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The Hypergraph Assignment Problem

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The Hypergraph Assignment Problem

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Abstract

The hypergraph assignment problem (HAP) is the generalization of assignments from directed graphs to directed hypergraphs. It serves, in particular, as a universal tool to model several train composition rules in vehicle rotation planning for long distance passenger railways. We prove that even for problems with a small hyperarc size and hypergraphs with a special partitioned structure the HAP is NP-hard and APX-hard. Further, we present an extended integer linear programming formulation which implies, e. g., all clique inequalities.

1 Directed Hypergraph Terminology

Definition 1.1. A *directed hypergraph* $D = (V, W, A)$ is a pair of two disjoint vertex sets V, W of the same size $|V| = |W|$ and a set of hyperarcs $A \subseteq 2^{V \cup W}$ such that every hyperarc $a \in A$ has the same size $|a \cap V| = |a \cap W| > 0$ of vertices in V and W . If $|a \cap V| = |a \cap W| = 1$, then we call the hyperarc a also an arc.

In the usual definition of a directed hypergraph we have only one vertex set and every hyperarc is a pair of two sets called its tail and its head [GLPN93]. Despite that we do not require a nonempty intersection of the tail and the head, our definition is equivalent to it. To get this form of the directed hypergraph identify for the vertex sets $V = \{v_1, \dots, v_n\}, W = \{w_1, \dots, w_n\}$ the vertex v_1 with w_1, \dots, v_n with w_n and take for each hyperarc $a \in A$, $a \cap V$ as its tail and $a \cap W$ as its head.

Definition 1.2. Let $D = (V, W, A)$ be a directed hypergraph. A *hyperassignment* in D is a subset $H \subseteq A$ of the hyperarcs such that for every $v \in V \cup W$ there exists exactly one $a \in H$ such that $v \in a$.

For an example of a directed hypergraph and a hyperassignment see figure 1.

Definition 1.3. Given a set S , a *cost function* is a function $c_S : S \rightarrow \mathbb{R}$. For $T \subseteq S$ we define

$$c_S(T) := \sum_{s \in T} c_S(s).$$

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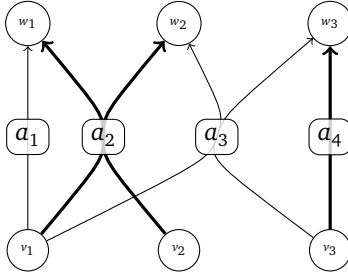


Figure 1: Visualization of the directed hypergraph $D = (V, W, A)$ with $V = \{v_1, v_2, v_3\}$, $W = \{w_1, w_2, w_3\}$, $A = \{a_1, a_2, a_3, a_4\}$, $a_1 = \{v_1, w_1\}$, $a_2 = \{v_1, v_2, w_1, w_2\}$, $a_3 = \{v_1, v_3, w_2, w_3\}$, $a_4 = \{v_3, w_3\}$ with the hyperassignment $\{a_2, a_4\}$ drawn with thick hyperarcs.

Problem 1.4 (Hypergraph Assignment Problem).

Input: A pair (D, c_A) consisting of a directed hypergraph $D = (V, W, A)$ and a cost function $c_A : A \rightarrow \mathbb{R}$.

Output: A minimum cost hyperassignment in D w. r. t. c_A , i. e., a hyperassignment H^* in D such that

$$c_A(H^*) = \min\{c_A(H) : H \text{ is a hyperassignment in } D\},$$

or the information that no hyperassignment in D exists if this is the case.

Analogically to the well-known minimum cost problem on directed graphs, the HAP can be viewed as a minimum cost flow problem on directed hypergraphs: We can add a source and a sink vertex, add arcs from the source to all vertices in V and from all vertices in W to the sink, all with cost 0, and then ask for a minimum cost flow from the source to the sink with $|V|$ units of flow which satisfies a maximum capacity of 1 on all hyperarcs.

However, the usual setting for the minimum cost flow problem on directed hypergraphs, the minimum cost hyperflow problem, [CGS92, CGS97, JMRW92] is a *B-hypergraph* $B = (N, E)$. There, we have one vertex set N and a set of so-called *B-hyperarcs* (backward hyperarcs) E . A B-hyperarc $e = (T_e, h_e) \in E$ is pair of a vertex set $T_e \subset N$ and a vertex $h_e \in N$. Further, a demand vector $b \in \mathbb{R}^N$ is given. Also, weights can be associated with the B-hyperarcs but we omit this here. A hyperflow $f \in \mathbb{R}^E$ is a function, which associates a flow value with each B-hyperarc such that the demand constraint

$$\sum_{e \in E: n = h_e} f_e - \sum_{e \in E: n \in T_e} f_e = b_n$$

is satisfied for all $n \in N$. The problem consists of finding a minimum cost hyperflow f^* , i. e.

$$\sum_{e \in E} c_E(e) f_e^* = \min \left\{ \sum_{e \in E} c_E(e) f_e : f \text{ is a hyperflow in } B \right\},$$

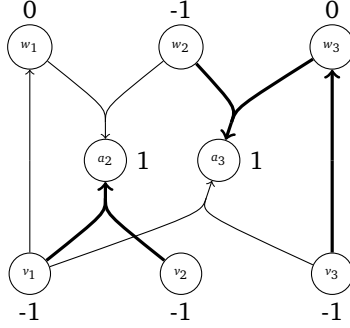


Figure 2: The corresponding B-hypergraph for the hypergraph in figure 1. The corresponding hyperflow is drawn with thick lines. The numbers next to the vertices are the values of the demand vector.

w. r. t. a given cost function $c_E : E \rightarrow \mathbb{R}$ on the B-hyperarcs.

We can state the HAP in (D, c_A) with $D = (V, W, A)$ as a minimum cost hyperflow problem in the following way. Let $A = A_1 \cup A_2$ where $A_1 = \{(h_a, t_a) : \{h_a\} = a \cap V, \{t_a\} = a \cap W, a \in A\}$ is the set of all arcs in A and $A_2 = A \setminus A_1$ are the other hyperarcs. We use the B-hypergraph $B = (N, E)$ with $N = V \cup W \cup A_2$ and $E = E_1 \cup E_2 \cup E'_2$, $E_1 = \{(\{t_a\}, h_a) : a \in A_1\}$, $E_2 = \{(T_a, a) : T_a = a \cap V, h_a = a, a \in A_2\}$, $E'_2 = \{(T_a, h_a) : T_a = a \cap W, h_a = a, a \in A_2\}$. In the cost function, we assign $c_{(\{t_a\}, h_a)} = c_a$ to the B-hyperarcs in E_1 , $c_{(T_a, h_a)} = c_a$ to the B-hyperarcs in E_2 and cost 0 to all B-hyperarcs in E'_2 . We define the demand vector b such that

$$b_n = \begin{cases} -1 & \text{if } n \in V \\ 1 & \text{if } n \in A \\ 1 - |\{a \in A_2 : w \in A\}| & \text{if } n \in W. \end{cases}$$

This leads to a cost-preserving bijection between the hyperassignments in D and hyperflows in B which assigns the following hyperflow values given a hyperassignment $H \subseteq A$ in D :

$$f_e = \begin{cases} 1 & \text{if } e = (\{t_a\}, h_a) \in E_1, a \in H \\ 0 & \text{if } e = (\{t_a\}, h_a) \in E_1, a \notin H \\ 1 & \text{if } e = (T_a, h_a) \in E_2, a \in H \\ 0 & \text{if } e = (T_a, h_a) \in E_2, a \notin H \\ 0 & \text{if } e = (T_a, h_a) \in E'_2, a \in H \\ 1 & \text{if } e = (T_a, h_a) \in E'_2, a \notin H. \end{cases}$$

For an example of the construction see figure 2.

To solve the minimum cost flow problem on directed hypergraphs, a hypergraph network simplex algorithm was introduced in [CGS92]. In contrast to the problem on graphs, for integral input optimal integral solutions do not have to exist. The generalization of the network simplex algorithm can produce non-integral

solutions. Special cases where the set of feasible solutions is integral can be found in [JMRW92], but these usually do not apply to the instances arisen from the HAP

Definition 1.5. For a vertex subset $U \subseteq V \cup W$ we define the *adjacent hyperarcs* $\delta(U) := \{a \in A : a \cap U \neq \emptyset\}$ to be the set of all hyperarcs having at least one vertex in U and write also $\delta(v)$ for $\delta(\{v\})$ if v is a vertex.

Definition 1.6. $D = (V, W, A)$ is called a *partitioned* directed hypergraph with *maximum part size* $d \in \mathbb{N}$ if additionally there exist pairwise disjoint $\leq d$ -element sets V_1, \dots, V_p and W_1, \dots, W_q called the *parts* of H such that $\bigcup_{i=1}^p V_i = V$, $\bigcup_{i=1}^q W_i = W$ and for every $a \in A$ there exist $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$ such that $a \cap V \subseteq V_i$, $a \cap V_k = \emptyset \forall k \neq i$, $a \cap W \subseteq W_j$, $a \cap W_k = \emptyset \forall k \neq j$.

We define the set of all *configurations* for some part $P \in \{V_1, \dots, V_p, W_1, \dots, W_q\}$ to be

$$\mathcal{C}_P = \left\{ C \subseteq A : a_1 \cap a_2 = \emptyset \forall a_1, a_2 \in C \text{ with } a_1 \neq a_2, \bigcup_{a \in C} a \in \{V \cup P, W \cup P\} \right\}.$$

We write \mathcal{C}^V for $\bigcup_{i=1}^p \mathcal{C}_{V_i}$ and \mathcal{C}^W for $\bigcup_{i=1}^q \mathcal{C}_{W_i}$, respectively. Finally, let $\mathcal{C} = \mathcal{C}^V \cup \mathcal{C}^W$.

For an example of a partitioned hypergraph see figure 3.

A hyperassignment in a partitioned hypergraph can be viewed as a system of disjoint representatives. To do this we can merge the all the vertices in each part of V to one vertex and use the configurations in \mathcal{C}^V to describe the HAP instead of hyperarcs. For this problem a generalization of Hall's theorem was proved in [AH00]. This could be used to check whether there exists a hyperassignment in a given partitioned hypergraph. But even more than Hall's theorem for bipartite matchings this is unusable for practical computations because of the very high exponential number of requirements one has to check.

2 Complexity and Structural Results

Theorem 2.1. *Given a directed hypergraph $D = (V, W, A)$ and a cost function $c_A : A \rightarrow \mathbb{R}$ the hypergraph assignment problem is NP-hard and APX-hard, even if H is partitioned with maximum part size 2.*

Proof. We will use the NP-complete [GJ79, page 46] and in its optimization version APX-hard [Kan91] 3-dimensional matching problem. The input of the 3-dimensional matching problem is an undirected hypergraph U , i. e., a pair $U = (X \cup Y \cup Z, E)$, $E \subseteq 2^{X \cup Y \cup Z}$ with $|X| = |Y| = |Z|$,

$$|e \cap X| = |e \cap Y| = |e \cap Z| = 1 \forall e \in E.$$

It asks whether a partitioning of U , i. e., $F \subseteq E$ such that each element from $X \cup Y \cup Z$

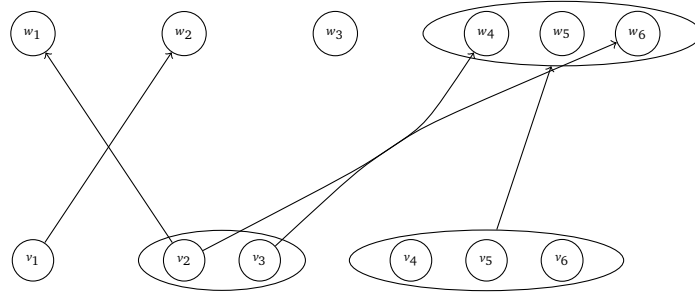


Figure 3: Visualization of a partitioned hypergraph with maximum part size $d = 3$ and parts $\{v_1\}$, $\{v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{w_1\}$, $\{w_2\}$, $\{w_3\}$, $\{w_4, w_5, w_6\}$ and hyperarcs $\{v_1, w_2\}$, $\{v_2, w_1\}$, $\{v_2, v_3, w_4, w_5, w_6\}$, $\{v_4, v_5, v_6, w_4, w_5, w_6\}$. The vertices of each part with more than one vertex are surrounded by an ellipse in the picture. For partitioned hypergraphs we visualize the hyperarcs which connect all the vertices from one part with all the vertices from another part by drawing just an arrow between the two ellipses surrounding the vertices of the part.

is contained in exactly one set in F , exists. Let

$$\begin{aligned}
 X &= \{x_1, \dots, x_n\}, \\
 Y &= \{y_1, \dots, y_n\}, \\
 Z &= \{z_1, \dots, z_n\}, \\
 E &= \left\{ \{x_{i_1}, y_{j_1}, z_{k_1}\}, \dots, \{x_{i_m}, y_{j_m}, z_{k_m}\} \right\}.
 \end{aligned}$$

To prove the theorem we construct an instance of the hypergraph assignment problem having polynomial size in the size of a given 3-dimensional matching problem such that there exists a hyperassignment if and only if there exists a partitioning of U .

Let $D = (V, W, A)$ be a partitioned hypergraph with parts

$$\begin{aligned}
 V_1^E &= \{v_{1x}^E, v_{1y}^E\}, \dots, V_m^E = \{v_{mx}^E, v_{my}^E\}, \\
 V_1^Z &= \{v_{1a}^Z, v_{1b}^Z\}, \dots, V_n^Z = \{v_{na}^Z, v_{nb}^Z\}
 \end{aligned}$$

in V and

$$\begin{aligned}
 W_1^E &= \{w_{1za}^E, w_{1zb}^E\}, \dots, W_m^E = \{w_{mza}^E, w_{mzb}^E\}, \\
 W_1^{XY} &= \{w_{1x}^{XY}, w_{1y}^{XY}\}, \dots, W_n^{XY} = \{w_{nx}^{XY}, w_{ny}^{XY}\}
 \end{aligned}$$

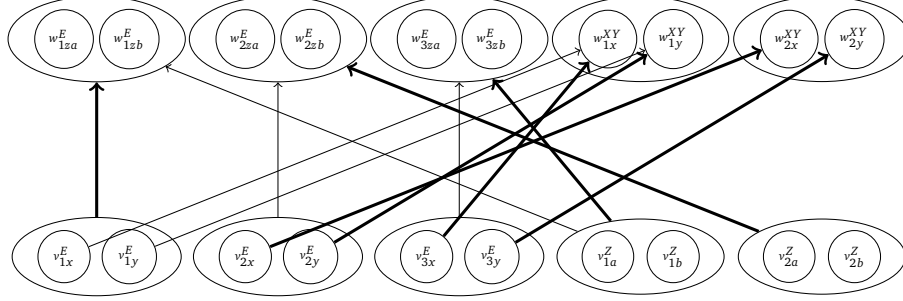


Figure 4: Example for the construction of D in the NP-hardness proof for $U = (X \cup Y \cup Z, E)$ with $n = 2$ and $E = \{\{x_1, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_1, y_2, z_1\}\}$. The partitioning $E = \{\{x_2, y_1, z_2\}, \{x_1, y_2, z_1\}\}$ implies a hyperassignment in D drawn with thick lines.

in W . Let

$$\begin{aligned}
E = & \left\{ \{v_{rx}^E, w_{i,x}^{XY}\} : r \in \{1, \dots, m\} \right\} \\
& \cup \left\{ \{v_{ry}^E, w_{j,y}^{XY}\} : r \in \{1, \dots, m\} \right\} \\
& \cup \left\{ \{v_{ra}^Z, v_{rb}^Z, w_{k,za}^E, w_{k,zb}^E\} : r \in \{1, \dots, m\} \right\} \\
& \cup \left\{ \{v_{rx}^E, v_{ry}^E, w_{rza}^E, w_{rzb}^E\} : r \in \{1, \dots, m\} \right\}
\end{aligned}$$

For an example of D see figure 4.

Now, let $H \subseteq A$ be a hyperassignment in D . The two vertices from each part V_r^Z are always contained in the same hyperarc and A contains exactly one hyperarc containing the vertices $w_{1x}^{XY}, \dots, w_{nx}^{XY}, w_{1y}^{XY}, \dots, w_{ny}^{XY}$ and those from V_1^Z, \dots, V_n^Z . All such hyperarcs contain also at least one vertex from one of the parts V_r^E, W_r^E for some r . Further, by construction for each $r \in \{1, \dots, m\}$ all vertices $v_{rx}^E, v_{ry}^E, w_{rza}^E, w_{rzb}^E$ from the parts V_r^E, W_r^E are contained either in the hyperarc $\{v_{rx}^E, v_{ry}^E, w_{rza}^E, w_{rzb}^E\}$ in A (first type) or, since there are no other hyperarcs containing these vertices, the hyperarcs $\{v_{rx}^E, w_{i,x}^{XY}\}, \{v_{ry}^E, w_{j,y}^{XY}\}, \{v_{ra}^Z, v_{rb}^Z, w_{k,za}^E, w_{k,zb}^E\}$ (second type) are in A . Thus, the set of all $\{x_{i_r}, y_{j_r}, z_{k_r}\} \in E$ for which r is such that H contains the second type of hyperarcs form a partitioning of U .

On the other hand, given a partitioning of U we get a hyperassignment by choosing for H the hyperarcs of the second type exactly for those r for which $\{x_{i_r}, y_{j_r}, z_{k_r}\}$ is in the partitioning and the hyperarc of the first type otherwise. \square

We work only with hypergraph assignment problems on partitioned hypergraphs. First of all, this is the type of hypergraph resulting in models of vehicle rotation planning problems for long distance passenger railways. Further, we can show that this also theoretically makes sense: Every directed hypergraph D with the maximum number of vertices d contained in a hyperarc can be transformed

to a partitioned hypergraph D' with maximum part size d such that there exists a bijection between the hyperassignments in D and D' .

The construction works as follows (for a formal proof see the next theorem). We duplicate each vertex v of D such that every hyperarc in D which contains v is transformed to a hyperarc of the same size in D' using a different duplicate v' of v . Then, we add arcs which ensure that we can always use only exactly one of the duplicates of the original vertex as an element of the corresponding original's adjacent hyperarcs in a hyperassignment in D' .

Theorem 2.2. *Let $D = (V, W, A)$ be a directed hypergraph with maximum hyperarc size d and c_A a cost function. Then there exists a partitioned hypergraph $D' = (V', W', A')$ with maximum part size d and a cost function $c_{A'}$ such that there is a bijection between the set of hyperassignments in D and the set of hyperassignments in D' preserving the costs w. r. t. the cost functions c_A and $c_{A'}$, respectively.*

Proof. Let

$$V' = \{(v, a) : v \in V, a \in A, v \in a\} \cup \{(w, i) : w \in W, i \in \{1, \dots, |\delta(w) - 1|\}\}$$

and

$$W' = \{(w, a) : w \in W, a \in A, w \in a\} \cup \{(v, i) : v \in V, i \in \{1, \dots, |\delta(v) - 1|\}\}.$$

For $v \in V$ define $\{v'_1, \dots, v'_{\delta(v)}\} := \{(v, a) \in V'\}$ to be an arbitrary but fixed order of the vertices constructed from v and, analogically, the w'_i .

For each $a \in A$ we construct a corresponding hyperarc $a' = \{(v, a), (w, a) : v \in V, w \in W\} \in A'$ with cost $c_{A'}(a') = c_A(a)$. Further, we construct arc sets to control that we always choose exactly one of the hyperarcs a' in a hyperassignment in D' which contain a vertex from $V'(v)$ or $W'(w)$. We assign to all these other arcs the cost 0 by $c_{A'}$. We have

$$\begin{aligned} A' := & \{a' : a \in A\} \cup \\ & \{\{v'_i, (v, j)\} : (i - j) \in \{0, 1\}, v \in V, i \in \{1, \dots, \delta(v)\}\} \cup \\ & \{\{(w, j), w'_i\} : (i - j) \in \{0, 1\}, w \in W, i \in \{1, \dots, \delta(w)\}\}. \end{aligned}$$

D' can be partitioned: Every vertex has by construction only one adjacent hyperarc which is not an arc. The tail and head of an arc are always contained in only one part. The maximum hyperarc size of D gives us the maximum part size of D' in the finest possible partitioning.

Now, let H be a hyperassignment in D . For every vertex $v \in V$ let $i(v)$ be the index of $v_{i(v)} = (v, a)$ where a is the unique hyperarc in H containing v and analogically define $i(w)$ for $w \in W$. Let

$$\begin{aligned} H' := & \{a' : a \in H\} \cup \\ & \{v'_i, (v, i) : v \in V, i < i(v)\} \cup \{v'_i, (v, i - 1) : v \in V, i > i(v)\} \cup \\ & \{(v, i), v'_i : w \in W, i < i(w)\} \cup \{(w, i - 1), w'_i : w \in W, i > i(w)\}. \end{aligned}$$

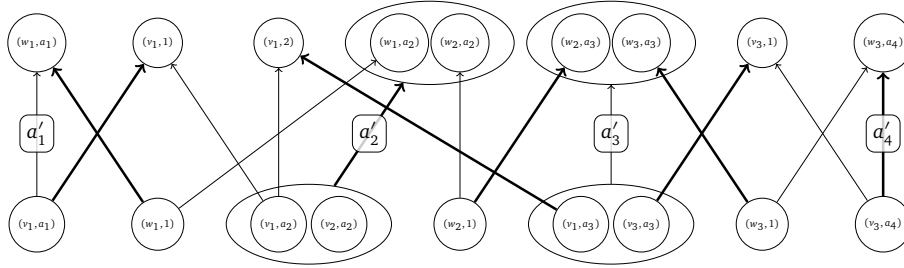


Figure 5: The corresponding partitioned hypergraph for the directed hypergraph from figure 1. The thick hyperarcs show a hyperassignment in both.

By construction, this is a hyperassignment in D' with the same cost w. r. t. $c_{A'}$ as H in D w. r. t. c_A .

On the other hand, given a hyperassignment H' in D' we can construct a hyperassignment H in D using exactly the corresponding hyperarcs from A for the chosen $a' \in H'$ and thus having the same cost. To prove this, we need to show that for every vertex $v \in V \cup W$ of D , H' has only one vertex (v, a) which is contained in a hyperarc a' in H' . For this purpose observe that by construction of D' there is exactly one vertex less of the type (v, i) than of the type (v, a) and the vertices of the first type have only adjacent hyperarcs which are arcs and whose other end is one of the vertices of the second type. The only other adjacent hyperarc to (v, a) is a' . \square

3 Clique Inequalities and an Extended Formulation

The canonical integer linear program for the HAP is the following.

$$\begin{aligned}
 & \underset{x \in \mathbb{R}^A}{\text{minimize}} && \sum_{a \in A} c_A(a) x_a && \text{(HAP)} \\
 & \text{subject to} && \sum_{a \in \delta(v)} x_a = 1 && \forall v \in V \cup W && \text{(i)} \\
 & && x \geq 0 && \text{(ii)} \\
 & && x \in \mathbb{Z}^A && \text{(iii)}
 \end{aligned}$$

It is easy to see that (HAP) is a valid formulation for the hypergraph assignment problem. For a formal proof see [Hei10].

Definition 3.1. A *clique* in (the conflict graph of) a directed hypergraph $D = (V, W, A)$ is a set $Q \subseteq A$ of hyperarcs such that every two hyperarcs $a_1, a_2 \in Q$ have at least one vertex in common, i. e., $a_1 \cap a_2 \neq \emptyset$. A clique Q is a *maximal clique* if there is no clique $Q' \supset Q$ containing Q and additionally other hyperarcs.

We call $\sum_{a \in Q} x_a \leq 1$ a clique inequality.

Every feasible solution of (HAP) fulfills every clique inequality since by definition no two elements of a clique can be both in a hyperassignment.

We want to state an extended integer linear programming formulation for the HAP. After proving its correctness we will show that it implies all clique inequalities. Clique inequalities are important since for HAPs with real application data in partitioned hypergraphs (there the maximum part size is 7) the gap between the optimum solution and the solution of the linear programming relaxation of (HAP) could be highly reduced by inserting clique inequalities [BH11].

$$\begin{aligned}
& \underset{x \in \mathbb{R}^A, y \in (\mathbb{R}^{\mathcal{C}^V} \times \mathbb{R}^{\mathcal{C}^W})}{\text{minimize}} && \sum_{a \in A} c_A(a) x_a && \text{(HAP_ext)} \\
& \text{subject to} && \sum_{C \in \mathcal{C}^V : a \in C} y_C = x_a && \forall a \in A && \text{(i)} \\
& && \sum_{C \in \mathcal{C}^W : a \in C} y_C = x_a && \forall a \in A && \text{(ii)} \\
& && \sum_{a \in \delta(v)} x_a = 1 && \forall v \in V \cup W && \text{(iii)} \\
& && x, y \geq 0 && && \text{(iv)} \\
& && x \in \mathbb{Z}^A && && \text{(v)} \\
& && y \in \mathbb{Z}^{\mathcal{C}^V} \times \mathbb{Z}^{\mathcal{C}^W} && && \text{(vi)}
\end{aligned}$$

Definition 3.2. Let

$$P_{\text{LP}}(\text{HAP_ext}) := \{(x, y) \in \mathbb{R}^A \times \mathbb{R}^{\mathcal{C}^V} \times \mathbb{R}^{\mathcal{C}^W} : (\text{HAP_ext}) \text{ (i)-(iv)}\}$$

be the polyhedron associated with the LP relaxation of (HAP). Further, let

$$\pi_x : \mathbb{R}^A \times \mathbb{R}^{\mathcal{C}^V} \times \mathbb{R}^{\mathcal{C}^W} \rightarrow \mathbb{R}^A, \quad (x, y) \mapsto x$$

be a mapping that produces a projection onto the coordinates of all the hyperarc variables and

$$\pi_{x^P} : \mathbb{R}^A \times \mathbb{R}^{\mathcal{C}^V} \times \mathbb{R}^{\mathcal{C}^W} \rightarrow \mathbb{R}^{\delta(P)}, \quad (x, y) \mapsto (x_a)_{a \in \delta(P)}$$

be a mapping that produces a projection onto the coordinates of all the variables for hyperarcs adjacent to the part P .

The main idea for the correctness proof is the following.

Lemma 3.3. *Let $D = (V, W, A)$ be a partitioned hypergraph. Then, for every hyperassignment H of D there exist sets $\mathcal{C}'_V \subseteq \mathcal{C}_V$ and $\mathcal{C}'_W \subseteq \mathcal{C}_W$ of configurations such that for all $a \in A$ $|\{C \in \mathcal{C}'_V : a \in C\}| = |\{C \in \mathcal{C}'_W : a \in C\}| = |H \cap \{a\}|$, i. e., \mathcal{C}'_V and \mathcal{C}'_W are sets of configurations whose disjoint union is exactly H , respectively.*

Proof. By the definition of a hyperassignment, for $\mathcal{C}'_V := \{C'_p : P \in \{V_1, \dots, V_p\}\}$ and $\mathcal{C}'_W := \{C'_p : P \in \{W_1, \dots, W_p\}\}$ with $C'_p = \delta(P) \cap H$ the required condition holds by construction. \square

This easily implies the correctness proof.

Theorem 3.4. *Given a directed hypergraph $D = (V, W, A)$ and a cost function $c_A : A \rightarrow \mathbb{R}$, there are bijections between the feasible solutions of (HAP_ext), and hyperassignments of D . The optimum value of (HAP_ext) is equal to the cost of the minimum cost hyperassignment in D w. r. t. c_A if it exists and to ∞ otherwise.*

Proof. Since the projection $\pi_x(x, y)$ of every feasible solution (x, y) of (HAP_ext) to its x -values is a feasible solution of (HAP) (the constraints from (HAP) are all contained in (HAP_ext)) and the objective functions are the same, the previous lemma implies that (HAP_ext) is a correct formulations for the HAP: For $C \in \mathcal{C}$ set $y_C = 1$ if $C \in \mathcal{C}'_V \cup \mathcal{C}'_W$ and $y_C = 0$ otherwise and use the same x -values as in (HAP). \square

Lemma 3.5. *Let $D = (V, W, A)$ be a partitioned hypergraph. Then for every clique Q in D there exists a part P such that $Q \subseteq \delta(P)$, i. e., every clique is a subset of the adjacent hyperarcs of the vertices of some part in D .*

Proof. Let Q be a nonempty clique in D (otherwise the lemma is trivial) containing some hyperarc a_1 where $a_1 \cap V$ is contained in some part V_1 and $a_1 \cap W$ is contained in some part W_1 . Thus, for every other hyperarc a in Q either $a \cap V \subseteq V_1$ or $a \cap W \subseteq W_1$, otherwise they would have an empty intersection. Assume that Q contains some hyperarc a_2 with $a_2 \cap V \not\subseteq V_1$ and some hyperarc a_3 with $a_3 \cap W \not\subseteq W_1$. Now, $a_2 \cap a_3$ must be empty since both $a_2 \cap V$, $a_3 \cap V$ and $a_2 \cap W$, $a_3 \cap W$ are contained in different parts, which contradicts the assumption that Q is a clique. Hence, either V_1 or W_1 contains $a \cap V$ or $a \cap W$ for all hyperarcs $a \in Q$. \square

Theorem 3.6. *Let $D = (V, W, A)$ be a partitioned hypergraph and $Q \subseteq A$ a clique. Then, the clique inequality*

$$\sum_{a \in Q} x_a \leq 1$$

is a valid inequality for every feasible solution of the LP relaxation of (HAP_ext).

Proof. First of all, observe that $|Q \cap C| \leq 1$ for every $C \in \mathcal{C}$ because if two hyperarcs are in C , then they are disjoint, which implies that they cannot be both in Q .

By Lemma 3.5, $Q \subseteq \delta(P)$ for some part P . W.l. o. g., let $P \subseteq V$ (the formulation is symmetric in V, W). Let v be some vertex in P . Then, by (HAP_ext) (iii)

$$1 = \sum_{a \in \delta(v)} x_a,$$

which is the sum of the right hand sides of (HAP_ext) (i) for $a \in \delta(v)$. Considering the left hand sides we get

$$= \sum_{C \in \mathcal{C}^V} |\delta(v) \cap C| \cdot y_C.$$

Since every $a \in \delta(v)$ is contained in V and exactly one such a is contained in every $C \in \mathcal{C}_p^V$, we get

$$= \sum_{C \in \mathcal{C}_p} y_C.$$

The observation from above implies

$$\begin{aligned} &= \sum_{a \in Q} \sum_{C \in \mathcal{C}_p: a \in C} y_C + \sum_{C \in \mathcal{C}_p: C \cap Q = \emptyset} y_C \\ &\geq \sum_{a \in Q} \sum_{C \in \mathcal{C}_p: a \in C} y_C \end{aligned}$$

by (HAP_ext) (iv). Finally, applying (HAP_ext) (i) again

$$= \sum_{a \in Q} x_a.$$

□

The configuration formulation implies much more than the clique inequalities. It implies all valid inequalities one can know from considering all adjacent hyperarcs to one part.

Theorem 3.7. *Let $D = (V, W, A)$ be a partitioned hypergraph and let P be some part. Then, $\pi_{x^p}(P_{LP}(\text{HAP_ext}))$ is a subset of*

$$X := \text{conv} \left\{ (x_a)_{a \in \delta(P)} \in \mathbb{Z}^{\delta(P)} : \sum_{a \in \delta(v) \cap \delta(P)} x_a \leq 1 \ \forall v \in V \cup W, \sum_{a \in \delta(v)} x_a = 1 \ \forall v \in P \right\}.$$

Proof. Let $(x, y) \in P_{LP}(\text{HAP_ext})$ be a feasible solution of the LP relaxation of (HAP_ext). For $C \in \mathcal{C}_p$, define $x(C) \in \mathbb{R}^{\delta(P)}$ by

$$x(C)_a = \begin{cases} 1 & \text{if } a \in C \\ 0 & \text{otherwise.} \end{cases}$$

This implies that every point in $\pi_{x^p}(P_{LP}(\text{HAP_ext}))$ is the linear combination of points $x(C) \in X$, since $x(C)$ has only integral coordinates 0 or 1 and satisfies by the definition of a configuration the equalities and inequalities defining X . □

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