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**Smaller compact formulation for  
lot-sizing with constant batches**

# Smaller compact formulation for lot-sizing with constant batches

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## Abstract

We consider a variant of the classical lot-sizing problem in which the capacity in each period is an integer multiple of some basic batch size. Pochet and Wolsey [3] presented an  $\mathcal{O}(n^2 \min\{n, C\})$  algorithm to solve this problem and a linear program with  $\mathcal{O}(n^3)$  variables and inequalities, where  $n$  is the number of periods and  $C$  the batch size. We provide a linear program of size  $\mathcal{O}(n^2 \min\{n, C\})$ , that is, in case that  $C < n$ , our formulation is smaller.

**Keywords:** compact formulation, lot-sizing, constant batch size

We study a variant of the classical lot-sizing problem which is called *lot-sizing problem with constant batches*. It can be defined by the following variables and constraints. We are given  $n$  time periods,  $t = 1, \dots, n$ , and a capacity  $C \in \mathbb{N}$ . In each period  $t$  there is a positive demand  $d_t \in \mathbb{N}$ , a stock cost  $h_t$ , a unit production cost  $p_t$ , and a fixed cost  $f_t$  per batch of capacity  $C$ . Introducing a period  $t = 0$ , the problem can be formulated as:

$$\begin{aligned} \text{(LCB)} \quad & \min \sum_{t=1}^n (h_t s_t + p_t x_t + f_t y_t) \\ & \text{s.t.} \quad s_0 = s_n = 0, \\ & \quad s_{t-1} + x_t = d_t + s_t, \quad t = 1, \dots, n, \\ & \quad x_t \leq C y_t, \quad t = 1, \dots, n, \\ & \quad x_t \geq 0, s_t \geq 0, y_t \in \mathbb{Z}_+, \quad t = 1, \dots, n, \end{aligned}$$

where  $s_t$  is the stock at the end of period  $t$ ,  $x_t$  is the production in period  $t$ , and  $y_t$  is the number of batches provided in period  $t$ .

Pochet and Wolsey [3] presented an  $\mathcal{O}(n^2 \min\{n, C\})$  dynamic program as well as a linear program of size  $\mathcal{O}(n^3)$  to solve the mixed integer program (LCB). Both approaches are based on the fact that (LCB) can be reduced to a shortest path problem over so-called *regeneration intervals*. Given a feasible solution  $(s, x, y)$  of (LCB), this is an interval  $[\alpha, \beta]$  of periods with  $\alpha \leq \beta$  such that  $s_{\alpha-1} = s_\beta = 0$  and  $s_t > 0$  for  $t = \alpha, \alpha + 1, \dots, \beta - 1$ . Let  $D_n = (V, A)$  be the acyclic digraph defined by  $V := \{0, 1, \dots, n\}$  and  $A := \{(i, j) \in V \times V \mid 0 \leq i <$

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$j \leq n\}$ . Then, a feasible solution  $(s, x, y)$  obviously corresponds to a  $(0, n)$ -path  $P$  in  $D_n$  such that  $(i, j) \in P$  if and only if  $[i+1, j]$  forms a regeneration interval for  $(s, x, y)$ . If  $(s, x, y)$  is an optimal extreme solution and  $[\alpha, \beta]$  a regeneration interval for  $(s, x, y)$ , then in all periods  $t \in \{\alpha, \dots, \beta\}$  except for some period  $j$ ,  $x_t = Cy_t$ , and in period  $j$ ,  $x_j = C(y_j - 1) + r_{\alpha\beta}$  if  $r_{\alpha\beta} > 0$  and  $x_j = Cy_j$  if  $r_{\alpha\beta} = 0$ , where  $r_{\alpha\beta}$  is the remainder of the integral division of  $d_{\alpha\beta} := \sum_{t=\alpha}^{\beta} d_t$  by  $C$ . Moreover, denoting by  $(\text{LCB}_{\alpha\beta})$  problem (LCB) restricted to the interval  $[\alpha, \beta]$  (this means,  $(\text{LCB}_{\alpha\beta})$  is obtained by replacing 1 by  $\alpha$  and  $n$  by  $\beta$  and by  $\phi_{\alpha\beta}$  the optimal cost of  $(\text{LCB}_{\alpha\beta})$ , then an optimal extreme solution of (LCB) corresponds to a  $(0, n)$ -path  $P$  in  $D_n$  minimizing  $\sum_{(i,j) \in P} \phi_{i+1,j}$ .

Pochet and Wolsey [3] presented a dynamic program that calculates  $\phi_{\alpha\beta}$  in time  $\mathcal{O}(n)$ . Since there are  $\frac{1}{2}n(n-1)$  different intervals  $[\alpha, \beta]$ , the calculation of all  $\phi_{\alpha\beta}$  requires at most  $\mathcal{O}(n^3)$  calculations. A shortest path can be found in time  $\mathcal{O}(n^2)$ , and hence, (LCB) can be solved in time  $\mathcal{O}(n^3)$ .

This solution scheme can be expressed by linear programming. For any interval  $[\alpha, \beta]$ , the cost  $\phi_{\alpha\beta}$  is the optimal objective value of the linear program

$$\begin{aligned}
(\text{LP}_{\alpha\beta}) \quad & \min \sum_{i=\alpha}^{\beta} \left[ (Ce_i + f_i)y_i^{\alpha\beta} + (r_{\alpha\beta}e_i + f_i)\epsilon_i^{\alpha\beta} \right] \\
\text{s.t.} \quad & \sum_{j=\alpha}^t y_j^{\alpha\beta} \geq \left\lceil \frac{d_{\alpha t} - r_{\alpha\beta}}{C} \right\rceil, \quad t = \alpha, \dots, \beta, \\
& \sum_{j=\alpha}^t (y_j^{\alpha\beta} + \epsilon_j^{\alpha\beta}) \geq \left\lceil \frac{d_{\alpha t}}{C} \right\rceil, \quad t = \alpha, \dots, \beta, \\
& y_j^{\alpha\beta}, \epsilon_j^{\alpha\beta} \geq 0, \quad j = \alpha, \dots, \beta,
\end{aligned}$$

where  $e_t := p_t + \sum_{j=t}^n h_j$  for  $t = 1, \dots, n$ . The inequality system of  $(\text{LP}_{\alpha\beta})$  is totally unimodular, and hence, its vertex solutions are integral. In any vertex solution,  $y_i^{\alpha\beta}$  is the number of batches provided in period  $i$  and  $\epsilon_i^{\alpha\beta}$  is a binary variable which is equal to one if and only if the remainder  $r_{\alpha\beta}$  is produced in period  $i$ .

Coupling the linear programs  $(\text{LP}_{\alpha\beta})$ ,  $1 \leq \alpha \leq \beta \leq n$ , by a shortest path formulation, we obtain a linear program of size  $\mathcal{O}(n^3)$  to solve (LCB):

$$\begin{aligned}
(\text{LP}) \quad & \min \sum_{\alpha=1}^n \sum_{\beta=\alpha}^n \sum_{i=\alpha}^{\beta} \left[ (Ce_i + f_i)y_i^{\alpha\beta} + (r_{\alpha\beta}e_i + f_i)\epsilon_i^{\alpha\beta} \right] \\
\text{s.t.} \quad & \sum_{j=\alpha}^t y_j^{\alpha\beta} \geq \left\lceil \frac{d_{\alpha t} - r_{\alpha\beta}}{C} \right\rceil z_{\alpha-1,\beta}, \quad t = \alpha, \dots, \beta, \quad 1 \leq \alpha \leq \beta \leq n, \quad (1) \\
& \sum_{j=\alpha}^t (y_j^{\alpha\beta} + \epsilon_j^{\alpha\beta}) \geq \left\lceil \frac{d_{\alpha t}}{C} \right\rceil z_{\alpha-1,\beta}, \quad t = \alpha, \dots, \beta, \quad 1 \leq \alpha \leq \beta \leq n, \quad (2) \\
& y_j^{\alpha\beta}, \epsilon_j^{\alpha\beta} \geq 0, \quad j = \alpha, \dots, \beta, \quad 1 \leq \alpha \leq \beta \leq n, \quad (3)
\end{aligned}$$

$$\sum_{\beta=1}^n z_{0\beta} = 1, \quad (4)$$

$$\sum_{\alpha=0}^{t-1} z_{\alpha t} - \sum_{\beta=t+1}^n z_{t\beta} = 0, \quad t = 1, \dots, n-1, \quad (5)$$

$$z_{\alpha-1, \beta} \geq 0, \quad 1 \leq \alpha \leq \beta \leq n. \quad (6)$$

Next, we give the smaller formulation. Pochet and Wolsey [3] showed that for any two regeneration intervals  $[\alpha, \beta]$ ,  $[\alpha, \gamma]$  with  $\beta < \gamma$  and  $r_{\alpha\beta} = r_{\alpha\gamma}$  one needs to run only the dynamic program to calculate  $\phi_{\alpha\gamma}$ , as one of the intermediate solutions of their algorithm provides  $\phi_{\alpha\beta}$ . There are at most  $C$  different remainders,  $0, 1, \dots, C-1$ . Thus, for each  $\alpha$  and each remainder  $r$ , we have to run the dynamic program at most once to calculate all  $\phi_{\alpha\beta}$  with  $r_{\alpha\beta} = r$ . Hence, (LCB) can be solved in time  $\mathcal{O}(n^2 \min\{n, C\})$ .

The above algorithmic shortcut can be reflected by linear programming as follows. For each  $\alpha$  and each remainder  $r$ , let  $T^{\alpha r}$  be the collection of all time periods  $\beta \geq \alpha$  such that  $\phi_{\alpha\beta} = r$ , let  $M$  be the set of all pairs  $(\alpha, r)$  for which  $T^{\alpha r}$  is nonempty, and for each  $(\alpha, r) \in M$ , let  $\gamma(\alpha, r) := \max_{\beta \in T^{\alpha r}} \beta$ . Consider the linear program (LP') which can be obtained by aggregating inequalities and identifying variables of (LP).

$$\begin{aligned} \text{(LP')} \quad & \min \sum_{(\alpha, r) \in M} \sum_{i=\alpha}^{\gamma(\alpha, r)} [(Ce_i + f_i)y_i^{\alpha r} + (re_i + f_i)\epsilon_i^{\alpha r}] \\ \text{s.t.} \quad & \left\{ \sum_{j=\alpha}^t y_j^{\alpha r} \geq \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta < t}} \left\lceil \frac{d_{\alpha\beta} - r}{C} \right\rceil z_{\alpha-1, \beta} + \left\lceil \frac{d_{\alpha t} - r}{C} \right\rceil \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta \geq t}} z_{\alpha-1, \beta} \right. \\ & \left. \sum_{j=\alpha}^t (y_j^{\alpha r} + \epsilon_j^{\alpha r}) \geq \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta < t}} \left\lceil \frac{d_{\alpha\beta}}{C} \right\rceil z_{\alpha-1, \beta} + \left\lceil \frac{d_{\alpha t}}{C} \right\rceil \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta \geq t}} z_{\alpha-1, \beta}, \right\} \\ & \text{for } t = \alpha, \alpha+1, \dots, \gamma(\alpha, r), (\alpha, r) \in M, \\ & y_j^{\alpha r}, \epsilon_j^{\alpha r} \geq 0 \text{ for } j = \alpha, \alpha+1, \dots, \gamma(\alpha, r), (\alpha, r) \in M, \\ & z \text{ satisfies (4)-(6)}. \end{aligned} \quad (7)$$

**Theorem 1.** (LP') solves (LBC).

*Proof.* We denote the inequality (1) associated with  $[\alpha, \beta]$  and  $t \in [\alpha, \beta]$  by  $(1:\alpha, \beta, t)$ . Similarly, we access to the inequalities  $(2:\alpha, \beta, t)$ ,  $(7:\alpha, r, t)$  and  $(8:\alpha, r, t)$ .

Let  $(y, \epsilon, z)$  be a feasible solution of (LP). Define  $(\bar{y}, \bar{\epsilon}, \bar{z})$  in the space of (LP') by

$$\begin{aligned} \bar{y}_j^{\alpha r} &:= \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta \geq j}} y_j^{\alpha\beta}, \\ \bar{\epsilon}_j^{\alpha r} &:= \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta \geq j}} \epsilon_j^{\alpha\beta} \quad \text{for } j = \alpha, \alpha+1, \dots, \gamma(\alpha, r), (\alpha, r) \in M, \\ \bar{z} &:= z. \end{aligned} \quad (10)$$

Clearly,  $(\bar{y}, \bar{\epsilon}, \bar{z})$  satisfies (4)-(6) and (9). To show that  $(\bar{y}, \bar{\epsilon}, \bar{z})$  also satisfies (7)-(8), consider any  $(\alpha, r) \in M$  and any  $t \in \{\alpha, \alpha+1, \dots, \gamma(\alpha, r)\}$ . Summing up the inequalities (1: $\alpha, \beta, \beta$ ) for  $\beta \in T^{\alpha r}$  with  $\beta < t$  and the inequalities (1: $\alpha, \beta, t$ ) for  $\beta \in T^{\alpha r}$  with  $\beta \geq t$ , and replacing in the resulting inequality  $(y, \epsilon, z)$  by  $(\bar{y}, \bar{\epsilon}, \bar{z})$  according to (10), we obtain the inequality (7: $\alpha, r, t$ ). Inequality (8: $\alpha, r, t$ ) can be obtained similarly. Thus,  $(\bar{y}, \bar{\epsilon}, \bar{z})$  satisfies (7)-(8), and hence,  $(\bar{y}, \bar{\epsilon}, \bar{z})$  is a feasible solution of (LP'). Since

$$\begin{aligned} & \sum_{\alpha=1}^n \sum_{\beta=\alpha}^n \sum_{i=\alpha}^{\beta} \left[ (Ce_i + f_i)y_i^{\alpha\beta} + (r_{\alpha\beta}e_i + f_i)\epsilon_i^{\alpha\beta} \right] \\ &= \sum_{(\alpha,r) \in M} \sum_{\beta \in T^{\alpha r}} \sum_{i=\alpha}^{\beta} \left[ (Ce_i + f_i)y_i^{\alpha\beta} + (re_i + f_i)\epsilon_i^{\alpha\beta} \right] \\ &= \sum_{(\alpha,r) \in M} \sum_{i=\alpha}^{\gamma(\alpha,r)} \left[ (Ce_i + f_i)\bar{y}_i^{\alpha r} + (re_i + f_i)\bar{\epsilon}_i^{\alpha r} \right], \end{aligned}$$

both  $(y, \epsilon, z)$  and  $(\bar{y}, \bar{\epsilon}, \bar{z})$  have the same objective value. This implies that the optimal objective value of (LP') is at most the optimal objective value of (LP).

To show the converse, let  $(y, \epsilon, z)$  be an optimal solution of (LP'). W.l.o.g. we assume that  $y_j^{\alpha r} = \epsilon_j^{\alpha r} = 0$  for  $j = \alpha, \alpha+1, \dots, \gamma(\alpha, r)$  if  $z_{\alpha-1, \beta} = 0$  for all  $\beta \in T^{\alpha r}$ . This implies that for all  $(\alpha, r) \in M$  and all  $t \in \{\alpha, \dots, \gamma(\alpha, r)\}$ , there exist  $\lambda(\alpha, r, t), \mu(\alpha, r, t) \geq 0$  such that

$$\begin{aligned} \sum_{j=\alpha}^t y_j^{\alpha r} &= \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta < t}} \left( \left\lceil \frac{d_{\alpha\beta} - r}{C} \right\rceil + \lambda(\alpha, r, \beta) \right) z_{\alpha-1, \beta} \\ &\quad + \left( \left\lceil \frac{d_{\alpha t} - r}{C} \right\rceil + \lambda(\alpha, r, t) \right) \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta \geq t}} z_{\alpha-1, \beta}, \\ \sum_{j=\alpha}^t (y_j^{\alpha r} + \epsilon_j^{\alpha r}) &= \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta < t}} \left( \left\lceil \frac{d_{\alpha\beta}}{C} \right\rceil + \mu(\alpha, r, \beta) \right) z_{\alpha-1, \beta} \\ &\quad + \left( \left\lceil \frac{d_{\alpha t}}{C} \right\rceil + \mu(\alpha, r, t) \right) \sum_{\substack{\beta \in T^{\alpha r}: \\ \beta \geq t}} z_{\alpha-1, \beta}. \end{aligned} \tag{11}$$

For each  $1 \leq \alpha \leq \beta \leq n$ , define  $(\tilde{y}_t^{\alpha\beta}, \tilde{\epsilon}_t^{\alpha\beta})$ ,  $t = \alpha, \dots, \beta$  as solution of the (nonsingular) system

$$\begin{aligned} \sum_{j=\alpha}^t \tilde{y}_j^{\alpha\beta} &= \left( \left\lceil \frac{d_{\alpha t} - r}{C} \right\rceil + \lambda(\alpha, r, t) \right) z_{\alpha-1, \beta}, \quad t = \alpha, \dots, \beta, \\ \sum_{j=\alpha}^t (\tilde{y}_j^{\alpha\beta} + \tilde{\epsilon}_j^{\alpha\beta}) &= \left( \left\lceil \frac{d_{\alpha t}}{C} \right\rceil + \mu(\alpha, r, t) \right) z_{\alpha-1, \beta}, \quad t = \alpha, \dots, \beta, \end{aligned} \tag{12}$$

where  $r = r_{\alpha\beta}$ . Then,  $(\tilde{y}, \tilde{\epsilon}, \tilde{z})$ , with  $\tilde{z} := z$ , is a feasible solution of (LP) whose objective value is at most the (optimal) objective value of  $(y, \epsilon, z)$ .  $\square$

The original variables  $x_i, y_i$  can be obtained by

$$x_i = \sum_{\substack{(\alpha,r) \in M: \\ \alpha \leq i}} (Cy_i^{\alpha r} + r\epsilon_i^{\alpha r}), \tag{13}$$

$$y_i = \sum_{\substack{(\alpha,r) \in M: \\ \alpha \leq i}} (y_i^{\alpha r} + \epsilon_i^{\alpha r}), \tag{14}$$

and hence (LCB) can be rewritten as the size  $\mathcal{O}(n^2 \min\{n, C\})$  linear program

$$\min \sum_{t=1}^n (e_t x_t + f_t y_t) \text{ s.t.} \\ ((x_t, y_t)_{t=1, \dots, n}, (y_t^{\alpha r}, \epsilon_t^{\alpha r})_{(\alpha, r) \in M, t=\alpha, \dots, \gamma(\alpha, r)}, z) \text{ satisfies (4)-(9), (13)-(14).}$$

## 1 Extensions

In this paper, we presented an  $\mathcal{O}(n^2 \min\{n, C\})$  compact formulation for the lot-sizing problem with constant batches. We would like to close the paper with three remarks on related topics and future research.

Let the polyhedron  $P$  be the convex hull of the feasible solutions  $(s, x, y)$  of (LCB). Pochet and Wolsey [3] introduced a rich class of facet defining inequalities for  $P$  and showed that they are the only inequalities of a certain type. It is, however, unknown whether there are further facet defining inequalities for  $P$  or not. Since our compact formulation (LP') is tighter than (LP), it maybe contributes to a better understanding of the facial structure of  $P$  via projection. For a description of the technique to derive facets by projection, see Balas [1].

The linear program (LP) is an example of *branched polyhedral systems* introduced by Kaibel and Loos [2], a framework for combining extended formulations. In our application, the extended formulations (LP $_{\alpha\beta}$ ) for the calculation of the costs  $\phi_{\alpha\beta}$  of the regeneration intervals  $[\alpha, \beta]$  are combined with a shortest path formulation. Further examples for combining extended formulations are given in [4].

In the literature, many variations and extensions of (LCB) are discussed, for instance, lot sizing with constant batch size and upper bounds on the  $y$ -variables and/or backlogging and/or Wagner-Within costs. The results of Pochet and Wolsey [3] indicate that in case of upper bounds  $y_t \leq b_t$  ( $t = 1, \dots, n$ ), (LP') becomes useless. Backlogging, however, can be incorporated in the linear program (LP') if no upper bounds on the  $y$ -variables are installed. For details about these variations, see Van Vyve [5].

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