

MARTIN GRÖTSCHEL<sup>1</sup> AND RÜDIGER STEPHAN<sup>2</sup>

# **Characterization of Facets of the Hop Constrained Chain Polytope via Dynamic Programming**

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<sup>1</sup>Email: [groetschel@zib.de](mailto:groetschel@zib.de)

<sup>2</sup>Email: [stephan@zib.de](mailto:stephan@zib.de)

Herausgegeben vom  
Konrad-Zuse-Zentrum für Informationstechnik Berlin  
Takustraße 7  
D-14195 Berlin-Dahlem

Telefon: 030-84185-0  
Telefax: 030-84185-125

e-mail: [bibliothek@zib.de](mailto:bibliothek@zib.de)  
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# Characterization of Facets of the Hop Constrained Chain Polytope via Dynamic Programming

Martin Grötschel and Rüdiger Stephan

## Abstract

In this paper, we study the hop constrained chain polytope, that is, the convex hull of the incidence vectors of  $(s, t)$ -chains using at most  $k$  arcs of a given digraph, and its dominant. We use extended formulations (implied by the inherent structure of the Moore-Bellman-Ford algorithm) to derive facet defining inequalities for these polyhedra via projection. Our findings result into characterizations of all facet defining  $0/\pm 1$ -inequalities for the hop constrained chain polytope and all facet defining  $0/1$ -inequalities for its dominant. Although the derived inequalities are already known, such classifications were not previously given to the best of our knowledge. Moreover, we use this approach to generalize so called jump inequalities, which have been introduced in a paper of Dahl and Gouveia in 2004.

## 1 Introduction

Let  $D = (V, A)$  be a directed graph without parallel arcs. An  $(s, t)$ -chain is a sequence of arcs  $C = (a_1, a_2, \dots, a_r)$  such that  $a_i = (i_{p-1}, i_p)$  for  $p = 1, \dots, r$ , with  $i_0 = s$  and  $i_r = t$ . The nodes  $i_1, i_2, \dots, i_{r-1}$  are the *internal nodes* of  $C$ . If all arcs  $a_i$  are distinct, then  $C$  is called a *walk*; If all nodes  $i_p$  are distinct, then  $C$  is called a *path*. In what follows, chains will be usually denoted only as a sequence of nodes, but their incidence vectors are defined in the arc space  $\mathbb{R}^A$ . Here, for any chain  $C$ , its incidence vector  $\chi^C \in \mathbb{R}^A$  is defined by

$$\chi_a^C := \text{number of times the arc } a \text{ is used by } C,$$

for  $a \in A$ . Note that different chains may have the same incidence vector.

Given a length function  $d : A \rightarrow \mathbb{R}$ , the *length* of a chain  $C = (i_0, i_1, i_2, \dots, i_q)$  is defined as  $d(C) := \sum_{p=1}^q d((i_{p-1}, i_p))$ . In the *hop constrained shortest chain (walk, path) problem* we are looking for a chain (walk, path) using at most  $k$  arcs of minimum length. The hop constrained shortest path problem, which is known to be NP-hard, arises, for instance, in the design of telecommunication networks when data have to be sent along paths that must not contain more than a certain number of intermediate nodes in order to guarantee a minimum level of service quality [14, 8].

The corresponding chain problem is a combinatorial relaxation of this problem which can be solved in polynomial time with the Moore-Bellman-Ford algorithm [3, 11, 23], see Algorithm 1. Using an integer programming approach for

the hop constrained path problem, valid inequalities for the easier chain problem are of interest, since they are also valid inequalities for the harder problem. Thus, a branch-and-cut algorithm for solving the path problem, for example, directly benefits from efficient separation routines for the polyhedron associated with the chain problem.

**Algorithm 1: Moore-Bellman-Ford**

**Input:** A digraph  $D = (V, A)$ , a fixed node  $s \in V$ , and a length function  $d : A \rightarrow \mathbb{R}$ .

**Output:** For each node  $j \in V$  and each number  $\ell \in \{0, \dots, |V| - 1\}$ , the length  $u_j^{(\ell)}$  of a shortest  $(s, j)$ -chain using at most  $\ell$  arcs and its predecessor  $p(j, \ell)$  on such a chain. If  $j$  is not reachable from  $s$ , then  $u_j^{(\ell)} = +\infty$  and  $p(j, \ell)$  is undefined for all  $\ell$ .

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- (1) Set  $u_s^{(0)} := 0$  and  $u_j^{(0)} := +\infty$  for all  $j \in V \setminus \{s\}$ .
- (2) **for**  $\ell := 1$  **to**  $|V| - 1$  **do**  
    Set  $t_j := u_j^{(\ell-1)}$  for all  $j \in V$ .  
    **forall**  $(i, j) \in A$  **do**  
      **if**  $t_j > u_i^{(\ell-1)} + d((i, j))$  **then**  
        Set  $t_j := u_i^{(\ell-1)} + d((i, j))$  and  $p(j, \ell) := i$ .  
    Set  $u_j^{(\ell)} := t_j$  for all  $j \in V$ .

In this paper, we present some results on the *hop constrained chain polytope*  $\mathcal{C}^{\leq k}$ , that is, the convex hull of the incidence vectors of chains using at most  $k$  arcs, and its *dominant*  $\text{dmt}(\mathcal{C}^{\leq k}) := \mathcal{C}^{\leq k} + \mathbb{R}_+^A$ , where  $\mathbb{R}_+^A$  is the nonnegative orthant. In the last years, closely related polyhedra have been investigated, see, for instance, [2, 6, 7, 9, 10, 12, 16, 17, 18, 19, 22, 24, 25], in particular the *hop constrained path polytope*  $\mathcal{P}^{\leq k}$  defined as the convex hull of the incidence vectors of hop constrained  $(s, t)$ -paths. Important for our context are the following three results.

**Fact 1** ([25]). *The integer points of  $\mathcal{P}^{\leq k}$  are characterized by the system*

$$x_{ii} = 0, \quad (i, i) \in A, \quad (1)$$

$$x(\delta^{\text{in}}(s)) = 0, \quad (2)$$

$$x(\delta^{\text{out}}(t)) = 0, \quad (3)$$

$$x(\delta^{\text{out}}(s)) = 1, \quad (4)$$

$$x(\delta^{\text{in}}(t)) = 1, \quad (5)$$

$$x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) = 0, \quad i \in V \setminus \{s, t\}, \quad (6)$$

$$x(A) \leq k, \quad (7)$$

$$x(\delta^{\text{out}}(i)) \leq 1, \quad i \in V \setminus \{s, t\}, \quad (8)$$

$$x(\delta^{\text{out}}(S)) - x(\delta^{\text{out}}(j)) \geq 0, \quad S \subset V, s, t \in S \quad (9)$$

$$j \in V \setminus S,$$

$$x_{ij} \in \{0, 1\}, \quad (i, j) \in A. \quad (10)$$

Here, for any  $S \subseteq V$ ,  $\delta^{\text{out}}(S) := \{(i, j) \in A : i \in S, j \in V \setminus S\}$  and  $\delta^{\text{in}}(S) := \{(i, j) \in A : i \in V \setminus S, j \in S\}$ . For nodes  $j \in V$ , we write  $\delta^{\text{out}}(j)$  and  $\delta^{\text{in}}(j)$  instead of  $\delta^{\text{out}}(\{j\})$  and  $\delta^{\text{in}}(\{j\})$ , respectively. Moreover, for any  $B \subseteq A$ ,  $x(B) := \sum_{a \in B} x_a$ .

**Fact 2** (Dahl and Gouveia [7]). *The nonnegativity constraints  $x_{ij} \geq 0$  for all  $(i, j) \in A$ , the equations (2)-(6), and the inequalities*

$$x_{si} - \sum_{j \in V \setminus \{s, t\}} x_{ij} \geq 0 \quad \text{for all } i \in V \setminus \{s, t\}$$

*provide a complete linear description of  $\mathcal{P}_{s,t\text{-path}}^{\leq 3}(D)$ .*

**Fact 3** (Dahl, Foldnes, and Gouveia [6]). *The 4-hop constrained walk polytope  $\mathcal{W}^{\leq 4}(D)$  is determined by the equations (2)-(6), the nonnegativity constraints  $x_{ij} \geq 0$  for all  $(i, j) \in A$ , and the inequalities*

$$\sum_{i \in I} x_{si} + \sum_{j \in J} x_{jt} - \sum_{i \in I, j \in J} x_{ij} \geq 0 \quad (11)$$

*for all  $I, J \subseteq V \setminus \{s, t\}$ .*

As the optimization problem corresponding to  $\mathcal{C}^{\leq k}$  can be solved in polynomial time, there is some hope to find a complete linear description of both  $\mathcal{C}^{\leq k}$  and  $\text{dmt}(\mathcal{C}^{\leq k})$ . Our results indicate, however, that these linear characterizations must be quite complicated.

Our contribution consists of four aspects, the first two of them are completeness results. For a better understanding, we first introduce two classes of facet defining inequalities for  $\text{dmt}(\mathcal{C}^{\leq k})$ : *cut inequalities*

$$x(C) \geq 1 \quad \text{for all } (s, t)\text{-cuts } C \quad (12)$$

and *r-jump inequalities* introduced by Dahl and Gouveia [7]. Given a partition  $\{S_p : p = 0, 1, \dots, k+r\}$  of  $V$ , where  $r \in \mathbb{N}$ ,  $1 \leq r \leq n-k$ ,  $S_0 = \{s\}$ , and  $S_{k+r} = \{t\}$ . The *r-jump inequality* associated with this partition is defined as

$$\sum_{p=0}^{k+r-2} \sum_{q=p+2}^{k+r} \alpha_{pq} x((S_p : S_q)) \geq r, \quad (13)$$

where for any  $U, W \subseteq V$ , we define  $(U : W) := \{(u, w) \in A : u \in U, w \in W\}$ . Moreover, for  $p < q$ ,  $\alpha_{pq} := \min\{q-p-1, r\}$ . The results are:

(i) Each 0/1-facet defining inequality for  $\text{dmt}(\mathcal{C}^{\leq k})$  is either a cut inequality (12) or an *r-jump inequality* (13) with  $r = 1$ .

(ii) For  $t \in V$ , an inequality  $\tau^T x \geq \tau_0$  (or equation  $\tau^T x = \tau_0$ ) with  $\tau \in \mathbb{R}^A$  is said to be *t-rooted* if  $\tau_{it} = 0$  for all  $i \in V \setminus \{t\}$ . Each 0/±1-facet defining *t-rooted inequality* for  $\mathcal{C}^{\leq k}$  has one of the two following forms:

$$\sum_{p=0}^{k-3} \sum_{q=p+2}^{k-1} x((S_p : S_q)) - x((S_{k-1} : S_1 \cup S_2)) \geq 0, \quad (14)$$

where  $\{S_p : p = 0, 1, \dots, k-1\}$  is a partition of  $V \setminus \{t\}$  with  $S_0 = \{s\}$ ;

$$x(\delta^{\text{out}}(s) \setminus \{(s, t)\}) + \sum_{p=1}^{k-4} \sum_{q=p+2}^{k-2} x((S_p : S_q)) - x((S_{k-2} : S_1)) \geq 0, \quad (15)$$

where  $\{S_p : p = 1, \dots, k-2\}$  is a partition of  $V \setminus \{s, t\}$ .

(iii) We present a class of facet defining inequalities for  $\text{dmt}(\mathcal{C}^{\leq k})$  that generalize *r-jump inequalities*.

(iv) We address an open problem (in terms of chains) raised by Dahl and Gouveia [7] how to transform an *r-jump inequality* (13) that defines a facet of the dominant of  $\mathcal{P}^{\leq k}$  into a facet defining inequality for  $\mathcal{P}^{\leq k}$  itself by decreasing coefficients. (Note that  $\text{dmt}(\mathcal{P}^{\leq k}) = \text{dmt}(\mathcal{C}^{\leq k})$ .) We provide a systematic way to transform *r-jump inequalities* (or the generalization in (iii)) into facet defining inequalities for  $\mathcal{C}^{\leq k}$ . Moreover, we give sufficient conditions for the resulting inequalities to be facet defining for  $\mathcal{P}^{\leq k}$ .

The above results are obtained on the basis of the following standard proof technique. We first provide extended formulations for both polyhedra and then derive facet defining inequalities for them by characterizing extreme rays of the associated projection cones. Here is a brief outline.

Given a polyhedron in *x-y-space*  $Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \geq a\}$ , the *projection of Q onto the x-space* is defined as  $\text{Proj}_x(Q) := \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q \text{ with } (x, y) \in Q\}$ . The polyhedral cone  $\mathcal{K} := \{v : v^T B = 0^T, v \geq 0\}$  is called the *projection cone*. Conversely, a system of the form  $Ax + Bx \geq a$  is said to be an *extended formulation* for a polyhedron  $P \subseteq \mathbb{R}^p$  if  $P = \text{Proj}_x(Q)$ , where  $Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + Bx \geq a\}$ . The system is said to be *compact* if the number of rows and columns of the matrix  $(A, B, a)$  and if the encoding length of each entry is polynomial in  $p$ .

Given an extended formulation for a polyhedron  $P \subseteq \mathbb{R}^p$ , the following theorem due to Balas [1] addresses the task to derive a complete linear description (in the space  $\mathbb{R}^p$ ) for this polyhedron.

**Theorem 1.1** (Balas [1]). *Let  $Q = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \geq a\}$  be a polyhedron. Then,  $\text{Proj}_x(Q) = \{x \in \mathbb{R}^p : (v^T A)x \geq v^T a \text{ for all } v \in \text{extr}(\mathcal{K})\}$ , where  $\text{extr}(\mathcal{K})$  denotes the set of extreme rays of the projection cone  $\mathcal{K}$ .  $\square$*

It is usually quite difficult to determine all extreme rays of  $\mathcal{K}$  or all those extreme rays  $v \in \text{extr}(\mathcal{K})$ , whose corresponding inequalities  $(v^T A)x \geq v^T a$  define facets of  $\text{Proj}_x(Q)$ . However, sometimes the extreme rays or a subset of them have a convenient structure.

Martin, Rardin, and Campbell [21] provided a framework to derive linear characterizations of dynamic programs, which can be used to derive (compact) extended formulations of the hop constrained chain polytope and its dominant. Our extended formulations are very similar to those given in [8, 13]. The authors there, however, do not distinguish between walks and chains which leads to somewhat misleading and wrong statements.

The results (iii) and (iv) are obtained as follows. We identify the extreme rays associated with the known  $r$ -jump inequalities (13) and derive new facets of the dominant by relaxing a certain structure common to all extreme rays that yield the  $r$ -jump inequalities. Between extreme rays for the chain polytope on the one hand and its dominant on the other hand seems to exist a strong relationship, which we, however, do not understand in general yet. We use this relationship to transform the known facet defining inequalities for the dominant into facet defining inequalities for the hop constrained chain polytope itself. This relationship and the easier access to facet defining inequalities for  $\text{dmt}(\mathcal{C}^{\leq k})$  is, in fact, the reason why we are not only interested in the facial structure of the hop constrained chain polytope but also in that of the dominant.

The remainder of this paper is structured as follows. In Section 2, we derive the extended formulations, give a partial characterization of the corresponding extreme rays, and introduce some well-known polyhedral techniques in order to simplify the presentation of our main results given in Section 3. We close the paper in Section 4 with some remarks regarding future research.

## 2 Compact extended formulations and preliminary results

Let  $D_n = (V_n, A_n)$  be the digraph obtained from the complete digraph including loops defined on the node set  $V_n := \{s, t, 1, 2, \dots, n\}$  by deleting the arcs  $(s, t)$  and  $a \in \delta^{\text{in}}(s) \cup \delta^{\text{out}}(t)$ . In what follows, our results will be presented on this graph. Moreover,  $\mathcal{C}^{\leq k}$  will be denoted by  $\mathcal{C}^{\leq k}(D_n)$ .

**Theorem 2.1.**

- (a)  $\dim \text{dmt}(\mathcal{C}^{\leq k}(D_n)) = |A_n|$ .
- (b) For  $k \geq 4$ ,  $\dim \mathcal{C}^{\leq k}(D_n) = \dim P_{\text{walk}}^{\leq k}(D_n) = |A_n| - (n + 1)$ .
- (c) For  $k \geq 4$ ,  $\dim \mathcal{P}^{\leq k}(D_n) = |A_n| - (2n + 1)$ .

*Proof.* (a) is clear; (b) and (c) follow from Theorem 2.3 in [19]. Notice that  $\dim \mathcal{P}^{\leq k}(D_n) = \dim \mathcal{C}^{\leq k}(D_n) - n$ , since paths do not use loops, by definition.  $\square$



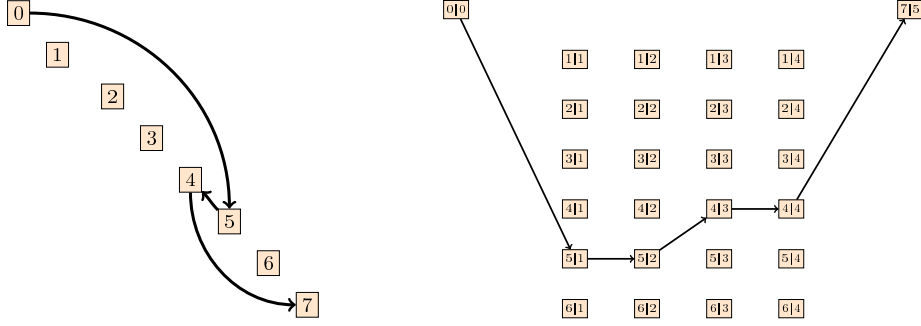


Figure 1: A digraph  $D = (V, A)$  on node set  $V = \{0, 1, \dots, 7\}$  and associated DP-graph  $\mathcal{D} = (\mathcal{V}, \mathcal{A})$  for  $(D, 0, 7, 5)$ ; arc sets are omitted. Illustration of a  $(0, 7)$ -chain and one of its counterparts in  $\mathcal{D}$ .

## 2.1 Compact formulations and the projection mechanism

Using the framework of Martin, Rardin, and Campbell [21] for linear characterizations of dynamic programs, we obtain a compact extended formulation of  $\mathcal{C}^{\leq k}(D_n)$  as follows. We define a so-called *dynamic programming digraph*  $\mathcal{D}_n = (\mathcal{V}_n, \mathcal{A}_n)$  associated with  $(D_n, s, t, k)$ . It consists of the nodes  $[s, 0]$ ,  $[t, k]$ , and  $[i, \ell]$  for  $i = 1, \dots, n$ ,  $\ell = 1, \dots, k - 1$ . With each node  $i \in \mathcal{V}_n \setminus \{s, t\}$ , we associate the set  $\mathcal{A}_n^i$  of arcs  $([i, \ell - 1], [i, \ell])$  and with each  $(i, j) \in A_n$ , we introduce the set  $\mathcal{A}_n^{ij}$  of arcs  $([i, \ell - 1], [j, \ell])$ , where  $\ell = 1$  if  $i = s$ ,  $\ell = k$  if  $j = t$ , and  $\ell \in \{2, 3, \dots, k - 1\}$  otherwise. The arc set  $\mathcal{A}_n$  is defined as the disjoint union of all arc sets  $\mathcal{A}_n^i$  and  $\mathcal{A}_n^{ij}$ . Notice that  $\mathcal{D}_n$  has two arcs connecting  $[i, \ell - 1]$  and  $[i, \ell]$ , for  $\ell = 2, 3, \dots, k - 1$ . For an illustration of the construction, see Figure 1.

$\mathcal{D}_n$  is acyclic. Hence, the convex hull of the incidence vectors of  $([s, 0], [t, k])$ -paths (or chains), denoted by  $\mathcal{P}(\mathcal{D}_n)$ , is determined by the flow conservation and nonnegativity constraints:

$$y(\delta^{\text{in}}([i, \ell])) - y(\delta^{\text{out}}([i, \ell])) = \begin{cases} -1 & \text{if } [i, \ell] = [s, 0], \\ 1 & \text{if } [i, \ell] = [t, k], \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

$$y \geq 0. \quad (17)$$

Moreover, connecting the arc sets  $A_n$  and  $\mathcal{A}_n$  via the set function  $\varphi : \mathcal{A} \rightarrow A \cup \emptyset$ ,

$$\varphi(([i, \ell - 1], [j, \ell])) = \begin{cases} \emptyset & \text{if } ([i, \ell - 1], [j, \ell]) \in \mathcal{A}^i, \\ (i, j) & \text{else,} \end{cases}$$

we see that each  $(s, t)$ -chain in  $D_n$  using at most  $k$  arcs corresponds to at least one  $([s, 0], [t, k])$ -path in  $\mathcal{D}_n$  and, conversely, each  $([s, 0], [t, k])$ -path in  $\mathcal{D}_n$  corresponds to an  $(s, t)$ -chain in  $D_n$  using at most  $k$  arcs. By coupling  $x$ - and  $y$ -variables via a transformation matrix  $T \in \mathbb{R}^{A \times \mathcal{A}}$  that represents the set function  $\varphi$ , we obtain compact extended formulation of the hop constrained chain polytope and its dominant. The proof of the following theorem, based on standard polyhedral techniques, will be omitted.

**Theorem 2.2.** *Let  $\mathcal{D}_n = (\mathcal{V}_n, \mathcal{A}_n)$  be the DP-graph associated with  $(D_n, s, t, k)$ . Moreover, define the polyhedra  $Q^\leq := \{(x, y) \in \mathbb{R}^A \times P_{[s,0],[t,k]\text{-path}}(\mathcal{D}) : x - Ty = 0\}$  and  $Q^\geq := \{(x, y) \in \mathbb{R}^A \times P_{[s,0],[t,k]\text{-path}}(\mathcal{D}) : x - Ty \geq 0\}$ . Then,  $\mathcal{C}^{\leq k}(D_n) = \text{Proj}_x(Q^\leq)$  and  $\text{dmt}(\mathcal{C}^{\leq k}(D_n)) = \text{Proj}_x(Q^\geq)$ .  $\square$*

Projecting out the  $y$ -variables, we obtain complete linear descriptions of the relaxations only in terms of  $x$ -variables. The projection cone associated with  $Q^\leq$  and  $\mathcal{C}^{\leq k}(D_n)$ , denoted by  $\mathcal{C}^\leq$ , is the set of all  $(\pi, \rho, \sigma) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{A}_n} \times \mathbb{R}^{\mathcal{A}_n}$  satisfying the equations

$$\rho_a \geq 0, a \in \mathcal{A}_n, \quad (18')$$

$$\begin{aligned} \pi_{i,\ell-1} - \pi_{i\ell} - \rho_a &= 0, a = ([i, \ell - 1], [i, \ell]) \in \mathcal{A}_n^i, \\ i &\in V_n \setminus \{s, t\}, \end{aligned} \quad (19')$$

$$\begin{aligned} \sigma_{ij} + \pi_{i,\ell-1} - \pi_{j\ell} - \rho_a &= 0, \\ a &= ([i, \ell - 1], [j, \ell]) \in \mathcal{A}^{ij}, (i, j) \in A_n, \end{aligned} \quad (20')$$

while that associated with  $Q^\geq$  and  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ , denoted by  $\mathcal{C}^\geq$ , is the set of all  $(\pi, \rho, \sigma) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{A}_n} \times \mathbb{R}^{\mathcal{A}_n}$  satisfying the equations (18')- (20') and

$$\sigma_{ij} \geq 0 \quad \forall (i, j) \in A_n. \quad (21)$$

By Theorems 1.1 and 2.2, it follows that  $\mathcal{C}^{\leq k}(D_n)$  is the set of all  $x \in \mathbb{R}^{\mathcal{A}_n}$  satisfying

$$\sum_{a \in \mathcal{A}_n} \sigma_a x_a \geq \pi_{tk} - \pi_{s0}$$

for all  $(\pi, \rho, \sigma) \in \mathcal{C}^\leq$ , and  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  is the set of all  $x \in \mathbb{R}^{\mathcal{A}_n}$  satisfying

$$\sum_{a \in \mathcal{A}_n} \sigma_a x_a \geq \pi_{tk} - \pi_{s0}$$

for all  $(\pi, \rho, \sigma) \in \mathcal{C}^\geq$ .

We now show that the essential part of the projection mechanism can be described only in terms of  $\pi$ -variables. The variables  $\rho_a$ ,  $a \in \mathcal{A}_n$ , only act as slack variables. Projecting them out, we see that every  $(\pi, \rho, \sigma) \in \mathcal{C}^\leq$  satisfies the inequalities

$$\begin{aligned} \pi_{i,\ell-1} - \pi_{i\ell} &\geq 0, & i &\in V_n \setminus \{s, t\}, \\ & & \ell &= 2, \dots, k-1, \end{aligned} \quad (19)$$

$$\begin{aligned} \sigma_{ij} + \pi_{ih} - \pi_{j\ell} &\geq 0, & a &= ([i, h], [j, \ell]) \in \mathcal{A}_n^{ij}, \\ & & (i, j) &\in A_n, \end{aligned} \quad (20)$$

while every  $(\pi, \rho, \sigma) \in \mathcal{C}^\geq$  satisfies the inequalities (19)-(21).

For fixed  $\pi$  satisfying (19), denote by  $\mathcal{C}_\pi^\leq$  the set of all  $\sigma \in \mathbb{R}^{\mathcal{A}_n}$  satisfying (20). By construction, all  $\sigma \in \mathcal{C}_\pi^\leq$  provide a valid inequality for  $\mathcal{C}^{\leq k}(D_n)$  with the same right hand side  $\pi_{tk} - \pi_{s0}$ :

$$\sigma^T x \geq \pi_{tk} - \pi_{s0}.$$

Moreover, (20) implies that each coefficient  $\sigma_{ij}$  of such an inequality satisfies

$$\sigma_{ij} \geq \max\{\pi_{j\ell} - \pi_{ih} : a = ([i, h], [j, \ell]) \in \mathcal{A}_n^{ij}\}.$$

Conversely, define  $\sigma^\pi$  by

$$\sigma_{ij}^\pi := \max\{\pi_{j\ell} - \pi_{ih} : a = ([i, h], [j, \ell]) \in \mathcal{A}_n^{ij}\}$$

for all  $(i, j) \in A_n$ . Then,  $\sigma^\pi \in \mathcal{C}_\pi^\leq$  and  $\sigma^\pi \leq \sigma$  for all  $\sigma \in \mathcal{C}_\pi^\leq$ . Thus,  $\sigma^\pi$  provides the strongest valid inequality under all inequalities  $\sigma^T x \geq \pi_{tk} - \pi_{s0}$  with  $\sigma \in \mathcal{C}_\pi^\leq$  unless the inequality

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq \pi_{tk} - \pi_{s0}$$

is an implicit equation. If so, it could be the case that, for some  $\sigma \in \mathcal{C}^\leq$  with  $\sigma \neq \sigma^\pi$ , the inequality  $\sigma^T x \geq \pi_{tk} - \pi_{s0}$  induces a facet of  $\mathcal{C}^{\leq k}(D_n)$ .

An analogous argumentation holds for  $\mathcal{C}^\geq$ . For fixed  $\pi$  satisfying (19), denote by  $\mathcal{C}_\pi^\geq$  the set of all  $\sigma \in \mathbb{R}^{A_n}$  satisfying (20) and (21). Then, the inequality

$$\sum_{a \in A_n} \sigma_a^{\pi,+} x_a \geq \pi_{tk} - \pi_{s0},$$

where  $\sigma^{\pi,+} \in \mathbb{R}^A$  is defined by

$$\sigma_a^{\pi,+} := \max\{\sigma_a^\pi, 0\}$$

for all  $a \in A_n$ , is the strongest valid inequality w.r.t.  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  under all inequalities  $\sigma^T x \geq \pi_{tk} - \pi_{s0}$  with  $\sigma \in \mathcal{C}_\pi^\geq$ , unless it is an implicit equation.

Denote by  $\bar{\Pi}_n$  the set of  $\pi \in \mathbb{R}^{V_n}$  satisfying (19). For any  $\pi \in \bar{\Pi}_n$ , we denote by  $F(\pi)$  and  $F(\pi, +)$  the faces of  $\mathcal{C}^{\leq k}(D_n)$  and  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  induced by the inequalities

$$\sum_{(i,j) \in A_n} \sigma_{ij}^\pi x_{ij} \geq \pi_{tk} - \pi_{s0}$$

and

$$\sum_{(i,j) \in A_n} \sigma_{ij}^{\pi,+} x_{ij} \geq \pi_{tk} - \pi_{s0},$$

respectively.

A facet of  $\mathcal{C}^{\leq k}(D_n)$  or  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  is called *nontrivial* if it cannot be induced by a nonnegativity constraint.

**Lemma 2.1.** *Let  $\pi \in \bar{\Pi}_n$ .*

(a1) *If  $F(\pi, +) = \text{dmt}(\mathcal{C}^{\leq k}(D_n))$ , then, for each  $\sigma \in \mathcal{C}_\pi^\geq$ , the inequality  $\sigma^T x \geq \pi_{tk} - \pi_{s0}$  is a conical combination of nonnegativity constraints.*

(a2) *Let  $F(\pi, +)$  be a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ . Then,  $F(\pi, +)$  is nontrivial if and only if  $\pi_{tk} - \pi_{s0} > 0$ .*

(b) *If  $F(\pi) = \mathcal{C}^{\leq k}(D_n)$ , then for each internal node  $i \in V_n$ , there exists  $\lambda_i \in \mathbb{R}$  such that  $\pi_{i\ell} = \lambda_i$  for all  $\ell \in \{1, 2, \dots, k-1\}$ . Moreover, in this case, the equation  $\sum_{a \in A_n} \sigma_a^\pi x_a = \pi_{tk} - \pi_{s0}$  is a linear combination of the equations (4)-(6), and the inequality  $\sigma^T x \geq \pi_{tk} - \pi_{s0}$  is the sum of this equation and a conical combination of nonnegativity constraints  $x_{ij} \geq 0$ , for every  $\sigma \in \mathcal{C}_\pi^\leq$ .*

*Proof.* (a1) Let  $F(\pi, +) = \text{dmt}(\mathcal{C}^{\leq k}(D_n))$ . This means that the inequality  $\sum_{a \in A_n} \sigma_a^{\pi, +} x_a \geq \pi_{tk} - \pi_{s0}$  is an implicit equation. On the other hand, the only equation satisfied by all  $x \in \text{dmt}(\mathcal{C}^{\leq k}(D_n))$  is  $0^T x = 0$ , since  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  is fulldimensional. This implies  $\pi_{i1} = \pi_{s0}$  and  $\pi_{i, k-1} = \pi_{tk}$  for all internal nodes  $i$  of  $V_n$ . Moreover, since  $\pi$  satisfies (19), it follows that  $\pi = \lambda \mathbb{1}$  for some  $\lambda \in \mathbb{R}$ , where  $\mathbb{1}$  denotes the vector of all ones. Thus, for every  $\sigma \in \mathcal{C}_{\pi}^{\geq}$ , the inequality  $\sigma^T x \geq \pi_{tk} - \pi_{s0}$  is a conical combination of nonnegativity constraints, since  $\pi_{tk} - \pi_{s0} = 0$  and  $\sigma \geq 0$ .

(a2) is clear.

(b) Assume that the inequality  $\sum_{a \in A_n} \sigma_a^{\pi} x_a \geq \pi_{tk} - \pi_{s0}$  is an implicit equation and  $\pi_{i, \ell-1} \neq \pi_{i\ell}$  for some  $i \in V_n \setminus \{s, t\}$  and  $\ell \in \{2, \dots, k-1\}$ . Since  $\pi \in \overline{\Pi}_n$ , it follows that  $\pi_{i, \ell-1} > \pi_{i\ell}$ . Consider the paths  $P := ([s, 0], [i, 1], [i, 2], \dots, [i, k-1], [t, k])$  and  $P' := ([s, 0], [i, 1], \dots, [i, \ell-1], \emptyset, [i, \ell], \dots, [i, k-1], [t, k])$ . Then,  $\sigma^{\pi}(\varphi(P)) < \sigma^{\pi}(\varphi(P'))$ , since  $\sigma_{ii}^{\pi} < 0$ . This is a contradiction. Thus, for each  $i \in V_n \setminus \{s, t\}$ , there exists  $\lambda_i \in \mathbb{R}$  such that  $\pi_{i\ell} = \lambda_i$  for all  $\ell \in \{1, 2, \dots, k-1\}$ .

Next, it follows from Theorem 2.1 (b) that each equation, which is satisfied by all  $x \in \mathcal{C}^{\leq k}(D_n)$ , is a linear combination of the equations (4)-(6), and hence also the equation  $\sum_{a \in A_n} \sigma_a^{\pi} x_a = \pi_{tk} - \pi_{s0}$ .

Finally, let  $\sigma \in \mathcal{C}_{\pi}^{\geq}$ . Since  $\sigma \geq \sigma^{\pi}$ , the inequality  $\tau^T x \geq 0$ , where  $\tau := \sigma - \sigma^{\pi}$ , is a conical combination of nonnegativity constraints. Moreover, by construction, the inequality  $\sigma^T x \geq \pi_{tk} - \pi_{s0}$  is the sum of the equation  $\sum_{a \in A_n} \sigma_a^{\pi} x_a = \pi_{tk} - \pi_{s0}$  and the inequality  $\tau^T x \geq 0$ .  $\square$

Let  $F$  be a face of  $\mathcal{C}^{\leq k}(D_n)$  or its dominant. Then, a chain  $C \in \mathcal{C}_{s,t}^{\leq k}(D_n)$  is said to be *tight w.r.t. F* if  $\chi^C \in F$ . Moreover, if  $C$  is tight, it will be also called tight w.r.t to any valid inequality  $b^T x \geq b_0$  defining  $F$ .

The  $i$ th component vector  $\pi_i := (\pi_{i1}, \dots, \pi_{i, k-1})$  associated with any internal node  $i \in V$ , is called a *row of  $\pi$* . Furthermore, define the sets:  $\Pi_n := \{\pi \in \overline{\Pi}_n : 0 \leq \pi \leq \pi_{tk}, \pi_{s0} = 0, \pi_{tk} > 0\}$  and  $\Pi_n^0 := \{\pi \in \overline{\Pi}_n : 0 \neq \pi \geq 0, \pi_{s0} = \pi_{tk} = \pi_{i, k-1} = 0 \text{ for all } i \in V_n \setminus \{s, t\}\}$ .

Two valid inequalities for  $\mathcal{C}^{\leq k}(D_n)$  are said to be *equivalent* if one can be obtained from the other by multiplication with a positive scalar and adding appropriate multiples of the equations (4)-(6). Equivalent inequalities induce the same face of  $\mathcal{C}^{\leq k}(D_n)$ . Due to the equations (4)-(6), every valid inequality for  $\mathcal{C}^{\leq k}(D_n)$  can be brought in  $t$ -rooted form. Two vectors  $\pi, \tilde{\pi} \in \overline{\Pi}$  are said to be *equivalent w.r.t.  $\mathcal{C}^{\leq k}(D_n)$*  if the corresponding inequalities  $\sum_{a \in A_n} \sigma_a^{\pi} x_a \geq \pi_{tk} - \pi_{s0}$  and  $\sum_{a \in A_n} \sigma_a^{\tilde{\pi}} x_a \geq \tilde{\pi}_{tk} - \tilde{\pi}_{s0}$  are equivalent. It is easy to see that two vectors  $\pi, \tilde{\pi} \in \overline{\Pi}$  are equivalent if exist  $\lambda > 0$  and  $\mu_i \in \mathbb{R}$ ,  $i \in V_n \setminus \{s, t\}$  such that  $\pi_i = \lambda \tilde{\pi}_i + \mu_i \mathbb{1}^T$  for each internal node  $i \in V_n$ .

**Theorem 2.3.** (a)  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  is the set of all  $x \in \mathbb{R}_+^{A_n}$  satisfying

$$\sum_{a \in A_n} \sigma_a^{\pi, +} x_a \geq \pi_{tk}, \quad \forall \pi \in \Pi_n,$$

or equivalently,

$$\sum_{a \in A_n} \sigma_a^{\pi, +} x_a \geq \pi_{tk} - \pi_{s0}, \quad \forall \pi \in \overline{\Pi}_n.$$

(b)  $\mathcal{C}^{\leq k}(D_n)$  is the set of all  $x \in \mathbb{R}_+^{A_n}$  satisfying equations (4)-(6) and the inequalities

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq 0 \quad \forall \pi \in \Pi_n^0,$$

or equivalently,

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq \pi_{tk} - \pi_{s0} \quad \forall \pi \in \bar{\Pi}_n.$$

*Proof.* (a) By Theorems 1.1 and 2.2, Lemma 2.1 (a1) and (a2), and the remarks made in the previous paragraphs, we see that  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  is the set of all  $x \in \mathbb{R}_+^{A_n}$  satisfying

$$\sum_{a \in A_n} \sigma_a^{\pi,+} x_a \geq \pi_{tk} - \pi_{s0} \quad \forall \pi \in \bar{\Pi}'_n,$$

where  $\bar{\Pi}'_n := \{\pi \in \bar{\Pi}_n : \pi_{s0} < \pi_{tk}\}$ . Next, observing that  $\sigma^{\pi,+}$  is invariant under shifting  $\pi$  by a constant, it follows that the above inequalities can be replaced by

$$\sum_{a \in A_n} \sigma_a^{\pi,+} x_a \geq \pi_{tk} \quad \forall \pi \in \bar{\Pi}''_n,$$

where  $\bar{\Pi}''_n := \{\pi \in \bar{\Pi}_n : 0 = \pi_{s0} < \pi_{tk}\}$ . Finally, consider the case that  $\pi \in \bar{\Pi}''_n$  and  $\pi_{i\ell} > \pi_{tk}$  for some internal node  $[i, \ell] \in \mathcal{V}$ . Then, inequalities (19) imply  $\pi_{i1} > \pi_{tk}$ , and hence,  $\sigma_{si}^{\pi,+} > \pi_{tk}$ , that is, greater than the right hand side of the inequality

$$\sum_{a \in A_n} \sigma_a^{\pi,+} x_a \geq \pi_{tk}.$$

Thus, none of the chains  $C \in \mathcal{C}_{s,t}^{\leq k}(D_n)$  using the arc  $(s, i)$  is tight, which implies that  $F(\pi, +)$  is strictly contained in the face induced by the nonnegativity constraint  $x_{si} \geq 0$ . Similarly, the inequality does not induce a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  if  $\pi_{i\ell} < 0$  for some internal node  $[i, \ell] \in \mathcal{V}$ . Hence, we may assume that  $\pi \in \Pi_n$ .

The second equality in Theorem 2.3 holds, since  $\Pi_n \subseteq \bar{\Pi}_n$ .

(b) Using Theorems 1.1 and 2.2, Lemma 2.1 (b), and the remarks made in the previous paragraphs, it follows that  $\mathcal{C}^{\leq k}(D_n)$  is the set of all  $x \in \mathbb{R}_+^{A_n}$  satisfying equations (4)-(6) and the inequalities

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq \pi_{tk} - \pi_{s0} \quad \forall \pi \in \bar{\Pi}_n.$$

Next,  $\sigma^\pi$  is invariant under shifting  $\pi$  by a constant. Moreover, adding a linear combination of the equations (4)-(6) to the inequality

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq \pi_{tk} - \pi_{s0},$$

results in an equivalent inequality. Hence, it is easy to see that  $\mathcal{C}^{\leq k}(D_n)$  is determined by (4)-(6),  $x \geq 0$ , and

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq 0 \quad \forall \pi \in \bar{\Pi}_n^0.$$

□

We summarize: the task to identify extreme rays of  $\mathcal{C}^=$  ( $\mathcal{C}^\geq$ ) turns out to be the task to find those  $\pi \in \overline{\Pi}$  such that  $F(\pi)$  ( $F(\pi, +)$ ) is a facet of  $\mathcal{C}^{\leq k}(D_n)$  ( $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ ).

## 2.2 Preliminary results

In this subsection, we use some techniques in order to present  $\pi$ -vectors in a canonical form.

Let  $\pi \in \mathbb{R}^{\mathcal{V}_n}$ . The restriction of  $\pi$  to any subset  $S$  of  $\mathbb{R}^{\mathcal{V}_n}$  will be denoted by  $\pi|_S$ . The restriction of  $\pi \in \mathbb{R}^{\mathcal{V}_n}$  to  $\mathbb{R}^{\mathcal{V}_n \setminus \{[s,0],[t,k]\}}$  can be perceived as a matrix in space  $\mathbb{R}^{n \times (k-1)}$ . Let  $\prec$  be the lexicographic ordering of  $\mathbb{R}^{\mathcal{V}_n}$  with respect to the ordering

$$\begin{aligned} [s, 0] &< [1, 1] < [1, 2] < \dots < [1, k-1] < [2, 1] \\ &< [2, 2] < \dots < [n, k-1] < [t, k] \end{aligned}$$

of vector positions. This means,  $\pi \prec \tilde{\pi}$  if and only if there is some position  $[j, \ell]$  such that  $\pi_{j\ell} < \tilde{\pi}_{j\ell}$  and  $\pi_{ih} = \tilde{\pi}_{ih}$  for all vector positions  $[i, h] < [j, \ell]$ . For any  $\pi \in \mathbb{R}^{\mathcal{V}_m}$ ,  $\tilde{\pi} \in \mathbb{R}^{\mathcal{V}_n}$ , we call  $\pi$  a *subvector* of  $\tilde{\pi}$  and write  $\pi \subseteq \tilde{\pi}$  if  $\pi_{s0} = \tilde{\pi}_{s0}$ ,  $\pi_{tk} = \tilde{\pi}_{tk}$ , and  $\pi|_{\mathcal{V}'_n}$  is a submatrix of  $\tilde{\pi}|_{\mathcal{V}'_n}$ . Here,  $\mathcal{V}'_n := \mathcal{V}_n \setminus \{[s, 0], [t, k]\}$ . Clearly,  $\pi \subseteq \tilde{\pi}$  implies  $m \leq n$ . If  $\pi \subseteq \tilde{\pi}$  and  $\tilde{\pi}$  is obtained by copying rows of  $\pi$ ,  $\tilde{\pi}$  is said to be a *clone* of  $\pi$ .

A vector  $\pi \in \mathbb{R}^{\mathcal{V}_n}$  is said to be *primitive* if all rows of  $\pi$  are different. Each  $\pi \in \mathbb{R}^{\mathcal{V}_n}$  contains a unique primitive subvector  $\pi^{\text{prim}} \subseteq \pi$  such that  $\pi$  is a clone of  $\pi^{\text{prim}}$ . A vector  $\tilde{\pi} \in \mathbb{R}^{\mathcal{V}_n}$  is called a (*minimal*) *primitive representative* of  $\pi$  if  $\tilde{\pi}$  (is lexicographic minimal and) can be obtained by permuting rows of  $\pi^{\text{prim}}$ . The unique minimal primitive representative of  $\pi$  is denoted by  $\pi^{\text{mpr}}$ .

Given  $\pi$ , the way to derive  $\pi^{\text{mpr}}$  can be explained with well-known polyhedral techniques. To obtain  $\pi^{\text{mpr}}$ , we introduced two operations: row permutation and row deletion.

Permuting the rows of  $\pi$  equates to permuting the internal nodes of  $V_n$ . Thus, if  $\tilde{\pi}$  is a permutation of  $\pi$ , then the vectors  $\sigma^\pi$  and  $\sigma^{\tilde{\pi}}$  (as well as  $\sigma^{\pi,+}$  and  $\sigma^{\tilde{\pi},+}$ ) have the same coefficient structure.

**Proposition 2.1.** *Let  $\pi, \tilde{\pi} \in \overline{\Pi}_n$  such that  $\tilde{\pi}$  is a permutation of the rows of  $\pi$ .*

(a)  *$F(\tilde{\pi}, +)$  is a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  if and only if  $F(\pi, +)$  is a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ .*

(b)  *$F(\tilde{\pi})$  is a facet of  $\mathcal{C}^{\leq k}(D_n)$  if and only if  $F(\pi)$  is a facet of  $\mathcal{C}^{\leq k}(D_n)$ .*

□

Next, row deletion, as used here, equates to the deletion of those internal nodes  $i \in V_n$  for which exists another internal node  $j$  with the same coefficient structure:  $\sigma_{ir}^\pi = \sigma_{jr}^\pi$  for all nodes  $r \in V_n \setminus \{s\}$  and  $\sigma_{ri}^\pi = \sigma_{rj}^\pi$  for all nodes  $r \in V_n \setminus \{t\}$ . The reverse operation is known as “lifting by node cloning”. To make use of row deletion and cloning in facet proofs, we need deeper polyhedral insights as for row permutation.

**Lemma 2.2.** *For any  $\pi \in \overline{\Pi}_n$  and any internal node  $i$  of  $V_n$ ,  $\sigma_{si}^{\pi,+} = \sigma_{si}^\pi = \pi_{i1}$ ,  $\sigma_{it}^{\pi,+} = \sigma_{it}^\pi = \pi_{tk} - \pi_{i,k-1}$ ,  $\sigma_{ii}^{\pi,+} = 0$ , and  $\sigma_{ii}^\pi \leq 0$ .* □

For any  $j \in V_n$ , we denote by  $\delta(j) := \delta^{\text{in}}(j) \cup \delta^{\text{out}}(j)$  the set of arcs entering and leaving  $j$ . Moreover, for any  $a \in A_n$ , we denote by  $u^{a,n}$  the  $a$ th unit vector.

**Lemma 2.3.** *Let  $j$  be an internal node of  $V_n$ , and let  $\pi \in \Pi_n$  such that  $F(\pi, +)$  is a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ . Then, there are  $x^1, \dots, x^{2n+1} \in F(\pi, +)$  such that  $x^1|_{\delta(j)}, \dots, x^{2n+1}|_{\delta(j)}$  are linearly independent.*

*Proof.* We adapt the proof of Lemma 7 of Hartmann and Özlük [16].

Consider the following procedure.

1. Set  $p := 1$ ,  $T := \{a \in \delta(j) : \sigma_a^{\pi,+} > 0\}$ , and label all the arcs in  $T$  *unmarked*.
2. Find a point  $x \in F(\pi, +) \cap \mathcal{C}^{\leq k}(D_n)$  such that  $x_a > 0$  for exactly one unmarked arc  $a \in T$ .
3. If no such  $x$  exists, STOP. Otherwise, set  $x^p := x$ ,  $a^p := a$ , and label arc  $a$  *marked*. Set  $p := p + 1$  and go back to 2.

Our claim is that this algorithm stops only if all arcs in  $T$  are *marked*. Suppose not, and let  $\emptyset \neq T^* \subset T$  be the set of arcs labeled *unmarked* at the end of the algorithm. Clearly,  $(j, j) \notin T^*$ , since  $(j, j) \notin T$  by Lemma 2.2. Moreover,  $P \cap \delta(j) \subseteq T^*$  or  $P \cap \delta(j) \subseteq \delta(j) \setminus T^*$  for each  $P \in \mathcal{P}_{s,t}^{\leq k}(D_n)$  with  $\chi^P \in F(\pi, +)$ . Consequently, each  $x \in F(\pi, +)$  satisfies the equation

$$x(T^* \cap \delta^{\text{in}}(j)) - x(T^* \cap \delta^{\text{out}}(j)) = 0.$$

However, none of the inequalities obtained by replacing the equality sign with “ $\leq$ ” or “ $\geq$ ” is valid for  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  unless  $T^* \cap \delta^{\text{in}}(j) = \emptyset$  or  $T^* \cap \delta^{\text{out}}(j) = \emptyset$ . W.l.o.g. assume that  $T^* \cap \delta^{\text{in}}(j) = \emptyset$ . Let  $a \in T^* \cap \delta^{\text{out}}(j)$ . Since  $F(\pi, +)$  is a nontrivial facet by Lemma 2.1 (a2), there is some tight path  $P \in \mathcal{P}_{s,t}^{\leq k}(D_n)$  using  $a$ . By construction,  $\chi^P \in F(\pi, +) \cap \mathcal{C}^{\leq k}(D_n)$  and  $\chi_a^P > 0$  for exactly one unmarked arc  $a \in T$ , a contradiction. Thus,  $T^* = \emptyset$ .

The procedure generates a sequence of points  $x^p \in F \cap \mathcal{C}^{\leq k}(D_n)$ ,  $p = 1, \dots, |T|$  such that  $x_{a^p}^p > 0$  and  $x^p|_{T^p} = 0$ , where  $T^p := T \setminus \{a^1, \dots, a^p\}$ . Next, define  $y^a := x^1 + u^{a,n}$  for each  $a \in \delta(j) \setminus T$ . Clearly, since  $\sigma_a = 0$  for all  $a \in \delta(j) \setminus T$ , it follows that  $y \in F(\pi, +)$ . Furthermore, the points  $x^p|_{\delta(j)}$ ,  $p = 1, \dots, |T|$ ,  $y^a|_{\delta(j)}$ ,  $a \in \delta(j) \setminus T$ , are linearly independent.  $\square$

For any internal node  $j$  of  $V_n$ , we say that a chain  $C = (v_0, v_1, \dots, v_r) \in \mathcal{C}_{s,t}^{\leq k}(D_n)$  has the *loop-path property in node  $j$*  if  $v_{\ell-1} = v_\ell = j$  for some index  $\ell \in \{2, 3, \dots, r-1\}$  and  $v_h \neq j$  for the remaining indices.

**Lemma 2.4.** *Let  $\pi \in \bar{\Pi}_n$  such that  $F(\pi)$  is a nontrivial facet of  $\mathcal{C}^{\leq k}(D_n)$ , and let  $j$  be any internal node of  $V_n$ .*

- (i) *For each  $a = (i, j) \in \delta^{\text{in}}(j) \setminus \{(j, j)\}$ , there exists a tight chain  $C = (v_0, v_1, \dots, v_r) \in \mathcal{C}_{s,t}^{\leq k}(D_n)$  such that  $(i, j) = (v_{\ell-1}, v_\ell)$  and  $(v_\ell, v_{\ell+1}) = (j, j)$  for some  $\ell \in \{1, 2, \dots, r-1\}$ .*
- (ii) *For each  $a = (j, i) \in \delta^{\text{out}}(j) \setminus \{(j, j)\}$ , there exists a tight chain  $C = (v_0, v_1, \dots, v_r) \in \mathcal{C}_{s,t}^{\leq k}(D_n)$  such that  $(j, j) = (v_{\ell-1}, v_\ell)$  and  $(v_\ell, v_{\ell+1}) = (j, i)$  for some  $\ell \in \{1, 2, \dots, r-1\}$ .*

(iii) There is a tight chain  $C \in \mathcal{C}_{s,t}^{\leq k}(D_n)$  visiting  $j$  that does not have the loop-path property in  $j$ .

*Proof.* (i) and (ii) Suppose, for the sake of contradiction, that the set of arcs  $T \subseteq \delta(j) \setminus \{(j, j)\}$ , for which statements (i) or (ii) are not true, is nonempty. Then, the incidence vectors of all tight chains  $C \in \mathcal{C}_{s,t}^{\leq k}(D_n)$  satisfy the equation

$$x(\delta^{\text{out}}(j) \cap T) - x(\delta^{\text{in}}(j) \cap T) = 0. \quad (*)$$

Assume that  $\delta^{\text{out}}(j) \cap T = \emptyset$ . Then,  $F(\pi)$  is contained in the face induced by the inequality  $x(\delta^{\text{in}}(j) \cap T) \geq 0$ , in contradiction to the assumption that  $F(\pi)$  is nontrivial. Similarly, we obtain a contradiction if we assume that  $\delta^{\text{in}}(j) \cap T = \emptyset$ . Thus,  $\delta^{\text{in}}(j) \cap T \neq \emptyset \neq \delta^{\text{out}}(j) \cap T$ . Next assume that  $T = \delta(j) \setminus \{(j, j)\}$ , which implies that  $(*)$  is the flow conservation constraint associated with  $j$ . Then, by definition of  $T$ , no tight chain uses the loop  $(j, j)$ , and hence,  $F$  is contained in the face induced by the nonnegativity constraint  $x_{jj} \geq 0$ . Thus,  $T \subsetneq \delta(j) \setminus \{(j, j)\}$ . However, this means that none of the inequalities obtained by replacing the equality sign in  $(*)$  with “ $\leq$ ” or “ $\geq$ ” is valid, which implies that  $F$  intersects  $\mathcal{C}^{\leq k}(D_n)$  in its relative interior, again a contradiction.

(iii) Assume that each tight chain in  $\mathcal{C}_{s,t}^{\leq k}(D_n)$  that visits  $j$  has the loop-path property. This implies that the incidence vectors of all tight chains in  $\mathcal{C}_{s,t}^{\leq k}(D_n)$  satisfy the equation

$$x(\delta(j) \setminus \{(j, j)\}) - 2x_{jj} = 0. \quad (**)$$

This means that  $F$  is contained in a hyperplane that intersects  $\mathcal{C}^{\leq k}(D_n)$  in its relative interior, a contradiction.  $\square$

Let  $\pi \in \overline{\Pi}_n$  and  $P \in \mathcal{P}_{s,t}(\mathcal{D}_n)$ . Let either  $(F, \sigma) = (F(\pi), \sigma^\pi)$  or  $(F, \sigma) = (F(\pi, +), \sigma^{\pi,+})$ .  $P$  is said to be *tight* w.r.t.  $F$  if  $\varphi(P)$  is tight w.r.t.  $F$ . Let  $P \in \mathcal{P}_{s,t}(\mathcal{D}_n)$  be tight w.r.t.  $F$ , and let  $a = ([i, \ell-1], [j, \ell]) \in P$ . Then, it follows that  $\sigma_{\varphi(a)} = \pi_{j\ell} - \pi_{i, \ell-1}$  if  $a \in \mathcal{A}_n^{ij}$ , and  $\pi_{j\ell} - \pi_{i, \ell-1} = 0$  if  $a \in \mathcal{A}_n^i$ . Consequently, if  $P \in \mathcal{P}_{s,t}(\mathcal{D}_n)$  is a tight path, we always may assume that all arcs of  $P$  are in  $\mathcal{A}_n \setminus \bigcup \mathcal{A}_n^i$ . Let  $P = ([v_0, 0], [v_1, 1], \dots, [v_k, k]) \in \mathcal{P}_{s,t}(\mathcal{D}_n)$  be a tight path. Then, each subpath  $P^{q,r} = ([v_q, q], \dots, [v_r, r])$  satisfies  $\sigma(\varphi(P^{q,r})) = \pi_{v_r, r} - \pi_{v_q, q}$ .

For any  $x \in \mathbb{R}^{A_n}$  and any permutation  $\alpha = (\alpha(s), \alpha(t), \alpha(1), \alpha(2), \dots, \alpha(n))$  of  $V_n$  with  $\alpha(s) = s$  and  $\alpha(t) = t$ , we define  $\alpha(x) \in \mathbb{R}^{A_n}$  by

$$\alpha(x)_{ij} := x_{\alpha^{-1}(i), \alpha^{-1}(j)} \quad \text{for all } (i, j) \in A_n.$$

**Proposition 2.2** (cf. Theorems 8 in [16], 6 in [25]). *Let  $\tilde{\pi} \in \overline{\Pi}_n$  be a clone of  $\pi \in \overline{\Pi}_m$ .*

- (a)  $F(\tilde{\pi}, +)$  is a nontrivial facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  if and only if  $F(\pi, +)$  is a nontrivial facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_m))$ .
- (b)  $F(\tilde{\pi})$  is a nontrivial facet of  $\mathcal{C}^{\leq k}(D_n)$  if and only if  $F(\pi)$  is a nontrivial facet of  $\mathcal{C}^{\leq k}(D_m)$ .

*Proof.* W.l.o.g. we may assume that  $n = m + 1$ ,  $\tilde{\pi}|_{V_m} = \pi$ , and  $\tilde{\pi}_{m+1} = \pi_m$ .

(a) To show the sufficiency, let  $F(\pi, +)$  be a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_m))$ . By Theorem 2.1 (a), the polyhedron  $\text{dmt}(\mathcal{C}^{\leq k}(D_{m+1}))$  is fulldimensional. Hence, we have to show that there are  $|A_{m+1}|$  affinely independent points in  $F(\tilde{\pi}, +)$ .



Since  $\text{dmt}(\mathcal{C}^{\leq k}(D_m))$  is fulldimensional, there are  $|A_m| = |A_{m+1}| - 2m - 3$  affinely independent points  $x^{ij} \in F(\pi, +) \cap \mathbb{Z}^{A_m}$ ; each point  $x^{ij}$  corresponds to an arc  $(i, j) \in A_m$ . Define the points  $\tilde{x}^{ij} \in \mathbb{R}^{A_{m+1}}$  by  $\tilde{x}^{ij}|_{A_m} := x^{ij}$  and  $\tilde{x}^{ij}|_{A_{m+1} \setminus A_m} := 0$  for  $(i, j) \in A_m$ . By construction, the points  $\tilde{x}^{ij}$  with  $(i, j) \in A_m$  are affinely independent, and they are in  $F(\tilde{\pi}, +)$ .

To construct the remaining points, let  $B := A_m \setminus A_{m-1}$ . By Lemma 2.3, we may assume w.l.o.g. that the points  $x^{ij}|_B$ ,  $(i, j) \in B$ , are linearly independent. Define the permutation  $\alpha := (s, t, 1, 2, \dots, m-1, m+1, m)$ . Note that  $\alpha$  is idempotent, that is,  $\alpha^2 = \text{id}$ . It follows that the points

$$\tilde{x}^{ij} := \alpha(\tilde{x}^{\alpha(i), \alpha(j)}) \in F(\tilde{\pi}, +), \quad (i, j) \in \tilde{B},$$

where  $\tilde{B} := (A_{m+1} \setminus A_m) \setminus \{(m, m+1), (m+1, m)\}$ . Moreover, they are linearly independent, and they are affinely independent of the former points. Finally, due to condition (19),  $\sigma_{m, m+1}^{\tilde{\pi}, +} = \sigma_{m+1, m}^{\tilde{\pi}, +} = 0$ . This implies the existence of two further linearly independent points, for instance,  $\tilde{x}^{m, m+1} := \tilde{x}^{s1} + u^{(m, m+1), m+1}$  and  $\tilde{x}^{m+1, m} := \tilde{x}^{s1} + u^{(m+1, m), m+1}$ .

The necessity will be shown by contraposition. Suppose that the face  $F(\pi, +)$  is not a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_m))$ . Then, exists  $\rho \in \bar{\Pi}_m$  such that the face  $F(\rho, +)$  strictly contains  $F(\pi, +)$ . Define  $\tilde{\rho} \in \mathbb{R}^{\mathcal{V}_{m+1}}$  by  $\tilde{\rho}|_{\mathcal{V}_m} := \rho$  and  $\tilde{\rho}_{m+1} := \rho_m$ . We show that  $F(\tilde{\pi}, +) \subsetneq F(\tilde{\rho}, +)$ .

Let  $\tilde{x} \in F(\tilde{\pi}, +)$ . Then,

$$x := \tilde{x}|_{A_m} + \sum_{(i, j) \in B} \tilde{x}_{ij} u^{(\alpha(i), \alpha(j)), m} \in F(\pi, +),$$

which implies  $x \in F(\rho, +)$ . We conclude that

$$\sum_{a \in A_{m+1}} \sigma_a^{\tilde{\rho}, +} \tilde{x}_a = \sum_{a \in A_m} \sigma_a^{\rho, +} x_a = \rho_{tk} = \tilde{\rho}_{tk},$$

since  $\sigma_{m, m+1}^{\tilde{\rho}, +} = \sigma_{m+1, m}^{\tilde{\rho}, +} = 0$ . Consequently,  $\tilde{x} \in F(\tilde{\rho}, +)$ .

Finally, by hypothesis, there is some  $x \in F(\rho, +) \setminus F(\pi, +)$ . Then,  $\tilde{x} \in F(\tilde{\rho}, +) \setminus F(\tilde{\pi}, +)$ , where  $\tilde{x}$  is defined by  $\tilde{x}|_{A_m} := x$  and  $\tilde{x}|_{A_{m+1} \setminus A_m} := 0$ .

(b) To show the sufficiency, let  $F(\pi)$  be a facet of  $\mathcal{C}^{\leq k}(D_m)$ . By Theorem 2.1 (b), we have to construct  $|A_{m+1} - (m+2)|$  affinely independent points in  $F(\tilde{\pi})$ . By the same theorem, there are  $|A_m - (m+1)|$  tight chains  $C^i \in \mathcal{C}_{s, t}^{\leq k}(D_m)$  whose incidence vectors are affinely independent. Clearly, their incidence vectors defined in  $\mathbb{R}^{A_{m+1}}$  remain affinely independent.

Next, let  $B := A_m \setminus A_{m-1}$ . By Lemma 2.4 (i), for each arc  $a = (i, m) \in B \setminus \{(m, m)\}$ , there exists a tight chain  $(v_0^a, v_1^a, \dots, v_{r_a}^a) \in \mathcal{C}_{s, t}^{\leq k}(D_m)$  such that  $(i, m) = (v_{\ell-1}^a, v_{\ell}^a)$  and  $(v_{\ell}^a, v_{\ell+1}^a) = (m, m)$  for some  $\ell \in \{1, 2, \dots, r_a - 1\}$ . Thus,  $C^{(i, m+1)} := (v_0^a, v_1^a, \dots, v_{\ell-1}^a, m+1, v_{\ell+1}^a, \dots, v_{r_a}^a) \in \mathcal{C}_{s, t}^{\leq k}(D_{m+1})$  is tight w.r.t.  $F(\tilde{\pi})$ . Similarly, for each arc  $a = (m, i) \in B \setminus \{(m, m)\}$ , there exists a tight chain  $(v_0^a, v_1^a, \dots, v_{r_a}^a) \in \mathcal{C}_{s, t}^{\leq k}(D_m)$  such that  $(m, m) = (v_{\ell-1}^a, v_{\ell}^a)$  and  $(v_{\ell}^a, v_{\ell+1}^a) = (m, i)$  for some  $\ell \in \{1, 2, \dots, r_a - 1\}$ . Thus,  $C^{(m+1, i)} := (v_0^a, v_1^a, \dots, v_{\ell-1}^a, m+1, v_{\ell+1}^a, \dots, v_{r_a}^a) \in \mathcal{C}_{s, t}^{\leq k}(D_{m+1})$  is tight w.r.t.  $F(\tilde{\pi})$ . Furthermore, the chain  $C^{(m+1, m+1)} := (s, m+1, m+1, v_3^a, \dots, v_{r_a}^a)$  for  $a = (s, m)$  is tight. Clearly, the incidence vectors of all chains  $C^a$ ,  $a \in \delta(m+1) \setminus \{(m, m+1), (m+1, m)\}$

are linearly independent, and they are linearly independent from the incidence vectors of the chains  $C^i$ ,  $i = 1, 2, \dots, |A_m - (m + 1)|$ .

To finish this part of the proof, we have to give one further tight chain whose incidence vector is linearly independent of the previously constructed vectors. By Lemma 2.4 (iii), there exists a chain  $C = (v_0, v_1, \dots, v_r) \in \mathcal{C}_{s,t}^{\leq k}(D_m)$ , which is tight w.r.t.  $F(\pi)$ , visits node  $m$ , and does not has the loop-path property in  $m$ . Let  $J \subseteq \{1, 2, \dots, r - 1\}$  be the set of indices  $\ell$  with  $v_\ell = m$ . Then, at least one of the following cases holds:

( $\alpha$ )  $g \in J$ ,  $g - 1, g + 1 \notin J$  for some  $g \in \{1, 2, \dots, r - 1\}$ ,

( $\beta$ )  $g, g + 1, g + 2 \in J$  for some  $g \in \{1, 2, \dots, r - 3\}$ .

In both cases, we set  $C' := (v_0, v_1, \dots, v_{g-1}, m + 1, v_{g+1}, \dots, v_r)$ . Then,  $C'$  is a tight chain, which in case ( $\alpha$ ) does not use the arcs  $(m, m + 1)$  and  $(m + 1, m)$ , and in case ( $\beta$ ) uses only the arcs  $(m, m + 1)$  and  $(m + 1, m)$  among all arcs incident with  $m + 1$ . Thus, in either case, the incidence vector of  $C'$  is linearly independent of the previous vectors.

To show the necessity, we assume that  $F(\tilde{\pi})$  is a facet of  $\mathcal{C}^{\leq k}(D_{m+1})$ . Define the mapping  $\psi : V_{m+1} \rightarrow V_m$  by

$$\psi(v) := \begin{cases} v & \text{if } v \in V_m \\ m & v = m + 1. \end{cases}$$

By definition of  $\tilde{\pi}$  and  $\sigma^{\tilde{\pi}}$ ,  $\sigma_{\psi(v), \psi(w)}^{\tilde{\pi}} = \sigma_{vw}^{\tilde{\pi}}$  for all  $(v, w) \in A_{m+1}$ . Since  $F(\tilde{\pi})$  is a nontrivial facet, for any  $(i, j) \in A_m$ , there exists a tight chain  $C = (v_0, v_1, \dots, v_r) \in \mathcal{C}_{s,t}^{\leq k}(D_{m+1})$  w.r.t.  $F(\tilde{\pi})$  containing  $(i, j)$ . Then, the chain  $C' := (\psi(v_0), \psi(v_1), \dots, \psi(v_r)) \in \mathcal{C}_{s,t}^{\leq k}(D_m)$  also contains  $(i, j)$  and is tight w.r.t.  $F(\pi)$ . Thus,  $F(\pi)$  is not induced by a nonnegativity constraint.

Suppose now, for the sake of contradiction, that  $F(\pi)$  is not a facet of  $\mathcal{C}^{\leq k}(D_m)$ . Since  $F(\pi)$  is nontrivial, there exists  $\rho \in \overline{\Pi}_m$  such that  $F(\pi) \subsetneq F(\rho)$ . Define  $\tilde{\rho} \in \mathbb{R}^{V_{m+1}}$  by  $\tilde{\rho}|_{V_m} := \rho$  and  $\tilde{\rho}_{m+1} := \rho_m$ . It follows immediately that  $\sigma_{\psi(v), \psi(w)}^{\tilde{\rho}} = \sigma_{vw}^{\tilde{\rho}}$  for all  $(v, w) \in A_{m+1}$ . We show that  $F(\tilde{\pi}) \subsetneq F(\tilde{\rho})$ .

By hypothesis, there is some  $x \in F(\rho) \setminus F(\pi)$ . Then,  $\tilde{x} \in F(\tilde{\rho}) \setminus F(\tilde{\pi})$ , where  $\tilde{x}$  is defined by  $\tilde{x}|_{A_m} := x$  and  $\tilde{x}|_{A_{m+1} \setminus A_m} := 0$ .

Finally, let  $\tilde{x} \in F(\tilde{\pi})$ . W.l.o.g. we may assume that  $\tilde{x}$  is the incidence vector of a tight chain  $C = (v_0, v_1, \dots, v_r) \in \mathcal{C}_{s,t}^{\leq k}(D_{m+1})$ . As we have already mentioned, then  $C' := (\psi(v_0), \psi(v_1), \dots, \psi(v_r)) \in \mathcal{C}_{s,t}^{\leq k}(D_m)$  is tight w.r.t.  $F(\pi)$ , and since  $F(\pi) \subseteq F(\rho)$ , it is also tight w.r.t. to  $F(\rho)$ . This, in turn, implies that  $C$  is tight w.r.t. to  $F(\tilde{\rho})$ , which means that  $\tilde{x} \in F(\tilde{\rho})$ .  $\square$

By Propositions 2.1 and 2.2, to show that a class  $\mathcal{K}$  of nontrivial inequalities define facets of  $\mathcal{C}^{\leq k}(D)$  ( $\text{dmt}(\mathcal{C}^{\leq k}(D))$ ), it suffices to prove that its minimal primitive member induces a facet.

### 3 Main results

In this section, we use the result of Dahl, Foldnes, and Gouveia [6] that  $r$ -jump inequalities (13) define facets of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  in two respects.

First, identifying a certain structure of the extreme rays of the projection cone in terms of  $\pi$ -vectors corresponding to  $r$ -jump inequalities (13), we derive a rich class of facet defining inequalities for  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  that contains these inequalities. We call the resulting inequalities *shifting inequalities*.

Second, we show that the  $\pi$ -vectors corresponding to shifting inequalities give also facet defining inequalities

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq \pi_{tk} - \pi_{s0}$$

for  $\mathcal{C}^{\leq k}(D_n)$ . We also give sufficient conditions for these inequalities to be facet defining for  $\mathcal{P}^{\leq k}(D_n)$ .

Finally, we present a complete characterization of all  $0/\pm 1$ -facet defining inequalities for  $\mathcal{C}^{\leq k}(D_n)$  and one of all  $0/1$ -facet defining inequalities for its dominant.

### 3.1 Shifting inequalities

In what follows, we exploit the special diagonal structure of lexicographic minimal primitive members of  $\pi$ -vectors corresponding to  $r$ -jump inequalities (13) to derive a generalization of these inequalities.

Given a partition  $\{S_p : p = 0, 1, \dots, k+r\}$  of  $V_n$ , where  $r \in \mathbb{N}$ ,  $1 \leq r \leq n-k$ ,  $S_0 = \{s\}$ , and  $S_{k+r} = \{t\}$ . Consider the  $r$ -jump inequality (13) associated with this partition:

$$\sum_{p=0}^{k+r-2} \sum_{q=p+2}^{k+r} \alpha_{pq} x((S_p : S_q)) \geq r,$$

where for  $p < q$ ,  $\alpha_{pq} = \min\{q - p - 1, r\}$ . The vector  $\pi \in \Pi_n$ , such that

$$\sum_{a \in A_n} \sigma_a^{\pi,+} x_a \geq \pi_{tk}$$

is inequality (13), is called *jump vector*. It has the following configuration:

$$\begin{aligned} \pi_{s0} &= 0, \\ \pi_{tk} &= r, \\ \pi_{i\ell} &= \begin{cases} 0 & \text{if } p - \ell \leq 0, \\ r & \text{if } p - \ell \geq r, \\ p - \ell & \text{otherwise,} \end{cases} \\ &= \min\{r, \max\{p - \ell, 0\}\}, \\ & \quad i \in S_p, p = 1, \dots, k+r-1, \ell = 1, \dots, k-1. \end{aligned}$$

For example, the lexicographic minimal primitive vector for  $k = 6$  and  $r = 8$  is the vector  $\pi'$  given in Figure 2 (a).

The vector  $\pi'$  has two obvious properties. First, at the top and at the bottom it has triangle structure. Second, the intermediate positions have diagonal structure with entries  $p = 1, 2, \dots, 7 = r - 1$ . All intermediate diagonals have width one. As it turns out, one can widen these diagonals and obtains new facets of the dominant of the hop constrained chain polytope. So  $\pi'$  is primitive in a horizontal and a diagonal sense. Copying rows of  $\pi'$ , we obtain again an



is called *shifting inequality*. The class of shifting inequalities contains that of  $r$ -jump inequalities (13).

In the following theorem we show that shifting inequalities induce facets of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ . However, they do not induce facets of  $\mathcal{C}^{\leq k}(D_n)$ . Dahl and Gouveia [7] deal with the problem to strengthen  $r$ -jump inequalities (13) for  $\mathcal{P}^{\leq k}(D_n)$ . For  $r = 1$  and  $r = 2$ , they obtain stronger inequalities by decreasing some coefficients associated with partition and backward arcs. Here, for any partition  $\{S_i : i = 0, 1, \dots, m\}$  of  $V_n$ ,  $a \in A_n$  is called a *forward partition*, or *partition arc* if  $a \in (S_p : S_q)$  for some  $p < q$ ,  $p = q$ , or  $p > q$ , respectively. For the 1-jump inequality associated with the partition  $\{S_0 = \{s\}, S_1, S_2, \dots, S_k, S_{k+1} = \{t\}\}$ , Dahl and Gouveia [7] derive the inequality

$$\sum_{p=0}^{k-1} \sum_{q=p+2}^{k+1} x((S_p : S_q)) - x((S_{k-1} \cup S_k : S_1 \cup S_2)) \geq 1. \quad (\text{j1})$$

They also discuss the case  $r = 2$  on the polytope  $P_{s,t\text{-path}}^{\leq 4}(D_n)$ . Given the 2-jump inequality defined on the partition  $\{S_0 = \{s\}, S_1, S_2, S_3, S_4, S_5, S_6 = \{t\}\}$ . By decreasing the coefficients associated with some backward arcs and all arcs in  $A_n(S_3)$ , they derive the inequality

$$\begin{aligned} & \sum_{p=0}^5 \sum_{q=p+1}^6 \alpha_{pq} x((S_p : S_q)) \\ & - x(A_n(S_3)) - 2x((S_4 \cup S_5 : S_1 \cup S_2)) \\ & - x((S_3 : S_2 \cup S_1)) - x((S_4 \cup S_5 : S_3)) \geq 2. \end{aligned} \quad (\text{j2})$$

Moreover, they show that these inequalities are equivalent to the inequalities (11):

$$\sum_{i \in I} x_{si} + \sum_{j \in J} x_{jt} - \sum_{i \in I, j \in J} x_{ij} \geq 0, \quad I, J \subseteq V_n \setminus \{s, t\}.$$

With a view to the completeness result for the 4-hop constrained walk polytope  $\mathcal{W}^{\leq 4}(D_n)$  (see Fact 3), Dahl, Foldnes, and Gouveia [6] point to the interesting fact that to describe  $\text{dmt}(\mathcal{W}^{\leq 4}(D_n))$ , one needs the whole class of  $r$ -jump inequalities, while to describe  $\mathcal{W}^{\leq 4}(D_n)$ , one only needs a suitable class of inequalities derived from 2-jump inequalities.

All these phenomenons can be explained using the DP-approach. Given any vector  $\pi \in \Pi_n$ , we obtain an equivalent vector  $\text{dif}(\pi) \in \Pi_n^0$  w.r.t.  $\mathcal{C}^{\leq k}(D_n)$  by setting  $\text{dif}(\pi)_{s0} := \text{dif}(\pi)_{tk} := 0$  and  $\text{dif}(\pi)_i := \pi_i - \pi_{i,k-1} \mathbb{1}^T$ . Observe that if  $\pi \in \Pi_n$  is a shifting vector, then the first row of  $(\text{dif}(\rho))^{\text{mPr}}$  is the zero vector, that is,  $(\text{dif}(\rho))_1^{\text{mPr}} = 0^T$ . Now, any  $\pi \in \Pi_n^0$  is called a *difference shifting vector* if there exists a shifting vector  $\rho \in \Pi_m$  for some  $m$  such that  $\pi^{\text{mPr}} = (\text{dif}(\rho))^{\text{mPr}}$  or  $\pi^{\text{mPr}} = X$ , where  $X$  is the vector obtained from  $(\text{dif}(\rho))^{\text{mPr}}$  by deleting the first row. The inequality

$$\sum_{a \in A_n} \sigma_a^\pi x_a \geq 0$$

corresponding to a difference shifting vector  $\pi$  is called *difference shifting inequality*.

For  $k = 4$ , consider now the 6-jump inequality associated with any partition

$$\{S_0 = \{s\}, S_1, S_2, \dots, S_9, S_{10} = \{t\}\}$$

$$\begin{array}{ccc}
 \begin{pmatrix} 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \\ 6 & 5 & 4 \\ 6 & 6 & 5 \\ 6 & 6 & 6 \\ 6 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 \end{pmatrix} \\
 \text{(a)} & \text{(b)} & \text{(c)}
 \end{array}$$

Figure 3: Equivalent jump vectors in  $\Pi_9$  and their minimal primitive representative in  $\Pi_4^0$  for  $k = 4$ .

of  $V_n$  and the corresponding vector  $\rho \in \Pi_n$ , that is, this 6-jump inequality is the inequality  $\sum_{a \in A_n} \sigma_a^{\rho,+} x_a \geq 6$ . For simplicity, we assume that  $n = 9$  and  $S_i = \{i\}$  for  $i = 1, 2, \dots, 9$ . Then,  $\rho = \rho^{\text{mpr}} \in \Pi_9$  is the vector shown in Figure 3 (a). To strengthen the 6-jump inequality w.r.t. to  $P_{s,t\text{-chain}}^{\leq 4}(D_9)$ ,  $P_{s,t\text{-walk}}^{\leq 4}(D_9)$ , or  $P_{s,t\text{-path}}^{\leq 4}(D_9)$ , one only needs to write down the inequality  $\sum_{a \in A_9} \sigma_a^\rho x_a \geq 6$ . Then, making use of the fact that any shifting of the rows of  $\rho$  by a constant results in an equivalent inequality, we see that the vector given in Figure 3 (b), say  $\rho'$ , implies an equivalent inequality  $\sum_{a \in A_9} \sigma_a^{\rho'} x_a \geq 2$ . This inequality is of the form (j2). In Figure 3 (c) is shown the vector  $\rho'' := (\text{dif}(\rho))^{\text{mpr}} = (\text{dif}(\rho'))^{\text{mpr}} \in \Pi_4^0$ . From the following results we conclude that the inequalities  $\sum_{a \in A_9} \sigma_a^\rho x_a \geq 6$  and  $\sum_{a \in A_9} \sigma_a^{\rho'} x_a \geq 2$  induce facets of  $P_{s,t\text{-chain}}^{\leq 4}(D_9)$ ,  $P_{s,t\text{-walk}}^{\leq 4}(D_9)$ , and  $P_{s,t\text{-path}}^{\leq 4}(D_9)$ , while the inequality  $\sum_{a \in A_4} \sigma_a^{\rho''} x_a \geq 0$  only induces facets of  $P_{s,t\text{-chain}}^{\leq 4}(D_4)$  and  $P_{s,t\text{-walk}}^{\leq 4}(D_4)$ .

Observe that not only  $(\text{dif}(\rho))^{\text{mpr}}, ((\text{dif}(\rho'))^{\text{mpr}}) \subseteq \rho''$ , but  $(\text{dif}(\pi))^{\text{mpr}} \subseteq \rho''$  for every shifting vector  $\pi \in \Pi_n$ , for  $k = 4$ . But observe also that this is only a special case. For  $k \geq 5$ , the difference minimal primitive members of jump vectors are only a proper subset of that of shifting vectors.

Given  $\pi \in \Pi_n$ , one can associate a partition of  $V_n$  as follows. Let  $m$  be the row size of  $\pi^{\text{mpr}}$ . Then, define a partition  $\{S_i^\pi : i = 0, 1, \dots, m+1\}$  of  $V_n$  by  $S_0^\pi := \{s\}$ ,  $S_i^\pi := \{v \in V_n : \pi_v = \pi_i^{\text{mpr}}\}$ ,  $i = 1, 2, \dots, m$ , and  $S_{m+1}^\pi := \{t\}$ . We call this partition of  $V_n$  the *mpr-partition* of  $V_n$  w.r.t.  $\pi$ .

For any  $\tau \in \mathbb{R}^{A_n}$  and any  $c \in \mathbb{R}$ , we denote by  $A_n(\tau, c)$  the set of arcs  $a \in A_n$  with  $\tau_a = c$ . Clearly, the set  $\{\emptyset \neq A_n(\tau, c) : c \in \mathbb{R}\}$  defines a partition of  $A_n$ .

### Theorem 3.1.

(a) *Shifting inequalities induce facets of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ .*

(b) *Difference shifting inequalities induce facets of  $\mathcal{C}^{\leq k}(D_n)$ .*

(c) Let  $\pi \in \Pi_n$  be a shifting vector, and let  $\{S_i^\pi : i = 0, 1, 2, \dots, p\}$  be the mpr-partition of  $\pi$ . Then, the inequality  $\sum_{a \in A_n} \sigma_a^\pi x_a \geq \pi_{tk}$  induces a facet of  $\mathcal{P}^{\leq k}(D_n)$  if  $|S_i^\pi| \geq 2$  for  $i = 1, 2, \dots, p-1$ .

*Proof.* We prove (b) and indicate the necessary modifications for (a) and (c).

Let  $\rho \in \Pi_n^0$  be a difference shifting vector. By definition of  $\rho$ , there is some shifting vector  $\pi \in \Pi_m$  such that either (1)  $\rho^{\text{mpr}} = (\text{dif}(\pi))^{\text{mpr}}$  or (2)  $\rho^{\text{mpr}} = X$ , where  $X$  is the vector obtained from  $(\text{dif}(\pi))^{\text{mpr}}$  by removing its first row. Furthermore, by definition of  $\pi$ ,  $\pi^{\text{mpr}} = \pi^{(v)}$  for some  $v \in S[r, m, k]$ .

(1) By Proposition 2.2 (b),  $F(\rho)$  is a facet of  $\mathcal{C}^{\leq k}(D_n)$  if and only if  $F(\pi)$  is one of  $\mathcal{C}^{\leq k}(D_m)$ . W.l.o.g. we may assume that  $\pi$  is lexicographic minimal. We construct  $|A_m| - (m+1)$  tight chains in  $\mathcal{C}_{s,t}^{\leq k}(D_m)$  whose incidence vectors are linearly independent.

For each arc  $a = ([i, \ell-1], [j, \ell]) \in A_m \setminus \bigcup_{i=1}^m \mathcal{A}_m^i$ , we introduce a path

$$P^a := ([s, 0], [v_1, 1], [v_2, 2] \dots, [v_{k-1}, k-1], [t, k])$$

contained in  $\mathcal{P}_{s,t}(\mathcal{D}'_m \cup \{a\})$ , where

$$\mathcal{D}'_m := \mathcal{D}_m \setminus \bigcup_{i=1}^n \mathcal{A}^{ii}$$

and

$$v_h := \begin{cases} \max\{1, i - \ell + 1 + h\}, & h = 1, \dots, \ell - 1 \\ \min\{m, j - \ell + h\}, & h = \ell, \dots, k - 1. \end{cases}$$

In particular,  $v_{\ell-1} = i$ ,  $v_\ell = j$ , and the chain  $\varphi(P^a)$  does not use loops with the exception of the loop  $(i, i)$  in case that  $i = j$ . Moreover,  $\pi_{v_1 1} - \pi_{s 0} = \sigma_{sv_1}^\pi$ ,  $\pi_{tk} - \pi_{v_{k-1}, k-1} = \sigma_{v_{k-1}t}^\pi$ , and  $\pi_{v_h h} = \pi_{v_{h-1}, h-1}$  for  $h = 2, \dots, \ell-1, \ell+1, \dots, k-1$ . This means that  $P^a$  is tight if and only if  $\pi_{j\ell} - \pi_{i, \ell-1} = \sigma_{ij}^\pi$ . For  $(i, j) \in A_m$ , denote by  $\mathcal{P}_{s,t}^{ij}(\mathcal{D}_m)$  the set of tight paths  $P^a \in \mathcal{P}_{s,t}(\mathcal{D}_m)$  with  $\varphi(a) = (i, j)$ . Then, by definition of  $\sigma^\pi$ ,  $\mathcal{P}_{s,t}^{ij}(\mathcal{D}_m) \neq \emptyset$  for all  $(i, j) \in A_m$ . An illustration for the definition of paths  $P^a$  is given in Figure 4. Both depicted paths are tight.

Let  $\{Y^p : p = 0, 1, 2, 3, 4\}$  be a partition of  $A_m$ , where

$$Y^0 := \{(s, 1), (1, 2), \dots, (m-1, m), (m, t)\},$$

$$Y^1 := (\delta^{\text{out}}(s) \cup \delta^{\text{in}}(t)) \cap A_m(\sigma^\pi, r),$$

$$Y^2 := \{(i, j) \in A_m : 1 \leq j \leq i \leq m\},$$

and  $Y^3$  is a subset of internal arcs of  $A_m$ .  $Y^3$  and  $Y^4$  will be specified later. Then, for each  $(i, j) \in Y^p$ ,  $p = 1, 2, 3$ , let  $C^{ij} := \varphi(P)$  for any  $P \in \mathcal{P}_{s,t}^{ij}(\mathcal{D}_m)$ . By construction,  $x^{ij} := \chi^{C^{ij}} \in F(\pi)$ .

In the remainder of this proof, we construct  $|Y^4|$  points  $\tilde{x}^q$  with  $\tilde{x}_{Y^2 \cup Y^3}^q = 0$  such that their restrictions to  $Y^4$ ,  $\tilde{x}_{Y^4}^q$ , are linearly independent. Then, the points  $x^{ij}$ ,  $(i, j) \in Y^p$ ,  $p = 1, 2, 3$ ,  $\tilde{x}^q$ ,  $q = 1, \dots, |Y^4|$ , are linearly independent: Let

$$\sum_{p=1}^3 \sum_{(i,j) \in Y^p} \lambda_{ij} x^{ij} + \sum_{q=1}^{|Y^4|} \lambda_q \tilde{x}^q = 0.$$





Since each point  $x^{ij}$ ,  $(i, j) \in Y^2 \cup Y^3$ , is the only point with nonzero entry at position  $(i, j)$ , it follows that  $\lambda_{gh} = 0$  for all arcs  $(g, h) \in Y^2 \cup Y^3$ . Next, since  $x_{Y^4}^{ij} = 0$  for all  $(i, j) \in Y^1$ , and the points  $\tilde{x}_{Y^4}^q$ ,  $q = 1, \dots, |Y^4|$ , are linearly independent, we see that  $\lambda_q = 0$  for  $q \in \{1, \dots, |Y^4|\}$ . Finally, among all points  $x^{ij}$ ,  $(i, j) \in Y^1$ ,  $x^{gh}$  is the only one with a nonzero entry at position  $(g, h)$ . Thus,  $\lambda_{gh} = 0$  for all  $(g, h) \in Y^1$ . This implies that the  $|A_m| - (m + 1)$  constructed points are linearly independent.

To complete the proof, we have to specify  $Y^4$  and the points  $\tilde{x}^q$ . Since  $Y^3$  is a subset of the internal arcs of  $A_m$ ,  $Y^1 = (\delta^{\text{out}}(s) \cup \delta^{\text{in}}(t)) \cap A_m(\sigma^\pi, r)$ , and  $(i, j) \in A_m(\sigma^\pi, 0)$  if  $\sigma_{ij}^\pi = 0$  for  $(i, j) \in \delta^{\text{out}}(s) \cup \delta^{\text{in}}(t)$ , it follows that

$$\begin{aligned} Z &:= Y^4 \cap (\delta^{\text{out}}(s) \cup \delta^{\text{in}}(t)) \\ &= (\delta^{\text{out}}(s) \cup \delta^{\text{in}}(t)) \setminus (Y^1 \cup A_m(\sigma^\pi, 0)) \\ &= \bigcup_{p=1}^{r-1} [(\delta^{\text{out}}(s) \cup \delta^{\text{in}}(t)) \cap A_m(\sigma^\pi, p)]. \end{aligned}$$

For each  $(s, i) \in Z$  and each  $(j, t) \in Z$ , we see that  $\mathcal{P}_{s,t}^{si}(\mathcal{D}_m) = \{P^{([s,0],[i,1])}\}$  and  $\mathcal{P}_{s,t}^{jt}(\mathcal{D}_m) = \{P^{([j,k-1],[t,k])}\}$ , respectively. Moreover,

$$\begin{aligned} P^{([s,0],[i,1])} &= ([s, 0], [i, 1], \dots, [i+k-2, k-1], [t, k]) \\ &= P^{([i+k-2, k-1],[t,k])}. \end{aligned}$$

For each  $p \in \{0, \dots, r\}$ , the node set

$$V_m(s, p) := \{i \in V_m : (s, i) \in \delta^{\text{out}}(s) \cap A_m(\sigma^\pi, p)\}$$

is of the form

$$V_m(s, p) = \{i_p, i_p + 1, \dots, i_p + m_p - 1\},$$

for some node  $i_p \in V_m$ , where

$$m_p := |\delta^{\text{out}}(s) \cap A_m(\sigma^\pi, p)|.$$

For  $p \in \{1, \dots, r-1\}$  consider now the following  $2m_p - 1$  chains:

$$\begin{aligned} \tilde{C}^{p,1+2j} &:= \varphi\left(P^{([s,0],[i_p+j,1])}\right), & j = 0, 1, \dots, m_p - 1, \\ \tilde{C}^{p,2j} &:= \varphi(\bar{P}^{p,j}), & j = 1, 2, \dots, m_p - 1, \end{aligned}$$

where

$$\begin{aligned} \bar{P}^{p,j} &:= P^{([s,0],[i_p+j-1,1])} \\ &\setminus \{([s, 0], [i_p + j - 1, 1]), ([i_p + j - 1, 1], [i_p + j, 2])\} \\ &\cup \{([s, 0], [i_p + j, 1]), ([i_p + j, 1], [i_p + j, 2])\}. \end{aligned}$$

All these chains are tight w.r.t.  $F(\pi)$ , and they do not use any arc in  $Y$ . Consequently,  $\chi^{\tilde{C}^{p,j}} \in F(\pi)$  and  $\chi_Y^{\tilde{C}^{p,j}} = 0$  for  $p = 1, \dots, r-1$ ,  $j = 1, 2, \dots, 2m_p - 1$ .

Finally, we introduce for each  $p \in \{1, \dots, r-1\}$  an internal arc  $a_p \in A_m$  to be in  $Y^4$ , which completes the specification of  $Y^4$ . This means,

$$Y^4 = Z \cup \{a_p : p = 1, \dots, r-1\}.$$

The specification of  $a_p$  starts with the node  $[i_{p+1} + 1, 2]$ . By definition of  $\pi$ ,  $\pi_{i_{p+1}+1,2} = p+1$  and  $\pi_{i_{p+1}+1,\ell} = p$  for  $\ell = 3, \dots, j := \min\{m_p + 2, k - 1\}$ . Then, for  $h := i_p + j - 3$ , we have the following configuration:

$$\begin{aligned} \pi_{h1} &= p, & \pi_{h,j-1} &= p - 1 \\ \pi_{i_{p+1}+1,2} &= p + 1, & \pi_{i_{p+1}+1,j} &= p, \end{aligned}$$

which implies  $\sigma_{a_p}^\pi \geq 1$ , where  $a_p := (h, i_{p+1} + 1)$ . On the other hand,  $\pi_{i_{p+1}+1,\ell} - \pi_{h,\ell-1} \leq 1$  for  $\ell = 2, \dots, k - 1$ . Hence,  $\sigma_{a_p}^\pi = 1$ , and thus, the chains

$$\tilde{C}^{p,0} := \varphi \left( P^{([h,j-1],[i_{p+1}+1,j])} \right)$$

and

$$\tilde{C}^{p,2m_p} := \varphi \left( P^{([h,1],[i_{p+1}+1,2])} \right)$$

are tight w.r.t.  $F(\pi)$ . Moreover, they do not use any arc in  $Y^2 \cup Y^3$ . Figure 5 gives an illustration of  $Y^4$ .

We conclude that  $\tilde{x}^{p,j} := \chi^{\tilde{C}^{p,j}} \in F(\pi)$  and  $\tilde{x}_Y^{p,j} = 0$  for  $p = 1, 2, \dots, r - 1$ ,  $j = 0, 1, \dots, 2m_p$ . Moreover, let  $H$  be the matrix whose columns are the vectors  $\tilde{x}^{p,j}$ . Then, an appropriate rearrangement of the restriction of  $H$  to the row index subset  $Y^4$  is a matrix of the form shown in Figure 6. By the following lemma, this matrix has full rank, which implies that the points  $\tilde{x}_{Y^4}^{p,j}$  are linearly independent.

For statement (a), the above proof can be adapted as follows. For any arc  $(i, j) \in A$ , not all paths in  $\mathcal{P}_{s,t}^{ij}(\mathcal{D}_m)$  are tight w.r.t.  $F(\pi, +)$ . Hence, we redefine  $\mathcal{P}_{s,t}^{ij}(\mathcal{D}_m)$  to be the set of all paths  $P^a$ , with  $a \in \mathcal{A}_{m,m}^{ij}$ , which are tight w.r.t. to  $F(\pi, +)$ . It follows that  $\mathcal{P}_{s,t}^{ij}(\mathcal{D}_m) \neq \emptyset$  for all arcs  $(i, j) \in A_m$  with  $i > j$ . For the construction of  $|A_m|$  affinely independent points  $x \in F(\pi, +)$ , we consider the points  $\tilde{x}^{ij} := x^{1t} + u^{ij}$  for  $(i, j) \in Y^0 \cup Y^2$ , where  $u^{ij}$  denotes the  $ij$ th unit vector, the points  $x^{ij}$  for  $(i, j) \in Y^1$ , and the points  $\tilde{x}^{p,j}$  for  $p = 1, 2, \dots, r - 1$ ,  $j = 0, 1, \dots, 2m_p$ . Notice that  $\tilde{x}^{ij} \in F(\pi, +)$  for  $(i, j) \in Y^0 \cup Y^2$ , since  $\sigma_{ij}^{\pi,+} = 0$ .

(2) Let  $\rho^{\text{mpr}} = X$ . Recall that  $X$  is the vector obtained by removing the first row of  $(\text{dif}(\pi))^{\text{mpr}}$ , while  $\pi^{\text{mpr}} = \pi^{(v)}$  for some  $v \in S[r, m, k]$ .

First, let  $k = 4$ . Then,  $\rho^{\text{mpr}} = X$  is the vector obtained by deleting the first or the first and the last row of the vector shown in Figure 3 (c). In either case one easily verifies that  $F(\rho)$  is a facet of  $\mathcal{C}^{\leq k}(D_n)$  using Proposition 2.2 (b).

Next, let  $k \geq 5$ . W.l.o.g. we may assume that  $\pi = \pi^{\text{mpr}}$  and that the first and last row of  $\pi$  are the only rows which are multiples of the vector of all ones. The vector that results from deleting these both rows from  $\pi$  shall be denoted by  $\vartheta$ . Then, it follows that  $(\text{dif}(\vartheta))^{\text{mpr}} = X$ . This means that  $F(\vartheta)$  is a facet of  $\mathcal{C}^{\leq k}(D_{m-2})$  if and only if  $F(\rho)$  is one of  $\mathcal{C}^{\leq k}(D_n)$ , by Proposition 2.2 (b). In what follows, we argue on the subgraph  $\hat{D} = (\hat{V}, \hat{A})$  of  $D_m$  induced by the node set  $V_m \setminus \{1, m\}$ . The DP-graph associated with  $\hat{D}$  will be denoted by  $\hat{\mathcal{D}} = (\hat{\mathcal{V}}, \hat{\mathcal{A}})$ . Moreover,  $\vartheta$  will be perceived as a subvector of  $\pi$  with row indices  $2, 3, \dots, m - 1$ . By construction,  $\vartheta_2 = (1, 0, 0, \dots, 0)$  and  $\vartheta_{m-1} = (r, r, \dots, r, r - 1)$ .

The proof of part (1) can be modified as follows. First, consider for every arc  $(s, i)$  in

$$\hat{Y}^s := \delta^{\text{out}}(s) \cap \hat{A}(\sigma^\vartheta, r)$$

the chain  $\hat{C}^{si} := (s, i, i + 1, \dots, m - 1, 2, t)$  and for every arc  $(j, t)$  in

$$\hat{Y}^t := \delta^{\text{in}}(t) \cap \hat{A}(\sigma^\vartheta, r),$$

with  $j \neq 2$ , the chain  $\hat{C}^{jt} := (s, m-1, 2, 3, \dots, j, t)$ . Then, we derive  $|\hat{Y}^s| + |\hat{Y}^t| - 1 = 2(k-2) - 1$  linearly independent points  $\hat{x}^{ij} := \chi^{\hat{C}^{ij}} \in F(\vartheta)$  for  $(i, j) \in (\hat{Y}^s \cup \hat{Y}^t) \setminus \{2, t\}$ .

Next, for each arc

$$a = ([i, \ell-1], [j, \ell]) \in \hat{\mathcal{A}} \setminus \bigcup_{i=2}^{m-1} \hat{\mathcal{A}}^i$$

we associate a path

$$\hat{P}^a := ([s, 0], [v_1, 1], [v_2, 2], \dots, [v_{k-1}, k-1], [t, k])$$

in  $\mathcal{P}_{s,t}(\hat{\mathcal{D}})$ , where

$$v_h := ((i - (\ell - 1) + h - 2) \bmod (m - 2)) + 2$$

for  $h = 1, \dots, \ell - 1$  and

$$v_h := ((j - \ell + h - 2) \bmod (m - 2)) + 2$$

for  $h = \ell, \dots, k - 1$ . Then, we can make the same observations on  $\hat{P}^a$  as on its counterpart  $P^a$  in case (1):  $v_{\ell-1} = i$ ,  $v_\ell = j$ , the chain  $\varphi(\hat{P}^a)$  does not use loops with the exception of the loop  $(i, i)$  in case that  $i = j$ , and  $\hat{P}^a$  is tight if and only if  $\pi_{j\ell} - \pi_{i, \ell-1} = \sigma_{ij}^j$ . For  $(i, j) \in \hat{\mathcal{A}}$ , denote by  $\hat{\mathcal{P}}_{s,t}^{ij}(\hat{\mathcal{D}})$  the set of tight paths  $\hat{P}^a \in \mathcal{P}_{s,t}(\hat{\mathcal{D}})$  with  $\varphi(a) = (i, j)$ . Then,  $\hat{\mathcal{P}}_{s,t}^{ij}(\hat{\mathcal{D}}) \neq \emptyset$  for all  $(i, j) \in \hat{\mathcal{A}}$ .

Next, we consider the partition  $\{\hat{Y}^p : p = 0, 1, 2, 3, 4\}$  of  $\hat{\mathcal{A}}$  defined as follows:

$$\begin{aligned} \hat{Y}^0 &:= \{(m-1, 2), (2, 3), (3, 4), \dots, (m-2, m-1)\}, \\ \hat{Y}^1 &:= \hat{Y}^s \cup \hat{Y}^t, \\ \hat{Y}^2 &:= \{(i, j) \in \hat{\mathcal{A}} : 2 \leq j \leq i \leq m-1\} \\ &\quad \setminus \{(m-1, 2)\}, \\ \hat{Y}^3 &:= \hat{\mathcal{A}} \setminus (\hat{Y}^0 \cup \hat{Y}^1 \cup \hat{Y}^2 \cup \hat{Y}^4), \\ \hat{Y}^4 &:= Y^4. \end{aligned}$$

Then, for each  $(i, j) \in \hat{Y}^2 \cup \hat{Y}^3$ , let  $\hat{C}^{ij} := \varphi(P)$  for any  $P \in \hat{\mathcal{P}}_{s,t}^{ij}(\hat{\mathcal{D}})$ . By construction, the points  $\hat{x}^{ij} := \chi^{\hat{C}^{ij}}$ ,  $(i, j) \in \hat{Y}^1 \cup \hat{Y}^2$ , are in  $F(\vartheta)$ , they are linearly independent, and they are linearly independent of the former points. In the following, we distinguish the two cases  $r = 1$  and  $r > 1$ . When  $r = 1$ , it follows that  $Y^4 = \emptyset$  and  $m = k$ . Thus,  $|\hat{Y}^1| = 2(k-2)$  and  $\dim \mathcal{C}^{\leq k}(\hat{\mathcal{D}}) = |\hat{\mathcal{A}}| - k + 1$ . Consequently, we have constructed  $|\hat{\mathcal{A}}| - k + 1$  linearly independent points in  $F(\vartheta)$ , which finishes the proof in case that  $r = 1$ .

When  $r > 1$ , it follows that  $\hat{Y}^4 \neq \emptyset$ . In this case, we use the points  $\tilde{x}^{p,j}$  for  $p \in \{1, 2, \dots, r-1\}$ ,  $j \in \{0, 1, \dots, 2m_p\}$  with  $(1, 0) \neq (p, j) \neq (r-1, 2m_{r-1})$  from part (1), and replace  $\tilde{x}^{1,0}$  and  $\tilde{x}^{r-1, 2m_{r-1}}$  by the points  $\hat{x}^{1,0}$  and  $\hat{x}^{r-1, 2m_{r-1}}$  defined as the incidence vectors of the chains

$$\varphi\left(\hat{P}^{([h_1, j_1-1], [i_2+1, j])}\right) \text{ and } \varphi\left(\hat{P}^{([h_{r-1}, 1], [i_r+1, 2])}\right),$$





Moreover, vectors  $\pi \in B(\Pi)$  and nontrivial 0/1-facets of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  are in 1-1-correspondence. This means,

- (a) if  $\pi \in B(\Pi_n)$ , then  $F(\pi, +)$  is a nontrivial 0/1-facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ ;
- (b) if  $F$  is a nontrivial 0/1-facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ , then  $F = F(\pi, +)$  for some  $\pi \in B(\Pi_n)$ ;
- (c) for any  $\pi, \tilde{\pi} \in B(\Pi_n)$ ,  $F(\pi, +) = F(\tilde{\pi}, +)$  implies  $\pi = \tilde{\pi}$ .

*Proof.* The first statement is obvious. We only prove the 1-1-correspondence between vectors  $\pi \in B(\Pi_n)$  and nontrivial 0/1-facets of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ .

- (a) Let  $\pi \in B(\Pi_n)$  be a minimal primitive vector. If

$$\pi = \begin{pmatrix} 0 & & \\ & 0^T & \\ & & 1 \end{pmatrix} \quad \left( \pi = \begin{pmatrix} 0 & & \\ & \mathbb{1}^T & \\ & & 1 \end{pmatrix} \right),$$

then  $\sum_{a \in A_n} \sigma_a^{\pi, +} x_a \geq 1$  is the inequality  $x_{1t} \geq 1$  ( $x_{s1} \geq 1$ ), which obviously induces a nontrivial 0/1-facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_1))$ . If  $\pi = X^1$ , then  $\sum_{a \in A_n} \sigma_a^{\pi, +} x_a \geq 1$  is the min-cut inequality  $x_{s2} + x_{12} + x_{1t} \geq 1$ . One easily verifies that this inequality induces a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_2))$ . In case that  $\pi = X^2$ , that is,  $\sum_{a \in A_n} \sigma_a^{\pi, +} x_a \geq 1$  is a 1-jump inequality, the statement is implied by Theorem 1 of Dahl, Foldnes, and Gouveia [6], saying that  $r$ -jump inequalities (13) induce facets of the dominant of the  $k$ -hop constrained walk polytope which is equal to the dominant of the  $k$ -hop constrained chain polytope.

(b) Let  $F$  be a nontrivial facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  induced by some valid 0/1-inequality  $\tau^T x \geq \tau_0$ . Using Lemma 2.1 (a2) and Theorem 2.3, it follows that  $\tau = \sigma^{\pi, +}$  for some  $\pi \in \Pi_n$ , which implies  $F = F(\pi, +)$ . The integrality of  $\sigma^{\pi, +}$  and the fact that the incidence vector of any 2-path has to satisfy the inequality implies  $\pi_{tk} \in \{1, 2\}$ . Now,  $\pi_{tk} = 2$  would immediately imply  $\sigma_{si}^{\pi, +} = \sigma_{it}^{\pi, +} = 1$  for all internal nodes  $i \in V_n$ , and as a consequence,  $\sigma_a^{\pi, +} = 0$  for all internal arcs  $a \in A_n$ . Thus, the inequality

$$\sum_{a \in A_n} \sigma_a^{\pi, +} x_a \geq \pi_{tk}$$

is an implicit equation, a contradiction. Hence, we conclude that  $\pi_{tk} = 1$ . Furthermore, since  $\sigma^{\pi, +} \in \{0, 1\}^{A_n}$ , it follows that  $\pi \in \mathbb{Z}_+^{V_n}$ , and due to  $0 = \pi_{s0} \leq \pi \leq \pi_{tk} = 1$ , we conclude that  $\pi$  is a binary vector.

It remains to be shown that  $\pi \in B(\Pi_n)$ . Suppose not. W.l.o.g. we may assume that  $\pi = \pi^{\text{mpr}}$ . Let  $i$  be the minimal row index such that  $\pi_i \neq X_i^2$ , and let  $S := \{s, 1, 2, \dots, i-1\}$ . Denote by  $F'$  the face of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  induced by the inequality  $x(\delta^{\text{out}}(S)) \geq 1$ . Moreover, let  $\rho$  be the coefficient vector of the right hand side of this inequality. It follows that  $A_n(\sigma^{\pi, +}, 1) \supsetneq A_n(\rho, 1)$ , and hence,  $F(\pi) \subsetneq F'$ . Thus,  $F(\pi)$  is not a facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$ , a contradiction.

(c) For any  $\pi \in B(\Pi_n)$  and any  $i \in \{1, \dots, k\}$ , we denote by  $S_i^X$  the set of internal nodes  $j \in V_n$  such that  $\pi_j = X_i^2$ . Observe that  $v \in S_1^X$  if and only if  $\sigma_{sv}^{\pi, +} = 0$ . Next,  $v \in V_k^X$  if and only if  $\sigma_{vt}^{\pi, +} = 0$ . Finally, for any  $i, j \in \{1, \dots, k\}$  with  $i < j$  and any  $u \in V_i^X$  and  $v \in V_j^X$ , it follows that  $\sigma_{uv}^{\pi, +} = 0$  if and only if  $j = i+1$ . These observations imply that each nontrivial 0/1-facet of  $\text{dmt}(\mathcal{C}^{\leq k}(D_n))$  is induced by only one vector  $\pi \in B(\Pi_n)$ .  $\square$

Finally, we show that inequalities (14)

$$\sum_{p=0}^{k-3} \sum_{q=p+2}^{k-1} x((S_p : S_q)) - x((S_{k-1} : S_1 \cup S_2)) \geq 0,$$

where  $\{S_p : p = 0, 1, \dots, k-1\}$  is a partition of  $V \setminus \{t\}$  with  $S_0 = \{s\}$ , and (15)

$$\begin{aligned} & x(\delta^{\text{out}}(s) \setminus \{(s, t)\}) \\ & + \sum_{p=1}^{k-4} \sum_{q=p+2}^{k-2} x((S_p : S_q)) - x((S_{k-2} : S_1)) \geq 0, \end{aligned}$$

where  $\{S_p : p = 1, \dots, k-2\}$  is a partition of  $V \setminus \{s, t\}$ , are the only  $\{-1, 0, 1\}$ -facet defining inequalities for  $\mathcal{C}^{\leq k}(D_n)$  in  $t$ -rooted form.

**Theorem 3.3.** *For  $k \geq 4$ , denote by  $B(\Pi_n^0)$  the set of all 0/1-vectors  $\pi \in \Pi_n^0$  with  $\pi^{\text{mpr}} = X^3$  or  $\pi^{\text{mpr}} = X^4$ , where*

$$X^3 := \begin{pmatrix} 0 & & & & \\ & 1 & 0 & \cdots & 0 \\ & \vdots & \ddots & \ddots & \vdots \\ & 1 & \cdots & 1 & 0 \\ & & & & 0 \end{pmatrix}$$

and

$$X^4 := \begin{pmatrix} 0 & & & & \\ & 0 & \cdots & \cdots & 0 \\ & 1 & \ddots & & \vdots \\ & \vdots & \ddots & \ddots & \vdots \\ & 1 & \cdots & 1 & 0 \\ & & & & 0 \end{pmatrix}.$$

Given  $\pi \in B(\Pi_n^0)$ , the inequality  $\sum_{a \in A_n} \sigma_a^\pi x_a \geq 0$  is an inequality among (14) or (15). Moreover, vectors  $\pi \in B(\Pi_n^0)$  and nontrivial facets of  $\mathcal{C}^{\leq k}(D_n)$  induced by  $t$ -rooted  $\{-1, 0, 1\}$ -inequalities are in 1-1-correspondence. This means,

- (a) if  $\pi \in B(\Pi_n^0)$ , then  $F(\pi)$  is a nontrivial 0/±1-facet of  $\mathcal{C}^{\leq k}(D_n)$ ;
- (b) if  $F$  is a nontrivial facet of  $\mathcal{C}^{\leq k}(D_n)$  induced by a  $t$ -rooted  $\{-1, 0, 1\}$ -inequality, then  $F = F(\pi)$  for some  $\pi \in B(\Pi_n^0)$ ;
- (c) for any  $\pi, \tilde{\pi} \in B(\Pi_n^0)$ ,  $F(\pi) = F(\tilde{\pi})$  implies  $\pi = \tilde{\pi}$ .

*Proof.*

(a) This follows from Theorem 3.1 (a).

(b) Let  $\tau^T x \geq 0$  be a  $t$ -rooted inequality, with coefficients  $\tau_a \in \{-1, 0, 1\}$  for all  $a \in A_n$ , that induces a nontrivial facet  $F$  of  $\mathcal{C}^{\leq k}(D_n)$ . Define  $\pi \in \mathbb{R}^{\mathcal{V}_n}$  by  $\pi_{i\ell} := u_i^{(\ell)}$ ,  $[i, \ell] \in \mathcal{V}_n$ , where the numbers  $u_i^{(\ell)}$  are the values returned by the Moore-Bellman-Ford algorithm 1 for the length function  $d : A_n \rightarrow \mathbb{R}$ ,  $(i, j) \mapsto \tau_{ij}$ . By construction,  $\pi \in \overline{\Pi}_n$ ,  $\tau \in \mathcal{C}_\pi^-$ , and  $F \subseteq F(\pi)$ . Since  $F$  is a nontrivial facet, it follows by Lemma 2.1 (b2) that  $F = F(\pi)$ .

It remains to be shown that  $\pi \in B(\Pi_n^0)$ . Since  $s$  has no ingoing arcs, it follows that  $\pi_{s0} = 0$ . Next, any tight chain implies  $\pi_{tk} = 0$ . Moreover, since  $\tau^T x \geq 0$  is  $t$ -rooted, it immediately follows that  $\pi_{i,k-1} = 0$  for all internal nodes  $i$ . Since all values  $\tau_a$  are integer, we see that  $\pi \in \mathbb{Z}_+$ . Moreover, assuming  $\tau_{0i} = -1$  for some internal node  $i$  implies that the incidence vector of the chain  $(s, i, t)$  violates the inequality. Thus,  $\tau_a \in \{0, 1\}$  for all  $a \in \delta^{\text{out}}(s)$ , which implies  $\pi_{i1} \in \{0, 1\}$  for all internal nodes  $i \in V_n$ , which implies that  $\pi$  is a binary vector.

W.l.o.g. we may assume that  $\pi = \pi^{\text{mpr}}$ . Suppose, for the sake of contradiction, that  $X^3 \neq \pi \neq X^4$ . First, consider the case that  $\pi_1 \succ (1, 0, 0, \dots, 0)$  or  $\pi_n \prec (1, 1, \dots, 1, 0)$ . Then, we see that  $A_n = A_n(\sigma^\pi, 0) \cup A_n(\sigma^\pi, 1)$ . Hence, the inequality  $\sum_{a \in A_n} \sigma_a^\pi x_a \geq 0$  is a conical combination of nonnegativity constraints, a contradiction. In this context, notice that  $\sigma_{ii}^\pi = 0$  for all  $i \in V_n \setminus \{s, t\}$ , since  $k \geq 4$ .

From the previous considerations we conclude that

$$\pi_1 \in \{(0, 0, \dots, 0), (1, 0, \dots, 0)\}$$

and  $\pi_n = (1, 1, \dots, 1, 0)$ . In what follows, we consider only the case  $\pi_1 = 0^T$ . The second case can be similarly handled. Since  $\pi \subsetneq X^4$ , it follows that  $n < k - 1$ . To simplify the notation, we introduce the mapping  $\varepsilon : V_n \rightarrow V_{k-1}$  defined by  $\varepsilon(s) := s$ ,  $\varepsilon(t) := t$ , and  $\varepsilon(i) := j$  if  $\pi_i = X_j^4$ , for  $i = 1, 2, \dots, n$ . In particular,  $\varepsilon(1) = 1$  and  $\varepsilon(n) = k - 1$ . Then, the digraph  $D_\varepsilon = (V_\varepsilon, A_\varepsilon)$  defined by  $V_\varepsilon := \{\varepsilon(i) : i \in V_n\}$  and  $A_\varepsilon := \{(\varepsilon(i), \varepsilon(j)) : (i, j) \in A_n\}$  is a subgraph of  $D_{k-1}$ . The row size of  $X^4$  is  $k - 1$ . The only tight paths of  $\mathcal{P}_{s,t}(D_{k-1})$  using the arc  $([k - 1, k - 1], [t, k])$  are

$$P := ([s, 0], [1, 1], [2, 2], \dots, [k - 1, k - 1], [t, k])$$

and

$$P' := ([s, 0], [k - 1, 1], [2, 2], \dots, [k - 1, k - 1], [t, k]).$$

Weaker formulated, this means that every tight chain  $C \in \mathcal{C}_{s,t}^{\leq k}(D_{k-1})$  using the arc  $(k - 1, t)$  visits each node in  $\{2, 3, \dots, k - 2\}$  exactly one time. However, since  $V_\varepsilon \cap \{2, 3, \dots, k - 2\}$  is a proper subset of  $\{2, 3, \dots, k - 2\}$ , there is no tight chain in  $\mathcal{C}_{s,t}^{\leq k}(D_\varepsilon)$  using arc  $(k - 1, t)$ . Therefore, the face induced by the inequality

$$\sum_{a \in A_\varepsilon} \sigma_a^{X^4} x_a \geq 0$$

is contained in the face induced by the nonnegativity constraint  $x_{k-1,t} \geq 0$ , a contradiction.

(c) This statement can be shown along the lines of the proof to Theorem 3.2 (c).  $\square$

## 4 Concluding remarks

The presented approach to derive results on the hop constrained chain polytope and its dominant is a nice example, where a higher representation of a polyhedron significantly helps to explore and understand its facial structure. Although we are currently not able to give a complete linear description of these polyhedra, the paper shows that quite complicated facet defining inequalities can be



explained much easier in terms of the corresponding extreme rays of the projection cone. In this context, we note that the coefficient vector of a shifting inequality, which is defined in a space of dimension  $O(n^2)$ , is decoded by the first  $O(n - k)$  entries of the first column of the corresponding  $\pi$ -vector.

From the compact formulations given in Section 2 one can derive compact linear programs that represent the separation problems for the hop constrained chain polytope and its dominant (see [4, 20]). Thus, one can solve the separation problems for both polyhedra in polynomial time using the ellipsoid method or interior point methods. Efficient combinatorial separation routines are however unknown. The development of such algorithms is an important issue for future research. The separation problem for the  $r$ -jump inequalities (13) was known to be polynomial time solvable for  $r = 1$  (see [8]) and assumed to be NP-hard for  $r \geq 2$  (see [5, 15]). This piece of work shows that they are contained in a class of inequalities for which the separation problem is polynomial time solvable, since we obtained them by projection from a compact formulation.

We would like to point out an interesting aspect of the technique used in the proof to Theorem 3.1. Whenever useful we have argued in terms of the higher representation to derive certain properties of chains (e.g. tightness or affine independence of the corresponding incidence vectors). To draw the same conclusions in the natural formulation, had been much harder.

The presented framework can probably be used to derive deeper results on the hop constrained chain polytope than that presented in this paper. Furthermore, we believe that there are other combinatorial optimization problems, which can be solved with dynamic programming in polynomial time, for which a similar approach contributes for a better understanding of the respective polyhedra.

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