



NIELS LINDNER¹
CHRISTIAN LIEBCHEN²

A two-stage model for periodic timetabling with fixed line activities

¹  0000-0002-8337-4387
²  0000-0002-4311-2024

Zuse Institute Berlin
Takustr. 7
14195 Berlin
Germany

Telephone: +49 30 84185-0
Telefax: +49 30 84185-125

E-mail: bibliothek@zib.de
URL: <http://www.zib.de>

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

A two-stage model for periodic timetabling with fixed line activities

Niels Lindner^[0000-0002-8337-4387] and
Christian Liebchen^[0000-0002-4311-2024]

Abstract The timetable is a central pillar of any public transportation system. Constructing and optimizing periodic timetables in terms of passenger comfort and operational efficiency leads to NP-hard optimization problems that are also computationally challenging in applications. The Periodic Event Scheduling Problem (PESP) as standard mathematical tool benefits from its succinct formulation and rich combinatorial structure, but suffers from poor linear programming relaxations and weak dual bounds.

These difficulties persist in a reduced version, where driving and dwelling activities of the lines are assumed to be fixed. In this case, fixing the initial departure time of each line fully determines the timetable, and for each pair of lines, the resulting (weighted) transfer durations can be expressed in terms of a piecewise linear non-convex function in terms of the difference of the initial times. When the number of activities between two lines is bounded, this function can be computed in polynomial time.

By inserting precomputed piecewise linear functions into a mixed-integer program with the initial departure times as variables, we introduce an equivalent formulation for reduced PESP instances. The model bears analogies with quadratic semi-assignment approaches and offers alternative ways to compute primal and dual bounds. We evaluate the computational behavior of our approach on realistic benchmarking instances.

1 Introduction

Periodic timetabling is a central task in the planning process of public transport. It affects both the service quality for the passengers (in particular transfer times) and

Niels Lindner
Freie Universität Berlin, Institut für Mathematik, Arnimallee 3, 14195 Berlin, Germany,
Zuse Institute Berlin, Takustr. 7, 14195 Berlin, Germany, e-mail: lindner@zib.de

Christian Liebchen
Technical University of Applied Sciences Wildau, Hochschulring 1, 15745 Wildau, Germany, e-mail:
liebchen@th-wildau.de

the operational efficiency of the company (in particular, using few vehicles). The general problem, as well as its practical applications for railway networks, makes use of feasibility constraints, e.g., to ensure safe operations on single tracks. The Periodic Event Scheduling Problem (PESP) [6] is the first choice to compute periodic timetables. Sometimes, periodic timetabling comes without feasibility constraints, as for bus networks or in the prominent collection of publicly available benchmarking instances PESPLib [5].

We are considering an even more restricted variant, namely, where the dwell times at intermediate stations are always fixed to their minimum durations. For some companies, this is actually their policy, in particular during peak hours. Apart from that, computing solutions for this restricted variant can serve as a heuristic for the PESPLib instances.

We propose an alternative formulation for this variant of periodic timetabling. The key variables are only the departure times of the directed lines at their first stops. The shape of the classical objective function, which emerges essentially from – occasionally (anti-)parallel – transfer arcs, is expressed by piecewise linear functions.

The role of the departure time variables at the first stations resembles early applications of the quadratic semi-assignment problem (QSAP) to periodic timetabling. However, in a recent survey on quadratic assignment problem variants [7] there are no more recent references than 1992 are listed.

2 Two-Stage Models

In this work, we consider the *Periodic Event Scheduling Problem* (PESP) [6]. We focus on “reduced” instances (G, T, ℓ, w) , where $G = (V, A)$ is a directed multi-graph, $T \in \mathbb{N}$ is a period time, $\ell \in \mathbb{Z}^A$ are lower bounds with $0 \leq \ell < T$, and $w \in \mathbb{R}_{\geq 0}$ are weights. The aim of PESP is to find a *periodic timetable* $\pi \in \{0, 1, \dots, T-1\}^V$ and a *periodic tension* $x \in \mathbb{Z}^A$ with $\ell \leq x < \ell + T$ such that $x_a \equiv (\pi_j - \pi_i - \ell_a) \pmod{T + \ell_a}$ holds for all arcs $a = (i, j) \in A$, and the weighted *periodic slack* $\sum_{a \in A} w_a(x_a - \ell_a)$ is minimized. While finding a feasible solution is trivial, this optimization problem is NP-hard [4].

In the context of periodic timetabling in public transport, it is standard to consider *event-activity networks*, whose vertices model departures or arrivals of lines at stations, and whose arcs represent activities such as *driving*, *waiting*, *transferring*, or further operational requirements. Transfer activities are often only constrained by a minimum transfer time requirements. When driving and waiting times are fixed, the corresponding arcs may be contracted [3], so that vertices correspond to (directed) lines, which are – when no further operational constraints exist – connected by a set of transfer arcs. A periodic timetable then assigns departure times modulo T to the lines at their initial station, implicitly determining the times at later stations that have been contracted. The periodic tension consists of all the transfer durations. If the weights represent the number of passengers using a specific transfer, then the weighted slack measures the total time that passengers “waste” beyond the minimum transfer time.

We will call the following straightforward formulation of PESP on (G, T, ℓ, w) as a mixed-integer program (MIP) the *classical* formulation ([6, 3]). The actual modulo constraints are resolved by integer *periodic offset* variables p . For a fixed solution

vector p , it is well known that due to the total unimodularity of the coefficient matrix, there is no need to put any explicit integrality requirements on π and x in the case of integral bounds.

$$\text{Minimize} \quad \sum_{a \in A} w_a(x_a - \ell_a), \quad (1)$$

$$\text{s.t.} \quad \pi_j - \pi_i + T p_a = x_a \quad a = (i, j) \in A, \quad (2)$$

$$\ell_a \leq x_a \leq \ell_a + T - 1, \quad a \in A, \quad (3)$$

$$0 \leq \pi_i \leq T - 1, \quad i \in V, \quad (4)$$

$$p_a \in \{0, 1, 2\}, \quad a \in A. \quad (5)$$

To develop our two-stage model, we consider an unordered pair $u = \{i, j\}$ of distinct vertices in V . We denote by A_u^+ the set of all activities (i, j) and by A_u^- the set of all activities (j, i) . In any integer solution, there is a number x such that $x_a \equiv x \pmod T$ for all $a \in A_u^+$ and $-x_a \equiv x \pmod T$ for all $a \in A_u^-$: By (2), we can simply choose $x := \pi_j - \pi_i$. For $x \in \mathbb{R}$, we define

$$f_u(x) := \min \left\{ \sum_{a \in A_u^+} w_a(x_a - \ell_a) + \sum_{a \in A_u^-} w_a(x_a - \ell_a) \left| \begin{array}{l} \forall a \in A_u^+ : x_a \equiv x \pmod T \\ \forall a \in A_u^- : -x_a \equiv x \pmod T \\ \forall a \in A_u^+ \cup A_u^- : \\ \ell_a \leq x_a < \ell_a + T \end{array} \right. \right\},$$

so that $f_u(x)$ computes the minimum weighted slack of all activities between i and j when $\pi_j - \pi_i$ is fixed to x modulo T . We summarize a few properties without proof:

Lemma 1 *The function f_u is piecewise linear and periodic with a period of T . Any breakpoint is at $x \equiv \ell_a \pmod T$ for some $a \in A_u^+ \cup A_u^-$. All line segments have the same slope s_u , namely $s_u := \sum_{a \in A_u^+} w_a - \sum_{a \in A_u^-} w_a$.*

If we set $\ell_u := \ell_a$ for some $a \in A_u^+ \cup A_u^-$, then the restriction of f_u to $x \in [\ell_u, \ell_u + T)$ has at most $|A_u^+ \cup A_u^-|$ line segments. An example is given in Figure 1.

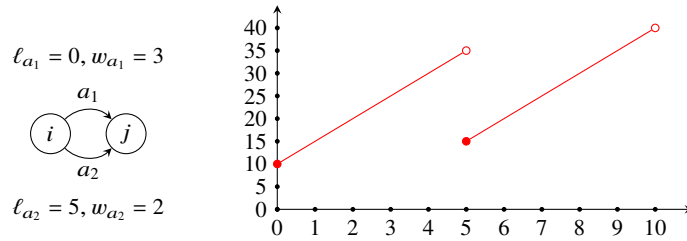


Fig. 1 Illustration of f_u for two transfer arcs a_1, a_2 , $\ell_u = 0$, and $T = 10$. The slope is $s_u = w_{a_1} + w_{a_2} = 5$, the breakpoints within the interval $[0, T)$ are at $\ell_{a_1} = 0$ and $\ell_{a_2} = 5$, and the intervals $[a_u^k, b_u^k)$ where f_u is linear are $[0, 5)$ and $[5, 10)$.

We can therefore derive a *two-stage* program that is equivalent to (1)-(5). In this, denote by \mathcal{U} denote the set of unordered pairs $\{i, j\}$ of distinct vertices of V . We

contract the multiple arcs between i and j to a single one, thereby reducing the number of first-stage integer variables. In turn, we shift the difficulty to the new non-PESP objective (6), to be computed in a second stage. The structural constraints (7)-(10) essentially keep the same structure as in the classical restricted PESP.

$$\text{Minimize} \quad \sum_{u \in \mathcal{U}} f_u(x_u) \quad (6)$$

$$\text{s.t.} \quad \pi_j - \pi_i + T p_u = x_u \quad u = \{i, j\} \in \mathcal{U}, \quad (7)$$

$$\ell_u \leq x_u \leq \ell_u + T - 1, \quad u \in \mathcal{U}, \quad (8)$$

$$0 \leq \pi_i \leq T - 1, \quad i \in V, \quad (9)$$

$$p_u \in \{0, 1, 2\}, \quad u \in \mathcal{U}. \quad (10)$$

To obtain proper mixed-integer linear programs, it remains to model the piecewise linear function f_u . We propose three variants, all of which introduce continuous variables f_u to compute $f_u(x_u)$, and binary variables $\delta_u^k \in \{0, 1\}$ to activate precisely one interval $[a_u^k, b_u^k)$ where f_u is linear, $k \in [K_u] := \{1, \dots, K_u\}$, with $\sum_{k=1}^{K_u} \delta_u^k = 1$.

1. *big-M*: For $M \gg 0$ and the slope s_u of f_u :

$$\begin{aligned} f_u &\geq s_u(x_u - a_u^k) + f_u(a_u^k) - M(1 - \delta_u^k), & u \in \mathcal{U}, k \in [K_u], \\ \ell_u + (a_u^k - \ell_u)\delta_u^k &\leq x_u \leq b_u^k + (\ell_u + T - 1 - b_u^k)(1 - \delta_u^k), & u \in \mathcal{U}, k \in [K_u]. \end{aligned}$$

2. *convex combination* or *multiple choice* model [1]:

$$\begin{aligned} f_u &= \sum_{k=1}^{K_u} \left((f_u(b_u^k) - f_u(a_u^k))\lambda_u^k + f_u(a_u^k)\delta_u^k \right), & u \in \mathcal{U}, \\ x_u &= \sum_{k=1}^{K_u} (b_u^k - a_u^k)\lambda_u^k + a_u^k\delta_u^k, & u \in \mathcal{U}, \\ 0 &\leq \lambda_u^k \leq \delta_u^k, & u \in \mathcal{U}, k \in [K_U]. \end{aligned}$$

3. *alternative convex combination* model:

$$\begin{aligned} f_u &= \sum_{k=1}^{K_u} s_u(\mu_u^k - a_u^k\delta_u^k) + f_u(a_u^k)\delta_u^k, & u \in \mathcal{U}, \\ x_u &= \sum_{k=1}^{K_u} \mu_u^k, & u \in \mathcal{U}, \\ a_u^k\delta_u^k &\leq \mu_u^k \leq b_u^k\delta_u^k, & u \in \mathcal{U}, k \in [K_U]. \end{aligned}$$

Due to space limitations, we only mention the following here without proof:

Theorem 1 *Using the convex combination models, the value v of the LP relaxation of (6)-(10) is at least $\sum_{u \in \mathcal{U}} \min\{f_u(x) \mid x \in \mathbb{R}\}$. Moreover, v is at least the value of the LP relaxation of (1)-(5), even when all flip inequalities for 2-cycles are included.*

In particular, the two-stage model has in general a non-trivial LP relaxation, in contrast to the classical model, where the value is always 0.

3 Computational Experiments

We test the two-stage formulations on the 16 railway instances RxLy of the PESPlib [5]. With the information presented in the TimPassLib library [5], we can identify driving and waiting activities, fix them to their lower bounds, and contract them. For all these reduced instances, we consider the classical PESP formulation as a MIP and the three two-stage formulations as described in Section 2. In addition to the incidence-based formulations (1)-(5) and (6)-(10), we also investigate a cycle-based formulation (see, e.g., [3]). This results in 8 equivalent MIPs per instance. We run these models for one hour wall time using Gurobi 12.0.1 [2] with default settings on a high throughput cluster, allocating 8 threads and 50 GB of shared memory to Intel Xeon Gold 6338 CPUs.

We summarize the results in 3 heatmaps for the primal bound (Figure 2a), the dual bound (Figure 2b), and the primal-dual gap (Figure 2c), respectively. We also show the evolution of the primal and dual bounds for one instance (R2L2) in Figure 3. What is stunning is that the cycle-based formulation performs significantly worse than the incidence-based formulation and sometimes does not even find a primal solution within one hour. We will therefore only consider the incidence-based models from now on. For 15 out of 16 instances, our two-stage models have superior dual bound performance, while primal bounds are typically better with the classical formulation. However, with the exception of the instance R2L4, there is always a two-stage formulation with a tighter primal-dual gap than the classical formulation, with up to 18 % reduction for the instance R1L4. When comparing the three two-stage variants among themselves, there is no clear winner. The patterns are similar and also the big- M formulation surprisingly achieves comparable and sometimes better bounds.

4 Conclusion

We conclude that our two-stage formulations perform especially well on the PESPlib instances concerning dual bounds and primal-dual gaps. This stimulates to investigate how to extend our procedure to more general instances and to more far-reaching contractions. For future work, it would be interesting to determine the trade-off between computational effectiveness and modeling potentially higher-dimensional piecewise linear functions, and to derive new theoretical insights, e.g., new cutting planes.

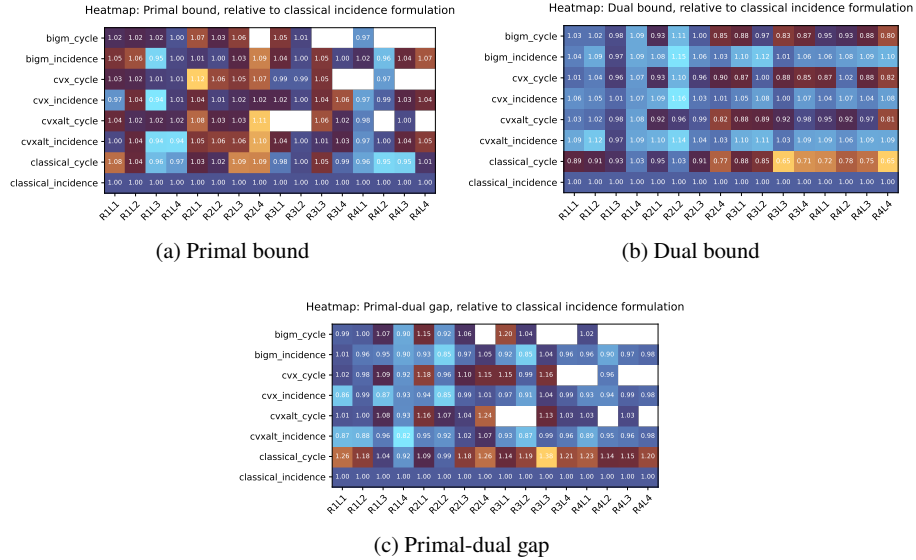


Fig. 2 Heatmaps after 1 hour for all 16 instances and all formulations, relative to the classical incidence formulation. Light blue means better, yellow and red mean worse. For example, for the instance R2L2, the two-stage incidence-based convex combination model finds a 16 % higher dual bound compared to the one computed by the standard PESP incidence-based model.

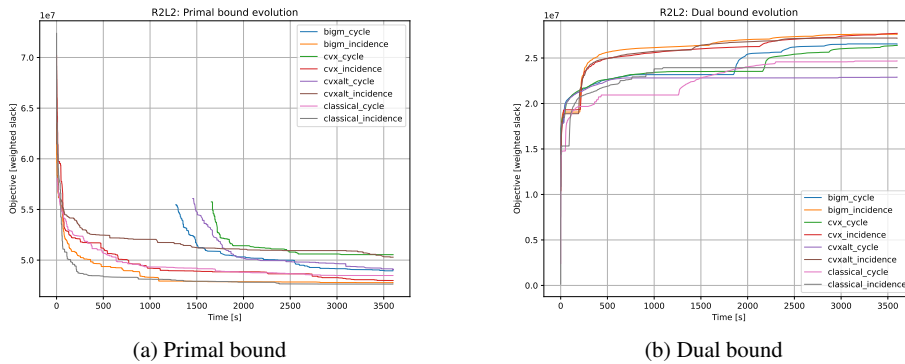


Fig. 3 Evolution of primal and dual bound for R2L2.

References

1. Balakrishnan, A., Graves, S.C.: A composite algorithm for a concave-cost network flow problem. *Networks* **19**(2), 175–202 (1989). DOI 10.1002/net.3230190202
2. Gurobi Optimization, LLC: Gurobi Optimizer Reference Manual (2025)
3. Liebchen, C.: Periodic timetable optimization in public transport. Ph.D. thesis, Technische Universität Berlin (2006)

4. Nachtigall, K.: Exact solution methods for periodic programs. Tech. Rep. 14/93, Hildesheimer Informatikberichte (1993)
5. Schiewe, P., Goerigk, M., Lindner, N.: Introducing TimPassLib – A Library for Integrated Periodic Timetabling and Passenger Routing. *Operations Research Forum* **4** (2023). DOI 10.1007/s43069-023-00244-1. See also <https://timpasslib.aalto.fi/pesplib.html>
6. Serafini, P., Ukovich, W.: A Mathematical Model for Periodic Scheduling Problems. *SIAM Journal on Discrete Mathematics* **2**(4), 550–581 (1989). DOI 10.1137/0402049
7. Silva, A., Coelho, L.C., Darvish, M.: Quadratic assignment problem variants: A survey and an effective parallel memetic iterated tabu search. *Eur. J. Oper. Res.* **292**(3), 1066–1084 (2021). DOI 10.1016/J.EJOR.2020.11.035