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Minimum Cost Hyperassignments with Applications to ICE/IC Rotation Planning

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Abstract

Vehicle rotation planning is a fundamental problem in rail transport. It decides how the railcars, locomotives, and carriages are operated in order to implement the trips of the timetable. One important planning requirement is operational regularity, i. e., using the rolling stock in the same way on every day of operation. We propose to take regularity into account by modeling the vehicle rotation planning problem as a minimum cost hyperassignment problem (HAP). Hyperassignments are generalizations of assignments from directed graphs to directed hypergraphs. Finding a minimum cost hyperassignment is $\mathcal{NP}$-hard. Most instances arising from regular vehicle rotation planning, however, can be solved well in practice. We show that, in particular, clique inequalities strengthen the canonical LP relaxation substantially.

1 Introduction

Vehicle rotation planning, also known as rolling stock roster planning, deals with the allocation of vehicles to trips in a timetable, see [4]. We focus in this article on a basic version of the problem that deals with the construction of a cyclic schedule for a standard week, disregarding maintenance, train composition, and some other side constraints. In this setting, we are looking for an assignment of each trip to a follow-on trip which will be serviced by the same vehicle.

Such an assignment is considered operationally *regular*, if many timetabled trips are followed by the same timetabled trips on as many days of the standard week as possible, i. e., if trip 4711 is followed by trip 4712 on Monday, this should also be the case on Tuesday, Wednesday, etc. (provided that these trips exist on these days). In practice, most trips appear on almost every day of operation. In other words, the weekly timetable is largely regular, such that there is a good chance to also construct a regular vehicle rotation plan.

Regular vehicle rotation plans are easier to communicate and understand than non-regular ones. They standardize operations, increase robustness, and facilitate real-time scheduling. It is therefore essential to include regularity in vehicle rotation planning models. This can be achieved by considering a suitable concept of hyperassignments, as we will show in the following sections.

2 Notions on Directed Hypergraphs

We start by recalling some basic notions on directed hypergraphs, see also [1] for an introduction (with slightly different requirements for hyperarcs).

Definition 2.1 (Directed Hypergraph, Directed Graph)**.** A *directed hypergraph* D is a pair (V, A) consisting of a *vertex* set V and a set $A \subseteq 2^V \times 2^V$ of *hyperarcs* (T_a, H_a) such that $T_a, H_a \neq \emptyset$. We call T_a the *tail* of the hyperarc $a \in A$ and H_a the *head* of a. A hyperarc a is called an *arc* if $|H_a| = |T_a| = 1$. If all hyperarcs are arcs, we call D a *directed graph.*

Definition 2.2 (Outgoing and Ingoing Hyperarcs)**.** Let $D = (V, A)$ be a directed hypergraph. For $W \subseteq V$, $B \subseteq A$ we define

$$\delta_B^{\text{out}}(W) := \{a \in B : T_a \cap W \neq \emptyset\} \quad \text{and} \quad \delta_B^{\text{in}}(W) := \{a \in B : H_a \cap W \neq \emptyset\}$$

to be the *outgoing* and *ingoing hyperarcs* of W, respectively. For $\delta_B^{\text{out}}(\{v\})$ and $\delta_B^{\text{in}}(\{v\})$ we simply write $\delta_B^{\text{out}}(v)$ and $\delta_B^{\text{in}}(v)$, respectively. If no set B is given in the index of δ^{out} or δ^{in}, B is assumed to be the full hyperarc set of the hypergraph in question.

Definition 2.3 (Cost Function)**.** Given a set S, a *cost function* is a function $c_S : S \to \mathbb{R}$. For $T \subseteq S$ we define

$$c_S(T) := \sum_{s \in T} c_S(s).$$

We are now ready to propose hyperassignments.

Definition 2.4 (Circulation, Hyperassignment)**.** Let $D = (V, A)$ be a directed hypergraph. $Z \subseteq A$ is called *circulation* in D if $|\delta_Z^{\text{out}}(v)| = |\delta_Z^{\text{in}}(v)|$ for every $v \in V$. A circulation $H \subseteq A$ in D is called a *hyperassignment* if for each $v \in V$ $|\delta_H^{\text{out}}(v)| = 1$.

Problem 2.5 (Hyperassignment Problem (HAP))**.**

Input: A pair (D, c_A) consisting of a directed hypergraph $D = (V, A)$ and a cost function $c_A : A \to \mathbb{R}$.

Output: minimum cost hyperassignment in D w. r. t. c_A, i. e., a hyperassignment H^* in D such that $c_A(H^*) = \min\{c_A(H) : H \text{ is a hyperassignment in } D\}$, or the information that no hyperassignment in D exists if this is the case.

An ILP formulation of the hyperassignment problem is as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{a \in A} c_A(a) x_a & & \text{(HAP)} \\
\text{subject to} \quad & \sum_{a \in \delta^{\text{in}}(v)} x_a - \sum_{a \in \delta^{\text{out}}(v)} x_a = 0 & \forall v \in V \quad & \text{(i)} \\
& \sum_{a \in \delta^{\text{out}}(v)} x_a = 1 & \forall v \in V \quad & \text{(ii)} \\
& x \geq 0 & & \text{(iii)} \\
& x \in \mathbb{Z}^A & & \text{(iv)}
\end{aligned}
$$

3 Complexity of the Hyperassignment Problem (HAP)

We study in this section the complexity of the HAP. Despite its simple form, the problem turns out to be $\mathcal{NP}$-hard even for directed hypergraphs with head and tail cardinality 2. In fact, already the LP-relaxation can be numerically complex.

Theorem 3.1. *Given a directed hypergraph $D = (V, A)$ satisfying $|T_a| = |H_a| \leq 2$ for all $a \in A$ and a cost function $c_A : A \to \mathbb{R}$, HAP with input (D, c_A) is $\mathcal{NP}$-hard.*

Proof. The 3-dimensional matching problem is $\mathcal{NP}$-complete (see [2], page 46). Given an undirected hypergraph $U = (N \cup O \cup P, E)$, $|N| = |O| = |P|$, $|e \cap N| = |e \cap O| = |e \cap P| = 1 \quad \forall e \in E$, it asks whether a partitioning of U into a subset of elements of E exists.

Construct a directed hypergraph $D = (V, A)$ with $V = N \cup O \cup \{\{p\} \times \{0, 1\} : p \in P\}$ and

$$\begin{aligned} A &= A_1 \cup A_2, \\ A_1 &= \Big\{ \big((e \cap N) \cup (e \cap O), \{(e \cap P, 0), (e \cap P, 1)\}\big) : e \in E \Big\}, \\ A_2 &= \Big\{ \big(\{(e \cap P, 0)\}, e \cap N\big), \big(\{(e \cap P, 1)\}, e \cap O\big) : e \in E \Big\}. \end{aligned}$$

This can be done in polynomial time and the resulting hypergraph satisfies $|T_a| = |H_a| \leq 2$ for all $a \in A$. Choose $c_A : A \to \mathbb{R}, c_A \equiv 0$. Then HAP with input (D, c_A) returns a hyperassignment with cost 0 if and only if a partitioning of U exists. The chosen hyperarcs from A_1 correspond to the edges $e \in E$ in the partitioning of U. This proves the theorem. □ □

The determinant of submatrices of the coefficient matrix of an ILP is a complexity indicator. For example, if the coefficient matrix is totally unimodular, the LP relaxation is integral. In general, by Cramer's rule, the denominator of the variable values in a basic solution of an LP is at most the determinant of the basis matrix (if the numerators and denominators are relatively prime). For the LP relaxation of the (HAP), the denominators, and therefore also the determinants of basis matrices, can be arbitrarily large. This is the case even if one allows only hyperarcs with head and tail cardinality at most two. The following example illustrates this fact.

Let s be a positive integer and consider the following directed and head cardinality at most two. We want $V = \{u, v_i, w_i, : i \in \{0, \ldots, s-1\}\}$ and $A = A_1 \cup A_2$ with

$$\begin{aligned} A_1 &= \{(\{v_i, w_i\}, \{v_i, w_i\}) : i \in \{0, \ldots, s-1\}\}, \\ A_2 &= \big\{ \big(\{u, v_i\}, \{w_{(i+1) \mod s}, u\}\big), (\{w_i\}, \{u\}), (\{u\}, \{v_i\}) : i \in \{0, \ldots, s-1\} \big\}. \end{aligned}$$

The only feasible solution of the LP relaxation of (HAP) is $x_a = \frac{2s-1}{2s}$ for all $a \in A_1$ and $x_a = \frac{1}{2s}$ for all $a \in A_2$. Thus the determinant of the basis matrix is at least $2s$.

An upper bound on the modulus of the determinant is $\prod_{a \in A} |T_a|$ if the hypergraph $D = (V, A)$ can be extended to a graph based hypergraph by adding arcs (this is also true for the hypergraph of the example). This is the case if head and tail cardinalities are equal for each hyperarc. Since every column of the basis matrix can be represented as the sum of columns for the corresponding arcs and basis matrices of (HAP) for directed graphs are totally unimodular, i. e., have determinant with modulus 0 or 1, we can apply the multilinearity of the determinant until we get only such matrices and obtain the bound.

We remark that one can also prove that the gap between the optimum solution of the LP relaxation of (HAP) and the minimum cost hyperassignment can be arbitrarily large. An example of such a HAP instance on a hypergraph with only 6 vertices is given in [3]. Moreover, the reduction from the 3-dimensional matching problem implies that HAP is APX-complete.

4 Hyperassignment Model for Regular Vehicle Rotation Planning

Considering regularity in vehicle rotation planning leads to the minimum cost hyperassignment problem, as we will show now.

Suppose we are given a weekly repeating schedule for long distance trains with all trips that a railway company wants to operate. A *trip* is characterized by its departure day, departure time, departure location, arrival location, and its duration.

Every trip has to be serviced by a vehicle. Between arrival and departure there may be several intermediate stops, but the vehicle must not change during the trip.

After servicing a trip, a vehicle does a *deadhead trip* (possibly of distance zero) from the arrival location of the trip to the departure location of the next trip it services. This deadhead trip has some duration. Afterwards, when the weekday and departure time of the next trip has come, the vehicle services this next trip.

A *vehicle rotation plan* is an assignment of each trip to another follow-on trip. This assignment tells every vehicle which trip it has to service next. Since the schedule is periodic, the sequence of trips for every vehicle is periodic, too. The period is a positive integral multiple of a week.

The *cost* of a pair of trips in the assignment depends on the duration and distance of the associated deadhead trip and on the duration of the breaks before and after the deadhead. Clearly, the longer the deadheads and the breaks are, the more vehicles the railway company needs.

The aim is to find an assignment of minimum cost. It is apparent that vehicle rotation planning as explained so far can be formulated in terms of an assignment problem in a directed graph $D = (V, A)$, where the vertices V are the trips and there is an arc $a \in A$ from every trip to every possible follow-on trip.

Now we include operational regularity. We associate with each trip $v \in V$ a departure weekday $d_v \in \{\mathrm{Mo}, \ldots, \mathrm{Su}\}$. We group all trips that differ only in the departure weekday and call such a set a *train*.

Given an assignment, we can count for each train the number of *non-regular deadheads* in the trips assigned to the trips of the train. Two deadheads are non-regular if the trains the next trips belong to are different or the breaks have a different length. Otherwise they are *regular*. The less the number of unequal deadheads in the vehicle rotation plan, the higher the operational regularity.

This criterion can be modeled in terms of a hyperassignment problem. To this purpose, we define hyperarcs as follows. For each possible set of deadhead trips $a_1, \ldots, a_d \in A$ with the same length between the trips of two trains we introduce a hyperarc $a \in A$ where $T_a = \bigcup_{i=1}^{d} T_{a_i}$ are the timetabled trips of the first train and $H_a = \bigcup_{i=1}^{d} H_{a_i}$ are the timetabled trips of the second train between which the deadhead trips take place. To reward regularity, we set $c_a < \sum_{i=1}^{d} c_{a_i}$. Then we can choose the hyperarc a for the hyperassignment if we would use $a_1, \ldots, a_d$ in the assignment to get lower costs. The difference $c_a - \sum_{i=1}^{d} c_{a_i}$ is the bonus for operational regularity.

Table 1: Computational results with real-world vehicle rotation planning problems using (HAP) and CPLEX 12.1.0. The LP-IP gap is given by $1 - \frac{L}{I}$, where L is the optimum value of the LP relaxation and I is the best integral solution found by CPLEX or Gurobi 3.0.0. The root gap is $1 - \frac{R}{I}$, where R is the optimum value of the LP relaxation before branching but after applying the cuts described in the seventh and eighth column of the table. The root improvement is $\frac{R}{L} - 1$. (*) means that the calculation was aborted.

# rows ($2 \cdot \|V\|$)	# columns ($\|A\|$)	nonzeros	LP-IP gap	root gap	root improvement	# clique cuts	# other cuts	root run time (sec.)
534	52056	140081	11.16 %	6.81 %	4.90 %	160	14	8
620	80477	236020	8.72 %	0.00 %	9.54 %	120	2	29
812	102375	216566	0.38 %	0.18 %	0.20 %	24	16	40
1128	267542	732134	4.59 %	0.26 %	4.55 %	263	0	160
1310	363513	1006024	7.85 %	0.22 %	8.28 %	378	2	270
1496	469932	1369224	18.70 %	1.86 %	20.71 %	809	0	971
1696	618348	1787078	5.17 %	0.16 %	5.28 %	925	0	1705
1746	649525	1859898	7.52 %	4.88 %	2.86 %	563	0	1129
1798	647650	1822718	13.60 %	0.95 %	14.65 %	537	0	1099
1798	647650	1822718	13.35 %	0.62 %	14.69 %	604	0	873
2006	855153	2491372	5.76 %	0.68 %	5.39 %	1025	0	2490
2260	1079535	3138752	9.89 %	2.03 %	8.73 %	954	0	5483
2502	1290750	3680124	7.06 %	0.76 %	6.79 %	801	0	4583
2620	1432355	4187296	9.05 %	1.15 %	8.68 %	1068	0	7910
2624	1439453	4087042	14.17 %	5.23 %	10.41 %	951	0	(*) 14400

5 Computational Results

Solving practical instances of HAP using the ILP (HAP) showed that the separation of clique inequalities associated with this formulation is very important. Our computational results (on an Intel Core i7-870) are summarized in Table 1. It can be seen that the duality gap between the LP solution and the ILP solution is very small if one adds enough clique inequalities, and that it can be large without them. The LP bound improved up to 20 % by adding clique inequalities and the LP-IP gap was reduced in many cases to less than 1 %.

All instances stem from a project with DB Fernverkehr AG, which deals with the optimization of long distance passenger railway transport in Germany. More precisely, they arise from cyclic weekly schedules of ICE 1 trains. Our results show that the HAP is computationally well-behaved. This model therefore provides an excellent basis for incorporating regularity requirements into more complex large-scale real-world vehicle rotation planning models.

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