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The Hypergraph Assignment Problem*

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Abstract

The hypergraph assignment problem (HAP) generalizes the assignment problem from bipartite graphs to bipartite hypergraphs; it is motivated by applications in railway vehicle rotation planning. The HAP is NP-hard and APX-hard even for small hyperedge sizes and hypergraphs with a special partitioned structure. We show that an algorithmically tractable model providing a strong LP relaxation which implies all clique inequalities can be derived from a suitable extended formulation of polynomial size.

Keywords: 90C27; hypergraph; bipartite hypergraph; assignment; extended formulation.

1 Introduction

The assignment problem is fundamental in combinatorial optimization, see [Burkard et al., 2012] for a survey. We propose and investigate a hypergraph generalization.

In the assignment problem, we are looking for a minimum cost perfect matching in a bipartite graph, i. e., we assign to each vertex on one side a vertex on the other side. Likewise, the hypergraph assignment problem looks for a minimum cost perfect matching in a bipartite hypergraph, i. e., we assign sets of vertices on one side to sets of vertices on the other side; we assume that each hyperedge contains the same number of vertices on both sides, see Fig. 1 for an illustration.

The HAP is motivated by applications in railway vehicle rotation planning, see [Borndörfer et al., 2011], [Borndörfer et al., 2012] for details. It can be formulated as a set partitioning problem, but, to the best of our knowledge, its special structure has not been studied yet. There is a relation to flow problems on directed hypergraphs, see [Gallo et al., 1993],[Cambini et al., 1997], but this theory does unfortunately also not provide positive results for our case.

The assignment problem can be solved in polynomial time, e. g., with the well-known Hungarian algorithm, see [Kuhn, 1955], [Munkres, 1957],

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[Burkard et al., 2012]. In contrast, the hypergraph assignment problem is NP-hard and APX-hard even in very simple cases. On the positive side, a model providing a strong LP relaxation that implies all clique inequalities can be derived from a suitable extended formulation of polynomial size. The construction works for hypergraphs with a special structure, namely, for what we call partitioned hypergraphs. This, however, is no loss of generality, as the general HAP can be reduced to the HAP on partitioned hypergraphs. A computational study on the impact of clique inequalities can be found in [Borndörfer and Heismann, 2011].

This paper is structured as follows. Section 2 introduces the hypergraph assignment problem. Section 3 discusses the relation to hyperflows. Section 4 contains the complexity results. In Section 5 we will introduce an extended IP formulation and prove that it implies all clique inequalities of the canonical formulation. Finally, you can find in Section 6 a proof that every HAP can be reduced to a HAP in a partitioned hypergraph.

2 Terminology

We start with basic notions for the hypergraph assignment problem, state the problem, and discuss its relation to hyperflows.

Definition 2.1. A bipartite hypergraph $G = (U, V, E)$ is a triple of two disjoint vertex sets U, V and a set of hyperedges $E \subseteq 2^{U \cup V}$. We assume that the vertex sets have the same size $|U| = |V|$, and that every hyperedge $e \in E$ has the same number $|e \cap U| = |e \cap V| > 0$ of vertices in U and V . We denote by $|e|$ the size of the hyperedge $e \in E$, and call a hyperedge of size 2 an *edge*.

Definition 2.2. For a vertex subset $W \subseteq U \cup V$ we define the *incident hyperedges* $\delta(W) := \{e \in E : e \cap W \neq \emptyset, e \setminus W \neq \emptyset\}$ to be the set of all hyperedges having at least one vertex in both W and $(U \cup V) \setminus W$. We also write $\delta(v) = \delta(\{v\})$ if v is a vertex.

Note that for $W \subseteq U$ or $W \subseteq V$, $e \in \delta(W)$ is equivalent to $e \cap W \neq \emptyset$. We are interested in hyperedge sets that we call hyperassignments.

Definition 2.3. Let $G = (U, V, E)$ be a bipartite hypergraph. A *hyperassignment* in G is a subset $H \subseteq E$ of hyperedges such that every $v \in U \cup V$ is contained in exactly one hyperedge $e \in H$.

Figure 1 illustrates a hyperassignment in a bipartite hypergraph. Our aim is to find a hyperassignment having minimum cost.

Definition 2.4. A *cost function* $c_S : S \rightarrow \mathbb{R}$ maps a set S to the reals. For $T \subseteq S$ let

$$c_S(T) := \sum_{s \in T} c_S(s).$$

Problem 2.5 (Hypergraph Assignment Problem).

Input: A pair (G, c_E) consisting of a bipartite hypergraph $G = (U, V, E)$ and a cost function $c_E : E \rightarrow \mathbb{R}$.

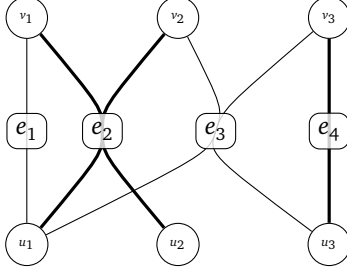


Figure 1: Visualization of the bipartite hypergraph $G = (U, V, E)$ with $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2, e_3, e_4\}$, $e_1 = \{u_1, v_1\}$, $e_2 = \{u_1, u_2, v_1, v_2\}$, $e_3 = \{u_1, u_3, v_2, v_3\}$, $e_4 = \{u_3, v_3\}$. Vertices are circles, edges have square labels, the hyperedges of the hyperassignment $\{e_2, e_4\}$ are drawn with thick lines.

Output: A minimum cost hyperassignment in G w. r. t. c_E , i. e., a hyperassignment H^* in G such that

$$c_E(H^*) = \min\{c_E(H) : H \text{ is a hyperassignment in } G\},$$

or the information that no hyperassignment exists.

Unlike in the graph case, bipartite hypergraphs can have a complex structure, which, of course, cannot be avoided. What we can do, however, is to study a certain “normal form” with a “graph-type appearance” which we find easier to analyze. Our normal form is based on a partitioning of the vertex set that allows to capture the local structure of a hyperassignment in terms of what we call “configurations”. We will show in Section 6 that every hypergraph can be polynomially transformed into a partitioned hypergraph in such a way that there is a one-to-one correspondence between the hyperassignments in the associated HAP instances.

Definition 2.6. A bipartite hypergraph $G = (U, V, E)$ is called *partitioned* with *maximum part size* $d \in \mathbb{N}$ if there exist pairwise disjoint $\leq d$ -element sets U_1, \dots, U_p and V_1, \dots, V_q called the *parts* of H such that $\bigcup_{i=1}^p U_i = U$, $\bigcup_{i=1}^q V_i = V$, and

$$E \subseteq \bigcup_{i=1}^p \bigcup_{j=1}^q 2^{U_i \cup V_j},$$

i. e., every hyperedge intersects exactly one part in U and one part in V . In other words, every hyperedge in a partitioned bipartite hypergraph runs from a part of G on the U -side to a part on the V -side. We shortly call a partitioned bipartite hypergraph a partitioned hypergraph.

For an example of a partitioned hypergraph see Figure 2.

We now introduce the notion of a configuration to describe the local structure of a hyperassignment H at a part Π , i. e., the possible sets $H \cap \delta(\Pi)$.

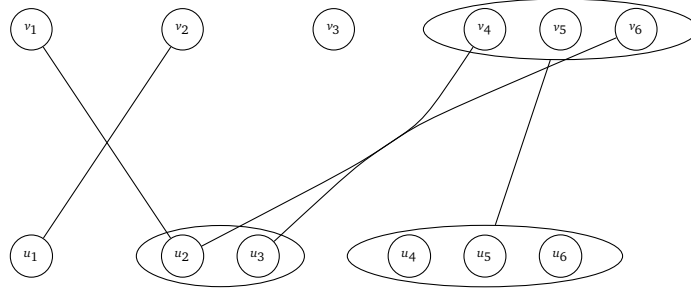


Figure 2: Visualization of a partitioned hypergraph with maximum part size $d = 3$ and parts $\{u_1\}, \{u_2, u_3\}, \{u_4, u_5, u_6\}$ and $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4, v_5, v_6\}$, and hyperedges $\{u_1, v_2\}, \{u_2, v_1\}, \{u_2, u_3, v_4, v_6\}, \{u_4, u_5, u_6, v_4, v_5, v_6\}$. The vertices of each part with more than one vertex are surrounded by an ellipse in the picture. For partitioned hypergraphs we visualize the hyperedges which connect all the vertices from one part with all the vertices from another part by drawing just a line between the two ellipses surrounding the vertices of the part.

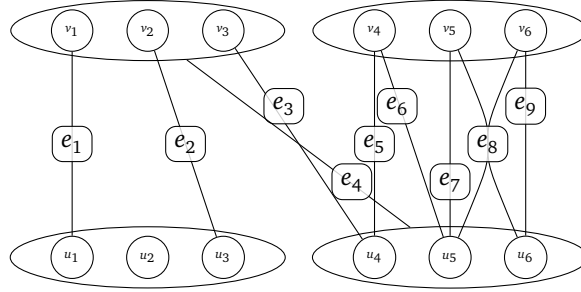


Figure 3: In this partitioned hypergraph, the set of all configurations for the part $\Pi = \{u_4, u_5, u_6\}$ is $\mathcal{C}_\Pi = \{\{e_3, e_6, e_9\}, \{e_3, e_7, e_9\}, \{e_3, e_8\}, \{e_5, e_7, e_9\}, \{e_5, e_8\}, \{e_4\}\}$.

Definition 2.7. Let $\Pi \in \{U_1, \dots, U_p, V_1, \dots, V_q\}$ be a part of a partitioned hypergraph. We define the set of all configurations associated with Π to be

$$\mathcal{C}_\Pi = \left\{ C \subseteq \delta(\Pi) : \Pi \subseteq \bigcup_{e \in C} e \text{ and } e_1 \cap e_2 = \emptyset \forall e_1, e_2 \in C \text{ with } e_1 \neq e_2 \right\}.$$

We write $\mathcal{C}_U := \bigcup_{i=1}^p \mathcal{C}_{U_i}$, $\mathcal{C}_V := \bigcup_{i=1}^q \mathcal{C}_{V_i}$, and $\mathcal{C} := \mathcal{C}_U \cup \mathcal{C}_V$.

A configuration $C \in \mathcal{C}_\Pi$ associated with part Π , w.l.o.g. $\Pi \subseteq U$, is a subset of disjoint hyperedges that connect all and only the vertices in Π on the U -side with some vertices on the V -side of G , see Fig. 3 for an illustration. A hyperassignment H has the same property and therefore induces a configuration $H \cap \delta(\Pi)$ at every part Π .

3 Relation to Other Problems

The hypergraph assignment problem can be viewed as a set partitioning problem with a special structure; the hypergraphs are bipartite. In Section 5, we show how this property can be used to derive a strong LP formulation, which cannot be stated for general set partitioning problems.

The problem to decide whether a bipartite hypergraph contains a hyperassignment can be related to the theory of systems of disjoint representatives, see [Aharoni and Haxell, 2000]. Namely, a hyperassignment in a partitioned hypergraph selects for each part U_i exactly one configuration $C \in \mathcal{C}_{U_i}$ that covers some vertices $C \cap V$. The vertex sets $C \cap V$ are disjoint (they actually form a partition of V) and therefore can be seen as a system of disjoint representatives in the hypergraph system $\{\{C \cap V : C \in \mathcal{C}_{U_i}\} : i = 1, \dots, p\}$. Conversely, every system of disjoint representatives in this hypergraph system gives rise to a hyperassignment since the number of covered vertices in V must be equal to $|U| = |V|$. The existence of a system of disjoint representatives and hence the existence of a hyperassignment can be checked using a generalization of Hall's theorem, see again [Aharoni and Haxell, 2000], which, however, involves a super-exponential number of conditions.

As far as we know, the optimization problem HAP has not been investigated in the literature before. It can be related to the more general minimum cost hyperflow problem with integrality constraints, as we will show now.

The hypergraph assignment problem can be stated as a *minimum cost hyperflow problem* with integrality constraints on a so-called (directed) *B-hypergraph*, see [Cambini et al., 1992],[Cambini et al., 1997],[Jeroslow et al., 1992].

A B-hypergraph (backward hypergraph) $D = (N, A)$ consists of a vertex set N and a set of *B-hyperarcs* (backward hyperarcs) A . A B-hyperarc $a = (T_a, h_a) \in A$ is pair of a vertex set $T_a \subset N$ (the *tail*) and a vertex $h_a \in N \setminus T_a$ (the *head*); it is supposed to be directed from the tail to the head. "Flow multipliers" can be associated with B-hyperarcs, but we omit them here. We are further given a *demand vector* $b \in \mathbb{R}^N$ and cost function $c_A : A \rightarrow \mathbb{R}$ on the B-hyperarcs. A *hyperflow* $f \in \mathbb{R}_{\geq 0}^A$ is a vector, which associates a flow value with each B-hyperarc such that for all vertices $n \in N$ the demand constraint

$$\sum_{a \in A: n = h_a} f_a - \sum_{a \in A: n \in T_a} f_a = b_n$$

is satisfied. Note that the flow at the head of a B-hyperarc is the same as the flow at *each* of the tail vertices. The problem consists of finding a (not necessarily integral) minimum cost hyperflow f^* , i. e.,

$$\sum_{a \in A} c_A(a) f_a^* = \min \left\{ \sum_{a \in A} c_A(a) f_a : f \text{ is a hyperflow in } D \right\}.$$

We can state the HAP in (G, c_E) with $G = (U, V, E)$ as a minimum cost hyperflow problem with integrality constraints in the following way. Let $E = E_1 \cup E_2$ where $E_1 = \{e \in E : |e| = 2\}$ is the set of all edges in E and $E_2 = E \setminus E_1$ the set of all other hyperedges. For $e \in E_1$, let $\{t_e\} = e \cap U$ and $\{h_e\} = e \cap V$; for $e \in E_2$, let

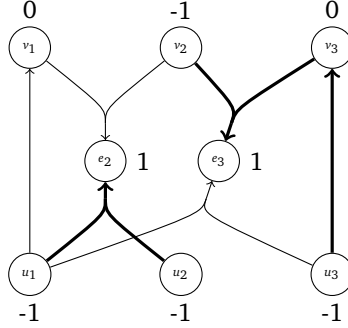


Figure 4: The B-hypergraph and the hyperflow (hyperarcs with value 1 are drawn with thick lines) corresponding to the hypergraph and the hyperassignment in Figure 1. The numbers next to the vertices are the values of the demand vector.

$U_e = U \cap e$ and $V_e = V \cap e$. We construct a B-hypergraph $D = (N, A)$ with vertex set $N = U \cup V \cup E_2$ and B-hyperarc set $A = A_1 \cup A_2 \cup A'_2$, $A_1 = \{(\{t_e\}, h_e) : e \in E_1\}$, $A_2 = \{(U_e, e) : e \in E_2\}$, $A'_2 = \{(V_e, e) : e \in E_2\}$. In the cost function, we assign $c_A(\{t_e\}, h_e) = c_E(e)$ to the B-hyperarcs in A_1 , $c_A(U_e, e) = c_E(e)$ to the B-hyperarcs in A_2 , and cost 0 to all B-hyperarcs in A'_2 . We define the demand vector b such that

$$b_n = \begin{cases} -1 & \text{if } n \in U \\ 1 & \text{if } n \in E \\ 1 - |\{e \in E_2 : n \in e\}| & \text{if } n \in V. \end{cases}$$

The idea of this construction is that B-hyperarcs $(\{t_e\}, h_e)$ and (U_e, e) have flow value 1 if e is contained in the hyperassignment, while a B-hyperarc (V_e, e) has flow value 1 if e is *not* contained in the hyperassignment; all other flow values are 0. It can be verified that there is a cost-preserving bijection between hyperassignments in G and 0/1 hyperflows in D . Namely, the following 0/1 hyperflow corresponds to a hyperassignment $H \subseteq E$ in G :

$$f_a = \begin{cases} 1 & \text{if } a = (\{t_e\}, h_e) \in A_1, e \in H \\ 0 & \text{if } a = (\{t_e\}, h_e) \in A_1, e \notin H \\ 1 & \text{if } a = (U_e, e) \in A_2, e \in H \\ 0 & \text{if } a = (U_e, e) \in A_2, e \notin H \\ 0 & \text{if } a = (V_e, e) \in A'_2, e \in H \\ 1 & \text{if } a = (V_e, e) \in A'_2, e \notin H. \end{cases}$$

For an example see Figure 4.

In contrast to the minimum cost flow problem on graphs, the hyperflow problem does not necessarily have integer solutions for integral inputs. A hypergraph network simplex algorithm to compute a (not necessarily integer) optimal solution was proposed in [Cambini et al., 1992], while sufficient conditions for ensuring

integrality in terms of so-called gain-free Leontief substitution flows have been investigated in [Jeroslow et al., 1992]. Similar results are not known in our setting; in fact, we will show now that the HAP is NP-hard even in very simple cases.

4 Complexity

We will now prove that the HAP is NP-hard and APX-hard using a reduction to the 3-dimensional matching problem. This results already hold for bipartite hypergraphs with a very simple structure, namely, for partitioned hypergraphs with part size at most two.

Theorem 4.1. *The hypergraph assignment problem is NP-hard and APX-hard, even for partitioned hypergraphs with maximum part size 2.*

Proof. We will use the NP-complete and in its optimization version APX-hard 3-dimensional matching problem [Garey and Johnson, 1979, page 46][Kann, 1991]. The input of the 3-dimensional matching problem is an undirected hypergraph $(X \cup Y \cup Z, T)$, $T \subseteq 2^{X \cup Y \cup Z}$ such that $|X| = |Y| = |Z|$ and

$$|t \cap X| = |t \cap Y| = |t \cap Z| = 1 \quad \forall t \in T.$$

It asks whether a partitioning of this hypergraph exists, i. e., a set $F \subseteq T$ such that each element from $X \cup Y \cup Z$ is contained in exactly one set in F . Let

$$\begin{aligned} X &= \{x_1, \dots, x_n\}, \\ Y &= \{y_1, \dots, y_n\}, \\ Z &= \{z_1, \dots, z_n\}, \\ T &= \{t_1, \dots, t_m\}, \end{aligned}$$

with $t_r = \{x_{i_r}, y_{j_r}, z_{k_r}\}$, $r = 1, \dots, m$.

To prove the theorem we construct an instance of the hypergraph assignment problem having a size that is polynomial in the size of the given 3-dimensional matching problem such that there exists a hyperassignment in the HAP if and only if there exists a partitioning in the 3-dimensional matching problem.

Let $G = (U, V, E)$ be a partitioned hypergraph with parts

$$\begin{aligned} U(z_1) &= \{u(z_1), u'(z_1)\}, \dots, U(z_n) = \{u(z_n), u'(z_n)\}, \\ U(t_1) &= \{u(t_1), u'(t_1)\}, \dots, U(t_m) = \{u(t_m), u'(t_m)\} \end{aligned}$$

in U and

$$\begin{aligned} V(x_1, y_1) &= \{v(x_1), v(y_1)\}, \dots, V(x_n, y_n) = \{v(x_n), v(y_n)\}, \\ V(t_1) &= \{v(t_1), v'(t_1)\}, \dots, V(t_m) = \{v(t_m), v'(t_m)\}, \end{aligned}$$

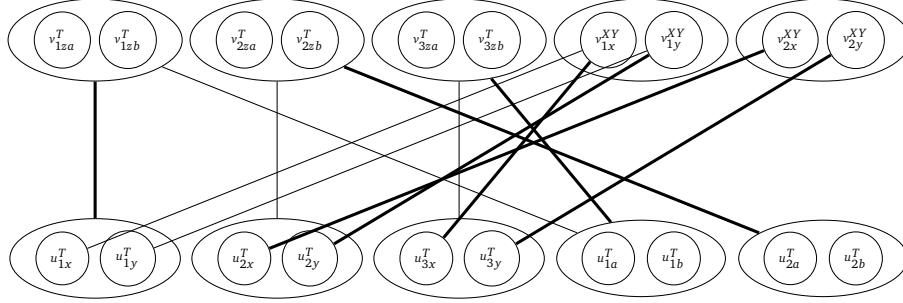


Figure 5: Example for the construction of G in the NP-hardness proof for $(X \cup Y \cup Z, T)$ with $n = 2$ and $T = \{\{x_1, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_1, y_2, z_1\}\}$. The partitioning $F = \{\{x_2, y_1, z_2\}, \{x_1, y_2, z_1\}\}$ gives rise to a hyperassignment in G drawn with thick lines.

in V . Let

$$\begin{aligned}
 E = & \{ \{u(t_r), v(x_{i_r})\} : r \in \{1, \dots, m\} \} \\
 & \cup \{ \{u'(t_r), v(y_{j_r})\} : r \in \{1, \dots, m\} \} \\
 & \cup \{ \{u(z_{k_r}), u'(z_{k_r}), v(t_r), v'(t_r)\} : r \in \{1, \dots, m\} \} \\
 & \cup \{ \{u(t_r), u'(t_r), v(t_r), v'(t_r)\} : r \in \{1, \dots, m\} \}.
 \end{aligned}$$

For an example of this construction see Figure 5.

Let $H \subseteq E$ be a hyperassignment in G . Each of the vertices $v(x_1), \dots, v(x_n), v(y_1), \dots, v(y_n), u(z_1), \dots, u(z_n), u'(z_1), \dots, u'(z_n)$ is contained in exactly one hyperedge in H . All such hyperedges contain at least one vertex from one of the parts $U(t_r)$ or $V(t_r)$ for some r . The four vertices $u(t_r), u'(t_r), v(t_r), v'(t_r)$ from the parts $U(t_r), V(t_r), r = 1, \dots, m$, are contained either in the hyperedge $\{u(t_r), u'(t_r), v(t_r), v'(t_r)\}$ in E (case one) or in the three hyperedges $\{u(t_r), v(x_{i_r})\}, \{u'(t_r), v(y_{j_r})\},$ and $\{u(z_{k_r}), u'(z_{k_r}), v(t_r), v'(t_r)\}$ (case two) in E . Thus, the set of all $\{x_{i_r}, y_{j_r}, z_{k_r}\} \in T$ for which r is associated with case two form a partitioning of $(X \cup Y \cup Z, T)$.

On the other hand, given a partitioning of $(X \cup Y \cup Z, T)$ we get a hyperassignment H in G by choosing the hyperedges associated with case two exactly for those r for which $\{x_{i_r}, y_{j_r}, z_{k_r}\}$ is in the partitioning and the hyperedge associated with case one otherwise. \square

5 Clique Inequalities and an Extended Formulation

The hypergraph assignment problem has a canonical integer linear programming formulation of the set partitioning type. Such a model can be strengthened by clique inequalities, which are usually difficult to separate. We will show in this section that partitioned hypergraph assignment problems give rise to an extended

formulation of polynomial size which implies all clique inequalities. This result will be proved by looking at the special structure of cliques in partitioned hypergraphs.

The canonical integer linear program for the HAP is the following:

$$\begin{aligned}
& \underset{x \in \mathbb{R}^E}{\text{minimize}} && \sum_{e \in E} c_E(e) x_e && \text{(HAP)} \\
& \text{subject to} && \sum_{e \in \delta(v)} x_e = 1 && \forall v \in U \cup V && \text{(i)} \\
& && x \geq 0 && \text{(ii)} \\
& && x \in \mathbb{Z}^E. && \text{(iii)}
\end{aligned}$$

It involves a binary variable x_e for the choice of hyperedge $e \in E$. The equations (HAP) (i) guarantee that every vertex is covered by exactly one hyperedge. (HAP) (ii) are the non-negativity and (iii) the integrality constraints. It is easy to see that (HAP) is a valid formulation for the hypergraph assignment problem. For a formal proof see [Heismann, 2010].

Definition 5.1. A *clique* in (the conflict graph of) a bipartite hypergraph $G = (U, V, E)$ is a set $Q \subseteq E$ of hyperedges such that every two hyperedges $e_1, e_2 \in Q$ have at least one vertex in common, i. e., $e_1 \cap e_2 \neq \emptyset$. A clique Q is a *maximal clique* if there is no clique $Q' \supset Q$ containing Q and in addition other hyperedges.

Associated with the clique Q is the clique inequality $\sum_{e \in Q} x_e \leq 1$.

Every feasible solution of (HAP) fulfills every clique inequality. These constraints are important for HAPs arising from real-world applications in railway vehicle rotation planning. Such instances are modeled using partitioned hypergraphs of maximum part size 7, and clique inequalities can significantly reduce the integrality gap of model (HAP), see [Borndörfer and Heismann, 2011].

Our extended integer linear programming formulation for the HAP is based on the notion of configurations introduced in Section 2. The configurations model the local incidence structure of hyperassignments at the parts of the hypergraph. It is as follows.

$$\begin{aligned}
& \underset{x \in \mathbb{R}^E, y \in \mathbb{R}^{\mathcal{C}}}{\text{minimize}} && \sum_{e \in E} c_E(e) x_e && \text{(HAP_ext)} \\
& \text{subject to} && \sum_{e \in \delta(v)} x_e = 1 && \forall v \in U \cup V && \text{(i)} \\
& && \sum_{C \in \mathcal{C}_v : e \in C} y_C = x_e && \forall e \in E && \text{(ii)} \\
& && \sum_{C \in \mathcal{C}_v : e \in C} y_C = x_e && \forall e \in E && \text{(iii)} \\
& && x, y \geq 0 && \text{(iv)} \\
& && x \in \mathbb{Z}^E && \text{(v)} \\
& && y \in \mathbb{Z}^{\mathcal{C}} && \text{(vi)}
\end{aligned}$$

The model uses binary variables x_e and y_C for the choice of hyperedge e and configuration C , respectively. Constraints (HAP_ext) (i) are copied from the canonical formulation. Equations (HAP_ext) (ii) and (iii) link the hyperedges to the configurations in parts in U resp. V that contain them. (HAP_ext) (iv), (v), and (vi) enforce non-negativity and integrality.

This section resorts to the following notation. For an index set I , a subset $J \subseteq I$, and a vector $x \in \mathbb{R}^I$, denote by $x|_J = x_J$ the projection of x onto the coordinates in J . Likewise, let $P|_J$ denote the projection of a polytope $P \subseteq \mathbb{R}^I$ onto \mathbb{R}^J .

Let

$$P_{\text{LP}}(\text{HAP_ext}) := \{(x, y) \in \mathbb{R}^E \times \mathbb{R}^{\mathcal{C}} : (\text{HAP_ext}) \text{ (i)–(iv)}\}$$

be the polytope associated with the LP relaxation of the integer program (HAP_ext). Then $P_{\text{LP}}(\text{HAP_ext})|_E$ and $P_{\text{LP}}(\text{HAP_ext})|_{\delta(\Pi)}$ project the LP relaxation of the extended formulation onto the original space of all hyperedge variables and those incident to some part Π , respectively.

The following theorem relates the integer program (HAP_ext) to formulation (HAP) and, in particular, proves the correctness of (HAP_ext).

Theorem 5.2. *Let $G = (U, V, E)$ be a partitioned hypergraph and $c_E : E \rightarrow \mathbb{R}$ a cost function. Then the projection*

$$\cdot|_E : \mathbb{R}^E \times \mathbb{R}^{\mathcal{C}}, (x, y) \mapsto x$$

is a bijection between feasible solutions of (HAP_ext) and (HAP), and therefore hyperassignments in G . The optimum value of (HAP_ext) is equal to the cost of the minimum cost hyperassignment in G w. r. t. c_E if it exists and to ∞ otherwise.

Proof. Let $x \in \mathbb{R}^E$ be the incidence vector of a hyperassignment H in G . We have to show that there is exactly one $y \in \mathbb{R}^{\mathcal{C}}$ such that (x, y) is feasible for (HAP_ext). Define

$$\mathcal{C}'_U := \{\delta(\Pi) \cap H : \Pi \in \{U_1, \dots, U_p\}\}, \mathcal{C}'_V := \{\delta(\Pi) \cap H : \Pi \in \{V_1, \dots, V_q\}\}$$

as the set of intersections of the hyperassignment H with the hyperedges incident to the parts in U and V , respectively. Then \mathcal{C}'_U and \mathcal{C}'_V are two sets of configurations, namely, $\mathcal{C}'_U \subseteq \mathcal{C}_U$ and $\mathcal{C}'_V \subseteq \mathcal{C}_V$, and we can define the incidence vector $y \in \mathbb{R}^{\mathcal{C}}$ of $\mathcal{C}'_U \cup \mathcal{C}'_V$.

We show next that the vector (x, y) is a solution of (HAP_ext). Equations (i) hold because x is the incidence vector of hyperassignment H . Now consider some hyperedge e incident to part Π , w.l.o.g. $\Pi \subseteq U$. Suppose $x_e = 1$, i.e., $e \in H$. Then e is contained in exactly one configuration in \mathcal{C}'_U , namely, $e \in H \cap \delta(\Pi)$, i.e., equation (HAP_ext) (ii) holds. If $x_e = 0$, i.e., $e \notin H$, then e is not contained in any configuration in \mathcal{C}'_U and (HAP_ext) (ii) also holds. The case $\Pi \subseteq V$ and (HAP_ext) (iii) is analogous. Constraints (iv)–(vi) are clear.

To see that there is only one possible choice for y consider some part Π containing a node $v \in \Pi$. Substituting for the variables x_e in the constraint (HAP_ext) (i) associated with node v yields

$$\sum_{C \in \mathcal{C}_\Pi} y_C \stackrel{(*)}{=} \sum_{e \in \delta(v)} \sum_{C \in \mathcal{C}'_U : e \in C} y_C = \sum_{e \in \delta(v)} x_e = 1;$$

i. e., y associates exactly one configuration with every part Π ; equation (*) holds because every configuration in \mathcal{C}_Π contains exactly one hyperedge incident to node v . Equations (HAP_ext) (ii) and (iii) ensure that this configuration contains exactly the edges in $H \cap \delta(\Pi)$, i. e., the edges such that $x_e = 1$. This means that $y_C = 1$ exactly for $C = H \cap \delta(\Pi)$.

The cost statement is obvious. \square

Corollary 5.3. *Let $G = (U, V, E)$ be a partitioned hypergraph and let Π be some part. Then $P_{LP}(\text{HAP_ext})|_{\delta(\Pi)}$ is a subset of*

$$\text{conv} \left\{ x \in \{0, 1\}^{\delta(\Pi)} : x \text{ is the incidence vector of a configuration in } \mathcal{C}_\Pi \right\}.$$

One class of inequalities implied in this way are the clique inequalities. This will be shown using the following lemma.

Lemma 5.4. *Let $G = (U, V, E)$ be a partitioned hypergraph and Q a clique in G . Then there exists a part Π such that $Q \subseteq \delta(\Pi)$, i. e., every clique is a subset of the set of hyperedges incident to some part Π in G .*

Proof. Let Q be a nonempty clique in G (otherwise the lemma is trivial) containing some hyperedge e_1 . Let $e_1 \cap U$ be contained in some part U_1 and $e_1 \cap V$ be contained in some part V_1 . Every other hyperedge e in Q must either satisfy $e \cap U \subseteq U_1$ or $e \cap V \subseteq V_1$, otherwise e_1 and e would have an empty intersection. Assume the statement does not hold and Q contains some hyperedge e_2 such that $e_2 \cap U \not\subseteq U_1$ and some hyperedge e_3 such that $e_3 \cap V \not\subseteq V_1$. Then $e_2 \cap e_3$ must be empty since both $e_2 \cap U$, $e_3 \cap U$ and $e_2 \cap V$, $e_3 \cap V$ are contained in different parts. This contradicts the assumption that Q is a clique. Hence, either U_1 or V_1 contains $e \cap U$ or $e \cap V$ for all hyperedges $e \in Q$. \square

Theorem 5.5. *Let $G = (U, V, E)$ be a partitioned hypergraph and $Q \subseteq E$ be a clique. Then, the clique inequality*

$$\sum_{e \in Q} x_e \leq 1$$

is satisfied by all feasible solutions of the LP relaxation of (HAP_ext).

Proof. First of all, observe that $|Q \cap C| \leq 1$ for every configuration $C \in \mathcal{C}$ because the hyperedges in a configuration are pairwise disjoint and those in Q are not.

By Lemma 5.4, $Q \subseteq \delta(\Pi)$ for some part Π . W.l. o. g., let $\Pi \subseteq U$ (the formulation is symmetric in U, V). Let v be some vertex in Π . Then, by (HAP_ext) (i)

$$1 = \sum_{e \in \delta(v)} x_e.$$

Substituting for x_e using equation (HAP_ext) (ii) yields

$$= \sum_{C \in \mathcal{C}_U} |\delta(v) \cap C| \cdot y_C.$$

Since $|\delta(v) \cap C| = 1$ if and only if $C \in \mathcal{C}_\Pi$ and zero otherwise, we get

$$= \sum_{C \in \mathcal{C}_\Pi} y_C.$$

Distinguishing the configurations with $|Q \cap C| = 1$ and $|Q \cap C| = 0$ leads to

$$\begin{aligned} &= \sum_{e \in Q} \sum_{C \in \mathcal{C}_\Pi: e \in C} y_C + \sum_{C \in \mathcal{C}_\Pi: C \cap Q = \emptyset} y_C \\ &\geq \sum_{e \in Q} \sum_{C \in \mathcal{C}_\Pi: e \in C} y_C. \end{aligned}$$

Finally, applying (HAP_ext) (ii) again yields

$$= \sum_{e \in Q} x_e.$$

□

6 Structural Results

Our main result works for hypergraph assignment problems on partitioned hypergraphs. We will show now that this is not a real restriction, because every bipartite hypergraph G with maximum hyperedge size $2d$ can be polynomially transformed into a partitioned hypergraph G' with maximum part size d in such a way that there exists a cost preserving bijection between the hyperassignments in G and G' .

The idea of the of the construction is to set up a hypergraph that consists of disjoint copies of the original hyperedges plus some “garbage collection” edges that will match superfluous vertices in a hyperassignment.

Theorem 6.1. *Let $G = (U, V, E)$ be a bipartite hypergraph with maximum hyperedge size $2d$ and c_E a cost function. Then there exists a partitioned hypergraph $G' = (U', V', E')$ with maximum part size d and a cost function $c_{E'}$ such that there is a hyperassignment H in G of cost c if and only if there is a hyperassignment H' in G' of the same cost. G' can be constructed in polynomial time.*

Proof. Let

$$U' = \{(u, e) : u \in U, e \in E, u \in e\} \cup \{u'_i(v) : v \in V, i \in \{1, \dots, |\delta(v)| - 1\}\}$$

and

$$V' = \{(v, e) : v \in V, e \in E, v \in e\} \cup \{v'_i(u) : u \in U, i \in \{1, \dots, |\delta(u)| - 1\}\}.$$

For every $u \in U$ and $v \in V$, order the vertices in $\{(u, e) \in U'\}$ as $\{u''_1, \dots, u''_{|\delta(u)|}\}$ and those in $\{(v, e) \in V'\}$ as $\{v''_1, \dots, v''_{|\delta(v)|}\}$, respectively.

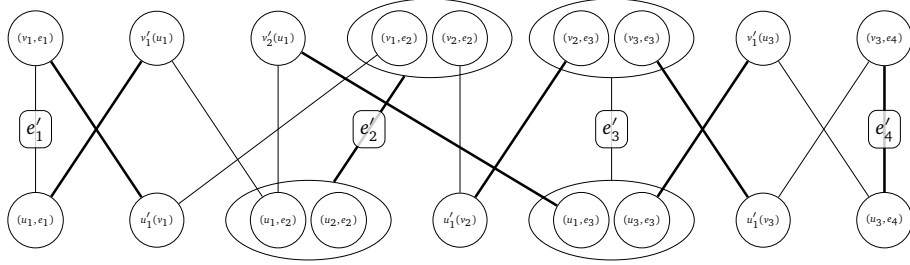


Figure 6: The corresponding partitioned hypergraph for the bipartite hypergraph from Figure 1. The thick hyperedges show a hyperassignment in both. In this construction the vertices of the type (u, e) or (v, e) for every vertex u or v of the original hypergraph are ordered by the index of the hyperedges.

For each hyperedge $e \in E$ we construct a “copy” $e' = \{(u, e), (v, e) : u \in U \cap e, v \in V \cap e\} \in E'$ with cost $c_{E'}(e') = c_E(e)$. Further, we construct edge sets of cost zero to control that exactly one of the copies (u, e) or (v, e) of vertex u or v , respectively, is covered by a hyperedge copy in every hyperassignment in G' . We have

$$E' := \{e' : e \in E\} \cup \left\{ \{u''_i, v'_j(u)\} : (i - j) \in \{0, 1\}, u \in U, i \in \{1, \dots, |\delta(u)|\} \right\} \cup \left\{ \{u'_j(v), v''_i\} : (i - j) \in \{0, 1\}, v \in V, i \in \{1, \dots, |\delta(v)|\} \right\}$$

and set $G' := (U', V', E')$. This construction can be done in polynomial time.

First of all, we show that G' can be partitioned. By construction, G' is bipartite (and well-defined). Choosing the sets $\{(u, e) : u \in e\}$, $\{(v, e) : v \in e\}$ for every $e \in E$ and the remaining vertices individually as parts produces a partitioned hypergraph. As $|\{U \cap e\}| = |\{V \cap e\}| \leq d$ for each hyperedge e and since every vertex in G' has at most one incident hyperedge that is not an edge, the maximum part size is d .

Let H be a hyperassignment in G . For every vertex $u \in U$ let $i(u)$ be the index of $u''_{i(u)} = (u, e)$ where e is the unique hyperedge in H containing u , and define $i(v)$ for $v \in V$ analogously; note that $i(u)$ and $i(v)$ depend on H . Let

$$H' := \{e' : e \in H\} \cup \left\{ u''_i, v'_i(u) : u \in U, i < i(u) \right\} \cup \left\{ u''_i, v'_{i-1}(u) : u \in U, i > i(u) \right\} \cup \left\{ u'_i(v), v''_i : v \in V, i < i(v) \right\} \cup \left\{ u'_{i-1}(v), v''_i : v \in V, i > i(v) \right\}.$$

H' contains the copies of the hyperedges in H , and matches the unused vertex copies to the garbage collection vertices $v'_i(u)$ and $u'_i(v)$, see Fig. 6 for an illustration. H' is a hyperassignment in G' with cost $c_{E'}(H') = c_E(H)$.

On the other hand, given a hyperassignment H' in G' we can construct a hyperassignment H in G using exactly the hyperedges from E corresponding to the hyperedges $e' \in H'$. Again, $c_{E'}(H') = c_E(H)$. To prove that H is a hyperassignment

we need to show that for every vertex $v \in U \cup V$ of G , H' has only one vertex (v, e) which is contained in a hyperedge e' in H' . For this purpose observe that G' contains $|\delta(v)| - 1$ vertices $u'_i(v)$ and $|\delta(v)|$ vertices (v, e) for every vertex v of G . The garbage collection vertices $u'_i(v)$ have to be matched with vertices (v, e) via garbage collection edges such that exactly one vertex copy (v, e) must be covered by a hyperedge. \square

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