \forall v \in V. \quad (4)$$

One can observe that subsystem (3), (4) has the special property that the  $\pi$ -values uniquely determine the  $q$ -values by (3), which in turn uniquely determine the net gas outflow values  $d$  by (4). So we may introduce the following notation.

**Definition 2.1.** For a pressure square vector  $\pi \in \mathbb{R}^V$ , we say that  $\pi$  **generates** the net gas outflow  $d$ , if Equations (3), (4) hold. Then we write  $h(\pi) = d$ .

Thus  $h : \mathbb{R}^V \rightarrow \mathbb{R}^V$  is a function mapping the (not necessarily nonnegative) pressure square vectors to the net gas outflow vectors they generate. Observe that  $\sum_{v \in V} h(\pi)_v = 0$  by (4) for any  $\pi \in \mathbb{R}^V$ . The approach we follow in this section is that we *relax the pressure bounds* (2), and for every nomination  $d$  we analyze the set of generating pressure square vectors  $\pi \in \mathbb{R}^V$ , that is for which  $h(\pi) = d$ .

The classical NVP in this sense was first solved by Maugis [13], who proved that for every nomination  $d$  the set of pressure square vectors  $\pi \in \mathbb{R}^V$  for which  $h(\pi) = d$  forms a line  $\{\pi^0 + \beta \cdot (1, \dots, 1) : \beta \in \mathbb{R}\}$  in  $\mathbb{R}^V$ . He gave a nice reduction to a certain strictly convex problem with a unique solution, where the Karush–Kuhn–Tucker multipliers correspond to the generating  $\pi$  variables. This result of Maugis has several applications in the field of gas transportation network optimization.

## 2.2 The Generalized Passive Gas Nomination Validation Problem

So far we described the classical PASSIVE GAS NOMINATION VALIDATION PROBLEM, or NVP, which we generalize in this section and introduce our main problem we consider, the GENERALIZED PASSIVE GAS NOMINATION VALIDATION PROBLEM, or GNVP. We replace Weymouth’s equation by the more general formula

$$Q_a(q_a) = \Phi_a(\pi_u, \pi_v) \quad \forall (u, v) = a \in A, \quad (5)$$

where we require the following.

**Requirement 2.2.** For all  $a \in A$  the following  $\mathbb{R} \rightarrow \mathbb{R}$  functions are continuous, strictly monotone increasing and unbounded from above and below:

- $Q_a$ ,
- $\Phi_a(\cdot, x)$  for all  $x \in \mathbb{R}$ , and
- $-\Phi_a(x, \cdot)$  for all  $x \in \mathbb{R}$ .

The GNVP is defined to consist of Equations (4), (5) and the pressure bounds (2). As the  $Q$ ’s are strictly monotone increasing, it still holds that  $\pi$  uniquely determines  $q$ , and thus the net gas outflow vector  $d$  is also unique. So we use Definition 2.1 on the generating function  $h : \mathbb{R}^V \rightarrow \mathbb{R}^V$  further on. Our goal now

is to analyze the set of (not necessarily nonnegative) pressure square vectors  $\pi$  that generate a given nomination  $d$  in the sense of Equations (4), (5).

We call a gas flow formula of type (5) **uniform** if  $\Phi_a(\pi_u, \pi_v) = \Phi_a^0(\pi_u - \pi_v)$  for some continuous, strictly monotone increasing function  $\Phi_a^0 : \mathbb{R} \rightarrow \mathbb{R}$  unbounded from above and below. For example, the Weymouth gas flow formula (3) is uniform.

### 2.3 Main result

**Theorem 2.3.** *If  $G$  is connected, then for every nomination  $d \in \mathbb{R}^V$  the set of (not necessarily nonnegative) pressure square vectors  $\pi \in \mathbb{R}^V$  generating  $d$  in the sense of (4), (5) forms a 1-dimensional connected and continuous curve in  $\mathbb{R}^V$ . Moreover, this curve is a line with direction  $(1, \dots, 1)$  in the uniform case.*

We remark that Maugis' proof [13] can also be generalized to the uniform case, but for the general case we need a different approach, based on the Hartman – Stampacchia fixed point theorem on finite variational inequalities.

**Theorem 2.4.** (Hartman, Stampacchia [11]) *Let  $X \subseteq \mathbb{R}^n$  be a nonempty, convex, and compact set and  $F : X \rightarrow \mathbb{R}^n$  a continuous function. Then there exists a point  $x^* \in X$  such that*

$$F(x^*)^T(y - x^*) \geq 0 \quad (6)$$

holds for all  $y \in X$ .

We split the proof of Theorem 2.3 into two parts, by showing that for every nomination  $d$ , node  $v_0 \in V$  and a (not necessarily nonnegative) pressure square value  $\pi_0 \in \mathbb{R}$  there exists at least one (Claim 2.5) and at most one (Claim 2.6)  $\pi \in \mathbb{R}^V$  generating  $d$  with  $\pi_{v_0} = \pi_0$ .

**Claim 2.5.** *If  $G$  is connected, then for every nomination  $d \in \mathbb{R}^V$ , node  $v_0 \in V$  and a (not necessarily nonnegative) pressure square value  $\pi_0 \in \mathbb{R}$  there exists at least one  $\pi \in \mathbb{R}^V$  with  $h(\pi) = d$  and  $\pi_{v_0} = \pi_0$ .*

*Proof.* The following proof is based on a similar approach by DeWolf and Smeers [5] to the classical NVP. However, that proof contained an error, and thus now an improved version, applied to the generalized problem follows.

First we show the existence of a (not necessarily continuous) function  $\underline{c} : \mathbb{R} \rightarrow \mathbb{R}$  such that

1.  $\underline{c}(x) < x$  for all  $x \in \mathbb{R}$ , and
2. for all  $x \in \mathbb{R}$  and partition  $V = V_1 \dot{\cup} V_2$  with  $V_1, V_2$  nonempty, if  $\pi \in \mathbb{R}^V$  is a pressure square vector with  $\max\{\pi_v : v \in V_1\} < \underline{c}(x)$  and  $\min\{\pi_v : v \in V_2\} > x$  then  $\sum_{v \in V_1} h(\pi)_v < \sum_{v \in V_1} d_v$  holds.

We denote the arcs between  $V_1$  and  $V_2$  by  $a^i \in A$  for  $1 \leq i \leq k$ , where we assume wlog. that  $a^i$  is directed from  $V_1$  to  $V_2$ .  $G$  is connected, so  $k \geq 1$ . Let  $x \in \mathbb{R}$ . We let  $\underline{c}^i \in \mathbb{R}$  be a number such that  $\Phi_{a^i}(\underline{c}^i, x) = s$  where  $s$  is defined as  $Q_{a^i}^{-1}(s) = \min\{0, \sum_{v \in V_1} d_v\}$ . By Requirement 2.2,  $Q$  and  $\Phi(\cdot, x)$  are bijective, so  $s$ , and thus  $\underline{c}^i$  exists. Let  $\underline{c}(x) = \min\{x - 1, \min\{\underline{c}^i : 1 \leq i \leq k\}\} < x$ . Let  $\pi' \in \mathbb{R}^V$  be the pressure square vector defined as  $\pi'_v = \underline{c}(x)$  for  $v \in V_1$ ,  $\pi'_v = x$  for  $v \in V_2$ . Now for any pressure square vector  $\pi \in \mathbb{R}^V$  with  $\max\{\pi_v :$



$v \in V_1\} < \underline{c}(x)$  and  $\min\{\pi_v : v \in V_2\} > x$  by Requirement 2.2 we have that  $\sum_{v \in V_1} h(\pi)_v < \sum_{v \in V_1} h(\pi')_v = k \cdot \min\{0, \sum_{v \in V_1} d_v\} \leq \sum_{v \in V_1} d_v$ , as required.

Similarly, a function  $\bar{c} : \mathbb{R} \rightarrow \mathbb{R}$  exists for which  $\bar{c}(x) > x$  and  $\sum_{v \in V_1} h(\pi)_v > \sum_{v \in V_1} d_v$  is implied whenever  $\min\{\pi_v : v \in V_1\} > \bar{c}(x)$  and  $\max\{\pi_v : v \in V_2\} < x$ .

Assume the nodes of  $G$  are indexed as  $V = \{v_0, v_1, \dots, v_{n-1}\}$ . Define

$$\bar{\pi}_0 = \underline{\pi}_0 = \pi_0,$$

$$\underline{\pi}_i = \underbrace{\underline{c}(\underline{c}(\dots \underline{c}(\underline{\pi}_{i-1}) \dots))}_{n \text{ times}}, \quad \bar{\pi}_i = \underbrace{\bar{c}(\bar{c}(\dots \bar{c}(\bar{\pi}_{i-1}) \dots))}_{n \text{ times}}$$

for  $1 \leq i \leq n-1$ , and let  $X = \times_{0 \leq i \leq n-1} [\underline{\pi}_i, \bar{\pi}_i] \subseteq \mathbb{R}^n$ ,  $F(\pi) = h(\pi) - d$ .

$X$  is compact and convex so Theorem 2.4 gives a point  $\pi^* \in X$  satisfying (6). It is enough to prove that  $\pi^*$  satisfies  $F(\pi^*)_i = 0$  for all  $0 \leq i \leq n-1$ , because then  $h(\pi^*) = dd$  and we are done. We prove this by backward induction on  $i$ . Given  $1 \leq i \leq n-1$ , assume that  $F(\pi^*)_j = 0$  for all  $i+1 \leq j \leq n-1$  (an empty assumption for  $i = n-1$ ).

First assume that  $\underline{\pi}_i < \pi_i^* < \bar{\pi}_i$  holds. Now we can choose an  $\varepsilon > 0$  such that  $y_+ := \pi^* + \varepsilon \cdot \chi_i \in X$  and  $y_- := \pi^* - \varepsilon \cdot \chi_i \in X$  ( $\chi_i \in \mathbb{R}^V$  is the unit vector with a 1 at coordinate  $i$  and 0 elsewhere), and so applying (6) to  $y^+$ ,  $y^-$  we get that  $F(\pi^*)_i = 0$ .

Next assume that  $\pi_i^* = \underline{\pi}_i$ . With similar trick as above  $F(\pi^*)_i \geq 0$  follows. Observe that by the choice of  $\underline{\pi}$  we can partition  $V$  into  $V_1 \dot{\cup} V_2$  such that  $v_i \in V_1 \subseteq \{v_i, v_{i+1}, \dots, v_{n-1}\}$ ,  $v_0 \in V_2$  and  $\max\{\pi_v : v \in V_1\} < \underline{c}(x)$ ,  $\min\{\pi_v : v \in V_2\} > x$  for some  $x \in \mathbb{R}$ . By the choice of  $\underline{c}$  we have that  $\sum_{v \in V_1} h(\pi)_v < \sum_{v \in V_1} d_v$ . However, this contradicts the fact that  $h(\pi^*)_{v_j} = d_{v_j}$  for all  $j > i$ ,  $h(\pi^*)_{v_i} \geq d_{v_i}$  and  $V_1 \subseteq \{v_i, v_{i+1}, \dots, v_{n-1}\}$ .

The case  $\pi_i^* = \bar{\pi}_i$  can be handled analogously.

Finally, if  $F(\pi^*)_{v_i} = 0$  for all  $1 \leq i \leq n-1$  then  $F(\pi^*)_{v_0} = 0$  as well, so  $h(\pi^*) = d$  as required.  $\square$

**Claim 2.6.** *If  $G$  is connected, then for every nomination  $d$ , node  $v_0 \in V$  and a (not necessarily nonnegative) pressure square value  $\pi_0 \in \mathbb{R}$  there exists at most one  $\pi \in \mathbb{R}^V$  with  $h(\pi) = d$  and  $\pi_{v_0} = \pi_0$ .*

*Proof.* Assume to the contrary that there exist two distinct pressure square vectors  $\pi^1, \pi^2 \in \mathbb{R}^V$  generating the same nomination  $d$  with  $\pi_{v_0}^1 = \pi_{v_0}^2 = \pi_0$ . Let  $V^< = \{v \in V : \pi_v^1 < \pi_v^2\}$ ,  $V^= = \{v \in V : \pi_v^1 = \pi_v^2\}$ , and  $V^> = \{v \in V : \pi_v^1 > \pi_v^2\}$ .  $v_0 \in V^=$  and as  $\pi^1 \neq \pi^2$ , either  $V^<$  or  $V^>$  is nonempty. Assume that  $V^< \neq \emptyset$ . By the properties of  $Q$  and  $\Phi$ , necessarily  $\sum_{v \in V^<} h(\pi_1)_v < \sum_{v \in V^<} h(\pi_2)_v$ , contradicting the fact that both equal  $\sum_{v \in V^<} d_v$ . The case  $V^> \neq \emptyset$  can be handled analogously.  $\square$

*Proof. (of Theorem 2.3)* Summarizing, by Claims 2.5 and 2.6, if  $G$  is connected then for every nomination  $d$ , node  $v_0 \in V$  and a (not necessarily nonnegative) pressure square value  $\pi_0 \in \mathbb{R}$  there exists exactly one  $\pi \in \mathbb{R}^V$  generating  $d$  with  $\pi_{v_0} = \pi_0$ . This function we denote by  $j_d : \pi_0 \mapsto \pi$ ,  $\mathbb{R} \rightarrow \mathbb{R}^V$ . Now we prove that  $j_d$  is continuous, which implies that the points  $\pi$  generating nomination  $d$  form a one-dimensional, connected and continuous curve in  $\mathbb{R}^V$ .

**Lemma 2.7.** *If  $G$  is connected, then for every nomination  $d$ ,  $\pi'_0 < \pi''_0 \in \mathbb{R}$  implies  $j_d(\pi'_0) < j_d(\pi''_0)$  (coordinate-wise).*

To verify Lemma 2.7, observe that if  $U := \{v \in V : j_d(\pi'_0) < j_d(\pi''_0)\} \neq V$  then by  $v_0 \in U$  and the connectivity of  $G$ , we have  $\sum_{v \in U} d_v = \sum_{v \in U} h(\pi'_0)_v < \sum_{v \in U} h(\pi''_0)_v = \sum_{v \in U} d_v$ , a contradiction.

First we show that whenever  $x_i \rightarrow x$  for  $x, x_i \in \mathbb{R}$  ( $i \in \mathbb{N}$ ) and  $j_d(x_i)$  is convergent, then  $j_d(x_i) \rightarrow j_d(x)$ . Indeed,  $h(j_d(x_i)) = d$  holds, thus  $h(\lim j_d(x_i)) = d$  as  $h$  is continuous and  $j_d(x_i)$  is convergent. As  $(\lim j_d(x_i))_0 = x$ ,  $\lim j_d(x_i) = j_d(x)$  follows by the definition of  $j_d$ .

Now assume that  $j_d$  is not continuous at a point  $x \in \mathbb{R}$ . By definition, there exists an  $\varepsilon > 0$ , and a sequence  $x_i \rightarrow x$  such that  $\|j_d(x_i) - j_d(x)\| > \varepsilon$ . By Lemma 2.7 and the existence of  $j_d$  itself, the set  $\{j_d(x_i)\}$  is bounded, so we can choose a convergent subsequent  $j_d(x_{i_k})$  from it. However, by our previous statement, now  $j_d(x_{i_k}) \rightarrow j_d(x)$  holds, which is a contradiction. So  $j_d$  is continuous and thus a 1-dimensional connected curve in  $\mathbb{R}^V$ .

That the  $j_d$  curve is a line with direction  $(1, \dots, 1)$  in the uniform case is obvious.  $\square$

We point out that an alternative proof to Theorem 2.3 can be given based on the Brouwer fixed point theorem.

## 2.4 Analyzing the solution curve

The same argument as in the proof of Claim 2.6 gives that curve  $j_d$  is coordinate-wise strictly monotone increasing. To calculate the gradient of  $j_d$ , defined in the proof of Theorem 2.3 as  $\{\pi \in \mathbb{R}^V : h(\pi) = d\}$ , we can sometimes apply the implicit function theorem.

**Theorem 2.8** (Implicit function theorem). *Let  $F : \mathbb{R}^{X \cup Y} \rightarrow \mathbb{R}^Y$  be a continuously differentiable function. Assume  $F(x_0, y_0) = 0$ . If  $\frac{\partial F}{\partial y} \Big|_{(x_0, y_0)}$  is regular then there exists an environment  $U$  of  $x_0$ , an environment  $V$  of  $y_0$ , and a unique continuously differentiable function  $G : U \rightarrow V$  such that*

$$\{(x, G(x)) | x \in U\} = \{(x, y) \in U \times V | f(x, y) = 0\}.$$

Moreover,

$$\frac{\partial G}{\partial x} \Big|_{x_0} = \left( \frac{\partial F}{\partial y} \Big|_{(x_0, y_0)} \right)^{-1} \cdot \frac{\partial F}{\partial x} \Big|_{x_0}.$$

Observe that the  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  function can be deemed an  $\mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  mapping, as  $\sum_{v \in V} h(\pi)_v = 0$  always holds for the gas net outflow vector  $h(\pi)$ . So next we try to apply the implicit function theorem to the  $\mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ ,  $\pi \mapsto (h(\pi) - d)|_{V-v_0}$  function. Call the point  $\pi \in \mathbb{R}^n$  **non-singular** if the function  $\pi' \mapsto \left( \frac{\partial h}{\partial(V-v_0)} \Big|_{\pi'} \right) \Big|_{V-v_0}$  exists and is continuous in a neighborhood of  $\pi'$ , and the Jacobian  $\left( \frac{\partial h}{\partial(V-v_0)} \Big|_{\pi} \right) \Big|_{V-v_0}$  is regular. If  $\pi$  is nonsingular we can apply the implicit function theorem and for the gradient of the curve  $j_d$  we get that

$$\frac{\partial j_d}{\partial v_0} \Big|_{\pi_0} = \left( \left( \frac{\partial h}{\partial(V-v_0)} \Big|_{\pi} \right) \Big|_{V-v_0} \right)^{-1} \cdot \frac{\partial h}{\partial v_0} \Big|_{\pi_0}.$$

Next we characterize when the Jacobian at  $\pi \in \mathbb{R}^V$  is regular.

**Theorem 2.9.** *If  $G$  is connected,  $\frac{\partial h}{\partial v}|_\pi$  exists and is finite for every  $v \in V - v_0$ , then  $\left(\frac{\partial h}{\partial(V-v_0)}\Big|_\pi\right)\Big|_{V-v_0}$  is regular.*

Note that the assumption in this theorem implies that  $\frac{\partial h}{\partial v_0}|_\pi$  (with  $v = v_0$ ) exists and is finite, because  $\sum_{v \in V} h(\pi)_v = 0$

*Proof.* Let us denote the node sets of the connected components of  $G - v_0$  by  $V_1, \dots, V_k$ ,  $k \geq 1$ . We recall some definitions from linear algebra. Let  $A \in \mathbb{R}^{X \times X}$  be a square matrix.  $A$  is called **diagonally dominant** if  $|A_{ii}| \geq \sum_{j \neq i} |A_{ij}|$  for all  $i \in X$ ,  $A$  is **irreducible** if the  $(X, \{(i, j) : A_{ij} \neq 0\})$  directed graph is strongly connected, and  $A$  is **irreducibly diagonally dominant** if irreducible, diagonally dominant, and in at least one row  $i \in X$  is strictly diagonally dominant, that is  $|A_{ii}| > \sum_{j \neq i} |A_{ij}|$ . A result of Horn and Johnson [12] states that irreducibly diagonally dominant matrices are regular.

Clearly,

$$\left(\frac{\partial h}{\partial v}\Big|_\pi\right)_u = \begin{cases} 0 & \text{if } u \neq v \text{ and they are not adjacent in } G, \\ < 0 & \text{if } u \neq v \text{ and they are adjacent in } G, \\ (-1) \cdot \sum_{w \neq v} \left(\frac{\partial h}{\partial v}\Big|_\pi\right)_w > 0 & \text{if } u = v. \end{cases}$$

As a consequence,  $h' \in \mathbb{R}^{V \times V}$  is diagonally dominant, and even more, the  $\left(\frac{\partial h}{\partial V_k}\Big|_\pi\right)\Big|_{V_k}$  matrices are irreducibly diagonally dominant. Putting these matrices together, we get that  $\left(\frac{\partial h}{\partial(V-v_0)}\Big|_\pi\right)\Big|_{V-v_0}$  is also regular.  $\square$

Although at singular  $\pi$ 's we cannot specify the gradient by this approach, in some cases it is still possible to get the gradient. For example, for the uniform case, the curve guaranteed by Theorem 2.3 is a line  $\{\pi + \beta(1, \dots, 1) : \beta \in \mathbb{R}\}$ , and thus has gradient  $(1, \dots, 1)$  everywhere, even if  $\pi$  is singular. Note that for the classical Weymouth's equation (3),  $\pi$  is singular if and only if  $\pi_u = \pi_v$  for an adjacent node pair  $u, v$ .

From an algorithmic point of view, we briefly mention that the Newton-Raphson method is quite efficient for finding the line in the classical Weymouth case (3). This is because the Jacobian can be explicitly calculated, and as it is negative semidefinite, one can use Cholesky-decomposition in calculating a Newton-step.

## 2.5 Examples

We enumerate some examples to models of gas network components fitting into our framework.

**Example 2.10.** The classical Weymouth's equation (3) clearly satisfies Requirement 2.2. There exists a more complex and exact version of Weymouth's equation:

$$\text{sgn}(q_a)q_a^2 = \alpha_a \cdot (\pi_u - b_a \cdot \pi_v) \quad \forall a = (u, v) \in A \quad (7)$$

where  $\alpha_a$  and  $b_a > 0$  are constants and  $b_a$  accumulates the height difference between nodes  $u$  and  $v$ . Although not uniform, this gas flow formula also obeys Requirement 2.2.

The Panhandle-formulas, which have the form

$$\operatorname{sgn}(q_a)q_a^c = \alpha_a \cdot (\pi_u - \pi_v) \quad \forall a = (u, v) \in A \quad (8)$$

with  $c \approx 2$  constant also fit into our framework.

In practice the gas transportation networks, in which nomination validation problems are solved, almost always contain active devices as well. These active devices are to regulate the flow of gas in the network, and they are modeled as directed edges  $a = (u, v) \in A$ . They involve discrete decisions to make during operation, for example, should a compressor be used or bypassed? Because of this discrete nature, usually the nomination validation problem and several other optimization problems in gas transportation networks are solved by Branch&Bound. We call a node of the Branch&Bound tree where the discrete decisions at the active devices are already made, and possibly other discretizations and linearizations already took place, a *fractional node*. Fractional nodes often involve a continuous mathematical program with constraints of the form (4), (5), together with pressure square bounds (2). By Theorem 2.3, the set of  $\pi$  vectors generating the nomination  $d$  in the sense of (4), (5) forms a curve, possibly violating the pressure bounds (2). If we can explicitly describe this curve, and for a uniform GNVP this is the case by Theorem 2.3, we can decide if also a solution satisfying the pressure bounds (2) exists at that particular fractional node. If yes, the fractional node is feasible, otherwise not. For the uniform case, the set of pressure square vectors  $\pi \in \mathbb{R}^V$  generating the nomination  $d$  is the line  $\{\pi + \beta(1, \dots, 1) : \beta \in \mathbb{R}\}$ , and this line intersects the pressure bound box  $[\underline{\pi}, \overline{\pi}] \subseteq \mathbb{R}^V$  if and only if

$$\max_{v \in V}(\underline{\pi}_v - \pi_v) \leq \min_{v \in V}(\overline{\pi}_v - \pi_v).$$

Thus, in the uniform case, though model (2), (4), (5) is neither linear, nor convex, a certificate for the infeasibility of a nomination still exists. This certificate consists of an arbitrary point  $\pi$  in the preimage line and a pair of nodes  $u, v$  for which  $\underline{\pi}_u - \pi_u > \overline{\pi}_v - \pi_v$ . Naturally, there can be many such node pairs.

Even if we cannot describe the solution curve or cannot calculate its gradient, the knowledge of having always one curve as a solution can be of theoretical help when NLP solvers are applied to the nonlinear program present at the fractional node.

Now we list some models of active devices, which are either models of them in themselves, or they result from appropriate discretization or linearization at a fractional node of the Branch&Bound tree. More details on modeling the active devices in a gas network can be found in [9].

**Example 2.11.** *Compressors* are the most important active devices in a gas transportation network, being able to increase the pressure between their two end nodes. They are a central source of nonlinear behavior of gas networks, and due to discrete decisions on activating or bypassing a compressor they introduce a further source of discontinuity. The most basic features of a compressor are described in its characteristic diagram. This is an area in the  $\mathbb{R}^2$  plane, bordered

by four parabolas, consisting of the allowed  $(p_v/p_u, q_a/p_v)$  value pairs, where  $p_u$  is the entry pressure,  $p_v$  is the exit pressure and  $q_a$  is the amount of gas flowing through the compressor. One way to circumvent the high nonlinearity of the characteristic diagram is to discretize it in an appropriate way such that for every value of a certain integral parameter, the relation between  $\pi_v = p_v^2$ ,  $\pi_u = p_u^2$  and  $q_a$  is expressed by the linear equality

$$d \cdot q_a = \pi_u - \pi_v + c. \quad (9)$$

Here  $d$  and  $c$  are appropriate positive constants depending on the integral parameter and the characteristic diagram of the compressor. As a uniform gas flow formula, it nicely fits into the GNVP.

**Example 2.12.** A *control valve*  $a = (u, v) \in A$  is an active device reducing the pressure between  $u$  and  $v$ . It can be closed or open. Being closed is equivalent to  $q_a = 0$ , and is tantamount to deleting  $a$  from the network. Being open means

$$\underline{q}_a \leq q_a \leq \bar{q}_a, \quad \underline{\pi}_a \leq \pi_u - \pi_v \leq \bar{\pi}_a$$

with appropriate bounds  $\underline{q}_a, \bar{q}_a, \underline{\pi}_a, \bar{\pi}_a$ . One possible discretization of this model is to define a range of possible pressure reductions, and for every fixed reduction level  $D_a$  to take

$$D_a = \pi_u - \pi_v \quad (10)$$

as a constant. Equation (10) does not directly fit into the GNVP, as the left hand side is not strictly monotone in  $q_a$ . Still, if the gas network contains such an edge, then one can contract it to a new node  $w$ , and define  $d_w = d_u + d_v$  and  $\pi_w = \pi_v$ ,  $\pi_u = \pi_w + D_a$ . Now, for every solution  $\pi, q$  of this contracted system, there is only one way to define  $q_a$ , determined by  $d_u$  and  $d_v$  (unless  $u = v$ , which is indeed possible if other such edge contracting steps already took place). Finally, we check whether  $\underline{q}_a \leq q_a \leq \bar{q}_a$  is also satisfied.

**Example 2.13.** A *valve*  $a = (u, v) \in A$  can be open or closed. Being closed is equivalent to requiring  $q_a = 0$  and to deleting  $a$  from the network, and being open means  $p_v = p_u$ . This latter state is equivalent to shrinking  $u$  and  $v$  to a single node.

**Example 2.14.** There are two types of *resistors*. A simplistic model of the first type leads to the formula

$$\text{sgn}(q_a)q_a^2 = \alpha_a \cdot (p_u - p_v), \quad (11)$$

where  $p_u, p_v$  are the pressures at the end nodes. Clearly, this formulation fits into the GNVP, by replacing  $p_u$  by  $\text{sgn}\sqrt{\pi_u}$  and  $p_v$  by  $\text{sgn}\sqrt{\pi_v}$  with the notation

$$\text{sgn}\sqrt{x} = \begin{cases} \sqrt{x} & \text{if } x \geq 0, \\ -\sqrt{-x} & \text{if } x < 0. \end{cases}$$

This is a strictly monotone increasing function, thus obeying Requirement 2.2. Recall that we temporarily ignore the pressure bounds (2), so  $p$  and  $\pi$  may have arbitrary negative value as well.

A more refined model of this type of resistor is as follows. Let  $\zeta_a$  denote a resistor specific constant. First,  $p_u \geq p_v$  is required, and the amount of gas flowing through the resistor is given by

$$\text{sgn}(q_a)q_a^2 = \frac{p_u^2 - p_u p_v}{1 + \zeta_a p_u}. \quad (12)$$

In the region  $p_u \geq p_v$ , with the transformation  $p_u = \sqrt[\text{sgn}]{\pi_u}$ ,  $p_v = \sqrt[\text{sgn}]{\pi_v}$  the functions  $\Phi(\cdot, x)$ ,  $\Phi(x, \cdot)$  given by (12) satisfy Requirement 2.2. So we can extend  $\Phi$  to the whole  $\mathbb{R}^{V \times V}$  keeping Requirement 2.2, and thus Theorem 2.3 can be applied.

The other type of a resistor is in fact a measuring system that causes a fixed pressure decrease

$$D_a = p_u - p_v, \quad (13)$$

where  $D_a$  is a constant, and in addition, also  $q_a \geq 0$  is required. Equation (13) does not directly fit into the GNVP, but we can use the idea of contracting the edge, in exactly the same way as we did at the control valve (Example 2.12). However, in this case we also need to apply the bijective transformations  $p_u = \sqrt[\text{sgn}]{\pi_u}$ ,  $p_v = \sqrt[\text{sgn}]{\pi_v}$ , keeping Requirement 2.2. Then for every solution  $\pi$ ,  $q$  of the contracted system we check whether the uniquely determined variable  $q_a$  is nonnegative, similarly as we did at a control valve.

### 3 Rubber band view

In this section we describe an efficient way for visualizing system (4), (5), making it possible to analyze the process of how a pressure square vector generates a gas net outflow vector. We call it the rubber band view, as pipelines will be modeled by rubber bands with a certain property to be defined in (14). The rubber band view may prove to be helpful in the visualization and thus in the design of gas distribution networks; it directly gives rise to a combinatorial algorithm yielding a solution to system (4), (5); and in Subsection 3.1 we apply it to explain the so called more pipeline – less throughput phenomenon, occurring in gas networks. For simplicity, in what follows we consider only the classical Weymouth’s equation (3), but the results of this section can also be applied to the GNVP involving gas flow formulas of the form (5).

We think of the network  $G = (V, A)$  as an undirected graph located in the real line  $\mathbb{R}$ , in such a way that the position of a node  $v$  in the real line corresponds to the pressure square  $\pi_v \in \mathbb{R}$  at node  $v$ . We think of the pipelines of the network as rubber bands with force constant  $\alpha$ . In physics Hooke’s law determines the restoring force of a rubber band, which we replace now by a reformulation of Weymouth’s equation:

$$q = \sqrt{\alpha} \cdot \sqrt{\ell}, \quad (14)$$

where  $q$  is the restoring force of the rubber band  $(u, v)$  and  $\ell = |\pi_u - \pi_v|$  is its length. Finally, let every node  $v$  be pulled by an additional force of  $d_v$ .

We call a position vector  $\pi : V \rightarrow \mathbb{R}$  a **balanced state** if under the additional force  $d$  and the restoring force vector  $q : A \rightarrow \mathbb{R}$  every node is in balance, i. e.,

$$\sum_{a=(u,v) \in A, \pi_u < \pi_v} q_a = \sum_{a=(u,v) \in A, \pi_u > \pi_v} q_a + d_v \quad (15)$$

is satisfied for all  $v \in V$ . Clearly, any shift of a balanced state on the real line is also a balanced state, because Weymouth’s equation is shift-invariant. As (15) is equivalent to (4), the fact that position vector  $\pi \in \mathbb{R}^V$  is a balanced state under the additional force  $d$  is tantamount to that  $\pi$  as a pressure square vector

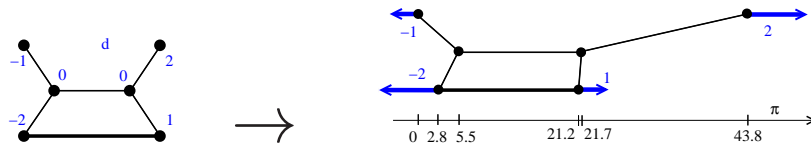


Figure 1: Balanced state of a network.

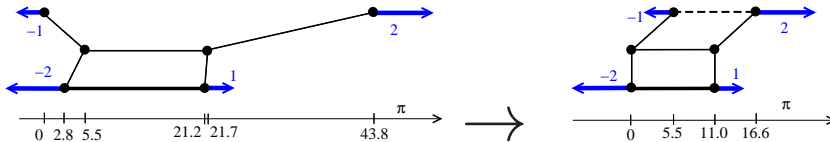


Figure 2: Balanced states when adding a new (dashed) pipeline

generates nomination  $d$  in the sense of system (3), (4). Furthermore, it is easy to see that Theorem 2.3 is equivalent to that for every additional force vector  $d$  there exists exactly one balanced state of the system (plus its shifts).

An example of a balanced state can be seen in Figure 1. The values of a nomination  $d$  are shown near the nodes, and the force determined by  $d$  are shown as arrows.

### 3.1 The more pipeline – less throughput phenomenon

Rubber bands make it possible to visualize the so-called more pipeline – less throughput phenomenon. Adding a new pipeline (a new rubber band) between nodes  $u$  and  $v$  to the network, it is intuitively clear that the whole network shrinks in the rubber band view – at least  $u$  and  $v$  get closer to each other, see Figure 2.

That the new rubber band nears its end nodes is really true, however, the **range** of the balanced state  $\pi$ , defined to be  $r(\pi) = \max \pi - \min \pi$ , may increase. This is shown in Figure 3.

This means that if, say, the pressure square bounds were  $[0, 100]$  at all nodes, then the nomination in Figure 3 is feasible in the left hand side network, but becomes infeasible after building a new pipeline in the right hand side network. However, we prove that this range increment is bounded. The next theorem is formulated in the terms of the NVP instead of rubber bands.

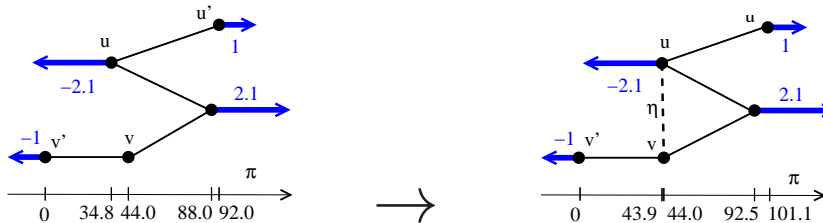


Figure 3: The more pipeline – less throughput phenomenon

**Theorem 3.1.** *Let the pressure square vector  $\pi$  generate nomination  $d$  in network  $G = (V, A)$  in the sense of (3), (4). Assume that we add to  $G$  a new pipeline  $\eta$  from node  $v$  to  $u$  with some capacity constant  $\alpha > 0$ , and let pressure square vector  $\pi^\alpha$  generate  $d$  in the new network  $G^+ := G + \eta$ . Then*

$$|r(\pi^\alpha) - r(\pi)| \leq |\pi_v - \pi_u|.$$

*This bound is strict.*

The more pipeline – less throughput phenomenon resembles to Braess’s paradox on traffic flow [8], the first paradox of this type, which states that building a new road may increase the traveling time for everyone participating in the traffic. Similar results were proved by Calvert and Keady on water flows [2].

We mention that Theorem 3.1 holds for any uniform GNVP as well.

*Proof.* Let  $\delta(x) = \pi(x) - \pi^\alpha(x)$  for  $x \in V$ . As the balanced state pressure square vectors can be shifted, we may assume that  $\pi_u = \pi_u^\alpha = 0$ . Wlog. we may also assume that  $\pi_v > 0$ . With these assumptions it is enough to prove that  $0 \leq \delta(x) \leq \pi_v - \pi_u$  for all nodes  $x \in V$ .

First we prove that if  $x \neq u, v$  is a node maximizing  $\delta$  over  $V$  then every neighbor of  $x$  also maximizes  $\delta$ . Indeed, as  $\pi$  generates  $d$  in  $G$ , we have that

$$\sum_{a=(x,y) \in A(G)} q_a - \sum_{a=(y,x) \in A(G)} q_a = d(x) \quad (16)$$

holds, with the flow  $q$  defined as in Weymouth’s equation (3). In addition, as  $\pi^\alpha$  generates  $d$  in  $G^+$ , we have that

$$\sum_{a=(x,y) \in A(G^+)} q_a^\alpha - \sum_{a=(y,x) \in A(G^+)} q_a^\alpha = d(x), \quad (17)$$

with the flow  $q^\alpha$  defined as in (3) (with  $\pi^\alpha$  instead of  $\pi$ ). As Weymouth’s equation (3) is shift-invariant and the function  $\text{sgn} \sqrt{a} = \text{sgn}(a) \sqrt{|a|}$  is monotone increasing, we have that  $q_a^\alpha \geq q_a$  for  $a = (y, x)$  and  $q_a^\alpha \leq q_a$  for  $a = (x, y)$ . Thus, as the pipelines in (16) and (17) are the same, we have that  $q_a = q_a^\alpha$  for all pipelines incident to  $y$ , and hence that  $\delta(y) = \delta(x)$  for all neighbors  $y$  of  $x$  in  $G$ .

Similarly, it can be proved that if  $y \neq u, v$  is a node minimizing  $\delta$  over  $V$  then every neighbor of  $y$  also minimizes  $\delta$ .

Thus, as  $G$  is connected, both the maximum and the minimum of  $\delta$  is achieved at the set  $\{u, v\}$ . By our choice,  $\delta_u = 0$ . As for  $\delta_v$ , there are three possibilities.

- $\pi_v^\alpha > \pi_v$ , that is  $\delta_v < 0$ . Now  $q_\eta^\alpha > 0$ . Note that  $v$  minimizes  $\delta$  over  $V$ , so by the same argument as above, we have that

$$\begin{aligned} d_v &= \sum_{a=(v,x) \in A(G^+)} q_a^\alpha - \sum_{a=(x,v) \in A(G^+)} q_a^\alpha = \\ & \sum_{a=(v,x) \in A(G)} q_a^\alpha - \sum_{a=(x,v) \in A(G)} q_a^\alpha + q_\eta^\alpha > d_v, \end{aligned}$$

a contradiction.



- $0 \leq \pi_v^\alpha \leq \pi_v$ . Now  $v$  maximizes and  $u$  minimizes  $\delta$  over  $V$ , so  $0 \leq \delta(x) \leq \pi_v - \pi_u$  for all  $x \in V$ , as required.
- $\pi_v^\alpha < 0$ , that is  $\delta_v > \pi_v$ . Now  $q_\eta^\alpha < 0$ . Note that  $v$  maximizes  $\delta$  over  $V$  so by the same argument as above, we have that

$$d_v = \sum_{a=(v,x) \in A(G^+)} q_a^\alpha - \sum_{a=(x,v) \in A(G^+)} q_a^\alpha = \sum_{a=(v,x) \in A(G)} q_a^\alpha - \sum_{a=(x,v) \in A(G)} q_a^\alpha + q_\eta^\alpha < d_v,$$

a contradiction.

Next we show that the bound of  $|\pi_v - \pi_u|$  is strict, and can be arbitrarily approximated if  $\alpha \rightarrow \infty$ . First, we show that  $\alpha \rightarrow \infty$  implies  $\pi_v^\alpha \rightarrow \pi_u^\alpha = 0$ . Indeed, otherwise  $\pi_v^\alpha > \varepsilon$  by some  $\varepsilon > 0$  and so  $q_\eta^\alpha \rightarrow \infty$ . However, as we saw,  $q_a^\alpha$  is bounded for all  $a \in A$ , and so  $q_\eta^\alpha \rightarrow \infty$  implies that the net outflow at  $v$  tends to infinity, which is impossible. Now consider the network in Figure 3. As  $\pi_u^\alpha - \pi_u^\alpha$  and  $\pi_v^\alpha - \pi_v^\alpha$  are constant functions of  $\alpha$ ,  $\pi_v^\alpha \rightarrow \pi_u^\alpha$  implies that  $r(\pi^\alpha) = \pi_u^\alpha - \pi_v^\alpha \rightarrow r(\pi) + \pi_v - \pi_u$ , as required.  $\square$

## 4 The frontier

In this section we consider passive gas transportation networks  $G = (V, A)$  containing only pipelines where the flow of gas is determined by Weymouth's equation (3), and give observations and statements on the convexity and connectedness of the set of feasible nominations and bookings. As before, a nomination  $d : V \rightarrow \mathbb{R}$  is feasible if there exist pressure square and flow vectors  $\pi \in \mathbb{R}^V$ ,  $q \in \mathbb{R}^A$  such that (2), (3), (4) are satisfied. Bookings, already mentioned in the introduction, are formally defined as follows. For vectors  $d, d' \in \mathbb{R}^V$  we write  $d' \preceq d$ , and say that  $d'$  is **closer to 0 than**  $d$ , if there exists a vector  $\sigma : V \rightarrow [0, 1]$  such that  $d'_v = d_v \cdot \sigma_v$  for all  $v \in V$ . We call  $d$  a **booking** if  $d'$  is feasible for all nominations  $d' \preceq d$ .

Next we show that

- the set of bookings is not always convex,
- if  $d' \preceq d$  are nominations, then feasibility of  $d$  does not imply feasibility of  $d'$ , in other words, not every nomination is a booking,
- the set of feasible nominations is not always convex,
- the set of feasible nominations and bookings are connected.

Consider the directed network in Figure 4. What is shown next to every node are two nominations separated by a '/' sign, and the allowed pressure square interval. Let  $\alpha = 1$  for all three edges. It is easy to check that both nominations are actually bookings. However, their average, denoted by  $d$ , which is clearly a feasible nomination, is not a booking. Indeed,  $d' \preceq d$  with  $d'_u = d'_y = 0$ ,  $d'_v = -d'_x = q + 1/2$  is not a feasible nomination. Especially, in this example, the set of bookings is not convex, and the feasibility of  $d$ , together with  $d' \preceq d$

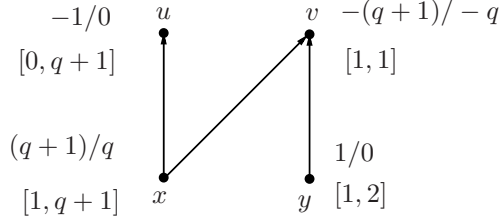


Figure 4: The set of bookings is not convex, and an example to a feasible nomination which is not a booking

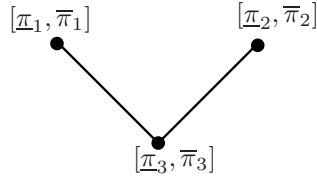


Figure 5: The set of feasible nominations is not convex

does not imply the feasibility of  $d'$  – in other words, not every nomination is a booking.

Note that in this example one can choose  $q$  arbitrary big, yielding a network with a very large throughput. On the other hand, choosing  $q = 0$  gives an example where the union of two nominations, which are bookings in their own networks if considered separately, is not a booking any more if we put the two networks together.

The set of feasible nominations is not necessarily convex, as shown by the example in Figure 5. Assuming that  $\alpha \equiv 1$  and  $\underline{\pi}_i = \bar{\pi}_i \geq \bar{\pi}_3$  for  $i = 1, 2$ , we get that the set of feasible nominations is a curve

$$(\sqrt{\pi_1 - \pi_3}, \sqrt{\pi_2 - \pi_3}, -(\sqrt{\pi_1 - \pi_3} + \sqrt{\pi_2 - \pi_3})),$$

parametrized by  $\underline{\pi}_3 \leq \pi_3 \leq \bar{\pi}_3$ . The image of this curve is clearly not a linear segment, and hence not convex.

Turning to connectivity, the continuity of the  $h$  function in Section 2 implies that the set of feasible nominations is connected. Moreover, the set of bookings is connected, as every booking can by definition be connected by a path to the 0-booking.

## 5 The gas nomination validation problem with active devices is $\mathcal{NP}$ -complete

Beneath pipelines, a real life gas transportation network also contains active devices. In the gas nomination validation problem for active networks, the network operator has to find pressures at the nodes and appropriate configurations for all active components such that exactly the nominated amount of gas is transmitted through the network. The active devices are modeled as edges of the network. Every active device can be either closed or open, and being closed is equivalent to that the flow on it is 0. There are several models for open active devices in the literature, but as we need only valves in the proof to follow, we

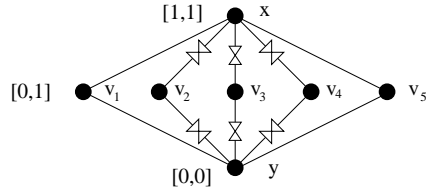


Figure 6: The ANVP is  $\mathcal{NP}$ -complete

only consider the modeling of an open valve. In this regard there is consensus in the literature, and an open valve  $(u, v)$  is always modeled as

$$\pi_u = \pi_v. \quad (18)$$

Thus we call the ACTIVE GAS NOMINATION VALIDATION PROBLEM (ANVP) the problem of deciding whether a nomination is feasible in a gas transportation network with active devices, *with arbitrary modeling of the active devices*, except the valve, whose open state is modeled as in (18), and closed state is tantamount to having 0 flow.

**Theorem 5.1.** *The ANVP is  $\mathcal{NP}$ -complete.*

*Proof.* We reduce the  $\mathcal{NP}$ -complete SUBSET SUM PROBLEM to the ANVP in a network containing pipelines and valves only. Let  $0 < a_i \in \mathbb{Z}_+$ ,  $i \in I = \{1, \dots, k\}$  and  $n, m \in \mathbb{Z}$  with  $n + m = \sum_{i \in I} a_i$  be an instance of the SUBSET SUM PROBLEM. The question is whether  $I$  has a partition  $\{I_x, I_y\}$  such that  $\sum_{i \in I_x} a_i = n$  and  $\sum_{i \in I_y} a_i = m$ . Now we create an instance of the ANVP as seen in Figure 6.

The gas transportation network has  $k+2$  nodes,  $x, y, v_i, i \in I$ , and  $2k$  valves  $xv_i, yv_i, i \in I$  (not all are denoted as a valve in the figure due to clarity). The pressure square bounds are written near the nodes, that is  $\pi_u^- = 0$  for every  $u \in V$  except that  $\pi^-(x) = 1$ , and  $\bar{\pi}_u = 1$  for every  $u \in V$  except that  $\bar{\pi}_y = 0$ . The nomination is as follows:  $d_x = n$ ,  $d_y = m$  and  $d_{v_i} = -a_i$  for  $i \in I$ . Assume that this instance of the ANVP has a feasible configuration, that is a pressure square vector  $\pi$  and a configuration of the valves which generates the given nomination  $d$ . As  $\pi_u = \pi_v$  for an open valve between  $u$  and  $v$ , every node  $v_i$  is incident to at most one open valve. On the other hand, as  $a_i \neq 0$ , it is incident to at least one open valve. So with the notation  $I_x = \{i \in I : v_ix \text{ is open, } v_iy \text{ is closed}\}$  and  $I_y = \{i \in I : v_iy \text{ is open, } v_ix \text{ is closed}\}$  we have that  $\{I_x, I_y\}$  is a partition of  $I$ , and by the feasibility of the configuration, we have that  $\sum_{i \in I_x} a_i = n$  and  $\sum_{i \in I_y} a_i = m$ . Thus the answer to the given instance of the SUBSET SUM PROBLEM is “yes”. On the other hand, using the same argument, it is easy to see that if the the answer to the given instance of the SUBSET SUM PROBLEM is “yes” then the nomination  $d$  has a feasible configuration in the network.  $\square$

## Acknowledgment

The author would like to thank Armin Fügenschuh for discovering the first example to the more pipeline – less throughput phenomenon.

## References

- [1] Timo Berthold, Stefan Heinz, and Stefan Vigerske. Extending a CIP framework to solve MIQCPs. In *Mixed Integer Nonlinear Programming*, volume 154, part 6 of *The IMA Volumes in Mathematics and its Applications*, pages 427–444. Springer, 2012. Also available as ZIB-Report 09-23.
- [2] B. Calvert and G. Keady. Braess’s paradox and power-law nonlinearities in networks. *The ANZIAM Journal*, 35(01):1–22, 1993.
- [3] R.G. Carter. Pipeline optimization: Dynamic programming after 30 years. In *Proceedings of the 30th PSIG Annual Meeting*, 1998.
- [4] D. De Wolf and Y. Smeers. Optimal dimensioning of pipe networks with application to gas transmission networks. *Operations Research*, pages 596–608, 1996.
- [5] D. DeWolf and Y. Smeers. Mathematical properties of formulations of the gas transmission problem. Technical Report SMG Preprint 94/12, Université libre de Bruxelles, 1994.
- [6] P. Domschke, B. Geißler, O. Kolb, J. Lang, A. Martin, and A. Morsi. Combination of nonlinear and linear optimization of transient gas networks. *INFORMS Journal on Computing*, 23(4):605–617, 2011.
- [7] K. Ehrhardt and M. C. Steinbach. Nonlinear optimization in gas networks. *Modeling, simulation and optimization of complex processes*, pages 139–148, 2005.
- [8] C. Fisk. More paradoxes in the equilibrium assignment problem. *Transportation Research Part B: Methodological*, 13(4):305–309, 1979.
- [9] A. Fügenschuh, B. Hiller, J. Humpola, T. Koch, T. Lehmann, R. Schwarz, J. Schweiger, and J. Szabó. Gas network topology optimization for upcoming market requirements. Technical Report 11-09, ZIB, Takustr.7, 14195 Berlin, 2011.
- [10] A. Fügenschuh, H. Homfeld, H. Schülldorf, and S. Vigerske. Mixed-Integer Nonlinear Problems in Transportation Applications. In *Proceedings of the 2nd International Conference on Engineering Optimization (+CD-rom)*, 2010. H. Rodrigues et al. (Hrsg.), ISBN 978-989-96264-3-0.
- [11] P. Hartman and G. Stampacchia. On some non-linear elliptic differential-functional equations. *Acta Mathematica*, 115(1):271–310, 1966.
- [12] R. A. Horn and C. R. Johnson. *Matrix analysis*, 1985.
- [13] J. J. Maugis. Etude de réseaux de transport et de distribution de fluide. *RAIRO Operations Research*, 11(2):243–248, 1977.
- [14] E. S. Menon. *Gas pipeline hydraulics*. CRC Press, May 2005.
- [15] A. Osiadacz. *Simulation and analysis of gas networks*. 1987.
- [16] A. J. Osiadacz and M. Gorecki. Optimization of pipe sizes for distribution gas network design. In *Proceedings of the 27th PSIG Annual Meeting*, 1995.

- [17] M. C. Steinbach. On pde solution in transient optimization of gas networks. *Journal of computational and applied mathematics*, 203(2):345–361, 2007.
- [18] T. van der Hoeven. Constrained Network Simulation. In *PSIG Conference, Bern, Switzerland*, pages 15–17, 2002.
- [19] Stefan Vigerske. *Decomposition in Multistage Stochastic Programming and a Constraint Integer Programming Approach to Mixed-Integer Nonlinear Programming*. PhD thesis, Humboldt-Universität zu Berlin, 2012.
- [20] T. R. Weymouth. Problems in Natural Gas Engineering. *Transactions of the American Society of Mechanical Engineers*, 34:185–231, 1912.