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Smaller compact formulation for lot-sizing with constant batches
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Abstract

We consider a variant of the classical lot-sizing problem in which the capacity in each period is an integer multiple of some basic batch size. Pochet and Wolsey [3] presented an $O(n^2 \min\{n, C\})$ algorithm to solve this problem and a linear program with $O(n^3)$ variables and inequalities, where $n$ is the number of periods and $C$ the batch size. We provide a linear program of size $O(n^2 \min\{n, C\})$, that is, in case that $C < n$, our formulation is smaller.

Keywords: compact formulation, lot-sizing, constant batch size

We study a variant of the classical lot-sizing problem which is called lot-sizing problem with constant batches. It can be defined by the following variables and constraints. We are given $n$ time periods, $t = 1, \ldots, n$, and a capacity $C \in \mathbb{N}$. In each period $t$ there is a positive demand $d_t \in \mathbb{N}$, a stock cost $h_t$, a unit production cost $p_t$, and a fixed cost $f_t$ per batch of capacity $C$. Introducing a period $t = 0$, the problem can be formulated as:

\begin{equation}
\text{(LCB)} \quad \min \sum_{t=1}^{n}(h_ts_t + p_tx_t + f_ty_t)
\end{equation}

s.t.

\begin{align*}
    s_0 &= s_n = 0, \\
    s_{t-1} + x_t &= d_t + s_t, \quad t = 1, \ldots, n, \\
    x_t &\leq Cy_t, \quad t = 1, \ldots, n, \\
    x_t &\geq 0, \quad s_t \geq 0, \quad y_t \in \mathbb{Z}_+, \quad t = 1, \ldots, n,
\end{align*}

where $s_t$ is the stock at the end of period $t$, $x_t$ is the production in period $t$, and $y_t$ is the number of batches provided in period $t$.

Pochet and Wolsey [3] presented an $O(n^2 \min\{n, C\})$ dynamic program as well as a linear program of size $O(n^3)$ to solve the mixed integer program (LCB). Both approaches are based on the fact that (LCB) can be reduced to a shortest path problem over so-called regeneration intervals. Given a feasible solution $(s, x, y)$ of (LCB), this is an interval $[\alpha, \beta]$ of periods with $\alpha \leq \beta$ such that $s_{\alpha-1} = s_{\beta} = 0$ and $s_t > 0$ for $t = \alpha, \alpha + 1, \ldots, \beta - 1$. Let $D_n = (V, A)$ be the acyclic digraph defined by $V := \{0, 1, \ldots, n\}$ and $A := \{(i, j) \in V \times V | 0 \leq i <$
Then, a feasible solution \((s, x, y)\) obviously corresponds to a \((0, n)\)-path \(P\) in \(D_n\) such that \((i, j)\) is in \(P\) if and only if \([i + 1, j]\) forms a regeneration interval for \((s, x, y)\). If \((s, x, y)\) is an optimal extreme solution and \([\alpha, \beta]\) a regeneration interval for \((s, x, y)\), then in all periods \(t \in \{\alpha, \ldots, \beta\}\) except for some period \(j\), \(x_t = Cy_t\), and in period \(j\), \(x_j = Cy_j\) if \(r_{\alpha\beta} > 0\) and \(x_j = Cy_j\) if \(r_{\alpha\beta} = 0\), where \(r_{\alpha\beta}\) is the remainder of the integral division of \(d_{\alpha\beta} := \sum_{i=\alpha}^{\beta} d_i\) by \(C\). Moreover, denoting by \((LCB, \alpha\beta)\) problem \((LCB)\) restricted to the interval \([\alpha, \beta]\) (this means, \((LCB, \alpha\beta)\) is obtained by replacing 1 by \(\alpha\) and \(n\) by \(\beta\) and by \(\phi_{\alpha\beta}\) the optimal cost of \((LCB, \alpha\beta)\), then an optimal extreme solution of \((LCB)\) corresponds to a \((0, n)\)-path \(P\) in \(D_n\) minimizing \(\sum_{(i,j) \in P} \phi_{i+1,j}\).

Pochet and Wolsey [3] presented a dynamic program that calculates \(\phi_{\alpha\beta}\) in time \(O(n)\). Since there are \(\frac{1}{2}n(n-1)\) different intervals \([\alpha, \beta]\), the calculation of all \(\phi_{\alpha\beta}\) requires at most \(O(n^3)\) calculations. A shortest path can be found in time \(O(n^2)\), and hence, \((LCB)\) can be solved in time \(O(n^3)\).

This solution scheme can be expressed by linear programming. For any interval \([\alpha, \beta]\), the cost \(\phi_{\alpha\beta}\) is the optimal objective value of the linear program

\[
(LP_{\alpha\beta}) \quad \min \sum_{i=\alpha}^{\beta} \left[ (Ce_i + f_i)y_{i,\alpha}^{\beta} + (r_{\alpha\beta} e_i + f_i)\epsilon_{i,\alpha}^{\beta} \right]
\]

\[
\text{s.t. } \sum_{j=\alpha}^{\beta} y_{j,\alpha}^{\beta} \geq \left\lfloor \frac{d_{\alpha t} - r_{\alpha\beta}}{C} \right\rfloor, \quad t = \alpha, \ldots, \beta,
\]

\[
\sum_{j=\alpha}^{\beta} (y_{j,\alpha}^{\beta} + \epsilon_{j,\alpha}^{\beta}) \geq \left\lfloor \frac{d_{\alpha t}}{C} \right\rfloor, \quad t = \alpha, \ldots, \beta,
\]

\[
y_{j,\alpha}^{\beta}, \epsilon_{j,\alpha}^{\beta} \geq 0, \quad j = \alpha, \ldots, \beta,
\]

where \(e_i := p_t + \sum_{j=t}^{n} h_j\) for \(t = 1, \ldots, n\). The inequality system of \((LP_{\alpha\beta})\) is totally unimodular, and hence, its vertex solutions are integral. In any vertex solution, \(y_{j,\alpha}^{\beta}\) is the number of batches provided in period \(i\) and \(\epsilon_{i,\alpha}^{\beta}\) is a binary variable which is equal to one if and only if the remainder \(r_{\alpha\beta}\) is produced in period \(i\).

Coupling the linear programs \((LP_{\alpha\beta})\), \(1 \leq \alpha \leq \beta \leq n\), by a shortest path formulation, we obtain a linear program of size \(O(n^3)\) to solve \((LCB)\):

\[
(LP) \quad \min \sum_{\alpha=1}^{n} \sum_{\beta=\alpha}^{n} \sum_{i=\alpha}^{\beta} \left[ (Ce_i + f_i)y_{i,\alpha}^{\beta} + (r_{\alpha\beta} e_i + f_i)\epsilon_{i,\alpha}^{\beta} \right]
\]

\[
\text{s.t. } \sum_{j=\alpha}^{\beta} y_{j,\alpha}^{\beta} \geq \left\lfloor \frac{d_{\alpha t} - r_{\alpha\beta}}{C} \right\rfloor z_{\alpha-1,\beta}, \quad t = \alpha, \ldots, \beta, 1 \leq \alpha \leq \beta \leq n, \quad (1)
\]

\[
\sum_{j=\alpha}^{\beta} (y_{j,\alpha}^{\beta} + \epsilon_{j,\alpha}^{\beta}) \geq \left\lfloor \frac{d_{\alpha t}}{C} \right\rfloor z_{\alpha-1,\beta}, \quad t = \alpha, \ldots, \beta, 1 \leq \alpha \leq \beta \leq n, \quad (2)
\]

\[
y_{j,\alpha}^{\beta}, \epsilon_{j,\alpha}^{\beta} \geq 0, \quad j = \alpha, \ldots, \beta, 1 \leq \alpha \leq \beta \leq n, \quad (3)
\]
\[ \sum_{\beta=1}^{n} z_{0,\beta} = 1, \quad (4) \]

\[ \sum_{\alpha=0}^{t-1} z_{\alpha t} - \sum_{\beta=t+1}^{n} z_{t,\beta} = 0, \quad t = 1, \ldots, n - 1, \quad (5) \]

\[ z_{\alpha-1,\beta} \geq 0, \quad 1 \leq \alpha \leq \beta \leq n. \quad (6) \]

Next, we give the smaller formulation. Pochet and Wolsey [3] showed that for any two regeneration intervals \([\alpha, \beta], [\alpha, \gamma]\) with \(\beta < \gamma\) and \(r_{\alpha \beta} = r_{\alpha \gamma}\) one needs to run only the dynamic program to calculate \(\phi_{\alpha \gamma}\), as one the intermediate solutions of their algorithm provides \(\phi_{\alpha \beta}\). There are at most \(C\) different remainders, \(0, 1, \ldots, C - 1\). Thus, for each \(\alpha\) and each remainder \(r\), we have to run the dynamic program at most once to calculate all \(\phi_{\alpha \beta}\) with \(r_{\alpha \beta} = r\). Hence, (LCB) can be solved in time \(O(n^2 \min\{n, C\})\).

The above algorithmic shortcut can be reflected by linear programming as follows. For each \(\alpha\) and each remainder \(r\), let \(T_{\alpha r}\) be the collection of all time periods \(\beta \geq \alpha\) such that \(\phi_{\alpha \beta} = r\), let \(M\) be the set of all pairs \((\alpha, r)\) for which \(T_{\alpha r}\) is nonempty, and for each \((\alpha, r) \in M\), let \(\gamma(\alpha, r) := \max_{\beta \in T_{\alpha r}} \beta\). Consider the linear program \((LP')\) which can be obtained by aggregating inequalities and identifying variables of \((LP)\).

\[ \text{(LP')} \quad \min_{(\alpha,r) \in M} \sum_{i=\alpha}^{\gamma(\alpha,r)} \left((Ce_i + f_i)y_{i,r} + (re_i + f_i)e_{i,r}\right) \]

s.t.

\[ \sum_{\beta \in T_{\alpha r}, \beta < t} \left\lceil \frac{d_{\alpha \beta} - r}{C} \right\rceil z_{\alpha-1,\beta} + \left\lfloor \frac{d_{at} - r}{C} \right\rfloor \sum_{\beta \geq t} z_{\alpha-1,\beta} - \sum_{\beta \in T_{\alpha r}, \beta < t} \left\lfloor \frac{d_{\alpha \beta} - r}{C} \right\rfloor z_{\alpha-1,\beta} \geq 0 \quad \text{for } t = \alpha, \alpha + 1, \ldots, \gamma(\alpha, r), (\alpha, r) \in M, \]

\[ y_{j,r}, e_{j,r} \geq 0 \quad \text{for } j = \alpha, \alpha + 1, \ldots, \gamma(\alpha, r), (\alpha, r) \in M, \]

\[ z \text{ satisfies (4)-(6)}. \quad (9) \]

**Theorem 1.** \((LP')\) solves \((LBC)\).

**Proof.** We denote the inequality (1) associated with \([\alpha, \beta]\) and \(t \in [\alpha, \beta]\) by \((1;\alpha, \beta, t)\). Similarly, we access to the inequalities \((2;\alpha, \beta, t), (7;\alpha, r, t)\) and \((8;\alpha, r, t)\).

Let \((y, e, z)\) be a feasible solution of \((LP)\). Define \((\bar{y}, \bar{e}, \bar{z})\) in the space of \((LP')\) by

\[ \bar{y}_{j,r} := \sum_{\beta \geq j, \beta \in T_{\alpha r}} y_{j,r}^{\alpha \beta}, \quad \text{for } j = \alpha, \alpha + 1, \ldots, \gamma(\alpha, r), (\alpha, r) \in M, \]

\[ \bar{e}_{j,r} := \sum_{\beta \geq j, \beta \in T_{\alpha r}} e_{j,r}^{\alpha \beta}, \quad \text{for } j = \alpha, \alpha + 1, \ldots, \gamma(\alpha, r), (\alpha, r) \in M, \]

\[ \bar{z} := z. \]
Clearly, \((\bar{y}, \bar{\epsilon}, \bar{z})\) satisfies (4)-(6) and (9). To show that \((\bar{y}, \bar{\epsilon}, \bar{z})\) also satisfies (7)-(8), consider any \((\alpha, r) \in M\) and any \(t \in \{\alpha, \alpha+1, \ldots, \gamma(\alpha, r)\}\). Summing up the inequalities \((1: \alpha, \beta, t)\) for \(\beta \in T^{\alpha t}\) with \(\beta < t\) and the inequalities \((1: \alpha, \beta, t)\) for \(\beta \in T^{\alpha t}\) with \(\beta \geq t\), and replacing in the resulting inequality \((y, \epsilon, z)\) by \((\bar{y}, \bar{\epsilon}, \bar{z})\) according to (10), we obtain the inequality \((7: \alpha, r, t)\). Inequality \((8: \alpha, r, t)\) can be obtained similarly. Thus, \((\bar{y}, \bar{\epsilon}, \bar{z})\) satisfies (7)-(8), and hence, \((\bar{y}, \bar{\epsilon}, \bar{z})\) is a feasible solution of \((LP')\).

Since

\[
\sum_{\alpha=1}^{n} \sum_{\beta=\alpha}^{n} \sum_{i=\alpha}^{\beta} \left[(Ce_i + f_i)\alpha^\beta + (r_{\alpha\beta} e_i + f_i)\alpha^\beta\right] = \sum_{(\alpha, r) \in M} \sum_{\beta \in T^{\alpha r}} \sum_{\gamma(\alpha, r)} (Ce_i + f_i)\gamma^\alpha \beta + (r_{\alpha\beta} e_i + f_i)\gamma^\alpha \beta
\]

both \((y, \epsilon, z)\) and \((\bar{y}, \bar{\epsilon}, \bar{z})\) have the same objective value. This implies that the optimal objective value of \((LP')\) is at most the (optimal) objective value of \((LP)\).

To show the converse, let \((y, \epsilon, z)\) be an optimal solution of \((LP')\). W.l.o.g. we assume that \(y_j^{\alpha r} = e_j^{\alpha r} = 0\) for \(j = \alpha, \alpha+1, \ldots, \gamma(\alpha, r)\) if \(z_{\alpha-1} = 0\) for all \(\beta \in T^{\alpha t}\). This implies that for all \((\alpha, r) \in M\) and all \(t \in \{\alpha, \ldots, \gamma(\alpha, r)\}\), there exist \(\lambda(\alpha, r, t), \mu(\alpha, r, t) \geq 0\) such that

\[
\sum_{j=\alpha}^{t} y_j^{\alpha r} = \sum_{\beta \in T^{\alpha r}} \left(\left[\frac{d_{\alpha \beta} - r}{e}\right] + \lambda(\alpha, r, \beta)\right) z_{\alpha-1, \beta} + \left(\left[\frac{d_{\epsilon} - r}{e}\right] + \lambda(\alpha, r, t)\right) \sum_{\beta \in T^{\alpha t}} z_{\alpha-1, \beta},
\]

\[
\sum_{j=\alpha}^{t} (y_j^{\alpha r} + e_j^{\alpha r}) = \sum_{\beta \in T^{\alpha r}} \left(\left[\frac{d_{\alpha \beta}}{e}\right] + \mu(\alpha, r, \beta)\right) z_{\alpha-1, \beta} + \left(\left[\frac{d_{\epsilon}}{e}\right] + \mu(\alpha, r, t)\right) \sum_{\beta \in T^{\alpha t}} z_{\alpha-1, \beta}.
\]

For each \(1 \leq \alpha \leq \beta \leq n\), define \((\bar{y}^{\alpha r}_t, \bar{e}^{\alpha r}_t)\), \(t = \alpha, \ldots, \beta\) as solution of the (nonsingular) system

\[
\sum_{j=\alpha}^{t} \bar{y}^{\alpha r}_j = \left(\left[\frac{d_{\alpha \beta} - r}{e}\right] + \lambda(\alpha, r, t)\right) \sum_{\beta \in T^{\alpha t}} \bar{z}_{\alpha-1, \beta}, \quad t = \alpha, \ldots, \beta,
\]

\[
\sum_{j=\alpha}^{t} (\bar{y}^{\alpha r}_j + \bar{e}^{\alpha r}_j) = \left(\left[\frac{d_{\beta}}{e}\right] + \mu(\alpha, r, t)\right) \sum_{\beta \in T^{\alpha t}} \bar{z}_{\alpha-1, \beta}, \quad t = \alpha, \ldots, \beta,
\]

where \(r = r_{\alpha r}\). Then, \((\bar{y}, \bar{\epsilon}, \bar{z})\), with \(\bar{z} := z\), is a feasible solution of \((LP)\) whose objective value is at most the (optimal) objective value of \((y, \epsilon, z)\).

The original variables \(x_i, y_i\) can be obtained by

\[
x_i = \sum_{(\alpha, r) \in M, \alpha \leq i} (Cy_i^{\alpha r} + re_i^{\alpha r}),
\]

\[
y_i = \sum_{(\alpha, r) \in M, \alpha \leq i} (y_i^{\alpha r} + e_i^{\alpha r}),
\]
and hence (LCB) can be rewritten as the size $O(n^2 \min\{n, C\})$ linear program

$$\min \sum_{t=1}^{n} (e_t x_t + f_t y_t) \text{ s.t.}$$

$$(x_t, y_t)_{t=1, \ldots, n}, (y_{\alpha r}, e_{\alpha r})_{(\alpha, r) \in \mathcal{M}, t=\alpha, \ldots, \gamma(\alpha, r), z} \text{ satisfies (4)-(9), (13)-(14).}$$

## 1 Extensions

In this paper, we presented an $O(n^2 \min\{n, C\})$ compact formulation for the lot-sizing problem with constant batches. We would like to close the paper with three remarks on related topics and future research.

Let the polyhedron $P$ be the convex hull of the feasible solutions $(s, x, y)$ of (LCB). Pochet and Wolsey [3] introduced a rich class of facet defining inequalities for $P$ and showed that they are the only inequalities of a certain type. It is, however, unknown whether there are further facet defining inequalities for $P$ or not. Since our compact formulation (LP') is tighter than (LP), it might contribute to a better understanding of the facial structure of $P$ via projection. For a description of the technique to derive facets by projection, see Balas [1].

The linear program (LP) is an example of branched polyhedral systems introduced by Kaibel and Loos [2], a framework for combining extended formulations. In our application, the extended formulations (LP$^{\alpha \beta}$) for the calculation of the costs $\phi_{\alpha \beta}$ of the regeneration intervals $[\alpha, \beta]$ are combined with a shortest path formulation. Further examples for combining extended formulations are given in [4].

In the literature, many variations and extensions of (LCB) are discussed, for instance, lot sizing with constant batch size and upper bounds on the $y$-variables and/or backlogging and/or Wagner-Within costs. The results of Pochet and Wolsey [3] indicate that in case of upper bounds $y_t \leq b_t$ ($t = 1, \ldots, n$), (LP') becomes useless. Backlogging, however, can be incorporated in the linear program (LP') if no upper bounds on the $y$-variables are installed. For details about these variations, see Van Vyve [5].

## References


