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# A Note on Menger's Theorem for Hypergraphs\*

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## Abstract

We prove the companion Theorem to Menger's Theorem for hypergraphs. This result gives rise to a new class of blocking pairs of ideal matrices, that generalize the incidence matrices of cuts and paths.

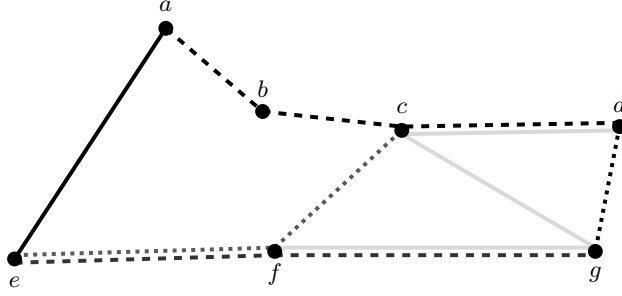
## 1 Introduction

Let  $H = (V, \mathcal{E})$  be an undirected *hypergraph* with  $n$  vertices  $V = \{v_1, \dots, v_n\}$  and  $m$  hyperedges  $\mathcal{E} = \{e_1, \dots, e_m\}$ , where  $e_i \subseteq V$ ,  $i = 1, \dots, m$ . We want to allow for *parallel* edges, i.e., it is possible that  $e_i = e_j$  for two edges  $e_i$  and  $e_j$ ,  $i \neq j$ ; we say that edges  $e_i$  and  $e_j$ ,  $i \neq j$ , are *distinct*. Let  $s$  and  $t$  be two different vertices of  $H$ . An *st-path*  $(v_{i_0}, e_{j_1}, v_{i_1}, \dots, e_{j_k}, v_{i_k})$  in  $H$  is an alternating sequence of mutually different nodes  $v_{i_h}$ ,  $h = 0, \dots, k$ , and mutually distinct hyperedges  $e_{j_h}$ ,  $h = 1, \dots, k$ , such that  $v_{i_0} = s$ ,  $v_{i_{h-1}}, v_{i_h} \in e_{j_h}$  for all  $h = 1, \dots, k$ , and  $v_{i_k} = t$ . The sequence  $\mathcal{P} = (e_{j_1}, \dots, e_{j_k})$  of hyperedges in an *st-path* is an *st-hyperpath*; we write  $e_{j_i} \in \mathcal{P}$ ,  $i = 1, \dots, k$ , and we say that an *st-hyperpath connects*  $s$  and  $t$ . A set of hyperedges  $\mathcal{E}' \subseteq \mathcal{E}$  is an *st-hypercut* if  $s$  and  $t$  are connected in  $H = (V, \mathcal{E})$ , but not in  $H' = (V, \mathcal{E} \setminus \mathcal{E}')$ . Let  $\delta_H(W)$ ,  $W \subset V$ , be the set of hyperedges that contain at least one node in  $W$  and one node in  $V \setminus W$ , i.e.,  $\delta_H(W) = \{e \in \mathcal{E} \mid e \cap W \neq \emptyset, e \cap (V \setminus W) \neq \emptyset\}$ . Then  $\delta_H(W)$  is an *st-hypercut* for every  $s \in W$  and  $t \notin W$ , provided that  $s$  and  $t$  are connected. A hypergraph is *connected* if each pair of nodes  $s, t \in V$  is connected. A hypergraph  $H = (V, \mathcal{E})$  is *k-hyperedge connected* if  $H = (V, \mathcal{E} \setminus \mathcal{E}')$  is connected for every set of  $k - 1$  hyperedges  $\mathcal{E}' \subseteq \mathcal{E}$ , i.e., if the removal of  $k - 1$  arbitrary hyperedges does not disconnect  $H$ . Let  $c : \mathcal{E} \rightarrow \mathbb{N}$  be a weight function on the hyperedges with non-negative integer values. The *capacity* of a hypercut  $\mathcal{E}'$  w.r.t.  $c$  is  $c(\mathcal{E}') = \sum_{e \in \mathcal{E}'} c_e / \text{length}$  of a hyperpath  $\mathcal{P}$  w.r.t.  $c$  is  $c(\mathcal{P}) = \sum_{e \in \mathcal{P}} c_e$ . An *st-hypercut packing/st-hyperflow* w.r.t.  $c$  is a set  $\{\mathcal{E}_1, \dots, \mathcal{E}_k\}$  of *st-hypercuts*/a set  $\{p_1, \dots, p_k\}$  of *st-hyperpaths* such that each hyperedge  $e$  is contained in at most  $c_e$  hypercuts/hyperpaths; again, we allow for parallel hypercuts/hyperpaths. Two hypercuts/hyperpaths are *disjoint* if their edges are mutually distinct. The *value* or *cardinality* of an *st-hypercut packing/st-hyperflow* is the number  $k$  of hypercuts/hyperpaths it contains.

Figure 1 gives an example of a hypergraph with six hyperedges. Here, we illustrate a hyperedge by connecting its nodes in an arbitrary order (i.e., we represent

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**Figure 1:** Example of a hypergraph  $G = (V, \mathcal{E})$  with six hyperedges ( $\mathcal{E} = \{e_1 = \{a, b, c, d\}, e_2 = \{e, f, g\}, e_3 = \{a, e\}, e_4 = \{e, f, c\}, e_5 = \{g, d\}, e_6 = \{f, g, c, d\}\}$ ). An  $ed$ -hyperpath  $\mathcal{P} = (e_4, e_1)$  with associated  $ed$ -path  $(e, e_4, c, e_1, d)$  contains the hyperedges  $e_4$  and  $e_1$ ; an example of an  $ed$ -hypercut is  $\{e_2, e_3, e_4\}$  (hyperedges depicted as paths).

hyperedges as paths).

Menger's theorem is known to hold for hypergraphs ([1, 2]).

**Proposition 1 (Menger's Theorem for Hypergraphs).** *The minimum cardinality of an  $st$ -hypercut is equal to the maximum number of hyperedge-disjoint  $st$ -hyperpaths.*

**Corollary 2.** *A hypergraph is  $k$ -hyperedge connected if and only if there are  $k$  hyperedge-disjoint hyperpaths between each pair of nodes.*

Multiplying hyperedges yields the following weighted version of Proposition 1.

**Corollary 3 (Max-Flow-Min-Cut Theorem for Hypergraphs).** *The minimum capacity of an  $st$ -hypercut is equal to the maximum value of an  $st$ -hyperflow.*

## 2 A Companion to Menger's Theorem

We will prove in this section a companion theorem to Proposition 3, which arises from interchanging the roles of hyperpaths and hypercuts.

**Proposition 4.** *The length of a shortest  $st$ -hyperpath is equal to the maximum value of an  $st$ -hypercut packing.*

Considering unit costs yields our main result, a companion to Menger's Theorem for hypergraphs.

**Proposition 5.** *The minimum cardinality of an  $st$ -hyperpath is equal to the maximum number of hyperedge-disjoint  $st$ -hypercuts.*

To prove Proposition 4 we first consider the following linear program

$$\begin{aligned}
 \min \quad & \sum_{e \in \mathcal{E}} c_e x_e \\
 \text{s.t.} \quad & \sum_{e \in \delta_H(W)} x_e \geq 1 \quad \forall s \in W \subseteq V \setminus \{t\} \\
 & x_e \geq 0 \quad \forall e \in \mathcal{E}.
 \end{aligned} \tag{1}$$

Here,  $x_e$  is a variable which indicates how often the hyperedge  $e$  is chosen. The inequalities guarantee that at least one hyperedge crosses each  $st$ -hypercut. Proposition 4 follows from showing that the inequality system of program (1) plus the upper bounds  $x \leq 1$  is TDI.

**Proposition 6.** *The inequality system of program (1) is TDI.*

*Proof.* It suffices to consider  $st$ -connected hypergraphs, because otherwise program (1) is infeasible, and a nonnegative cost vector  $c$ , because otherwise program (1) is unbounded. Consider the dual of program (1):

$$\begin{aligned} \max \quad & \sum_{W \in \mathcal{W}} y_W \\ \text{s.t.} \quad & \sum_{W \in \mathcal{W}: e \in \delta_H(W)} y_W \leq c_e \quad \forall e \in \mathcal{E} \quad (2) \\ & y_W \geq 0 \quad \forall W \in \mathcal{W}, \end{aligned}$$

where  $\mathcal{W} = \{W \subseteq V \setminus \{t\} \mid s \in W\}$ . We use the primal-dual *shortest hyperpath Algorithm 1* to construct optimal solutions  $x$  and  $y$  for the linear programs (1) and (2), respectively. The algorithm generalizes Dijkstra's algorithm to the hypergraph setting. It computes a shortest hyperpath from node  $s$  to node  $t$  with respect to the cost function  $c$ . Note that the algorithm does not compute a tree in a hypergraph (in contrast to Dijkstra's algorithm for graphs).

The distances from node  $s$  are stored in node labels  $d(v)$ , and the nodes  $v_i$  are marked in the order of increasing distance from the root; their unions  $W_i = \bigcup_{1 \leq j \leq i} \{v_j\}$  produce a sequence of nested hypercuts  $\delta_H(W_i)$ . The shortest  $st$ -hyperpath and the set of nested hypercuts give rise to primal and dual solutions  $x$  and  $y$  for programs (1) and (2), respectively. In the following we show that  $x$  and  $y$  are integral, feasible, and that the associated objective values are equal.

Consider the nodes  $s = v_1, v_2, \dots, v_h = t$  as marked by Algorithm 1 in line 18 and the node sets  $W_0, \dots, W_h$  as constructed in line 20. The following properties are easy to see:

1.  $W_i = \{v_1, \dots, v_i\}$  for  $i = 1, \dots, h$  and  $\emptyset = W_0 \subset W_1 \subset \dots \subset W_h$ . For each  $W_i$ ,  $i = 1, \dots, h-1$ , we have  $s \in W_i$  and  $t \notin W_i$ , i.e.,  $\delta_H(W_i)$  is an  $st$ -hypercut. (For  $i = h$  we have  $s, t \in W_h$ , i.e.,  $\delta_H(W_h)$  is not an  $st$ -hypercut.)
2.  $d(v_{i-1}) \leq d(v_i)$  for  $i = 1, \dots, h$ . (For  $i = 1$ , note that  $v_0 := s$  and  $d(s) := 0$  is set in line 2, and  $v_1 := s$  is set in the first pass through line 18.)
3.  $d(t) < \infty$ . ( $H$  is  $st$ -connected.)

We first show that  $y$  is a solution of program (2). Property 2 implies  $y \geq 0$ . In fact, the variables  $y_W$  are zero for all  $W \neq W_1, \dots, W_{h-1}$ . It remains to show that

$$\sum_{W \in \mathcal{W}: e \in \delta_H(W)} y_W \leq c_e \quad \forall e \in \mathcal{E}. \quad (3)$$

Let  $e \in \mathcal{E}$ . If  $v_i \notin e$  for all  $i = 1, \dots, h-1$ , then  $e \notin \delta_H(W_i)$ ,  $i = 1, \dots, h-1$ , i.e.,  $\sum_{W \in \mathcal{W}: e \in \delta_H(W)} y_W = 0 \leq c_e$ . Otherwise let  $1 \leq i < h$  be the minimal

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**Algorithm 1:** Primal-dual shortest hyperpath algorithm.

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**Input** : An  $st$ -connected hypergraph  $H = (V, \mathcal{E})$ , with costs  $c \in \mathbb{R}_+^{\mathcal{E}}$  on the hyperedges, two nodes  $s, t \in V$ .

**Output:** A minimum cost  $st$ -hyperpath  $\mathcal{P} \subseteq \mathcal{E}$ . Values for  $x$  and  $y$  for programs (1) and (2).

```

1  $d(s) := 0, d(v) := \infty \forall v \in V \setminus \{s\}, p(v) := s, P(v) := \emptyset \forall v \in V$ 
2  $i := 0, v_0 := s, W_0 := \emptyset, y_W := 0 \forall W \in \mathcal{W}$ 
3  $\mathcal{P} := \emptyset, k := 1, u_1 := t, x_p := 0 \forall e \in \mathcal{E}, k = 1$ 
4 All nodes are unmarked. All hyperedges are unmarked.
5 while  $v_i \neq t$  and  $\exists$  unmarked node  $w$  with  $d(w) < \infty$  do
6   Find  $v$  with  $d(v) = \min\{d(w) \mid w \text{ unmarked}\}$ 
7   for all unmarked  $e \in \mathcal{E}$  with  $v \in e$  do
8     for all unmarked  $w$  with  $w \in e$  do
9       if  $d(w) > d(v) + c_e$  then
10         $d(w) := d(v) + c_e$ 
11         $p(w) := v$ 
12         $P(w) := e$ 
13      end
14    end
15    mark  $e$ 
16  end
17  mark  $v$ 
18   $v_{i+1} := v$ 
19   $y_{W_i} := d(v_{i+1}) - d(v_i)$ 
20   $W_{i+1} := W_i \cup \{v_{i+1}\}$ 
21   $i := i + 1$ 
22 end
23 while  $u_k \neq s$  do
24    $\mathcal{P} := \mathcal{P} \cup P(u_k)$ 
25    $x_{P(u_k)} := 1$ 
26    $u_{k+1} := p(u_k)$ 
27    $k := k + 1$ 
28 end
29 return  $\mathcal{P}, x, y$ 

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index smaller than  $h$  such that  $v_i \in e$ , i.e.,  $e \notin \delta_H(W_j)$  for  $1 \leq j < i < h$  but  $e \in \delta_H(W_i)$ . Let similarly  $i \leq \ell < h$  be the maximal index smaller than  $h$  such that  $e \notin W_j$ , i.e., we have  $e \in \delta_H(W_j)$  for  $i \leq j \leq \ell < h$  and we have  $e \notin \delta_H(W_j)$  for  $\ell < j < h$ . Then equation (3) becomes:

$$\begin{aligned}
\sum_{W \in \mathcal{W}: e \in \delta_H(W)} y_W &= \sum_{j=i}^{\ell} y_{W_j} = \sum_{j=i}^{\ell} d(v_{j+1}) - d(v_j) \\
&= d(v_{\ell+1}) - d(v_i) \leq d(v_i) + c_e - d(v_i) = c_e.
\end{aligned}$$

For the last inequality we distinguish the cases  $v_{\ell+1} \in e$  and  $v_{\ell+1} \notin e$ . In the first case  $d(v_{\ell+1}) \leq d(v_i) + c_e$ , because this value is considered in the computation of the distance label  $d(v_{\ell+1})$  in line 9 when  $v_i$  is marked. In the second case,  $v_{\ell+1} = v_h = t$  and there exists a node  $w \in e$  with  $w \notin W_{h-1}$ . Since  $v_{\ell+1}$  is

marked and  $w$  not, we have  $d(v_{\ell+1}) \leq d(w)$ . Since  $w$  can be reached from  $v_i$  via  $e$  we have  $d(w) \leq d(v_i) + c_e$ . Again,  $d(v_{\ell+1}) \leq d(v_i) + c_e$ , and inequality (3) is satisfied.

We now show that  $x$  is a solution of program (1). Due to the definition of  $x$  we have  $x \geq 0$ . We have to show that

$$\sum_{e \in \delta_H(W)} x_e \geq 1 \quad \forall s \in W \subseteq V \setminus \{t\}. \quad (4)$$

Consider the nodes  $t = u_1, \dots, u_k = s$  computed in the while loop starting in line 23 and an  $st$ -hypercut  $\delta_H(W)$ . Let  $i$  be the largest index with  $u_i \notin W$  and  $u_{i+1} \in W$ . This index exists since  $u_1 = t \notin W$  and  $u_k = s \in W$ . Then we have  $x_{P(u_i)} = 1$ ,  $P(u_i) \in \delta_H(W)$ , and inequality (4) is satisfied.

The objective value of program (2) is

$$\sum_{i=1}^{h-1} y_{W_i} = \sum_{i=1}^{h-1} d(v_{i+1}) - d(v_i) = d(v_h) - d(v_1) = d(t) - d(s) = d(t).$$

Using lines 23 to 28 and 9 to 13 in Algorithm 1 we get

$$\begin{aligned} d(t) &= d(u_1) = d(u_2) + c_{P(u_1)} = d(u_3) + c_{P(u_2)} + c_{P(u_1)} = \dots \\ &= d(u_k) + \sum_{i=1}^k c_{P(u_i)} = 0 + \sum_{e \in \mathcal{P}} c_e = \sum_{e \in \mathcal{P}} c_e x_e, \end{aligned}$$

i.e., the objective values of (1) and (2) are equal. The integrality of  $x$  is obvious. Since  $c_e$  is integral, it follows that  $d(v_i)$  is integral for  $i = 0, \dots, h-1$ . Therefore  $y_{W_i}$ ,  $i = 1, \dots, h-1$ , is also integral (line 19). This shows the claim.  $\square$

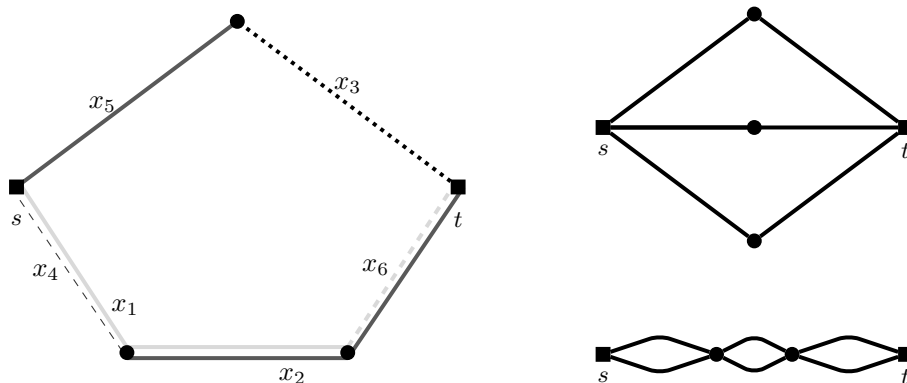
$$\begin{aligned} \min \quad & \sum_{e \in \mathcal{E}} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta_H(W)} x_e \geq 1 \quad \forall s \in W \subseteq V \setminus \{t\} \\ & 1 \geq x_e \geq 0 \quad \forall e \in \mathcal{E}. \end{aligned} \quad (5)$$

**Corollary 7.** *The inequality system of program (5) is TDI.*

*Proof.* Inequality system (5) adds upper bounds  $x \leq 1$  to the inequality system (1); this adds variables  $z$  to the dual program. For solutions  $x$  and  $y$  as constructed by Algorithm 1 (note that  $x \leq \mathbb{1}$ ) the vectors  $x$  and  $(y, 0)$  are primal and dual integer solutions of the extended systems with the same objective value.  $\square$

### 3 A New Class of Ideal Matrices

$st$ -hypercuts and  $st$ -hyperpaths form a blocking pair similar to  $st$ -cuts and  $st$ -paths. Likewise, the incidence matrices of all (inclusion wise) minimal  $st$ -hypercuts  $\mathcal{A}_c$  and the incidence matrices of all (inclusion wise) minimal  $st$ -hyperpaths  $\mathcal{A}_p$  form blocking pairs of matrices. By Propositions 3 and 4, these



**Figure 2:** Example of a hypergraph for which the incidence matrix of all (minimal)  $st$ -hypercuts is neither totally unimodular nor balanced (hyperedges are depicted as paths).

matrices are ideal, like the incidence matrices of  $st$ -cuts and  $st$ -paths. Note that these matrices are in general not totally unimodular and also not balanced, see Figure 2 for an example. The  $st$ -hypercut incidence matrix  $\mathcal{A}_c$  for the hypergraph associated with this example is

$$\begin{array}{c}
 \{s, a\} \\
 \{s, a, b, c\} \\
 \{s, b\} \\
 \{s\} \\
 \{s, a, b\} \\
 \{s, b, c\}
 \end{array}
 \begin{pmatrix}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 1 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 1 & 0 & 0 & 1 \\
 1 & 1 & 0 & 0 & 1 & 0 \\
 1 & 0 & 0 & 1 & 1 & 0 \\
 1 & 1 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & 1
 \end{pmatrix}
 = \mathcal{A}_c.$$

The  $3 \times 3$  matrix in the upper left corner has determinant  $-2$ .

The example also shows that the incidence matrices of  $st$ -hypercuts and  $st$ -hyperpaths form a new class of blocking pairs of ideal matrices which generalizes the class of incidence matrices of  $st$ -cuts and  $st$ -paths. This can be seen as follows. If  $\mathcal{A}_c$  would be an incidence matrix of  $st$ -cuts in an undirected graph  $G = (V, E)$ , the columns of  $\mathcal{A}_c$  have to correspond to the edges of  $G$ , i.e., such a graph would have six edges. Each cut of the graph contains exactly three edges, i.e., the edge-degrees of  $s$  and  $t$  have to be three. Furthermore, there can not be an edge connecting  $s$  and  $t$  since this edge would be contained in every cut. Therefore, the only possible graph is shown on the upper left of Figure 2. But this graph has seven minimal  $st$  cuts instead of the six in matrix  $\mathcal{A}_c$ . If  $\mathcal{A}_c$  would be an incidence matrix of  $st$ -paths in an undirected graph  $G = (V, E)$ , this graph would have six edges and each (minimal) path from  $s$  to  $t$  would have to contain exactly three edges. The only possible graph of this type is shown on the lower left of Figure 2. But this graph has eight minimal  $st$ -paths.



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