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**On the connectivity of k -clique
polytopes**

On the connectivity of k -clique polytopes

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Abstract

In this paper, we study the neighbourlicity of the polytope P_{kn}^2 constituted by the k -cliques of the complete graph K_n on n vertices. We prove that this polytope is 3-, but not 4-neighbourly. Following a remark of Pierre Duchet, we partially generalize this result to the k -clique polytopes of r -uniform complete hypergraphs, P_{kn}^r . We show that the neighbourlicity of P_{kn}^r is between r and $2^r - 1$ whenever $k \geq r + 1$ and $n \geq k + r + 1$. Computational results indicate that the upper bound is tight.

1 Introduction

Given a graph G , we denote its node set by $V(G)$ and its edge set by $E(G)$. In this paper, a *clique* of G is a complete subgraph of G . The collection of all k -cliques, that is, the cliques on k nodes, is denoted by \mathcal{K} . For any node set $W \subseteq V(G)$, we denote by $E(W)$ the set of edges $e \in E(G)$ whose both endnodes are in W . For any $F \subseteq E(G)$, its *characteristic vector* is a vector $\chi^F \in \{0, 1\}^{E(G)}$ defined by $\chi_e^F = 1$ if and only if $e \in F$. The *k -clique polytope* is the convex hull of the characteristic vectors of the edge sets of all k -cliques of G .

A polyhedron is said to be *h -neighbourly* if each subset of h vertices form a proper $(h - 1)$ -face of the polyhedron. This especially implies that every h vertices of an h -neighbourly polyhedron are affinely independent. Moreover, for each $j \leq h$, such a polyhedron is also j -neighbourly.

In this paper, we study the neighbourlicity of the k -clique polytope P_{kn}^2 defined on the complete graph K_n on n nodes. In Section 3, we will show that P_{kn}^2 is 3- but usually not 4-neighbourly.

In Section 4, we partially extend this result to the k -clique polytope P_{kn}^r defined on a complete r -uniform-hypergraph K_n^r on n nodes. Here, *r -uniform* means that the hyperedges of K_n^r consist of exactly r distinct nodes. Thus, as K_n^r is complete, it follows that $E(K_n^r) = \{e \subseteq V(K_n^r) \mid |e| = r\}$. Moreover,

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a k -clique of K_n^r is a hypersubgraph K_k^r of K_n^r , and P_{kn}^r is the convex hull of the characteristic vectors of the edge sets of all k -cliques of K_n^r . We show that the neighbourlicity of P_{kn}^r is between r and $2^r - 1$ whenever $k \geq r + 1$ and $n \geq k + r + 1$. Using an integer program, we compute the neighbourlicity of P_{kn}^r for small numbers k, n, r , which is $2^r - 1$ in each studied case. Hence, we assume that the upper bound is tight.

In the following section, we give some preliminary results that are independent of the special structure of the polytopes P_{kn}^r .

2 Preliminaries

The initial idea to prove the above-mentioned results is mainly based on the Farkas Lemma specialized to ℓ -COPs. An ℓ -COP is a combinatorial optimization problem whose feasible solutions have the same size (=cardinality) ℓ .

Let $\mathcal{I} \subseteq 2^E$ be the set of all feasible solutions of an ℓ -COP defined on a finite set E , and let $P_{\mathcal{I}}(E)$ be the polytope defined as the convex hull of the characteristic vectors of the solutions $I \in \mathcal{I}$. $P_{\mathcal{I}}(E)$ is h -neighbourly if and only if for the characteristic vectors of every subset $\mathcal{J} \subseteq \mathcal{I}$ of size h the following two conditions hold:

- (i) they are affinely independent;
- (ii) the system

$$\begin{aligned} \sum_{e \in E} \chi_e^I \alpha_e - \beta &= 0 && \text{for all } I \in \mathcal{J}, \\ \sum_{e \in E} \chi_e^I \alpha_e - \beta &< 0 && \text{for all } I \in \mathcal{I} \setminus \mathcal{J} \end{aligned} \quad (1)$$

is consistent, that is, there exists a valid inequality $\alpha x \leq \beta$ for $P_{\mathcal{I}}(E)$ such that the vectors χ^I , $I \in \mathcal{J}$, are the only vertices of $P_{\mathcal{I}}(E)$ contained in the induced face.

The conditions directly follow from the definition of neighbourlicity. However, condition (i) is redundant. For this, assume that condition (ii) holds but not (i). Then, there exist a set $\mathcal{J}' \subseteq \mathcal{I}$ of size h and an element $J' \in \mathcal{J}'$ such that its characteristic vector is an affine combination of the characteristic vectors of the other elements of \mathcal{J}' . Since (ii) holds for all sets $\mathcal{J} \subseteq \mathcal{I}$ of size h , and since $h \leq \dim P_{\mathcal{I}}(E)$, there exist $J'' \in \mathcal{I} \setminus \mathcal{J}'$ and a valid inequality $\alpha x \leq \beta$ for $P_{\mathcal{I}}(E)$ such that the characteristic vectors of J'' and $I \in \mathcal{J} \setminus \{J'\}$, are the only vertices of $P_{\mathcal{I}}(E)$ contained in the face induced by this inequality, contradicting the fact that the characteristic vector J' is an affine combination of the characteristic vectors of the other elements of \mathcal{J}' .

All $x \in P_{\mathcal{I}}(E)$ satisfy the equation $\sum_{e \in E} x_e = \ell$. Hence, we may assume that $\beta = 0$. Moreover, since the system (1) determines a (polyhedral) cone, we may assume, by a scaling argument, that (1) has a solution if and only if

$$\begin{aligned} \sum_{e \in E} \chi_e^I \alpha_e &= 0 && \text{for all } I \in \mathcal{J}, \\ \sum_{e \in E} \chi_e^I \alpha_e &\leq -1 && \text{for all } I \in \mathcal{I} \setminus \mathcal{J}. \end{aligned} \quad (2)$$

has one. By the Farkas Lemma [4], (2) does not have a solution if and only if

there exist $\lambda_I, I \in \mathcal{J}$ and $\mu_I, I \in \bar{\mathcal{J}} := \mathcal{I} \setminus \mathcal{J}$ with

$$\sum_{I \in \mathcal{J}} \lambda_I \chi_e^I - \sum_{I \in \bar{\mathcal{J}}} \mu_I \chi_e^I = 0 \quad \text{for all } e \in E, \quad (3)$$

$$\mu_I \geq 0 \quad \text{for all } I \in \bar{\mathcal{J}}, \quad (4)$$

$$\sum_{I \in \bar{\mathcal{J}}} \mu_I > 0. \quad (5)$$

It immediately follows from inequalities (3) and (4) that $\mu_I = 0$ for all $I \in \mathcal{I} \setminus \mathcal{J}$ containing an element not covered by the union of the sets $J \in \mathcal{J}$. Moreover, since all feasible solutions have the same size ℓ , we conclude that the sum of all equations (3) is

$$\sum_{I \in \mathcal{J}} \lambda_I - \sum_{I \in \bar{\mathcal{J}}} \mu_I = 0. \quad (6)$$

For easier reference, we summarize our observations:

Lemma 2.1. *Given any ℓ -COP with feasible solutions $\mathcal{I} \subseteq 2^E$ and a natural number $h \leq \dim P_{\mathcal{I}}(E)$, the polytope $P_{\mathcal{I}}(E)$ is h -neighbourly if and only if for every set $\mathcal{J} \subseteq \mathcal{I}$ of size h the system of constraints*

$$\sum_{I \in \mathcal{J}} \lambda_I \chi_e^I - \sum_{I \in \bar{\mathcal{J}}} \mu_I \chi_e^I = 0 \quad \text{for all } e \in E, \quad (7)$$

$$\bar{\mathcal{J}} \neq \emptyset$$

is inconsistent, where $\bar{\mathcal{J}} := \{I \in \mathcal{I} \setminus \mathcal{J} \mid I \subseteq \bigcup_{J \in \mathcal{J}} J, \mu_I > 0\}$.

3 Neighbourlicity of the k -clique polytope P_{kn}^2

Let us open this section by showing that P_{kn}^2 is 2-neighbourly. Let α be the characteristic vector of the union of the edge sets of two different cliques C_1 and C_2 . The inequality $\alpha x \leq \frac{k(k-1)}{2}$ is valid for P_{kn}^2 , and the characteristic vectors of C_1 and C_2 are the only characteristic vectors of k -cliques satisfying the inequality at equality.

After determining the dimension of P_{kn}^2 , we will show that P_{kn}^2 is 3-, but usually not 4-neighbourly.

Theorem 3.1. *For $2 \leq k \leq n - 2$, the dimension of the k -clique polytope P_{kn}^2 defined on a complete graph K_n is $\frac{1}{2}n(n-1) - 1 = |E(K_n)| - 1$.*

Proof. All $x \in P_{kn}^2$ satisfy the equation

$$\sum_{e \in E(K_n)} x_e = \frac{1}{2}k(k-1) \quad (8)$$

proving $\dim P_{kn}^2 \leq |E(K_n)| - 1$. To show equality, suppose that there is any equation $\sum_{e \in E(K_n)} \alpha_e x_e = \beta$ satisfied by all $x \in P_{kn}^2$. Considering any two distinct nodes $u, v \in V(K_n)$ and any node set $W \subseteq V(K_n) \setminus \{u, v\}$ of size $k-1$, we derive from $\alpha(E(W \cup \{u\})) = \alpha(E(W \cup \{v\})) = \beta$ the equation

$$\sum_{w \in W} \alpha_{\{u, w\}} - \sum_{w \in W} \alpha_{\{v, w\}} = 0. \quad (\text{eq}(u, v, W))$$

Furthermore, consider any four distinct nodes $t, u, v, w \in V(K_n)$. Then, for any node set $Z \subseteq V(K_n) \setminus \{t, u, v, w\}$ of size $k-2$, we derive the equation system

$$\begin{aligned} E(Z \cup \{t, u\}) - E(Z \cup \{u, w\}) &= 0 \\ E(Z \cup \{t, v\}) - E(Z \cup \{v, w\}) &= 0 \\ \Leftrightarrow \alpha_{\{t, u\}} + \sum_{z \in Z} \alpha_{\{t, z\}} - \alpha_{\{u, w\}} + \sum_{z \in Z} \alpha_{\{w, z\}} &= 0 \\ \alpha_{\{t, v\}} + \sum_{z \in Z} \alpha_{\{t, z\}} - \alpha_{\{v, w\}} + \sum_{z \in Z} \alpha_{\{w, z\}} &= 0. \end{aligned}$$

Subtracting the second equation from the first equation, we see that

$$\alpha_{\{t, u\}} + \alpha_{\{v, w\}} - \alpha_{\{t, v\}} - \alpha_{\{u, w\}} = 0 \quad (\text{eq}(t, u, v, w))$$

for all distinct nodes $t, u, v, w \in V(K_n)$.

Next, consider any three distinct nodes $t, u, v \in V(K_n)$ and any node set $W \subseteq V(K_n) \setminus \{t, u, v\}$ of size $k-2$. Adding the equations $\text{eq}((t, u, W'))$ and $\text{eq}((t, u, v, w))$ for all $w \in W'$, where $W' := W \cup \{v\}$, we conclude that $\alpha_{\{t, u\}} = \alpha_{\{u, v\}}$. Thus, for any two edges $e = \{t, u\}, f = \{v, w\} \in E(K_n)$, $\alpha_{\{t, u\}} = \alpha_{\{u, v\}}$ and $\alpha_{\{u, v\}} = \alpha_{\{v, w\}}$ implies $\alpha_e = \alpha_f$. Thus, the equation $\sum_{e \in E(K_n)} \alpha_e = \beta$ is a multiple of (8). \square

Theorem 3.2.

(i) The polytope P_{2n}^2 is m -neighbourly, where $m := |E(K_n)| - 1$.

Moreover, for $3 \leq k \leq n-3$, P_{kn}^2 is

(ii) 3-neighbourly,

(iii) but not 4-neighbourly.

Proof. (i) This follows from the fact that, for any subset $F \subseteq E(K_n)$ of m edges, the inequality $\sum_{e \in F} x_e \leq 1$ is valid for P_{2n}^2 and is satisfied at equality if and only if x is the characteristic vector of an edge (= 2-clique) belonging to F .

(ii) We have to prove that any three distinct k -cliques C_1, C_2, C_3 determine a 2-face of P_{kn}^2 . Since every k -clique has an edge not contained in the union of two other k -cliques, it follows that the characteristic vectors of the cliques C_i are affinely independent. Thus, by Lemma 2.1, it remains to show that the system

$$\sum_{C \in \mathcal{K}'} \mu_C \chi_e^{E(C)} - \sum_{i=1}^3 \lambda_i \chi_e^{E(C_i)} = 0 \quad \forall e \in E(K_n), \quad (9)$$

$$\mathcal{K}' \neq \emptyset \quad (10)$$

is inconsistent. Here,

$$\mathcal{K}' := \{C \in \mathcal{K} \mid \mu_C > 0, C \neq C_i \text{ for } i = 1, 2, 3, C \subseteq C_1 \cup C_2 \cup C_3\}.$$

For any edge $e \in E(K_n)$, denote the equation among (9) associated with e by (9e). Of course, summing up all equations (9e) for $e \in \mathcal{K}'$, we derive that

$$\sum_{C \in \mathcal{K}'} \mu_C = \lambda_1 + \lambda_2 + \lambda_3. \quad (11)$$

For any $i \in \{1, 2, 3\}$, each $C \in \mathcal{K}'$ contains an edge f not contained in the union of the both k -cliques C_j with $j \neq i$. Because of $\mu_C > 0$ and equation (9f), we conclude that $\lambda_i > 0$ for $i = 1, 2, 3$.

Next, suppose that, for some $j \in \{1, 2, 3\}$, there exists a node $v \in V(C_j)$ not contained in the node set of the union of the two other cliques C_i , $i \neq j$. This implies $\sum_{i=1}^3 \lambda_i \chi_e^{E(C_i)} = \lambda_j$ for all $e \in \delta(v) \cap E(C_j)$. Since $\lambda_j > 0$, for every such edge e , there exists some $C \in \mathcal{K}'$ with $\mu_C > 0$ and $e \in E(C)$. Since $E(C) \subseteq E(C_1) \cup E(C_2) \cup E(C_3)$, we conclude that $\delta(v) \cap E(C) \subseteq \delta(v) \cap E(C_j)$. Since C and C_j are k -cliques, we derive that $|\delta(v) \cap E(C)| = |\delta(v) \cap E(C_j)| = k - 1$, and hence $\delta(v) \cap E(C) = \delta(v) \cap E(C_j)$ which, in turn, implies $C = C_j$, a contradiction. Consequently, for any $j \in \{1, 2, 3\}$, $V(C_j)$ is contained in the union of the node sets of the two other cliques C_i , $i \neq j$.

For the remainder of this proof, let $V^{1,2,3} := V(C_1) \cap V(C_2) \cap V(C_3)$, and for distinct $i, j \in \{1, 2, 3\}$, let $V^{i,j} := (V(C_i) \cap V(C_j)) \setminus V^{1,2,3}$ and $E^{i,j} := \{\{u, v\} \in E(K_n) \mid u \in V^{i,j}, v \in V^{1,2,3}\}$. Moreover, for any $i \in \{1, 2, 3\}$, let $V^i := V(C_i) \setminus \bigcup_{j \neq i} V(C_j)$ and $E^i := E(C_i) \setminus \bigcup_{j \neq i} E(C_j)$.

By the last observation, $V^i = \emptyset$ for $i = 1, 2, 3$. Moreover, letting $\{1, 2, 3\} = \{i, j, \ell\}$, we see that

$$\begin{aligned} V(C_i) &= V(C_i) \cap (V(C_j) \cup V(C_\ell)) \\ &= (V(C_i) \cap V(C_j)) \cup (V(C_i) \cap V(C_\ell)) \\ &= V^{i,j} \cup V^{i,\ell} \cup V^{1,2,3}. \end{aligned}$$

This implies $k = |V(C_i)| = |V^{i,j}| + |V^{i,\ell}| + |V^{1,2,3}|$ for $i = 1, 2, 3$, and hence, $r := |V^{1,2}| = |V^{1,3}| = |V^{2,3}|$. This, in turn, means that k and $s := |V^{1,2,3}|$ have the same parity, since $|V^{i,j}| + |V^{i,\ell}| = 2r$ is even. Clearly, $r > 0$, because otherwise, it follows that $C_1 = C_2 = C_3$, a contradiction.

Let now $K \in \mathcal{K}'$. Define $p_K^{1,2,3} := |V^{1,2,3} \cap V(K)|$, and, for any distinct $i, j \in \{1, 2, 3\}$, define $p_K^{i,j} = |V^{i,j} \cap V(K)|$. Since $V(K)$ is contained in the union of the node sets of the cliques C_i , and since the node sets $V^{1,2}$, $V^{1,3}$, $V^{2,3}$, and $V^{1,2,3}$ are disjoint, we conclude that $p_K^{1,2,3} + p_K^{1,2} + p_K^{1,3} + p_K^{2,3} = k$. Letting $\{i, j, \ell\} = \{1, 2, 3\}$, we see, in addition, that the number of edges of K in E^i is $p_K^{i,j} p_K^{i,\ell}$, as $E^i = \{\{u, v\} \in E(C_i) \mid u \in V^{i,j}, v \in V^{i,\ell}\}$. Next, the number of edges of K in $E(V^{i,j})$ is $\frac{1}{2} p_K^{i,j} (p_K^{i,j} - 1)$, that in $E^{i,j}$ is $p_K^{i,j} p_K^{1,2,3}$, and that in $E(V^{1,2,3})$ is $\frac{1}{2} p_K^{1,2,3} (p_K^{1,2,3} - 1)$. Thus, we derive from the equations (9e) the following four

equations:

$$\begin{aligned}
& \sum_{i=1}^3 \sum_{e \in E^i} \sum_{K \in \mathcal{K}'} \mu_K \chi_e^{E(K)} = \sum_{i=1}^3 \sum_{e \in E^i} \sum_{j=1}^3 \lambda_j \chi_e^{E(C_j)} \\
& \Leftrightarrow \sum_{i=1}^3 \sum_{K \in \mathcal{K}'} \mu_K \left(\sum_{e \in E^i} \chi_e^{E(K)} \right) = \sum_{i=1}^3 \sum_{e \in E^i} \lambda_i \\
& \Leftrightarrow \sum_{K \in \mathcal{K}'} \mu_K \sum_{i=1}^3 |E(K) \cap E^i| = \sum_{i=1}^3 |E^i| \lambda_i \\
& \Leftrightarrow \sum_{K \in \mathcal{K}'} \mu_K \sum_{i=1}^3 p_K^{i,j} p_K^{i,\ell} = \sum_{i=1}^3 |V^{i,j}| |V^{i,\ell}| \lambda_i \\
& \Leftrightarrow \sum_{K \in \mathcal{K}'} \mu_K (p_K^{1,2} p_K^{1,3} + p_K^{1,2} p_K^{2,3} + p_K^{1,3} p_K^{2,3}) = r^2 (\lambda_1 + \lambda_2 + \lambda_3), \quad (12)
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{e \in E(V^{i,j})} \sum_{K \in \mathcal{K}'} \mu_K \chi_e^{E(K)} = \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{e \in E(V^{i,j})} \sum_{\ell=1}^3 \lambda_\ell \chi_e^{E(C_\ell)} \\
& \Leftrightarrow \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{K \in \mathcal{K}'} \mu_K \frac{1}{2} p_K^{i,j} (p_K^{i,j} - 1) = \sum_{i=1}^2 \sum_{j=i+1}^3 \frac{1}{2} r(r-1) (\lambda_i + \lambda_j) \\
& \Leftrightarrow \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{K \in \mathcal{K}'} \mu_K p_K^{i,j} (p_K^{i,j} - 1) = 2r(r-1) (\lambda_1 + \lambda_2 + \lambda_3), \quad (13)
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{e \in E^{i,j}} \sum_{K \in \mathcal{K}'} \mu_K \chi_e^{E(K)} = \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{e \in E^{i,j}} \sum_{\ell=1}^3 \lambda_\ell \chi_e^{E(C_\ell)} \\
& \Leftrightarrow \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{K \in \mathcal{K}'} \mu_K p_K^{i,j} p_K^{1,2,3} = rs \sum_{i=1}^2 \sum_{j=i+1}^3 (\lambda_i + \lambda_j) \\
& \Leftrightarrow \sum_{K \in \mathcal{K}'} \mu_K p_K^{1,2,3} (p_K^{1,2} + p_K^{1,3} + p_K^{2,3}) = 2rs (\lambda_1 + \lambda_2 + \lambda_3), \quad (14)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{e \in E(V^{1,2,3})} \sum_{K \in \mathcal{K}'} \mu_K \chi_e^{E(K)} = \sum_{e \in E(V^{1,2,3})} \sum_{\ell=1}^3 \lambda_\ell \chi_e^{E(C_\ell)} \\
& \Leftrightarrow \sum_{K \in \mathcal{K}'} \mu_K p_K^{1,2,3} (p_K^{1,2,3} - 1) = s(s-1) (\lambda_1 + \lambda_2 + \lambda_3). \quad (15)
\end{aligned}$$

Let us show that $p_K^{1,2,3} = s$ for all $K \in \mathcal{K}'$. By definition, $0 \leq p_K^{1,2,3} \leq s$. Thus, if $s = 0$, then $p_K^{1,2,3} = s$. Next, let $s = 1$. This means, $p_K^{1,2,3} \in \{0, 1\}$ for all $K \in \mathcal{K}'$ and (14) is

$$\sum_{K \in \mathcal{K}'} \mu_K p_K^{1,2,3} (p_K^{1,2} + p_K^{1,3} + p_K^{2,3}) = 2r (\lambda_1 + \lambda_2 + \lambda_3).$$

Since $2r+s = k = p_K^{1,2} + p_K^{1,3} + p_K^{2,3} + p_K^{1,2,3}$, we conclude that $p_K^{1,2} + p_K^{1,3} + p_K^{2,3} = 2r$ whenever $p_K^{1,2,3} = 1$. Thus, setting $\mathcal{K}'_1 := \{K \in \mathcal{K}' \mid p_K^{1,2,3} = 1\}$, we derive from equation (14) that $\sum_{K \in \mathcal{K}'_1} \mu_K = \lambda_1 + \lambda_2 + \lambda_3$, contradicting (11) if $\mathcal{K}'_1 \subsetneq \mathcal{K}'$. Thus, $\mathcal{K}'_1 = \mathcal{K}'$, that is, $p_K^{1,2,3} = 1 = s$ for all $K \in \mathcal{K}'$. Finally, let $s \geq 2$. Adding to both sides of equation (15) the sum $-\sum_{K \in \mathcal{K}'} \mu_K s(s-1)$, we obtain the equation

$$\sum_{K \in \mathcal{K}'} \mu_K (p_K^{1,2,3} (p_K^{1,2,3} - 1) - s(s-1)) = s(s-1)(\lambda_1 + \lambda_2 + \lambda_3 - \sum_{K \in \mathcal{K}'} \mu_K).$$

Using (11), we see that the right hand side of this equation vanishes, while $\mu_K > 0$ and $p_K^{1,2,3} \leq s$ imply that each summand of the left hand side is less than or equal to zero. Thus, $p_K^{1,2,3} = s$ for all $K \in \mathcal{K}'$.

To finish the proof, we derive a contradiction from equations (12) and (13). First, we see that $p_K^{1,2,3} = s$ implies $p_K^{1,2} + p_K^{1,3} + p_K^{2,3} = 2r$ for all $K \in \mathcal{K}'$. Next, we observe that, for any $K \in \mathcal{K}'$,

$$\begin{aligned} & \sum_{i=1}^2 \sum_{j=i+1}^3 p_K^{i,j} (p_K^{i,j} - 1) \\ &= \left(p_K^{1,2} + p_K^{1,3} + p_K^{2,3} \right)^2 - (p_K^{1,2} + p_K^{1,3} + p_K^{2,3}) \\ & \quad - 2(p_K^{1,2} p_K^{1,3} + p_K^{1,2} p_K^{2,3} + p_K^{1,3} p_K^{2,3}) \\ &= 4r^2 - 2r - 2t_K, \end{aligned}$$

where $t_K := p_K^{1,2} p_K^{1,3} + p_K^{1,2} p_K^{2,3} + p_K^{1,3} p_K^{2,3}$. Thus, multiplying (12) with $-2r(r-1)$, multiplying (13) with r^2 , and summing up the resulting equations, we obtain:

$$\begin{aligned} & \sum_{K \in \mathcal{K}'} \mu_K (r^2(4r^2 - 2r - 2t_K) - 2r(r-1)t_K) = 0 \\ & \Leftrightarrow 2r(2r-1) \sum_{K \in \mathcal{K}'} \mu_K (r^2 - t_K) = 0 \\ & \stackrel{r \geq 0}{\Leftrightarrow} \sum_{K \in \mathcal{K}'} \mu_K (r^2 - t_K) = 0 \end{aligned} \tag{16}$$

Finally, consider the real valued function

$$f(\varphi, \psi, \vartheta) := \varphi\psi + \varphi\vartheta + \psi\vartheta$$

under the condition that $0 \leq \varphi, \psi, \vartheta \leq r$ and $\varphi + \psi + \vartheta = 2r$. Since f is equal to r^2 at $(r, r, 0)$, $(r, 0, r)$, and $(0, r, r)$, and f is concave, we conclude that $r^2 - t_K \leq 0$ for all $K \in \mathcal{K}'$ and $r^2 - t_K = 0$ if and only if $(p_K^{1,2}, p_K^{1,3}, p_K^{2,3}) \in \{(r, r, 0), (r, 0, r), (0, r, r)\}$. Consequently, (16) implies that $K \in \{C_1, C_2, C_3\}$ for each $K \in \mathcal{K}'$ contradicting $\{C_1, C_2, C_3\} \cap \mathcal{K}' = \emptyset$.

(iii) In what follows, we construct four k -cliques such that, if their characteristic vectors are contained in the same face, then this face contains the characteristic vectors of further k -cliques. W.l.o.g. let $V(K_n) = \{1, 2, \dots, n\}$, and let $W \subseteq V(K_n) \setminus \{1, 2, \dots, 6\}$ be a node set of size $k-3$. Consider the k -cliques C_i , $i = 1, 2, 3, 4$, determined by the node sets

$$\begin{aligned} V_1 &:= W \cup \{1, 5, 6\}, & V_2 &:= W \cup \{2, 6, 4\}, \\ V_3 &:= W \cup \{3, 4, 5\}, & V_4 &:= W \cup \{1, 2, 3\}, \end{aligned}$$

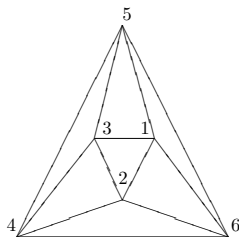


Figure 1: The union of the four k -cliques C_1, \dots, C_4 restricted to $\{1, 2, \dots, 6\}$.

respectively. The graph depicted in Figure 1 is the subgraph of the union of these four k -cliques induced by the node set $\{1, 2, \dots, 6\}$. Now, every face of P_{kn}^2 that contains the characteristic vectors of the cliques determined by V_i for $i = 1, 2, 3, 4$, also contains that of the k -cliques determined by

$$\begin{aligned} V_5 &:= W \cup \{1, 2, 6\}, & V_6 &:= W \cup \{2, 3, 4\}, \\ V_7 &:= W \cup \{3, 1, 5\}, & V_8 &:= W \cup \{4, 5, 6\}. \end{aligned}$$

Suppose not. This means, there exists a valid inequality $\sum_{e \in E(K_n)} \alpha_e x_e \leq \beta$ such that $\alpha(E(V_i)) = \beta$ for $i = 1, 2, 3, 4$ and $\alpha(E(V_i)) < \beta$ for $i = 5, 6, 7, 8$, which, in turn, implies that

$$\begin{aligned} \alpha_{\{1,5\}} + \alpha_{\{1,6\}} + \alpha_{\{5,6\}} + \alpha^1 + \alpha^5 + \alpha^6 &= \gamma, \\ \alpha_{\{2,6\}} + \alpha_{\{2,4\}} + \alpha_{\{6,4\}} + \alpha^2 + \alpha^4 + \alpha^6 &= \gamma, \\ \alpha_{\{3,4\}} + \alpha_{\{3,5\}} + \alpha_{\{4,5\}} + \alpha^3 + \alpha^4 + \alpha^5 &= \gamma, \\ \alpha_{\{1,2\}} + \alpha_{\{2,3\}} + \alpha_{\{1,3\}} + \alpha^1 + \alpha^2 + \alpha^3 &= \gamma, \end{aligned}$$

and

$$\begin{aligned} \alpha_{\{1,2\}} + \alpha_{\{1,6\}} + \alpha_{\{2,6\}} + \alpha^1 + \alpha^2 + \alpha^6 &< \gamma, \\ \alpha_{\{2,4\}} + \alpha_{\{2,3\}} + \alpha_{\{3,4\}} + \alpha^2 + \alpha^3 + \alpha^4 &< \gamma, \\ \alpha_{\{3,5\}} + \alpha_{\{1,3\}} + \alpha_{\{1,5\}} + \alpha^1 + \alpha^3 + \alpha^5 &< \gamma, \\ \alpha_{\{4,5\}} + \alpha_{\{5,6\}} + \alpha_{\{4,6\}} + \alpha^4 + \alpha^5 + \alpha^6 &< \gamma, \end{aligned}$$

where $\gamma := \beta - \alpha(E(W))$ and $\alpha^v := \sum_{w \in W} \alpha_{\{v,w\}}$ for $v \in \{1, 2, \dots, 6\}$. Adding the four inequalities and subtracting the four equations, we conclude that $0 < 0$, a contradiction. \square

4 Neighbourlicity of the k -clique polytope P_{kn}^r

In this section, we will partially transfer the previous result on the k -cliques of complete graphs, to r -uniform-hypergraphs. This study was initialized by a remark of Pierre Duchet [3] who proposed to us this generalization.

For any natural numbers p, q we denote by $B(p, q) := \binom{p}{q}$ the binomial coefficient of p over q . Recall that the set of hyperedges of a complete r -uniform hypergraph K_n^r on n nodes is given by $E(K_n^r) = \{e \subseteq V(K_n^r) \mid |e| = r\}$. K_n^r has $B(n, r)$ hyperedges, and a k -clique of K_n^r has $B(k, r)$ hyperedges. In the following, we assume that $k \geq r + 1$. Moreover, for any node v of K_n^r , we denote by $\delta(v) := \{e \in E(K_n^r) \mid v \in e\}$ the set of edges of K_n^r containing v .

Theorem 4.1. *Let K_n^r be the complete r -uniform hypergraph on n nodes, where $2 \leq r \leq n$, and let k be an integer with $r \leq k \leq n$.*

- (i) *For $k = r$, P_{kn}^r is m -neighbourly, where $m := |E(K_n^r)| - 1$.*
- (ii) *For $r + 1 \leq k \leq n - 1$, a lower bound for the neighbourlicity of P_{kn}^r is r .*
- (iii) *For $k \geq r + 1$ and $n \geq k + r + 1$, an upper bound for the neighbourlicity of P_{kn}^r is $2^r - 1$.*

Proof. (i) For $k = r$, each k -clique only consists of one hyperedge, since, in this case, $B(k, r) = B(k, k) = 1$. Thus, m k -cliques always form an $(m - 1)$ -face of P_{kn}^k .

(ii) Let C_1, C_2, \dots, C_r be distinct k -cliques of K_n^r . Consider the inequality $\sum_{e \in F} x_e \leq B(k, r)$ for $F := \bigcup_{i=1}^r E(C_i)$. This inequality is obviously valid for P_{kn}^r , since each k -clique of K_n^r consists of $B(k, r)$ edges. Moreover, the characteristic vectors of the cliques C_i are contained in the face induced by this inequality. Consider now any k -clique C different from the cliques C_i . Then, for each $i \in \{1, 2, \dots, r\}$, C has a node v_i not contained in $V(C_i)$. The nodes are not necessarily distinct, but this argument tells us that there exists a nonempty set $U \subseteq V(C)$ of size r such that for every clique C_i , there exists a node $v \in U$ such that $v \notin V(C_i)$. This means that U is an edge of C that is contained in none of the cliques C_i . Thus, for $x = \chi^{E(C)}$, the left hand side of the above inequality is strictly less than $B(k, r)$.

(iii) We have to show that P_{kn}^r is not 2^r -neighbourly. We prove this by induction on r by showing that there always exists a counterexample given by $2s_r$ k -cliques $C_i, \tilde{C}_i, i = 1, 2, \dots, s_r$ with $\sum_{i=1}^{s_r} \chi^{E(C_i)} = \sum_{i=1}^{s_r} \chi^{E(\tilde{C}_i)}$, where $s_r := 2^r$, disproving that P_{kn}^r is s -neighbourly. For $r = 2$, the claim directly follows from Theorem 3.2 (iii) and its proof.

So let $r \geq 3$, and let $C_i, \tilde{C}_i, i = 1, 2, \dots, s_{r-1}$ be $k - 1$ -cliques of K_{n-2}^{r-1} that generate a counterexample for $P_{k-1, n-2}^{r-1}$ being s_{r-1} -neighbourly. Define $2s_r = 4s_{r-1}$ k -cliques $C'_i, \tilde{C}'_i, C''_i, \tilde{C}''_i, i = 1, 2, \dots, s_{r-1}$, of K_n^r as subgraphs induced by the node sets $V'_i := V(C_i) \cup \{u\}$, $\tilde{V}'_i := V(\tilde{C}_i) \cup \{v\}$, $V''_i := V(C_i) \cup \{v\}$, and $\tilde{V}''_i := V(\tilde{C}_i) \cup \{u\}$, respectively, where $\{u, v\} = V(K_n^r) \setminus V(K_{n-2}^{r-1})$. By definition of the node sets, $E(C'_i) \setminus \delta(u) = E(C_i) \setminus \delta(v)$ and $E(\tilde{C}'_i) \setminus \delta(v) = E(\tilde{C}_i) \setminus \delta(u)$. Moreover, by hypothesis, $E(C'_i) \cap \delta(u) = E(\tilde{C}'_i) \cap \delta(u)$ and $E(\tilde{C}'_i) \cap \delta(v) = E(C''_i) \cap \delta(v)$, since $e \in E(C'_i) \cap \delta(u)$ if and only if $e \setminus \{u\} \in E(C_i)$, and an analogous statement holds for the other edge sets $E(\tilde{C}''_i) \cap \delta(u)$, $E(\tilde{C}'_i) \cap \delta(v)$, and $E(C''_i) \cap \delta(v)$. Hence, we conclude that

$$\sum_{i=1}^{s_{r-1}} (\chi^{E(C'_i)} + \chi^{E(\tilde{C}'_i)}) = \sum_{i=1}^{s_{r-1}} (\chi^{E(C''_i)} + \chi^{E(\tilde{C}''_i)}),$$

which means that every face of P_{kn}^r that contains the incidence vectors of the cliques C'_i, \tilde{C}'_i also contains that of the cliques C''_i, \tilde{C}''_i . \square

Our computational experiments indicate that the upper bound $2^r - 1$ is probably tight whenever $n \geq k + r + 1$ and $k \geq r + 1$, see Table 1. For the computation of these bounds, we used the following integer programming

formulation:

$$\min \sum_{C \in \mathcal{K}} h_C, \quad (17)$$

$$\sum_{C \in \mathcal{K}} \lambda_C \chi_e^{E(C)} - \sum_{C \in \mathcal{K}} \mu_C \chi_e^{E(C)} = 0 \quad \forall e \in E(K_n^r), \quad (18)$$

$$\lambda_C \leq h_C, \mu_C \leq \ell_C \quad \forall C \in \mathcal{K}, \quad (19)$$

$$h_C + \ell_C \leq 1 \quad \forall C \in \mathcal{K}, \quad (20)$$

$$\sum_{\substack{C \in \mathcal{K} \\ C \ni v}} h_C \geq r \quad \forall v \in U, \quad (21)$$

$$\lambda_C, \mu_C \geq 0, h_C, \ell_C \in \{0, 1\} \quad \forall C \in \mathcal{K}. \quad (22)$$

Here, U is some particular subset of $V(K_n^r)$ of size $k + 1$. This integer program computes the smallest number h such that the system (7) applied to P_{kn}^r is consistent. In other words, if $(\lambda^*, \mu^*, h^*, \ell^*)$ is an optimal solution of (17)-(22), then $\sum_C h_C^* - 1$ is the neighbourlicity of P_{kn}^r . Equations (18) are the equations of system (7) for P_{kn}^r . Next, the constraints (19), (20), and (22) express that both clique collections to be generated have to be disjoint. Finally, inequalities (21) exclude the trivial solution. To be more specific, we always can assume that the k -cliques of the set $\mathcal{K}_h := \{C \in \mathcal{K} \mid h_C = 1\}$ cover at least $k + 1$ nodes. Since the above integer programming formulation without inequalities (21) is symmetric, we can fix a particular subset of $V(K_n^r)$ of size $k + 1$. In the remainder of this section, we prove the validity of inequalities (21).

For any $L \subseteq \mathcal{K}_h$, let

$$V(L) := \bigcap_{C \in L} V(C) \setminus \bigcup_{C \subseteq \mathcal{K}_h \setminus L} V(C)$$

be the set of nodes contained in each clique in L but not contained in any other clique. For each clique $C \in \mathcal{K}_h$, the collection $T_C := \{V(L) \mid V(L) \cap V(C) \neq \emptyset\}$ obviously is a partition of $V(C)$.

Using a similar argumentation as in the proof to statement (ii) of Theorem 4.1, we see that $V(L) = \emptyset$ whenever $|L| < r$. Suppose not. Then, there is some $C' \in \mathcal{K}_\ell := \{C \in \mathcal{K} \mid \ell_C = 1\}$ such that $V(L) \cap V(C') \neq \emptyset$. Thus, $V(C') \subseteq V := \bigcup_{C \in L} V(C)$. Since C' has to be different to each clique in L ,

$n \geq 2(r + 1)$	k	r	Neighbourlicity
$\geq k + 3$	≥ 3	2	3
8	4	3	7
9	4	3	7
9	5	3	7
10	5	3	7
10	6	3	7
10	5	4	15
11	6	4	15
12	6	5	31

Table 1: Neighbourlicities of some k -clique polytopes

there exists a node set $U \subseteq V(C') \setminus V(L)$ of size at most $r-1$ such that for every clique $C \in L$, there is some node $u \in U$ not contained in C . Of course, U can be completed to a node set $W \subseteq V(C')$ of size r containing a node $v \in V(L)$. By construction, W is an edge of C' , W is contained in none of the cliques $C \in L$, since W contains U , and W is contained in none of the other cliques of \mathcal{K}_h , since W contains a node of $V(L)$. Hence, the integer program (17)-(22) is unfeasible. We conclude, in order to be feasible, $V(L) = \emptyset$ for all $L \subseteq \mathcal{K}_h$ with $|L| < r$. In other words, each node covered by the cliques in \mathcal{K}_h is contained in at least r cliques. This verifies the validity of inequalities (21).

5 Conclusion

In this paper, we studied the connectivity of the k -clique polytopes P_{kn}^r . We have shown that, for $r = 2$, P_{kn}^r is 3- but not 4-neighbourly, in general. Moreover, by adapting the proof for P_{kn}^2 not being 4-neighbourly to the case that $r \geq 3$, we have shown that P_{kn}^r is not 2^r -neighbourly in many cases (depending on the relations between k , n , and r), which implies that the neighbourlicity is at most $2^r - 1$. On the other hand, we have presented, in Theorem 4.1, a lower bound for the neighbourlicity, namely r . However, in view of our computational results indicating that the upper bound is tight, the lower bound seems to be quite weak.

It seems to be obvious to try to adapt the proof that P_{kn}^r is $2^r - 1$ -neighbourly for $r = 2$ to the case that $r \geq 3$. For $r = 2$, the proof is essentially based on the simple but important observation that the node sets $V^{1,2}$, $V^{1,3}$, $V^{2,3}$, constructed there, have all the same size. These node sets correspond to the node sets $V(L)$ defined in the previous section. If the proof for $r = 2$ can be extended to the general case, then one probably has to show that $|V(L)| = |V(L')|$ if and only if $|L| = |L'|$. Unfortunately, we are not able to determine the relations between the node sets $V(L)$. Currently, we only see that, in general, almost all sets $V(L)$ have to be empty, even more if $|\mathcal{K}_h| \approx 2^r$. This follows from the fact that, for each k -clique $C \in \mathcal{K}_h$, the partition $T_C = \{V(L) \mid V(L) \cap V(C) \neq \emptyset\}$ of $V(C)$ consists of at most k node sets. So at most $k|\mathcal{K}_h|$ of the node sets $V(L)$ are nonempty, which means that at least $2^{|\mathcal{K}_h|} - k|\mathcal{K}_h|$ of the sets $V(L)$ are empty.

References

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