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On the strength of cut-based inequalities for capacitated network design polyhedra

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On the strength of cut-based inequalities for capacitated network design polyhedra *

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Abstract

In this paper we study capacitated network design problems, differentiating directed, bidirected and undirected link capacity models. We complement existing polyhedral results for the three variants by new classes of facet-defining valid inequalities and unified lifting results. For this, we study the restriction of the problems to a cut of the network. First, we show that facets of the resulting cutset polyhedra translate into facets of the original network design polyhedra if the two subgraphs defined by the network cut are (strongly) connected. Second, we provide an analysis of the facial structure of cutset polyhedra, elaborating the differences caused by the three different types of capacity constraints. We present flow-cutset inequalities for all three models and show under which conditions these are facet-defining. We also state a new class of facets for the bidirected and undirected case and it is shown how to handle multiple capacity modules by mixed-integer rounding (MIR).

Keywords: cutset polyhedra, flow-cutset inequalities, capacitated network design, integer programming

MSC: 90C11, 90C35, 90C57, 90B18

1 Introduction

We address variants of the following capacitated network design problem. Given point-to-point demands between locations, a topology of a network connecting these locations has to be determined. All links have to be dimensioned by assigning capacity to them such that all demands can be realized by a network flow. In most practical applications the admissible capacities follow a discrete structure. We consider a finite set of capacity modules. Each module has a base capacity and cost and may be installed multiple times on every link of the network (modular capacity assignment). The routing of a demand from its source to its destination can be done by splitting the flow among several paths (bifurcated routing). A capacitated network design problem consists of finding such a modular capacity and flow assignment that minimizes the overall installation cost. This problem arises in the context of planning and dimensioning telecommunication or public transport networks.

We distinguish three different types of capacity usage. A link might offer its capacity for flow in one direction only (Directed link capacity model), the capacity of a link may be consumed by the flow of both directions independently (Bidirected link capacity model) or it is shared between them (Undirected link capacity model). In this paper, we unify the polyhedral results for these three cases and elaborate on the differences. Moreover, we allow for an arbitrary number of installable modules with no restriction on the capacity structure. We show that facets of the polyhedra defined by the restriction of the problems to a cut in the network translate to facets of the original network design polyhedra if the two (directed) subgraphs defining the cut are (strongly) connected. Further, we extend the class of flow-cutset inequalities for the Directed case to the Bidirected and Undirected case generalizing all known inequalities of this type. Finally, we present a new class of facet-defining inequalities for the Bidirected and Undirected case that has no counterpart in the Directed model.

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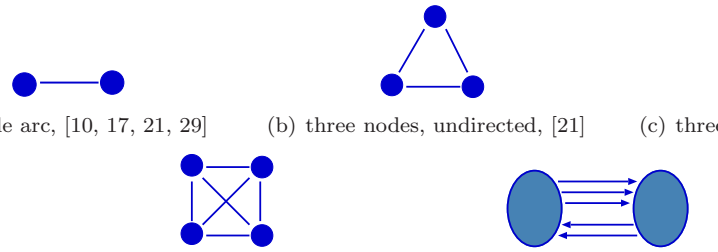
The network design polyhedra considered in this paper have already been studied for special cases by several authors. Magnanti and Mirchandani [20], Magnanti et al. [21, 22] consider Undirected link models and a modular capacity structure with one to three different modules. The base capacities are integer multiples of each other. Magnanti et al. initiate the study of network design polyhedra and introduce cutset inequalities, three-partition inequalities and arc residual capacity inequalities. Bienstock and Günlük [8] study polyhedra based on Bidirected problems with two modules also having divisible base capacities. In addition to cutset and partition inequalities they consider a generalization of cutset inequalities to simple flow-cutset inequalities. General flow-cutset inequalities for Directed models with a single module have been introduced by Chopra et al. [11] without showing their strength.

Most of the strong valid inequalities for network design polyhedra given in the literature are based on substructures of the network such as single arcs, cuts, three-partitions or, more general, k-partitions. These inequalities have been derived as facets of simple structured relaxations such as single-arcset polyhedra or network design polyhedra corresponding to simple k-node networks. Figure 1 shows simple networks and the corresponding polyhedral studies. Brockmüller et al. [10], van Hoesel et al. [17], Magnanti et al. [21] and Rajan and Atamtürk [29] consider single-arc sets. Magnanti et al. [21] and Bienstock et al. [9] study the capacity formulation that is obtained by



(a) single arc, [10, 17, 21, 29] (b) three nodes, undirected, [21] (c) three nodes, directed, [9]

(d) four nodes, undirected, [1] (e) cutset, directed, [3]

Figure 1: small network structures

projecting out all flow-variables in the classical multi-commodity link-flow formulation, see [6, 18]. They present a complete description of a three-node network design polyhedra. Agarwal [1] identifies facet-defining inequalities for the capacity formulation of the four-node Undirected problem. Atamtürk [3] gives a detailed analysis for Directed cutset polyhedra which are based on two node networks with parallel links. He studies the flow-cutset inequalities introduced in [11] and shows that they induce a complete description in the single-commodity, single-module case. Further, Atamtürk [3] generalizes flow-cutset inequalities to an arbitrary number of modules. cutset inequality

It is of interest to know under which conditions facets for polyhedra of k-node networks translate into facets for network design polyhedra considering a k-partition of the underlying nodeset. Magnanti et al. [21] as well as Bienstock and Günlük [8] answer this question for the special models and inequalities they consider. To verify that the considered cut and k-partition based inequalities are facet-defining they need the components defining the partition to be connected. A proof for arbitrary facets for the Undirected model with a single module was recently given by Agarwal [1]. He considers the capacity formulation and shows that every facet-defining inequality for the k-node problem is a facet of the original problem if the subgraph of each component of the k-partition is connected. This result applies to inequalities containing capacity variables only. Also the coefficients in these inequalities corresponding to parallel edges are identical since Agarwal aggregates parallel edges in the shrunken k-node network.

This article is closely related to the work of Agarwal [1] and Atamtürk [3]. In contrast to [1], we consider the link-flow formulation allowing for parallel links and an arbitrary modular link capacity structure. This is done for the Directed, Bidirected and Undirected models. Given a cut of the network, we show that any facet of the cutset polyhedron translates into a facet of the network design polyhedra if both network components defining the cut are connected (Bidirected and Undirected models) or strongly connected (Directed models). Our approach gives rise to more general classes of inequalities than those in [1]. They may also have flow variables and different coefficients for parallel

links in the cutset. For the Directed model we extend the work of Atamtürk [3] on cutset polyhedra to general network design polyhedra. Moreover, we provide a similar analysis for Bidirected and Undirected models by presenting a unified polyhedral study of cutset polyhedra for all three model type, having the focus on flow-cutset inequalities.

Cutsets (see Figure 1(e)) and cutset polyhedra are closely related to single node flow sets, which have been studied extensively in the literature. Cutset polyhedra can be seen as single node flow sets with unbounded integer capacity variables. The polyhedral study of single node flow sets was initiated by Padberg et al. [26]. They introduce a special case of so-called flow-cover inequalities with only outflow arcs. A generalization to inequalities with no-zero coefficients also for inflow arcs is from Van Roy and Wolsey [31]. Important work on the strengthening of flow-cover inequalities by superadditive lifting has been carried out by Gu et al. [15]. Reverse flow-cover inequalities were introduced by Stallaert [30] and investigated in detail by Atamtürk [2]. Single node flow sets with a very general capacity model based on additive variable upper bounds are studied in Atamtürk et al. [5]. Louveaux and Wolsey [19] recently showed how strong valid flow-cover inequalities can be obtained by a MIR procedure applied to single node flow sets that has been introduced by Marchand and Wolsey [23]. We apply the same procedure to cutset polyhedra in order to obtain flow-cutset inequalities with the difference that we do not complement capacity variables because these are unbounded.

This paper is structured as follows. After defining network design polyhedra and cutset polyhedra in Section 2 and Section 3, respectively, we show how facets of these polyhedra are related to each other in Section 3.1. A detailed analysis of the facial structure of cutset polyhedra for the three link capacity models is provided in Section 4. We conclude with some remarks and open questions in Section 5.

2 Network design polyhedra

A network design instance is given by a directed graph $G = (V, A)$ (Directed link model) or an undirected graph $H = (V, E)$ (Bidirected and Undirected link model), a set M of capacity modules installable on the network links, and a set K of commodities. The literature often refers to M as being the set of facilities or technologies. We will use the term modules throughout. We assume G to be strongly connected and H to be connected. Note that we explicitly allow for parallel arcs and edges. In order to handle flow on edges for the Bidirected and Undirected link model we define $G = (V, A)$ to be the digraph obtained by bidirecting all edges in E. The two arcs corresponding to edge $e \in E$ are denoted by e^+ and e^-, hence $A := \{e^+ = (i, j), e^- = (j, i) : e = \{i, j\} \in E\}$. A module $m \in M$ has a capacity $c^m \in \mathbb{Z}_+ \backslash \{0\}$. For simplicity we consider the same set of modules for all arcs or edges. With every $k \in K$ we associate a vector $d^k \in \mathbb{Z}^V$ of demand values such that $\sum_{v \in V} d^k_v = 0$. We call d^k_v the emanating demand of node v with respect to commodity k.

We define variables x^m_a, x^m_e to be the number of installed modules of type $m \in M$ on arc $a \in A$ or edge $e \in E$, respectively. We assume a fractional multi-commodity flow routing. Let $f^k \in \mathbb{R}^A_+$ be the vector of flow variables corresponding to commodity $k \in K$.

Consider the following inequalities:

$$\sum_{a \in \delta^+_G(v)} f^k_a - \sum_{a \in \delta^-_G(v)} f^k_a = d^k_v \qquad \forall v \in V,\, k \in K \tag{1}$$

$$\sum_{k \in K} f^k_a \leq \sum_{m \in M} c^m x^m_a \qquad \forall a \in A \qquad \text{(Directed)} \tag{2a}$$

$$\max\Big\{ \sum_{k \in K} f^k_{e^+}, \sum_{k \in K} f^k_{e^-} \Big\} \leq \sum_{m \in M} c^m x^m_e \qquad \forall e \in E \qquad \text{(Bidirected)} \tag{2b}$$

$$\sum_{k \in K} (f^k_{e^+} + f^k_{e^-}) \leq \sum_{m \in M} c^m x^m_e \qquad \forall e \in E \qquad \text{(Undirected)} \tag{2c}$$

The flow conservation constraints (1) ensure a feasible routing. Inequalities (2a), (2b), and (2c) are the capacity constraints for the three model types. The network design polyhedra for the link

capacity models Directed, Bidirected, and Undirected are given by

$$\begin{aligned}
\mathcal{ND}^{di} &= \operatorname{conv}\left\{(f,x) \in \mathbb{R}_+^{A\times K} \times \mathbb{Z}_+^{A\times M} : (f,x) \text{ satisfies (1) and (2a)}\right\},\\
\mathcal{ND}^{bi} &= \operatorname{conv}\left\{(f,x) \in \mathbb{R}_+^{A\times K} \times \mathbb{Z}_+^{E\times M} : (f,x) \text{ satisfies (1) and (2b)}\right\},\\
\mathcal{ND}^{un} &= \operatorname{conv}\left\{(f,x) \in \mathbb{R}_+^{A\times K} \times \mathbb{Z}_+^{E\times M} : (f,x) \text{ satisfies (1) and (2c)}\right\}.
\end{aligned}$$

The problem of optimizing a linear cost-function over one of these polyhedra is called a capacitated network design problem. This problem is known to be NP-hard already for special cases, see for instance Bienstock et al. [9] and Chopra et al. [11]. A valid inequality is called trivial if it is equivalent to one of the capacity or non-negativity constraints defining $\mathcal{ND}^{di}$, $\mathcal{ND}^{bi}$ or $\mathcal{ND}^{un}$. Notice that the capacity constraints (2b) for $\mathcal{ND}^{bi}$ can be expressed by two linear inequalities for each edge, that $\mathcal{ND}^{bi}$ is a relaxation of $\mathcal{ND}^{un}$, and that the constraint matrices and right-hand side vectors are integral. Since the capacity variables are not bounded and the underlying graphs are (strongly) connected we can construct a feasible flow for every demand vector (e. g., by applying a (min-cost) flow algorithm). Hence the given polyhedra are not empty.

For special cases of these network design polyhedra the following dimension result has been proven by Atamtürk [3], Bienstock and Günlük [8], and Magnanti et al. [22]. A generalization can be easily obtained. We present a proof here primarily because we will make use of the same arguments in the proof of Theorem 3.7.

Proposition 2.1. *The dimension of $\mathcal{ND}^{di}$ is $|K||A| + |M||A| - |K|(|V|-1)$. The dimension of $\mathcal{ND}^{bi}$ and $\mathcal{ND}^{un}$ is $2|K||E| + |M||E| - |K|(|V|-1)$.*

Proof. For $\mathcal{ND}^{di}$, there are $|K||A| + |M||A|$ variables and $|K|(|V|-1)$ linearly independent flow conservation constraints (1). We show that there are no additional implied equations. Let

$$\sum_{a\in A}\Big(\sum_{k\in K} \gamma_a^k f_a^k + \sum_{m\in M} \beta_a^m x_a^m\Big) = \pi \tag{3}$$

be an equation satisfied by all points in $\mathcal{ND}^{di}$ and let $\hat{p} = (\hat{f}, \hat{x}) \in \mathcal{ND}^{di}$. For all $a \in A$ and every $m \in M$ we can modify $\hat{p}$ by increasing the capacity variable $\hat{x}_a^m$, without leaving $\mathcal{ND}^{di}$. Hence, $\beta_a^m = 0$ for all $a \in A$ and $m \in M$. Now we choose a spanning arborescence $T \subseteq A$ of G with root $r \in V$, that is, for every node $v \in V\backslash\{r\}$ there exists a unique directed path in T from r to v. The arborescence T exists since G is strongly connected.

By adding a linear combination of the flow conservation constraints (1) to (3) we can assume that $\gamma_a^k = 0$ for all $a \in T, k \in K$. Now let (v_0, v_1) be an arc in $A\backslash T$ and $(v_1, v_2, ..., v_t = r)$ a directed path in G from v_1 to r with $t \geq 1$, which exists because G is strongly connected. Now, the unique path from r to v_{t-1} in T and the arc (v_{t-1}, r) define a circuit in G. For every commodity k we can modify $\hat{p}$ by sending a circulation flow along that circuit, increasing capacities on arcs if necessary. This way we get a new feasible point satisfying (3). It follows that $\gamma^k_{(v_{t-1},r)} = 0$ since $\gamma_a^k = 0$ for all $a \in T$. Similarly, there is a closed directed path defined by the unique path from r to v_{t-2} in T and the arcs (v_{t-2}, v_{t-1}) and (v_{t-1}, r). Again, sending a circulation flow on that circuit gives $\gamma^k_{(v_{t-2},v_{t-1})} = 0$. We proceed inductively and conclude that

$$\gamma^k_{(v_0,v_1)} = \gamma^k_{(v_1,v_2)} = \cdots = \gamma^k_{(v_{t-1},v_t)} = 0.$$

Since (v_0, v_1) was chosen arbitrarily we obtain $\gamma_a^k = 0$ for all $a \in A$ and $k \in K$. Hence, equation (3) is a linear combination of flow conservation constraints.

The corresponding result for $\mathcal{ND}^{bi}$ and $\mathcal{ND}^{un}$ can be proved analogously by using a spanning tree T in H and send circulation flows (in both directions) on undirected circuits defined by edges of T and single edges in $E\backslash T$ (see for instance Bienstock and Günlük [8, proof of Theorem 2.2]). □

3 Cutset polyhedra

Cutset polyhedra arise from the aggregation of flow conservation constraints for all nodes in a non-empty nodeset $S \subset V$. The network design problems are restricted to the two artificial nodes

S and $V\backslash S$ and the corresponding network cut. Since a significant part of the characteristics of cutset polyhedra carries over on the related network design polyhedra, their polyhedral structure is important. After introducing the cutset polyhedra $\mathcal{CS}_S^{di}$, $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$, we show that facet-defining inequalities for these cutset polyhedra define facets of $\mathcal{ND}^{di}$, $\mathcal{ND}^{bi}$ or $\mathcal{ND}^{un}$, provided that rather mild conditions on the structure of the underlying graphs are satisfied.

Let the network cut $A_S \subseteq A$ in G be the set of arcs with one endnode in S and one endnode in $V\backslash S$ where $A_S^+ \subseteq A_S$ is the set of arcs with source in S and $A_S^- := A_S\backslash A_S^+$. Similar E_S denotes the undirected cut in H, containing all edges with one endnode in S and one endnode in $V\backslash S$. For the Bidirected and Undirected link model, every edge e in the cut E_S is represented by the two arcs e^+ and e^- in A_S such that $e^+ \in A_S^+$ and $e^- \in A_S^-$. The total flow with respect to $Q \subseteq K$ on $A^* \subseteq A$ is denoted by $f^Q(A^*) := \sum_{k\in Q} \sum_{a\in A^*} f_a^k$. We abbreviate $f^k(A^*)$ for $f^{\{k\}}(A^*)$. Notice that $A_S^+, A_S^- \neq \emptyset$ and $E_S \neq \emptyset$ since G is strongly connected and H is connected.

Let $d_S^k := \sum_{v\in S} d_v^k$ be the total demand with respect to commodity $k \in K$ over the cut given by S and define

$$K_S^+ := \left\{k \in K : d_S^k > 0\right\}, \qquad K_S^- := \left\{k \in K : d_S^k < 0\right\}, \qquad K_S := K_S^+ \cup K_S^-$$

and $K_S^0 := K\backslash K_S$ (see Figure 2). Considering the Undirected link capacity model, we may reverse the direction of every demand and exchange the corresponding flow variables without changing the problem. Hence, without loss of generality we assume that $K_S^- = \emptyset$ for Undirected models.

(a) 4-node network with emanating demands for three commodities ($|K| = 3$)

(b) cutset corresponding to S with two remaining commodities ($|K_S| = 2$)

Figure 2: Relation between network design polyhedra and cutset polyhedra

Definition 3.1. Given $S \subset V$, the cutset polyhedron $\mathcal{CS}_S^{di}$ ($\mathcal{CS}_S^{bi}$, $\mathcal{CS}_S^{un}$) is the network design polyhedron $\mathcal{ND}^{di}$ ($\mathcal{ND}^{bi}$,$\mathcal{ND}^{un}$) with respect to the graph $G_S := (\{S, V\backslash S\}, A_S)$ ($H_S := (\{S, V\backslash S\}, E_S)$) defined by the two artificial nodes S and $V\backslash S$ and the cut arcs A_S (cut edges E_S). while the commodity set is given by K_S with aggregated demand vector $(d_S^k, d_{V\backslash S}^k)$ for $k \in K_S$.

Note that from $\sum_{v\in V} d_v^k = 0$ follows $d_S^k = -d_{V\backslash S}^k$ for all $k \in K_S$. Multiplying all flow conservation constraints by -1 does not change the polyhedra, hence the cutset polyhedra of S and $V\backslash S$ are identical. Since cutset polyhedra are two-node network design polyhedra we conclude:

Corollary 3.2. *Let $S \subset V$. The dimension of $\mathcal{CS}_S^{di}$ is $|K_S||A_S| + |M||A_S| - |K_S|$. The dimension of $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ is $2|K_S||E_S| + |M||E_S| - |K_S|$.*

3.1 Strength of cut-based inequalities

In the following we will work out some statements about cutset polyhedra and their relation to network design polyhedra, preparing the main result of this section which is given in Theorem 3.7. For proofs we will concentrate on the Directed case, but we will mention how to vary the argumentation for the Bidirected and Undirected models, if not obvious.

Let $\hat{p} \in \mathcal{ND}^{di}$ and for $S \subset V$ consider the sub-vector $\hat{p}_S$ of $\hat{p}$ obtained by deleting all entries in $\hat{p}$ that correspond to $A\backslash A_S$ or K_S^0. Since the demand constraints for $\mathcal{CS}_S^{di}$ are aggregations of demand constraints for $\mathcal{ND}^{di}$, the point $\hat{p}_S$ is in $\mathcal{CS}_S^{di}$:

$$\hat{p} \in \mathcal{ND}^{di} \implies \hat{p}_S \in \mathcal{CS}_S^{di}$$

This property holds similarly for points in $\mathcal{ND}^{bi}$ and $\mathcal{ND}^{un}$. From here it follows directly that

Lemma 3.3. *Given $S \subset V$, a valid inequality for $\mathcal{CS}_S^{di}$ ($\mathcal{CS}_S^{bi}$, $\mathcal{CS}_S^{un}$) is also valid for $\mathcal{ND}^{di}$ ($\mathcal{ND}^{bi}$, $\mathcal{ND}^{un}$).*

The reverse of Lemma 3.3 is not true in general.

Example 3.4. Consider the strongly connected 4-node network given in Figure 2 and assume that there is sufficient capacity on all arcs. For the first commodity there is a demand of $d_S^1 = 1$ from the left to the right which has to be realized using arc a_1. Hence $f_{a_1}^1 \geq 1$. This inequality is not valid for the cutset polyhedron corresponding to S since for the problem on the cutset it is also valid to route the demand using arc a_2. The information about the structure of $G[S]$, which is not strongly connected, is lost in the aggregation process.

If both $G[S]$ and $G[V\backslash S]$ are strongly connected ($H[S]$ and $H[V\backslash S]$ are connected), then any point of the cutset polyhedron can be extended to a point of the network design polyhedron without changing the values on the cut. Hence, valid inequalities of network design polyhedra with non-zero coefficients only on the cut correspond to valid inequalities of the associated cutset polyhedra.

Lemma 3.5. *For some $S \subset V$, let the graphs $G[S]$ and $G[V\backslash S]$ be strongly connected. There exists a point $\hat{p} \in \mathcal{ND}^{di}$ for every point $\bar{p} \in \mathcal{CS}_S^{di}$, such that $\hat{p}_S = \bar{p}$. Any valid inequality for $\mathcal{ND}^{di}$ with zero coefficients for $A\backslash A_S$ and K_S^0 is also valid for $\mathcal{CS}_S^{di}$.*

For some $S \subset V$, let the graphs $H[S]$ and $H[V\backslash S]$ be connected. There exists a point $\hat{p} \in \mathcal{ND}^{bi}$ ($\hat{p} \in \mathcal{ND}^{un}$) for every point $\bar{p} \in \mathcal{CS}_S^{bi}$ ($\bar{p} \in \mathcal{CS}_S^{un}$), such that $\hat{p}_S = \bar{p}$. Any valid inequality for $\mathcal{ND}^{bi}$ ($\mathcal{ND}^{un}$) with zero coefficients for $E\backslash E_S$ and K_S^0 is also valid for $\mathcal{ND}^{bi}$ ($\mathcal{ND}^{un}$).

Proof. Let $\bar{p} = (\bar{f}, \bar{x}) \in \mathcal{CS}_S^{di}$. We can w. l. o. g. assume that $\bar{x} \in \mathbb{Z}_+^{|M||K_S|}$. We construct a point $\hat{p} = (\hat{f}, \hat{x}) \in \mathcal{ND}^{di}$ the following way:

$$\hat{x}_a^m := \begin{cases} \bar{x}_a^m & a \in A_S, m \in M \\ \mathcal{M} & a \in A\backslash A_S, m \in M \end{cases}, \quad \hat{f}_a^k := \begin{cases} \bar{f}_a^k & k \in K_S, a \in A_S \\ 0 & k \in K_S^0, a \in A_S \end{cases},$$

where $\mathcal{M} \in \mathbb{Z}_+$ is a large number. It remains to define $\hat{f}_a^k$, $k \in K$, for arcs a in $A\backslash A_S = A[S] \cup A[V\backslash S]$. For $k \in K_S^0$, we solve the (min-cost) flow problem (with arbitrary cost-function) on $G[S]$ respectively $G[V\backslash S]$ to determine $\hat{f}_a^k$. For $k \in K_S$, we define two (min-cost) flow problems on $G[S]$ respectively $G[V\backslash S]$ by using the demand-vector

$$\hat{d}_v^k := \begin{cases} d_v^k + \bar{f}^k(\delta_G^-(v) \cap A_S^-) - \bar{f}^k(\delta_G^+(v) \cap A_S^+) & v \in S \\ d_v^k + \bar{f}^k(\delta_G^-(v) \cap A_S^+) - \bar{f}^k(\delta_G^+(v) \cap A_S^-) & v \in V\backslash S. \end{cases}$$

Thus $\hat{d}_v^k$ is the emanating demand of the node v with respect to commodity k plus the flow that has to leave (or enter) v across the cut. Note that $\sum_{v \in S} \hat{d}_v^k = \sum_{v \in V\backslash S} \hat{d}_v^k = 0$. Since $G[S]$ and $G[V\backslash S]$ are strongly connected and the capacity is large enough, a min-cost flow (with arbitrary cost function) can be computed. The resulting flow values are completing the point $\hat{p} = (\hat{f}, \hat{x}) \in \mathcal{ND}^{di}$. By definition $\hat{p}$ meets all flow conservation constraints (1) and capacity constraints (2a) for $\mathcal{ND}^{di}$. Hence $\hat{p} \in \mathcal{ND}^{di}$ and $\hat{p}_S = \bar{p}$. □

The following lemma characterizes cut-based facets of network design polyhedra. Coefficients of flow-variables corresponding to K_S^0 are either all zero or they are all non-zero. Notice we can assume a zero coefficient for at least one of the cut arcs or edges by adding flow conservation constraints to the cut-based facet-defining inequality.

Lemma 3.6. *Consider $S \subset V$ such that $G[S]$ and $G[V\backslash S]$ are strongly connected and let*

$$\sum_{a \in A_S} \Big(\sum_{k \in K} \gamma_a^k f_a^k + \sum_{m \in M} \beta_a^m x_a^m \Big) \geq \pi \tag{4}$$

be a non-trivial facet-defining inequality of $\mathcal{ND}^{di}$. If for $k \in K_S^0$ there exists $\hat{a} \in A_S$ with $\gamma_{\hat{a}}^k = 0$ then $\gamma_a^k = 0$ for all $a \in A_S$.

Consider $S \subset V$ such that $H[S]$ and $H[V \backslash S]$ are connected and let

$$\sum_{e \in E_S} (\sum_{k \in K} \gamma^k_{e^+} f^k_{e^+} + \sum_{k \in K} \gamma^k_{e^-} f^k_{e^-} + \sum_{m \in M} \beta^m_e x^m_e \,) \geq \pi$$

be a non-trivial facet-defining inequality of $\mathcal{ND}^{bi}$ ($\mathcal{ND}^{un}$). If for $k \in K^0_S$ there exists $\hat{e} \in E_S$ with $\gamma^k_{\hat{e}^+} = 0$ or $\gamma^k_{\hat{e}^-} = 0$ then $\gamma^k_{e^+} = \gamma^k_{e^-} = 0$ for all $e \in E_S$.

Proof. Let $k \in K^0_S$ and let w. l. o. g. $\hat{a} \in A^+_S$ such that $\gamma^k_{\hat{a}} = 0$. Since (4) is different from non-negativity and capacity constraints and by Proposition 2.1, there is a point $\hat{p} = (\hat{f}, \hat{x})$ on the facet such that for all arcs $a \in A_S$ the capacity constraint of a is not tight and also $\hat{f}^k_a > 0$ (e. g. $\hat{p}$ is an inner point of the facet).

Now choose some $a \in A^+_S \backslash \{\hat{a}\}$. We construct a second point in $\mathcal{ND}^{di}$ from $\hat{p}$ by decreasing the flow on $\hat{a}$ by $\epsilon > 0$ and by increasing the flow on a by the same amount. Since $k \in K^0_S$ we get a point in $\mathcal{ND}^{di}$ by rerouting flow in $G[S]$ and $G[V \backslash S]$ only. Notice that these graphs are strongly connected and that capacity can be increased on the non-cut arcs if necessary. The resulting point fulfills (4), hence $\gamma^k_a \geq 0$. We can also modify $\hat{p}$ by increasing flow on $\hat{a}$ and decreasing it on a, implying $\gamma^k_a = 0$ for all $a \in A^+_S$. If $a \in A^-_S$ we change $\hat{p}$ by simultaneously decreasing or increasing flow on $\hat{a}$ and a which gives $\gamma^k_a = 0$ for all $a \in A^-_S$. □

We conclude with the central result of this section.

Theorem 3.7. *Let $S \subset V$ and*

$$\sum_{a \in A_S} (\sum_{k \in K_S} \gamma^k_a f^k_a + \sum_{m \in M} \beta^m_a x^m_a \,) \geq \pi \tag{5}$$

be a non-trivial facet-defining inequality of $\mathcal{CS}^{di}_S$. Then it also defines a facet of $\mathcal{ND}^{di}$ if both $G[S]$ and $G[V \backslash S]$ are strongly connected. Let $S \subset V$ and

$$\sum_{e \in E_S} (\sum_{k \in K_S} \gamma^k_{e^+} f^k_{e^+} + \sum_{k \in K_S} \gamma^k_{e^-} f^k_{e^-} + \sum_{m \in M} \beta^m_e x^m_e \,) \geq \pi$$

be a non-trivial facet-defining inequality of $\mathcal{CS}^{bi}_S$ ($\mathcal{CS}^{un}_S$). Then it also defines a facet of $\mathcal{ND}^{bi}$ ($\mathcal{ND}^{un}$) if both $H[S]$ and $H[V \backslash S]$ are connected.

Proof. We will first show that the related face

$$F := \Big\{ (f, x) \in \mathcal{ND}^{di} : (f, x) \text{ satisfies (5) at equality} \Big\}$$

is non-trivial, i. e., it is not empty and it does not equal $\mathcal{ND}^{di}$. Then we will show that it is inclusion-wise maximal. Let

$$F_S := \Big\{ (f, x) \in \mathcal{CS}^{di}_S : (f, x) \text{ satisfies (5) at equality} \Big\}$$

be the facet of $\mathcal{CS}^{di}_S$ defined by (5). Choose a point $\bar{p} \in F_S$. From Lemma 3.5 follows that there is a point $\hat{p} \in \mathcal{ND}^{di}$ with $\hat{p}_S = \bar{p}$. It follows that $\hat{p}$ fulfills (5) at equality and hence $\hat{p} \in F$. Since F_S is a facet of $\mathcal{CS}^{di}_S$, there is a point $\bar{q} \in \mathcal{CS}^{di}_S$ with $\bar{q} \notin F_S$. Again by Lemma 3.5 there is $\hat{q} \in \mathcal{ND}^{di}$ with $\hat{q}_S = \bar{q}$. Thus, this point is not on the face F. We conclude that F is a non-trivial face of $\mathcal{ND}^{di}$. It remains to show that F is inclusion-wise maximal. Choose a facet $\tilde{F}$ of $\mathcal{ND}^{di}$ with $F \subseteq \tilde{F}$ and let $\tilde{F}$ be defined by

$$\sum_{a \in A} (\sum_{k \in K} \tilde{\gamma}^k_a f^k_a + \sum_{m \in M} \tilde{\beta}^m_a x^m_a) \geq \tilde{\pi}. \tag{6}$$

Every point in F satisfies (6) at equality. We will show that (6) equals (5) up to a linear combination of flow conservation constraints. To see that

$$\tilde{\gamma}^k_a = \tilde{\beta}^m_a = 0 \;\; \forall a \notin A_S, m \in M, k \in K, \tag{7}$$

we apply the arguments of the proof of Proposition 2.1 to $G[S]$ and $G[V \backslash S]$ respectively. We can thus concentrate on coefficients of variables in the cut. For a fixed arc $\bar{a} \in A_S$ it can be assumed that

$$\tilde{\gamma}_{\bar{a}}^k = \gamma_{\bar{a}}^k = 0 \ \forall k \in K$$

by adding a linear combination of the flow conservation constraints to (5) and (6), respectively. First assume that (6) is non-trivial, i. e., it is not equivalent to a capacity constraint nor a non-negativity constraint. By Lemma 3.6 follows

$$\tilde{\gamma}_a^k = 0 \ \forall a \in A_S, k \in K_S^0.$$

Now, by Lemma 3.5, inequality (6) is valid for $\mathcal{CS}_S^{di}$. Let $\tilde{F}_S$ be the corresponding face of $\mathcal{CS}_S^{di}$. By construction every point in F_S also fulfills (6) at equality. Since F_S is a facet, it follows $\tilde{F}_S = F_S$. Hence, (6) is (5) up to a scalar multiple and a linear combination of flow conservation constraints. We conclude that also $F = \tilde{F}$, and hence F defines a facet of $\mathcal{ND}^{di}$.

Finally consider (6) being a non-negativity constraint or a capacity constraint for some $a \in A$. We can assume that $a \in A_S$ because of (7). It remains to be shown that (6) is not $f_a^k \geq 0$ for $k \in K_S^0$ and $a \in A_S$, since in all other remaining cases already the inequality defining F_S should have been trivial for $\mathcal{CS}_S^{di}$. We can assume that the point $\hat{p} \in F$ is given such that all capacity constraints for arcs in A_S are not tight, because $\hat{p}$ was constructed from $\bar{p} \in F_S$ and F_S is a facet not defined by a capacity constraint. The point $\hat{p}$ can be modified by sending circulation flows for $k \in K_S^0$ on the cut which shows that for $k \in K_S^0$ and $a \in A_S$ there are points on $F \subseteq \tilde{F}$ having $f_a^k > 0$. This completes the proof.

The proof for the Bidirected and Undirected case is analogous. □

4 Facets of cutset polyhedra

By Theorem 3.7, every facet of a cutset polyhedron translates into a facet of the corresponding network design polyhedron if both components of the cut are (strongly) connected. This result motivates the analysis of the facial structure of cutset polyhedra. In this section facet-defining inequalities for the polyhedra corresponding to the three capacity models Directed, Bidirected and Undirected are presented. We will present facet-proofs for classes of so-called flow-cutset inequalities generalizing known special cases. Their derivation by mixed-integer rounding is emphasized.

In Section 4.1 and Section 4.2 single-module problems, i. e., $|M| = 1$, are investigated. Flow-cutset inequalities for Directed problems have been introduced by Chopra et al. [11]. They were studied in detail by Atamtürk [3]. The most important results for $\mathcal{CS}_S^{di}$ will be summarized and supplemented in Section 4.1. In Section 4.2 we study $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$. We will define a class of flow-cutset inequalities, similar to those known for $\mathcal{CS}_S^{di}$, which contains cutset inequalities and simple flow-cutset inequalities, as a special case. The corresponding results of Magnanti and Mirchandani [20] and Bienstock and Günlük [8] will be extended. Additionally, a new class of facet-defining inequalities will be presented that has no counterpart in the Directed case. This new class reflects the special structure of the polyhedra $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$. In Section 4.3 we will investigate how facet-defining inequalities for cutset polyhedra with a single module can be generalized to strong valid inequalities in the multi-module case. The exact subadditive lifting function used by Atamtürk [3] for lifting is based on MIR. For $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ we will make use of the same function to obtain valid inequalities if $|M| > 1$.

The total demand over the cut defined by $S \subset V$ with respect to a non-empty commodity subset $Q \subseteq K$ is denoted by $d_S^Q := \sum_{k \in Q} d_S^k$. Given a module $m \in M$, we define $x^m(A^*) = \sum_{a \in A^*} x_a^m$ or $x^m(E^*) = \sum_{e \in E^*} x_e^m$ to be the total number of modules on arcs of $A^* \subseteq A$ or edges of $E^* \subseteq E$. Let

$$r(a, c) := a - c(\lceil \tfrac{a}{c} \rceil - 1) > 0 \tag{8}$$

be the remainder of the division of $a \in \mathbb{R}$ by $c \in \mathbb{R}_+ \backslash \{0\}$ if $\frac{a}{c} \notin \mathbb{Z}$, and c otherwise. The same operator has already been used by Bienstock and Günlük [8], Magnanti and Mirchandani [20] in the context of strong valid inequalities of network design polyhedra.

4.1 The single-module case for $\mathcal{CS}_S^{di}$

Let $|M| = 1$ and c denote the unique module capacity. We also omit the superscript m for the capacity variables. For fixed $S \subset V$, we define for every commodity subset Q

$$r^Q := r(d_S^Q, c), \quad \eta^Q := \left\lceil \frac{d_S^Q}{c} \right\rceil, \quad r_-^Q := r(-d_S^Q, c) \quad \text{and} \quad \eta_-^Q := \left\lceil \frac{-d_S^Q}{c} \right\rceil.$$

If $\frac{d_S^Q}{c} \notin \mathbb{Z}$ then the relations $\eta_-^Q = 1 - \eta^Q$ and $r_-^Q = c - r^Q$ hold.

We consider two subsets $A_1^+ \subseteq A_S^+$ and $A_2^- \subseteq A_S^-$ of the cut arcs A_S (see Figure 3). Define $\bar{A}_1^+ := A_S^+ \backslash A_1^+$.

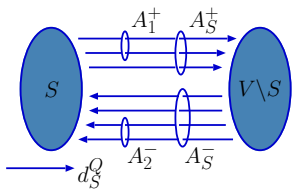


Figure 3: A directed cutset with selected cut arcs

Lemma 4.1 (Atamtürk [3], Chopra et al. [11], Raack et al. [28])**.** *The following flow-cutset inequality is valid for $\mathcal{CS}_S^{di}$:*

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + r^Q x(A_1^+) + (c - r^Q)x(A_2^-) \geq r^Q \eta^Q. \tag{9}$$

The flow-cutset inequality (9) is the $\frac{1}{c}$-MIR inequality (see [24, §II.1.7], [4], and [28]) for the base inequality

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + c\big(x(A_1^+) - x(A_2^-)\big) \geq d_S^Q$$

with $\bar{f}^Q(A_2^-) = cx(A_2^-) - f^Q(A_2^-)$. We call inequality (9) a simple flow-cutset inequality if $A_2^- = \emptyset$. A simple flow-cutset inequality will be called a cutset inequality if additionally $A_1^+ = A_S^+$. Cutset inequalities are given by $x(A_S) \geq \eta^Q$. They contain no flow-variables. Notice that if d_S^Q is an integer multiple of c, then $r^Q = c$ and (9) reduces to a trivial aggregation of flow conservation, capacity and non-negativity constraints (see [3, 28]). If not explicitly stated otherwise, we assume $r^Q < c$, i. e., c does not divide d_S^Q.

Flow-cutset inequalities have the nice property to be symmetric in S and $V\backslash S$ in the sense that for every flow-cutset inequality of $\mathcal{CS}_S^{di}$ there exists a unique flow-cutset inequality for $\mathcal{CS}_{V\backslash S}^{di}$. Adding the flow conservation constraint $f^Q(A_S^-) - f^Q(A_S^+) = -d_S^Q$ to (9) gives

$$\begin{aligned} f^Q(\bar{A}_2^-) - f^Q(A_1^+) + (c - r_-^Q)x(A_1^+) + r_-^Q x(A_2^-) \geq r^Q\eta^Q - d_S^Q &= r^Q\eta^Q - c\eta^Q + c - r^Q \\ &= r^Q(\eta^Q - 1) - c(\eta^Q - 1) \\ &= r_-^Q \eta_-^Q. \end{aligned}$$

It turns out that if $d_S^Q < 0$, then (9) is equivalent to a flow-cutset inequalities for $\mathcal{CS}_{V\backslash S}^{di}$ with positive right-hand side. Interchanging S and $V\backslash S$ we can assume w. l. o. g. that $d_S^Q > 0$. Moreover, we concentrate on commodity subsets with $Q \subseteq K_S^+$ in the sequel.

Before reviewing the results of Atamtürk [3], we will give some necessary conditions for flow-cutset inequalities to be facet-defining for $\mathcal{CS}_S^{di}$.

Lemma 4.2. *If (9) is a non-trivial facet-defining inequality for $\mathcal{CS}_S^{di}$, then*

i) $r^Q < c$ and $A_1^+ \neq \emptyset$.

ii) If (9) is a simple flow-cutset inequality and $A_1^+ \neq A_S^+$, then either $\eta^Q \geq 2$ or $|Q| = 1$.

iii) If (9) *is a cutset inequality, then* $\eta^Q = \eta^{K_S^+}$.

Proof. i) If $r^Q = c$, then inequality (9) reduces to $f^Q(\bar{A}_1^+) + cx(A_1^+) - f^Q(A_2^-) \geq d_S^Q$, which is the sum of $f^Q(A_S^+) - f^Q(A_S^-) \geq d_S^Q$, non-negativity constraints for $A_S^- \backslash A_2^-$ and capacity constraints for A_1^+. Hence it does not define a facet or it is trivial.

If $A_1^+ = \emptyset$ and $r^Q < c$, then inequality (9) can be written as

$$f^Q(A_S^+) - f^Q(A_2^-) + (c - r^Q)x(A_2^-) \geq r^Q \eta^Q = d_S^Q - (\eta^Q - 1)(c - r^Q),$$

and is dominated by $f^Q(A_S^+) - f^Q(A_2^-) \geq d_S^Q$ since $\eta^Q \geq 1$ and $c > r^Q$.

ii) Suppose $A_2^- = \emptyset$, $A_1^+ \subset A_S$, $\eta^Q = 1$ and $Q = \{q_1, ..., q_l\}$ with $l \geq 2$. It follows $d_S^{q_i} \leq c \ \forall i \in \{1, \ldots, l\}$, $d_S^Q = r^Q = \sum_{i=1}^l d_S^{q_i} = \sum_{i=1}^l r^{q_i}$ and $\eta^Q = \eta^{q_i} = 1$. Hence (9) is the sum of the following l valid simple flow-cutset inequalities:

$$f^{q_i}(\bar{A}_1^+) + r^{q_i} x(A_1^+) \geq r^{q_i}.$$

iii) By definition $d_S^Q \leq d_S^{K_S^+}$ and thus $\eta^Q \leq \eta^{K_S^+}$. If $\eta^Q < \eta^{K_S^+}$, then $x(A_S^+) \geq \eta^{K_S^+}$ dominates $x(A_S^+) \geq \eta^Q$. □

We will now give sufficient conditions for flow-cutset inequalities of type (9) to be facet-defining for $\mathcal{CS}_S^{di}$. We start with the proof of an important result for cutset inequalities, since it introduces most of the methodology needed for facet-proofs for cutset polyhedra without being too technical, and because Atamtürk [3] does not explicitly consider this well-known subclass of flow-cutset inequalities. These inequalities are crucial for the performance of cutting-plane-based algorithms for network design problems, see Barahona [7], Bienstock et al. [9], Bienstock and Günlük [8] or Raack et al. [28].

Theorem 4.3. *The cutset inequality* $x(A_S^+) \geq \eta^{K_S^+}$ *is non-trivial and facet-defining for* $\mathcal{CS}_S^{di}$ *if and only if* $r^{K_S^+} < c$.

Proof. Necessity was shown in Lemma 4.2. Notice that by definition $c\eta^{K_S^+} = d_S^{K_S^+} + c - r^{K_S^+}$ and that $r^{K_S^+} < c$ is equivalent to $\frac{d_S^{K_S^+}}{c} \notin \mathbb{Z}$. We will show that the related face

$$F := \left\{ (f, x) \in \mathcal{CS}_S^{di} : x(A_S^+) = \eta^{K_S^+} \right\}$$

is non-trivial i. e., it is not empty and it does not equal $\mathcal{CS}_S^{di}$. Then we will show that it is maximal (inclusion-wise). Choose $a_1 \in A_S^+$ and $a_2 \in A_S^-$ (G is strongly connected). Now we construct a feasible point p on the face F the following way. We install exactly $\eta^{K_S^+}$ capacity units on a_1. The capacity on a_2 is chosen sufficiently large. Every demand d_S^k for $k \in K_S^+$ is routed on a_1 and commodities in K_S^- are satisfied on a_2 such that the arcs a_1, a_2 are not saturated and carry a total flow of $d_S^{K_S^+}$ and $d_S^{K_S^-}$, respectively. By construction it holds $p \in F$. Modifying p by increasing capacity on A_S^+ gives a point that is in $\mathcal{CS}_S^{di}$ but not on the face F. It follows that F is non-trivial.

We will now prove that F is inclusion-wise maximal. Choose a facet $\tilde{F}$ of $\mathcal{CS}_S^{di}$ with $F \subseteq \tilde{F}$. Let $\tilde{F}$ be defined by

$$\sum_{a \in A_S} (\beta_a x_a + \sum_{k \in K_S} \gamma_a^k f_a^k) = \pi, \tag{10}$$

where $\beta_a, \gamma_a^k, \pi \in \mathbb{R}$. We will show that (10) is a multiple of $x(A_S^+) = \eta^{K_S^+}$ up to a linear combination of flow conservation constraints. Since multiples of the $|K_S|$ flow conservation constraints may be added to (10) without changing the induced face, we may assume that $\gamma_{a_1}^k = 0$ for all $k \in K_S$. We can also modify the point p by increasing capacity on A_S^- resulting in $\beta_a = 0$ for all $a \in A_S^-$. Modifying p by simultaneously increasing flow on a_1 and on some arc of A_S^- by a small amount for every commodity $k \in K_S$ gives new points on F. Hence $\gamma_a^k = 0$ for all $k \in K_S$, $a \in A_S^-$. The proof is

complete for $|A_S^+| = 1$. Otherwise, we choose $a \in A_S^+$ different from a_1 and construct another point on the face F the following way. We modify p by deleting one unit of capacity on a_1 and installing one unit of capacity on a. The total capacity on A_S^+ remains unchanged. We shift a total flow of at least $r^{K_S^+}$ from the arc a_1 to the arc a since $c(\eta^{K_S^+} - 1) = d_S^{K_S^+} - r^{K_S^+}$. The new point is on the face F. Note that the capacity on a is not saturated since $r^{K_S^+} < c$. We can increase the flow on a and on some arc in A_S^-, resulting in $\gamma_a^k = 0$ for all $a \in A_S^+$ since a was chosen arbitrarily. Finally, this implies $\beta_a = \beta_{a_1}$ for all $a \in A_S^+$.

Plugging in all coefficients in (10) shows that $\tilde{F}$ is given by a multiple of $x(A_S^+) = \eta^{K_S^+}$ (up to a linear combination of flow conservation constraints). Thus $F = \tilde{F}$, concluding the proof. □

The following two results by Atamtürk [3] give more classes of facet-defining flow-cutset inequalities. It turns out that flow-cutset inequalities capture a significant part of the facial structure of cutset polyhedra. Theorem 4.5 is crucial for the theory of strong valid inequalities for Directed network design polyhedra. In the single-commodity, single-module case, the trivial inequalities and all flow-cutset inequalities completely describe $\mathcal{CS}_S^{di}$. In the following section we will show that this does not hold for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$.

Theorem 4.4 (Atamtürk [3]). *Let $\emptyset \neq Q \subseteq K_S^+$ and $r^Q < c$. The flow-cutset inequality* (9) *is facet-defining for $\mathcal{CS}_S^{di}$ if $\emptyset \neq A_1^+ \subset A_S^+$ and $\emptyset \neq A_2^- \subset A_S^-$.*

Theorem 4.5 (Atamtürk [3]). *Let $|Q| = |K_S^+| = 1$. The flow-cutset inequality* (9) *is non-trivial and facet-defining for $\mathcal{CS}_S^{di}$ if and only if $r^{K_S^+} < c$ and $A_1^+ \neq \emptyset$. The inequalities* (1), (2a), (9), *and the non-negativity constraints completely describe $\mathcal{CS}_S^{di}$.*

In this paper we solely consider strongly connected digraphs, which implies $A_S^- \neq \emptyset$. If $A_S^- = \emptyset$, then Theorem 4.3 and Theorem 4.5 do not hold. It can be shown that if $x(A_S^+) \geq \eta^{K_S^+}$ defines a facet of $\mathcal{CS}_S^{di}$ and $|A_S^+| \geq 2$, then either $\eta^{K_S^+} \geq 2$ or $A_S^- \neq \emptyset$. In particular if $A_S^- = \emptyset$, $K_S^- = \emptyset$, $|A_S^+| \geq 2$, and $\eta^{K_S^+} = 1$ then the inequality $x(A_S^+) \geq 1$ is not a facet of $\mathcal{CS}_S^{di}$. Taking $A_1^+ \subset A_S^+$, it is the sum of the flow-cutset inequalities

$$f^{K_S^+}(A_1^+) + r^{K_S^+} x(\bar{A}_1^+) \geq r^{K_S^+} \eta^{K_S^+} \qquad \text{and} \qquad f^{K_S^+}(\bar{A}_1^+) + r^{K_S^+} x(A_1^+) \geq r^{K_S^+} \eta^{K_S^+}$$

because $r^{K_S^+} = d_S^{K_S^+}$ and $f^{K_S^+}(A_S^+) = d_S^{K_S^+}$. In this respect, Atamtürk [3, Theorem 2] is not correct. In the corresponding proof flow is routed using an arc $s \in A_S^+$ having capacity $\eta^{K_S^+} - 1$, which is possible only if $\eta^{K_S^+} \geq 2$. For $A_S^- \neq \emptyset$ this can be fixed by using the arguments presented in the proof of Theorem 4.3 (routing epsilon-flows using arcs of A_S^-).

4.2 The single-module case for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$

In the sequel we consider the two subsets E_1, E_2 of the undirected cut edges E_S (see Figure 4). Remember that to handle flow across the cut we bidirect all edges in E_S. Edge $e \in E_S$ of the cut corresponds to $e^+ \in A_S^+$ and $e^- \in A_S^-$. Let A_1^+ and A_1^- denote all forward and backward arcs with respect to E_1 (similar A_2^+ and A_2^- with respect to E_2), while $\bar{A}_1^+ := A_S^+ \backslash A_1^+$ and $\bar{A}_1^- := A_S^- \backslash A_1^-$.

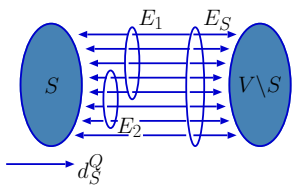


Figure 4: An undirected cutset with selected cut edges

Lemma 4.6 (Raack et al. [28]). *The following flow-cutset inequality is valid for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$:*

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + r^Q x(E_1) + (c - r^Q)x(E_2) \geq r^Q \eta^Q. \tag{11}$$

The flow-cutset inequality (11) is the $\frac{1}{c}$-MIR inequality (see [24, §II.1.7], [4], and [28]) for the base inequality

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + c\big(x(E_1) - x(E_2)\big) \geq d_S^Q.$$

with $\bar{f}^Q(A_2^-) = cx(E_2) - f^Q(A_2^-)$. It can be seen as the undirected analogon of (9). It was considered in this general form in [28] for the first time. Special cases have been studied in [7, 8, 20, 21, 22] (see Example 4.25). An important difference to the directed case is that the two edge-sets E_1 and E_2 are not necessarily disjoint. A simple flow-cutset inequality is a flow-cutset inequality with $E_2 = \emptyset$ and a cutset inequality is a simple flow-cutset inequality with $E_1 = E_S$, i. e., it reduces to $x(E_S) \geq \eta^Q$.

For Bidirected and Undirected models we consider a second class of strong valid inequalities that turns out to have no analogon in the Directed case. In contrast to Theorem 4.5, flow-cutset inequalities of type (11) alone do not provide a complete description for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ if $|K| = |M| = 1$.

Example 4.7. Define a cutset polyhedron with $|K| = 1$, $|E_S| = 2$ for the Bidirected link model:

$$\begin{aligned} P = \operatorname{conv}\Big\{x \in \mathbb{Z}_+^2, f \in \mathbb{R}_+^4 \ : f_1 + f_2 - f_3 - f_4 &= 7, \\ 0 \leq f_i &\leq 3x_1 \ \ \forall i \in \{1,3\}, \\ 0 \leq f_i &\leq 3x_2 \ \ \forall i \in \{2,4\}\Big\} \end{aligned}$$

When adding all flow-cutset inequalities (11) to the LP-relaxation of P, the resulting polyhedron still has the two fractional vertices $(\frac{1}{2}, \frac{15}{2}, 1, 0, \frac{1}{2}, \frac{5}{2})$ and $(\frac{15}{2}, \frac{1}{2}, 0, 1, \frac{5}{2}, \frac{1}{2})$, (PORTA [12]). But we can formulate two valid inequalities cutting off these points, namely:

$$3x_1 + 2x_2 + f_3 - f_1 \geq 2 \quad \text{and} \quad 3x_2 + 2x_1 + f_4 - f_2 \geq 2.$$

Additionally adding these inequalities results in a complete description of P (PORTA [12]).

The inequalities of the last example can be generalized to a large class of valid inequalities for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$.

Lemma 4.8. *The following new flow-cutset inequality is valid for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$:*

$$cx(E_1) + \big(c - r^Q\big)x(\bar{E}_1) + f^Q(A_1^-) - f^Q(A_1^+) \geq c - r^Q. \tag{12}$$

Proof. If $r^Q = c$ then inequality (12) reduces to $cx(E_1) - f^Q(A_1^+) + f^Q(A_1^-) \geq 0$, which is valid because of $cx(E_1) \geq f^Q(A_1^+)$ and $f^Q(A_1^-) \geq 0$. Now consider $r^Q < c$. First assume that $x(\bar{E}_1) = 0$. All flow has to be routed through E_1. It follows that

$$f^Q(A_1^+) - f^Q(A_1^-) = d_S^Q \quad \text{and} \quad x(E_1) \geq \left\lceil \frac{|d_S^Q|}{c} \right\rceil \geq \eta^Q.$$

Hence

$$cx(E_1) - (f^Q(A_1^+) - f^Q(A_1^-)) \geq c\eta^Q - d_S^Q = c - r^Q.$$

If, on the other hand, $x(\bar{E}_1) \geq 1$ then from $cx(E_1) - f^Q(A_1^+) + f^Q(A_1^-) \geq 0$ we conclude that

$$cx(E_1) + (c - r^Q)x(\bar{E}_1) + f^Q(A_1^-) - f^Q(A_1^+) \geq c - r^Q. \qquad \square$$

We have introduced two classes of valid inequalities for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$. The class of flow-cutset inequalities (11) is the analogon of (9). It is (in this general form) symmetric in S and $V \backslash S$, i. e., if $d_S^Q < 0$ we can exchange the two nodesets such that d_S^Q becomes positive and we find an equivalent flow-cutset inequality. The new class of inequalities (12) has no analogon in the Directed case. It turns out to be specific for the facial structure of $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$. It is not symmetric in

the sense above, but for $d_S^Q < 0$ it is weak because in this case inequality (12) is dominated by the sum of the capacity constraint $cx(E_1) - f^Q(A_1^+) \geq 0$ and the simple flow-cutset inequality $r_-^Q x(\bar{E}_1) + f^Q(A_1^-) \geq r_-^Q \eta_-^Q$. We assume

$$\emptyset \neq Q \subseteq K_S^+$$

for both classes throughout the rest of this article. In the following we will provide necessary and sufficient conditions for these classes to define facets for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$. We consider necessity first.

Lemma 4.9. *If* (11) *is a non-trivial and facet-defining inequality for* $\mathcal{CS}_S^{bi}$ *or* $\mathcal{CS}_S^{un}$*, then*

i) $r^Q < c$ *and* $E_1 \backslash E_2 \neq \emptyset$.

ii) If (11) *is a simple flow-cutset inequality with* $E_1 \neq E_S$*, then either* $\eta^Q \geq 2$ *or* $|Q| = 1$.

iii) If (11) *is a cutset inequality, then* $\eta^Q = \eta^{K_S^+} \geq \eta_-^{K_S^-}$. *If additionally* $|E_S| > 1$*, then* $\eta^{K_S^+} \geq 2$.

If (11) *is facet-defining for* $\mathcal{CS}_S^{un}$*, then* $E_1 \cap E_2 = \emptyset$.

Proof. For proving i) and ii) simply follow the proof of Lemma 4.2 i) and ii). For iii) consider the cutset inequality $x(E_S) \geq \eta^Q$. The largest right-hand side is obtained if $\eta^Q = \eta^{K_S^+}$. Also $x(E_S) \geq \eta_-^{K_S^-}$ is a valid cutset inequality for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ and hence $\eta^{K_S^+} \geq \eta_-^{K_S^-}$. Suppose $|E_S| > 1$ and $\eta^{K_S^+} = 1$. It follows that $d_S^{K_S^+} = r^{K_S^+}$. Choose $E^* \subset E_S$ such that $E^*, \bar{E}^* \neq \emptyset$. Then with Lemma 4.8

$$\begin{aligned} &cx(E^*) + (c - r^{K_S^+})x(\bar{E}^*) + f^Q(A_1^-) - f^Q(A_1^+) \geq c - r^{K_S^+} \quad \text{and} \\ &cx(\bar{E}^*) + (c - r^{K_S^+})x(E^*) + f^Q(\bar{A}_1^-) - f^Q(\bar{A}_1^+) \geq c - r^{K_S^+} \end{aligned}$$

are both valid inequalities for $\mathcal{CS}_S^{bi}$ (and $\mathcal{CS}_S^{un}$) of the form (12) different from flow conservation constraints. Adding them up gives

$$\begin{aligned} (2c - r^{K_S^+})x(E_S) + f^Q(A_S^-) - f^Q(A_S^+) \geq 2c - 2r^{K_S^+} &\iff (2c - r^{K_S^+})x(E_S) - d_S^{K_S^+} \geq 2c - 2r^{K_S^+} \\ &\iff x(E_S) \geq 1 = \eta^{K_S^+}. \end{aligned}$$

It turns out that the cutset inequality $x(E_S) \geq 1$ is the sum of non-trivial valid inequalities when $|E_S| > 1$. To prove the last statement, we show that (11) is the sum of valid inequalities for $\mathcal{CS}_S^{un}$ if $E_1 \cap E_2 \neq \emptyset$. Consider the aggregated capacity constraint $cx(E_1 \cap E_2) - f^Q(A_1^+ \cap A_2^+) - f^Q(A_1^- \cap A_2^-) \geq 0$ and the flow-cutset inequality

$$f^Q(\bar{A}_1^+) + f^Q(A_1^+ \cap A_2^+) - f^Q(A_2^- \backslash A_1^-) + r^Q x(E_1 \backslash E_2) + (c - r^Q)x(E_2 \backslash E_1) \geq r^Q \eta^Q,$$

for the two edge sets $E_1 \backslash E_2$ and $E_2 \backslash E_1$. Summing up these two constraints gives

$$\begin{aligned} f^Q(\bar{A}_1^+) - f^Q(A_2^- \backslash A_1^-) - f^Q(A_1^- \cap A_2^-) + r^Q x(E_1 \backslash E_2) + cx(E_1 \cap E_2) + (c - r^Q)x(E_2 \backslash E_1) \\ \geq r^Q \eta^Q \end{aligned}$$

which is (11). □

Lemma 4.10. *If* (12) *is a non-trivial facet-defining inequality for* $\mathcal{CS}_S^{bi}$ *or* $\mathcal{CS}_S^{un}$*, then it holds:*

i) $r^Q < c$

ii) If $E_1 = \emptyset$*, then* $\eta^{K_S^+} = 1$ *and* $|E_S| = 1$

iii) If $\bar{E}_1 = \emptyset$*, then either* $\eta^Q = \eta^{K_S^+} \geq 2$ *or* $|E_S| = 1$

Proof. If $r^Q = c$ then inequality (12) reduces to $cx(E_1) - (f^Q(A_1^+) - f^Q(A_1^-)) \geq 0$, which is the sum of capacity constraints and non-negativity constraints. Assume $r^Q < c$ in the sequel.

Suppose $E_1 = \emptyset$. Inequality (12) reduces to $x(E_S) \geq 1$, which is dominated by the cutset inequality $x(E_S) \geq \eta^{K_S^+}$ if $\eta^{K_S^+} \geq 2$. If on the other hand $\eta^{K_S^+} = 1$ and $|E_S| > 1$, then $x(E_S) \geq 1$ is the sum of two valid inequalities (see Lemma 4.9). Now suppose that $\bar{E}_1 = \emptyset$. We can write (12) as

$$cx(E_S) + f^Q(A_S^-) - f^Q(A_S^+) \geq c - r^Q \iff cx(E_S) \geq d_S^Q + c - r^Q = c\eta^Q$$

which is either dominated by the cutset inequality $x(E_S) \geq \eta^{K_S^+}$ or it is the sum of valid inequalities when $\eta^Q = \eta^{K_S^+} = 1$ and $|E_S| > 1$ (see Lemma 4.9). □

After stating necessary conditions, the following results provide sufficient conditions for flow-cutset inequalities of type (11) and (12) to be facet-defining for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$. We start with the well-known cutset inequalities [7, 8, 20, 21, 22] in Theorem 4.11. Theorem 4.12 can be seen as the analogon of Theorem 4.4. Theorem 4.13 extends Theorem 4.12 to the case $\bar{E}_1 \backslash E_2 = \emptyset$. Corollary 4.14 summarizes the results of this section for the single-commodity, single-module case for inequalities (11). Eventually, Theorem 4.15 is a facet theorem for the new flow-cutset inequalities (12). Recall that by reversing demand directions, $K_S^- = \emptyset$ can be assumed for Undirected models. Hence for $\mathcal{CS}_S^{un}$ the conditions $\eta^{K_S^+} \geq \eta_-^{K_S^-}$ and $d_S^{K_S^+} \geq |d_S^{K_S^-}|$ are trivially fulfilled.

Theorem 4.11 (Bienstock and Günlük [8], Magnanti et al. [22])**.** *The cutset inequality $x(E_S) \geq \eta^{K_S^+}$ is non-trivial and facet-defining for $\mathcal{CS}_S^{bi}$ ($\mathcal{CS}_S^{un}$) if and only if $r^{K_S^+} < c$, $\eta^{K_S^+} \geq \eta_-^{K_S^-}$ and if either $\eta^{K_S^+} \geq 2$ or $|E_S| = 1$.*

Proof. Necessity was shown in Lemma 4.9 iii). Related sufficiency-results were proven by Bienstock and Günlük [8] for $\mathcal{CS}_S^{bi}$ and Magnanti et al. [22] for $\mathcal{CS}_S^{un}$. □

Theorem 4.12. *Let $\emptyset \neq Q \subseteq K_S^+$ and $r^Q < c$. The flow-cutset inequality (11) is facet-defining for $\mathcal{CS}_S^{bi}$ if $E_1 \backslash E_2 \neq \emptyset$, $\bar{E}_1 \backslash E_2 \neq \emptyset$, and one of the following conditions holds:*

i) $E_2 = \emptyset$ and either $\eta^Q \geq 2$ or $|Q| = 1$

ii) $E_2 \neq \emptyset$

The same holds for $\mathcal{CS}_S^{un}$ if additionally $E_1 \cap E_2 = \emptyset$.

Proof. See Appendix A.1. □

Theorem 4.13. *Let $\emptyset \neq Q = K_S^+$ with $d_S^{K_S^+} \geq |d_S^{K_S^-}|$ and $r^{K_S^+} < c$. The flow-cutset inequality (11) is facet-defining for $\mathcal{CS}_S^{bi}$ if $E_1 \backslash E_2 \neq \emptyset$, $\bar{E}_1 \subseteq E_2 \neq \emptyset$, and one of the following conditions holds*

i) $E_1 \cap E_2 = \emptyset$

ii) $E_1 \cap E_2 \neq \emptyset$ and either $K_S^- = \emptyset$ or $d_S^{K_S^+} > \max(|d_S^{K_S^-}|, c)$

The flow-cutset inequality (11) is facet-defining for $\mathcal{CS}_S^{un}$ if $r^{K_S^+} < c$, $E_1, E_2 \neq \emptyset$ and $\bar{E}_1 = E_2$.

Proof. See Appendix A.2. □

Corollary 4.14. *Let $|K| = |K_S^+| = 1$. The flow-cutset inequality (11) is non-trivial and facet-defining for $\mathcal{CS}_S^{bi}$ if and only if $r^{K_S^+} < c$, $E_1 \backslash E_2 \neq \emptyset$ and one of the following conditions holds:*

i) $E_2 = \emptyset$, $\bar{E}_1 = \emptyset$ and either $\eta^{K_S^+} \geq 2$ or $|E_S| = 1$

ii) $E_2 \neq \emptyset$ or $\bar{E}_1 \neq \emptyset$

The same holds for $\mathcal{CS}_S^{un}$ if additionally $E_1 \cap E_2 = \emptyset$.

Theorem 4.15. *Let $\emptyset \neq Q = K_S^+$ with $d_S^{K_S^+} \geq |d_S^{K_S^-}|$. The new flow-cutset inequality* (12) *defines a non-trivial facet of $\mathcal{CS}_S^{bi}$ if and only if $r^{K_S^+} < c$ and one of the following conditions holds:*

i) $E_1, \bar{E}_1 \neq \emptyset$

ii) $E_1 = \emptyset$ *and* $\eta^{K_S^+} = 1$ *and* $|E_S| = 1$

iii) $\bar{E}_1 = \emptyset$ *and either* $\eta^{K_S^+} \geq 2$ *or* $|E_S| = 1$

Proof. See Appendix A.3. □

By Example 4.7 and Theorem 4.15 the flow-cutset inequalities (11) do not suffice to provide a complete description of Bidirected or Undirected cutset polyhedra in the single-module, single-commodity case, in contrast to the result for $\mathcal{CS}_S^{di}$. The authors presume that the new flow-cutset inequalities (12) are the only missing facets.

Conjecture 4.16. *Let $|Q| = |K_S^+| = 1$. The inequalities* (1), (2b), (11), (12), *and the non-negativity constraints completely describe $\mathcal{CS}_S^{bi}$. The inequalities* (1), (2c), (11), (12), *and the non-negativity constraints completely describe $\mathcal{CS}_S^{un}$.*

4.3 The multi-module case

In the sequel, we will state a generalization of the flow-cutset inequalities (9) and (11) to the case that $|M| \geq 1$. It turns out that there are $|M|$ different flow-cutset inequalities, one for every available capacity module.

Furthermore, the lifting of the flow-cutset inequalities (9) and (11) to the multi-module case can be done using a subadditive lifting function that is based on mixed-integer rounding [24, 25]. We start by introducing the concept of subadditivity and present lifted flow-cutset inequalities. We propose a strengthening for the case that $A_2^- = \emptyset$ or $E_2 = \emptyset$ and conclude by showing that our approach generalizes special cases considered in the literature on network design [8, 20].

Definition 4.17. A function $F : D \subseteq \mathbb{R} \to \mathbb{R}$ is called subadditive on D if $F(a) + F(b) \geq F(a + b)$ for all a, b such that a, b, $a + b \in D$. If the limes exists set

$$\bar{F}(a) := \lim_{t \searrow 0} \tfrac{F(at)}{t}.$$

Proposition 4.18 (Nemhauser and Wolsey [24, Theorem 7.4])**.** *Let N_1, N_2 be two finite index sets and*

$$X = \Big\{(f, x) \in \mathbb{R}_+^{N_1} \times \mathbb{Z}_+^{N_2} : \sum_{j \in N_1} \gamma_j f_j + \sum_{j \in N_2} \beta_j x_j \geq \pi\Big\},$$

where γ_j, β_j and π are rational numbers. If the function $F : \mathbb{R} \to \mathbb{R}$ is nondecreasing and subbadditive on $\mathbb{R}$ with $F(0) = 0$ and $\bar{F}$ exists for all $j \in N_1$, then

$$\sum_{j \in N_1} \bar{F}(\gamma_j) f_j + \sum_{j \in N_2} F(\beta_j) x_j \geq F(\pi)$$

is valid for X.

We will now introduce the subadditive function used to generalize flow-cutset inequalities to the multi-module case. Given $a, c, d \in \mathbb{R}$ with $c > 0$ and $\frac{d}{c} \notin \mathbb{Z}$, define $a^+ := \max(0, a)$ and consider the function $F_{d,c} : \mathbb{R} \to \mathbb{R}$ defined by

$$F_{d,c}(a) := \left\lceil \tfrac{a}{c} \right\rceil r(d, c) - (r(d, c) - r(a, c))^+.$$

The function $F_{d,c}$ is the $\frac{1}{c}$-MIR function for $\geq$-base-inequalities with right-hand side d scaled by the factor $r(d, c)$, see Nemhauser and Wolsey [24, §II.1.7], Atamtürk [4], and Raack et al. [28]. It has the following nice properties.

Lemma 4.19 (Raack et al. [28]). *Let $c, d \in \mathbb{R}$ with $c > 0$ and $\frac{d}{c} \notin \mathbb{Z}$. The function $F_{d,c}$ is subadditive on $\mathbb{R}$ and nondecreasing with $F_{d,c}(0) = 0$ and $\bar{F}_{d,c}(a) = a^+$ for all $a \in \mathbb{R}$. It holds that $|F_{d,c}(a)| \leq |a|$ for all $a \in \mathbb{R}$. If $a, c,$ and d are integral, then also $F_{d,c}(a), \bar{F}_{d,c}(a) \in \mathbb{Z}$.*

Now, we are prepared to generalize the inequalities (9) and (11) to the multi-module case. Let $S \subset V$ and $Q \subseteq K$ be fixed and let $d_S^Q > 0$ be the corresponding cut demand as defined in Section 4. By aggregating model inequalities and substituting $\bar{f}^Q(A_2^-) := \sum_{m \in M} c^m x^m(A_2^-) - f^Q(A_2^-)$,, the following base inequality is valid for $\mathcal{CS}_S^{di}$:

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{m \in M} c^m \big(x^m(A_1^+) - x^m(A_2^-)\big) \geq d_S^Q. \tag{13}$$

Similarly, the following base inequality is valid for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$:

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{m \in M} c^m \big(x^m(E_1) - x^m(E_2)\big) \geq d_S^Q. \tag{14}$$

For every module $s \in M$ with capacity $c^s \in \mathbb{Z}_+ \backslash \{0\}$ we consider the functions

$$F_s := F_{d_S^Q, c^s} \quad \text{and} \quad \bar{F}_s := \bar{F}_{d_S^Q, c^s}.$$

By Proposition 4.18 and Lemma 4.19 we can apply these function to the coefficients in (13) and (14) which results in valid inequalities for $\mathcal{CS}_S^{di}$, $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$, respectively. Resubstituting $\bar{f}^Q(A_2^-)$ gives the following result:

Proposition 4.20 (Atamtürk [3], Raack et al. [28]). *For every $s \in M$ the flow-cutset inequality*

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + \sum_{m \in M} F_s(c^m) x^m(A_1^+) + \sum_{m \in M} (c^m + F_s(-c^m)) x^m(A_2^-) \geq F_s(d_S^Q) \tag{15}$$

is valid for $\mathcal{CS}_S^{di}$, whereas the following flow-cutset inequality is valid for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$:

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + \sum_{m \in M} F_s(c^m) x^m(E_1) + \sum_{m \in M} (c^m + F_s(-c^m)) x^m(E_2) \geq F_s(d_S^Q). \tag{16}$$

Inequalities (15) and (16) generalize (9) and (11) since $F_s(c^s) = r(d_S^Q, c^s)$ and $F_s(-c^s) = -r(d_S^Q, c^s)$. From Lemma 4.19 we know that the coefficients in (15) and (16) are small compared to the given capacities c^m. In particular:

$$0 < F_s(c^m) \leq c^m \quad \text{and} \quad 0 \leq c^m + F_s(-c^m) < c^m$$

Note that $F_s(c^m) > 0$ and $F_s(-c^m) < 0$ because $c^m > 0$. Moreover, since all data is integral, all coefficients are integral. If d_S^Q is an integer multiple of c^s, then (15) and (16) reduce to the base inequalities (13) and (14) because in this case $F_s(a) = a$ for all $a \in \mathbb{R}$ (see Raack et al. [28]).

Atamtürk [3] has been the first to study the general multi-module case for Directed problems. His approach is based on lifting. Given a module $s \in M$, let $\mathcal{CS}_S^{di}(s)$ be the restriction of $\mathcal{CS}_S^{di}$ obtained by fixing all module variables to their lower bound zero that do not correspond to s:

$$\mathcal{CS}_S^{di}(s) = \Big\{(f, x) \in \mathcal{CS}_S^{di} : x_a^m = 0,\, a \in A_S,\, M \backslash \{s\}\Big\}.$$

Now, flow-cutset inequalities of type (9) with $c = c^s$ and $r^Q = r_s^Q := r(d_S^Q, c^s)$ are valid for $\mathcal{CS}_S^{di}(s)$ and can be facet-defining for $\mathcal{CS}_S^{di}(s)$ by Theorems 4.3, 4.4, and 4.5. Atamtürk considers the problem of exact lifting (9) to a valid inequality of $\mathcal{CS}_S^{di}$ and shows that lifting can be done in a sequence-independent way and for all coefficients simultaneously (see Atamtürk [4], Gu et al. [16], Wolsey [32]) by using subadditive lifting functions. The resulting inequalities are given by

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + \sum_{m \in M} \phi_s^+(c^m) x^m(A_1^+) + \sum_{m \in M} \phi_s^-(c^m) x^m(A_2^-) \geq \phi_s^+(d_S^Q) \tag{17}$$

where for $a \in \mathbb{R}$

$$\phi_s^+(a) := \begin{cases} a - k(c^s - r_s^Q) & \text{if } kc^s \le a < kc^s + r_s^Q \\ (k+1)r_s^Q & \text{if } kc^s + r_s^Q \le a < (k+1)c^s \end{cases} \qquad k \text{ integer}$$

and $\phi_s^-(a) = a + \phi_s^+(-a)$. Now by setting $k = \lceil \frac{a}{c^s} \rceil - 1$ it can be seen that $\phi_s^+(a) = F_s(a)$ and hence it turns out that the inequalities (15) and (17) are identical. Obviously flow-cutset inequalities can be obtained by lifting using the subadditive MIR-function F_s. In this context flow-cutset inequalities are closely related to the concept of complemented-MIR inequalities introduced by Marchand and Wolsey [23] and to the flow-cover inequalities presented by Louveaux and Wolsey [19] (besides the fact that we do not complement integer variables since they are not bounded). Notice that Louveaux and Wolsey also consider a lifting function different from MIR.

Considering the lifting problem (instead of simply applying Proposition 4.18) allows to make statements about the strength of the resulting inequalities [4, 16, 32]. This way Atamtürk proves that certain lifted flow-cutset inequalities (15) define facets for $\mathcal{CS}_S^{di}$ in the multi-module case. Theorem 4.21 is a generalization of Theorem 4.4 and Theorem 4.22 is a generalization of Theorem 4.5. Similar results for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ are not known although lifting using the function F_s is still valid.

Theorem 4.21 (Atamtürk [3])**.** *Let $Q \subseteq K_S^+$ and $s \in M$ with $r_s^Q < c^s$. The flow-cutset inequality* (15) *is facet-defining for $\mathcal{CS}_S^{di}$ if $A_1^+, \bar{A}_1^+, A_2^-, \bar{A}_2^- \neq \emptyset$.*

Theorem 4.22 (Atamtürk [3])**.** *Let $|Q| = |K_S^+| = 1$ and $s \in M$ with $r_s^Q < c^s$. The flow-cutset inequality* (15) *is facet-defining for $\mathcal{CS}_S^{di}$ if $A_1^+, \bar{A}_1^+, A_2^- \neq \emptyset$.*

Notice that Theorem 4.21 and Theorem 4.22 explicitly exclude simple flow-cutset inequalities. In fact, inequalities (15) cannot be facet-defining if $A_2^- = \emptyset$. In this case the left-hand side contains only non-negative coefficients which can be strengthened to the value of the right-hand side. This strengthening has already been proposed by Atamtürk [3]. Since F_s is non-decreasing, applying the strengthening and MIR can be exchanged, resulting in the same inequality. In particular, the coefficient $F_s(c^m)$ can be reduced to $\min(F_s(c^m), F_s(d_S^Q)) = F_s(\min(c^m, d_S^Q))$ for all $m \in M$. By setting $A_2^- = \emptyset$, the strengthened simple flow-cutset inequality for $\mathcal{CS}_S^{di}$ writes as

$$f^Q(\bar{A}_1^+) + \sum_{m \in M} F_s(\min(c^m, d_S^Q)) x^m(A_1^+) \ge F_s(d_S^Q) \tag{18}$$

for every $s \in M$, whereas the strengthened simple flow-cutset inequality for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ is given by

$$f^Q(\bar{A}_1^+) + \sum_{m \in M} F_s(\min(c^m, d_S^Q)) x^m(E_1) \ge F_s(d_S^Q). \tag{19}$$

To give a formal proof of the validity of (18) and (19) we consider the function $\mathcal{F}_{d,c} : \mathbb{R}_+ \to \mathbb{R}_+$ for $d, c \in \mathbb{R}_+ \backslash \{0\}$ given by

$$\mathcal{F}_{d,c}(a) = F_{d,c}(\min(a, d)).$$

Lemma 4.23. *Let $c, d \in \mathbb{R}_+ \backslash \{0\}$ with $c > 0$ and $\frac{d}{c} \notin \mathbb{Z}$. The function $\mathcal{F}_{d,c}$ is nondecreasing and subadditive on $\mathbb{R}_+$ with $\mathcal{F}_{d,c}(0) = 0$ and $\bar{\mathcal{F}}_{d,c}(a) = a$ for all $a \in \mathbb{R}_+$.*

Proof. Let $a_1, a_2 \in \mathbb{R}_+$ with $a_1 \le a_2$. It holds that $\mathcal{F}_{d,c}(a_1) \le \mathcal{F}_{d,c}(a_2)$ because $F_{d,c}$ is nondecreasing and $\min(a_1, d) \le \min(a_2, d)$. Using that $F_{d,c}$ is subadditive and nondecreasing gives

$$\begin{aligned} \mathcal{F}_{d,c}(a_1) + \mathcal{F}_{d,c}(a_2) &= F_{d,c}(\min(a_1, d)) + F_{d,c}(\min(a_2, d)) \\ &\ge F_{d,c}(\min(a_1, d) + \min(a_2, d)) \\ &\ge F_{d,c}(\min(a_1 + a_2, d)) \\ &= \mathcal{F}_{d,c}(a_1 + a_2). \end{aligned}$$

From $d \in \mathbb{R}_+ \backslash \{0\}$ and $F_{d,c}(0) = 0$ follows that $\mathcal{F}_{d,c}(0) = 0$, and for any $a \in \mathbb{R}_+$ it holds

$$\bar{\mathcal{F}}_{d,c}(a) = \lim_{t \searrow 0} \frac{F_{d,c}(\min(at, d))}{t} = \lim_{t \searrow 0} \frac{F_{d,c}(at)}{t} = a. \qquad \square$$

Assume $A_2^- = \emptyset$. Using Proposition 4.18 and applying the subadditive function $\mathcal{F}_{d,c}$ with $d = d_S^Q$ and $c = c^s$ to the valid base inequalities (13) and (14) shows the validity of (18) and (19). Note that we may extend $\bar{\mathcal{F}}_{d,c}$ to $\mathbb{R}$ by setting $\bar{\mathcal{F}}_{d,c}(a) := 0$ for $a < 0$.

Corollary 4.24. *The strengthened simple flow-cutset inequality* (18) *is valid for* $\mathcal{CS}_S^{di}$*, and the strengthened simple flow-cutset inequality* (19) *is valid for* $\mathcal{CS}_S^{bi}$ *and* $\mathcal{CS}_S^{un}$*.*

In the single-commodity case the strengthening has no effect since $F_s(c^s) = r(d_S^Q, c^s) \leq d_S^Q$. For $|M| > 1$, it is not known in general under which conditions strengthened simple flow-cutset inequalities define facets for $\mathcal{CS}_S^{di}$, $\mathcal{CS}_S^{bi}$ or $\mathcal{CS}_S^{un}$. There is also no analogon of Theorem 4.21 and Theorem 4.22 on the strength of flow-cutset inequalities for Bidirected or Undirected models in the multi-module case. For some special cases these inequalities define facets as shown by Bienstock and Günlük [8] as well as Magnanti and Mirchandani [20] and Magnanti et al. [21, 22]. In particular, these authors consider the network design polyhedra $\mathcal{ND}^{bi}$ and $\mathcal{ND}^{un}$, respectively. For $|M| \leq 3$ and divisible base capacities they provide conditions for simple flow-cutset inequalities and cutset inequalities to define facets. This may serve as an indication that our approach produces strong valid inequalities for cutset polyhedra (and by Theorem 3.7 also for network design polyhedra) in the multi-module case.

Example 4.25. Bienstock and Günlük [8] consider network design polyhedra with Bidirected capacity constraints and two modules, where $c^1 = 1$ and $c^2 = \lambda \in \mathbb{Z}_+, \lambda > 1$. Specializing (16) with $d = d_S^Q$, $c^s = \lambda$, $r = r(d, \lambda)$, $\eta = \lceil \frac{d}{\lambda} \rceil$ gives

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + x^1(E_1) + rx^2(E_1) + \min(1, \lambda - r)x^1(E_2) + (\lambda - r)x^2(E_2) \geq r\eta, \tag{20}$$

Setting $E_2 = \emptyset$ results in one of the simple flow-cutset inequality introduced in [8]. A strengthening as proposed above has no effect here. Bienstock and Günlük also consider other classes of inequalities of this type, all corresponding to the case that d is fractional, which we do not consider here (also see Atamtürk [3, Example 1]). By [8, Theorem 3.5], inequality (20) defines a facet of $\mathcal{ND}^{bi}$ if $r < \lambda$, $Q \subseteq K_S^+$, $\emptyset \neq E_1 \subset E_S$, $d_S > 1$ and if $H[S]$ and $H[V \backslash S]$ are connected.

Magnanti and Mirchandani [20] investigate single-commodity network design polyhedra with Undirected capacity constraints, three modules and one commodity, where $c^1 = 1$, $c^2 = C \in \mathbb{Z}_+, C > 1$ and $c^3 = \lambda C \in \mathbb{Z}_+, \lambda > 1$. We can formulate two non-trivial cutset inequalities of type (19) corresponding to $c^s = C$ and $c^s = \lambda C$, which are

$$x^1(E_S) + r_1 x^2(E_S) + \lambda r_1 x^3(E_S) \geq r_1 \left\lceil \frac{d}{C} \right\rceil \quad \text{and} \tag{21}$$

$$x^1(E_S) + \min(C, r_2)x^2(E_S) + r_2 x^3(E_S) \geq r_2 \left\lceil \frac{d}{\lambda C} \right\rceil. \tag{22}$$

where $r_1 = r(d, C)$ and $r_2 = r(d, \lambda C)$. These are two of the cutset inequalities considered in [20]. Inequality (21) is known to be facet-defining for $\mathcal{CS}_S^{un}$ under certain conditions [20]. Notice that if $d < \lambda C$, then (21) can be strengthened to

$$x^1(A_S) + r_1 x^2(A_S) + \min(\lambda, \lceil \tfrac{d}{C} \rceil) r_1 x^3(A_S) \geq r_1 \lceil \tfrac{d}{C} \rceil$$

by using Corollary 4.24. Inequality (22) defines a facet of $\mathcal{CS}_S^{un}$ if $H[S]$ as well as $H[V \backslash S]$ are connected and if $r_2 < \lambda C$ by [20, Proposition 5.2].

Magnanti and Mirchandani present a third facet-defining cutset inequality, which can be seen as a 2-step MIR-inequality or knapsack-partition inequality, see Pochet and Wolsey [27].

In Section 4.2, we presented (12) as a second class of valid inequalities for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ in the single-module case. To prove the validity of (12) we did not use MIR or any other subadditive function. A base inequality is missing such that we cannot apply the same approach based on subadditive lifting. To generalize (12) to the multi-module case the lifting problem has to be studied for these inequalities, which goes beyond the scope of this paper.

5 Concluding remarks

We have studied capacitated network design cutset polyhedra for the three different capacity models Directed, Bidirected and Undirected. We have shown that given a network cut, any facet of the corresponding cutset polyhedron translates into a facet of the network design polyhedron if both network components are (strongly) connected. In the single-module case we could state necessary and sufficient conditions for flow-cutset inequalities to be facet-defining. We worked out the differences caused by the different types of capacity constraints. For the models Bidirected and Undirected we identified a new class of facet-defining inequalities. Flow-cutset inequalities could be generalized to the multi-module case using mixed-integer rounding and it was shown that this approach gives known strong valid inequalities for network design polyhedra with restricted capacity module sets.

We want to conclude this article with some extensions and ideas on interesting future research topics. None of the considered inequalities in this paper does exploit the structure of the subgraphs defined by the cut components. This might be a drawback when optimizing sparse networks as they are common in practice. Consider the Undirected model and assume that the condition of Theorem 3.7 does not hold, i. e., the subgraph $H[S]$ is not connected but decomposes into the components $H[S_1]$ and $H[S_2]$, where (S_1, S_2) is a partition of S. Assume that some point-to-point commodity k with value $\bar{d} > 0$ has to be realized from $H[S_1]$ to $H[S_2]$. The corresponding flow has to cross the cut E_S twice. It follows that we can add the value $2\bar{d}$ to the right-hand side of the base inequality (14). Notice that the resulting inequality is valid for $\mathcal{ND}^{un}$ but not for $\mathcal{CS}_S^{un}$ as considered here, in particular $k \notin K_S$. For the capacity formulation and inequalities only containing capacity variables, this approach leads to the well known metric inequalities, see [6, 18]. It might now be of interest to study metric type flow-cutset inequalities and metric inequalities plus MIR.

Flow-cutset inequalities as presented in this article are rank-1-MIR inequalities. At least in the multi-module case it is a promising idea to apply the functions $F_{d,c}$ or $\mathcal{F}_{d,c}$ in a second MIR-step, where c is one of the left-hand side coefficients and d the right-hand side of a flow-cutset inequality. See Dash and Günlük [13, 14] for some recent results on 2-step MIR inequalities. The base cutset inequalities

$$\sum_{m \in M} c^m x^m(E_S) \geq d_S^{K_S^+}$$

can be seen as an integer knapsack inequality. The corresponding integer knapsack sets have been studied by Pochet and Wolsey [27] for the case that the base capacities c^m are integer multiples of each other. They prove that these sets are completely described by knapsack-partition inequalities that can be obtained by consecutively applying MIR to the base knapsack inequality. In particular, the (1-step) MIR cutset inequality

$$\sum_{m \in M} F_s(\min(c^m, d_S^Q)) x^m(E_S) \geq F_s(d_S^{K_S^+}).$$

that is obtained by setting $E_1 = E_S$ and $Q = K_S^+$ in (19), is a knapsack-partition inequality. Also the three cutset inequalities presented by [20] and mentioned in Example 4.25 for models with three modules are knapsack-partition inequalities. No result is known on the strength of an n-step MIR approach for network design polyhedra applied to cutset inequalities and flow-cutset inequalities. Also computational experience in this direction is missing.

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A Omitted proofs

A.1 Proof of Theorem 4.12

Proof. We will show that the related face

$$F =: \Big\{(f,x) \in \mathcal{CS}_S^{bi}\ (\mathcal{CS}_S^{un}) : (f,x) \text{ satisfies (11) with equality}\Big\}$$

is non-trivial and then by contradiction, we will show that it defines a facet. This will be done for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$ simultaneously. In the following we will construct points on the face F. Whenever we can ensure that $E_1 \cap E_2 \neq \emptyset$ these points may only be valid for $\mathcal{CS}_S^{bi}$ but not for $\mathcal{CS}_S^{un}$. Given $e \in E_S$, let b_e denote the unit vector in $\mathbb{R}^{|E_S|+2|K||E_S|}$ for the integer design variable of e and let $g_{e^+}^k, g_{e^-}^k$ be the unit vectors for the two continuous flow variables of e for commodity $k \in K$. We set $d := d_S^Q$, $\eta := \eta^Q$, $r := r^Q < c$, $\epsilon > 0$ small enough and $\mathcal{M}$ a large integral number. Let $\bar{Q} := K_S^+ \backslash Q$. Choose $l \in E_1 \backslash E_2$ and $\bar{l} \in \bar{E}_1 \backslash E_2$. We construct a point on the face F by sending all flow for Q on l and the flow for all other commodities on $\bar{l}$:

$$p := \eta b_l + \mathcal{M} b_{\bar{l}} + \sum_{k \in Q} d_S^k g_{l^+}^k + \sum_{k \in \bar{Q}} d_S^k g_{\bar{l}^+}^k + \sum_{k \in K_S^-} d_S^k g_{\bar{l}^-}^k.$$

The point p is on the face F by construction. Hence F is not empty. $p + b_l$ is a point that is in $\mathcal{CS}_S^{bi}$ ($\mathcal{CS}_S^{un}$) but not on the face F.

It remains to show that F is inclusion-wise maximal. Choose facet $\tilde{F}$ of $\mathcal{CS}_S^{bi}$ ($\mathcal{CS}_S^{un}$) with $F \subseteq \tilde{F}$ and let $\tilde{F}$ be defined by

$$\sum_{e \in E_S} (\beta_e x_e + \sum_{k \in K} \gamma_{e^+}^k f_{e^+}^k + \sum_{k \in K} \gamma_{e^-}^k f_{e^-}^k) = \pi \tag{23}$$

where $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k, \pi \in \mathbb{R}$. We will show that (23) is (11) up to a scalar multiple and a linear combination of flow conservation constraints, proving that $F = \tilde{F}$.

Adding multiples of the $|K|$ flow conservation constraints to (23) we can assume $\gamma_{l^+}^k = 0$ for all $k \in Q$, $\gamma_{\bar{l}^+}^k = 0$ for all $k \in \bar{Q}$ and $\gamma_{\bar{l}^-}^k = 0$ for all $k \in K_S^-$. Set $\beta := \beta_l$ and $\bar{\beta} := \beta_{\bar{l}}$. Since p lies on the hyperplane, we conclude that $\beta\eta + \bar{\beta}\mathcal{M} = \pi$. Now we modify p by installing a capacity of $\mathcal{M}+1$ on $\bar{l}$. This is another point on the face and thus $\bar{\beta} = 0$. It follows that

$$\beta\eta = \pi \tag{24}$$

The capacity on l is not saturated since $d < c\eta$. Modifying p by simultaneously increasing flow on l^+ and l^- by ϵ for $k \in Q$ gives new points on the face and thus

$$\gamma_{l^+}^k, \gamma_{l^-}^k = 0 \ \forall k \in Q$$

The same can be done on $\bar{l}^+$, $\bar{l}^-$ for $k \in K_S^- \cup \bar{Q}$, hence

$$\gamma_{\bar{l}^+}^k, \gamma_{\bar{l}^-}^k = 0 \qquad \forall k \in K_S^- \cup \bar{Q}.$$

Now consider the disjoint partition $E_S = (E_1 \cap E_2) \cup (E_1 \backslash E_2) \cup (\bar{E}_1 \cap E_2) \cup (\bar{E}_1 \backslash E_2)$. We will compute the coefficients $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k$ for e in each of the four sets by constructing new points on the face F. They will obviously fulfill the flow conservation constraint and satisfy inequality (11) with equality. To see that they meet the (Bidirected) capacity constraints just use that $r < c$ and remember the relation $c\eta = d + c - r$. Note that all the points to be defined for edges in $\bar{E}_1 \cap E_2$, $E_1 \backslash E_2$ and $\bar{E}_1 \backslash E_2$ additionally satisfy the Undirected capacity constraints. Hence with $E_1 \cap E_2 = \emptyset$ the theorem holds for $\mathcal{CS}_S^{un}$.

i) $\mathbf{\bar{E}_1 \cap E_2}$: For $e \in \bar{E}_1 \cap E_2$ and $k \in Q$ define the following three points on F:

$$p + b_e + (c-r)g_{l^+}^k + (c-r)g_{e^-}^k \implies \beta\eta + \beta_e + (c-r)\gamma_{e^-}^k = \pi \tag{25}$$

$$p + b_l + cg_{l^+}^k + b_e + cg_{e^-}^k \implies \beta\eta + \beta + \beta_e + c\gamma_{e^-}^k = \pi \tag{26}$$

$$p + (c-r)g_{l^+}^k + b_e + \tfrac{r}{2}g_{e^+}^k + (c - \tfrac{r}{2})g_{e^-}^k \implies \beta\eta + \beta_e + \tfrac{r}{2}\gamma_{e^+}^k + (c - \tfrac{r}{2})\gamma_{e^-}^k = \pi \tag{27}$$

Comparison of (25) and (26) shows that $-r\gamma^k_{e^-} = \beta$ for all $e \in \bar{E}_1 \cap E_2$, for all $k \in Q$. From (25) it follows then that $\beta_e = \frac{\beta}{r}(c-r)\ \forall e \in \bar{E}_1 \cap E_2$. From (27) we find that $\frac{\beta}{r}(c-r) - (c - \frac{r}{2})\frac{\beta}{r} + \frac{r}{2}\gamma^k_{e^+} = 0$, which implies that $r\gamma^k_{e^+} = \beta$ for all $e \in \bar{E}_1 \cap E_2$, for all $k \in Q$.

To conclude that $\gamma^k_{e^-} = 0$ for all $k \in K^-_S, \bar{Q}$ just modify the point in (25) by increasing flow for $k \in K^-_S, \bar{Q}$ on $\bar{l}^+$ by some ϵ and routing this ϵ-flow back on e^-. Simultaneously increasing flow on e^+, e^- gives $\gamma^k_{e^+} = 0$ for all $k \in K^-_S, \bar{Q}$.

ii) $\mathbf{E_1 \cap E_2}$: For $e \in E_1 \cap E_2$ and $k \in Q$ define:

$$v^k_e := p + b_e + cg^k_{e^+} + cg^k_{e^-} \implies \beta\eta + \beta_e + c\gamma^k_{e^+} + c\gamma^k_{e^-} = \pi \tag{28}$$

We can still increase flow on l^+ by a small amount for commodity k. Decreasing flow on e^+ at the same time gives another point on the face and thus $\gamma^k_{e^+} = 0$ for all $k \in Q$. When having changed v^k_e this way, some flow for a commodity in $K^-_S \cup \bar{Q}$ can be routed on e^+ while the same amount of flow increases on $\bar{l}^-$. Hence $\gamma^k_{e^+} = 0$ for all $k \in K^-_S \cup \bar{Q}$.

For $k_1, k_2 \in Q, e \in E_1 \cap E_2$ consider the point

$$v^{k_1}_e - \epsilon g^{k_1}_{l^+} + \epsilon g^{k_2}_{l^+} - \epsilon g^{k_1}_{e^-} + \epsilon g^{k_2}_{e^-}$$

It is well defined and feasible because flow on l^+ is positive for every $k \in Q$ and flow on e^+ is positive for k_1. It follows that $\gamma_{e^-} := \gamma^{k_1}_{e^-} = \gamma^{k_2}_{e^-}$ for all $k_1, k_2 \in Q$.

To construct another point p_e on the face F we modify p by deleting one unit of capacity on l and installing one unit of capacity on $e \in E_1 \cap E_2$. A total flow of r has to be rerouted since $c(\eta - 1) = d - r$. We do so by decreasing flow of Q on l^+ by a total of r and increasing it on e^+, e^- by a total of c and $(c-r)$ respectively. This can be done in such a way that flow is positive on e^+, e^- for all $k \in Q$. If $\varphi^k_{l^+}, \varphi^k_{e^+}, \varphi^k_{e^-} > 0$ denote the rerouted flows on l^+, e^+, e^-, then p_e can be written as

$$p_e := p - b_l + b_e - \sum_{k \in Q} \varphi^k_{l^+} g^k_{l^+} + \sum_{k \in Q} \varphi^k_{e^+} g^k_{e^+} + \sum_{k \in Q} \varphi^k_{e^-} g^k_{e^-}$$

with $\sum_{k \in Q} \varphi^k_{l^+} = r$, $\sum_{k \in Q} \varphi^k_{e^+} = c$ and $\sum_{k \in Q} \varphi^k_{e^-} = c - r$. From $p_e \in F$ and the fact that $\gamma^k_{l^+} = \gamma^k_{e^+} = 0$ and $\gamma^k_{e^-} = \gamma_{e^-}$ for all $k \in Q$ we conclude that

$$\beta\eta - \beta + \beta_e + (c-r)\gamma_{e^-} = \pi \tag{29}$$

Now comparing (28) and (29) gives

$$-r\gamma_{e^-} = -r\gamma^k_{e^-} = \beta\ \forall k \in Q.$$

From (28) and (24) follows then

$$\beta_e = c\frac{\beta}{r}\ \forall e \in E_1 \cap E_2.$$

Again considering the point p_e, the total flow on e^- is $c - r$, thus the capacity on e^- is not saturated. Increasing flow on $\bar{l}^+$ and e^- for commodities in $\bar{Q} \cup K^-_S$ gives $\gamma^k_{e^-} = 0$ for all $k \in \bar{Q} \cup K^-_S$.

iii) $\mathbf{\bar{E}_1 \backslash E_2}$: For $e \in \bar{E}_1 \backslash E_2$ consider the following point on F:

$$p + b_e \implies \beta\eta + \beta_e = \pi \tag{30}$$

The point can be modified by simultaneously increasing flow on l^+ and e^-. This can be done for every commodity in Q, thus $\gamma^k_{e^-} = 0$ for all $k \in Q$. Comparing (30) with (24) gives $\beta_e = 0$ for all $e \in \bar{E}_1 \backslash E_2$.

To construct a new point q_e on the face F we modify p by deleting one unit of capacity on l and installing one unit of capacity on $e \in \bar{E}_1 \backslash E_2$. We decrease flow of Q on l^+ by a total

of r and increase it by the same amount on e^+. This can be done in such a way that flow is positive on e^+ for all $k \in Q$. If $\varphi^k > 0$ denotes the rerouted flow with respect to $k \in Q$, then q_e can be written as

$$q_e := p - b_l + b_e - \sum_{k \in Q} \varphi^k g^k_{l^+} + \sum_{k \in Q} \varphi^k g^k_{e^+} \quad \Longrightarrow \quad \beta\eta - \beta + \sum_{k \in Q} \varphi^k \gamma^k_{e^+} = \pi \tag{31}$$

with $\sum_{k \in Q} \varphi^k = r$. Modifying q_e by simultaneously increasing flow on e^+ and $\bar{l}^-$ for $k \in K_S^- \cup \bar{Q}$ gives $\gamma^k_{e^+} = \gamma^k_{e^-} = 0$ for all $k \in K_S^- \cup \bar{Q}$. It remains to show that $\gamma^k_{e^+} = \frac{\beta}{r}$ for k in Q. We make use of the conditions i) and ii) of Theorem 4.12.

Assume first that $E_2 = \emptyset$. If $|Q| = 1$, it follows that $\beta\eta - \beta + r\gamma^k_{e^+} = \pi$ and $r\gamma^k_{e^+} = \beta$. If $|Q| > 1$ and $\eta \geq 2$, then $d > c > r$ and q_e can be constructed such that flows are positive both on l^+ and e^+ for every commodity in Q. We choose $k_1, k_2 \in Q$ and modify q_e by adding the flow $\epsilon g^{k_2}_{l^+} - \epsilon g^{k_1}_{l^+} + \epsilon g^{k_1}_{e^+} - \epsilon g^{k_2}_{e^+}$. This way we conclude that $\gamma^{k_1}_{e^+} = \gamma^{k_2}_{e^+}$. From (31) follows then $r\gamma^k_{e^+} = \beta$ for all $k \in Q$. Now let us assume that there is an edge $\bar{e}$ in $E_1 \cap E_2$. Modify q_e by installing one unit of capacity on $\bar{e}$ and sending a flow of c on $\bar{e}^+$ and $\bar{e}^-$ for a commodity $k_1 \in Q$. Now adding $\epsilon g^{k_2}_{\bar{e}^+} - \epsilon g^{k_1}_{\bar{e}^+} + \epsilon g^{k_1}_{e^+} - \epsilon g^{k_2}_{e^+}$ gives $\gamma^{k_1}_{e^+} = \gamma^{k_2}_{e^+}$ and $r\gamma^k_{e^+} = \beta$ for all $k \in Q$ again since $\gamma^{k_1}_{\bar{e}^+} = \gamma^{k_2}_{\bar{e}^+} = 0$, as shown above. Finally assume that there is $\bar{e}$ in $\bar{E}_1 \cap E_2$. For a commodity $k \in Q$ consider the following vector:

$$\begin{aligned} p + (c-r)g^k_{l^+} + b_{\bar{e}} + b_e + cg^k_{\bar{e}^-} + rg^k_{e^+} &\Longrightarrow \beta\eta + \beta_{\bar{e}} + \beta_e + c\gamma^k_{\bar{e}^-} + r\gamma^k_{e^+} = \pi \\ &\Longrightarrow \beta\eta + (c-r)\tfrac{\beta}{r} - c\tfrac{\beta}{r} + r\gamma^k_{e^+} = \pi \\ &\Longrightarrow \beta = r\gamma^k_{e^+} \ \ \forall k \in Q \end{aligned}$$

iv) $\mathbf{E_1 \backslash E_2}$: We construct the vector q_e again but for $e \in E_1 \backslash E_2$.

$$q_e := p - b_l + b_e - \sum_{k \in Q} \varphi^k g^k_{l^+} + \sum_{k \in Q} \varphi^k g^k_{e^+} \quad \Longrightarrow \quad \beta\eta - \beta + \beta_e + \sum_{k \in Q} \varphi^k \gamma^k_{e^+} = \pi \tag{32}$$

with $\sum_{k \in Q} \varphi^k = r$. For $k \in K$ add an ϵ-flow to e^+ and e^- to conclude that $\gamma^k_{e^+} = -\gamma^k_{e^-}$ for all $k \in K$. Now we modify the point q_e by simultaneously increasing flow on e^+ and $\bar{l}^-$ for a k in K. Hence $\gamma^k_{e^+} = -\gamma^k_{\bar{l}^-} = 0$. It follows that $\gamma^k_{e^-} = 0$ for all $k \in K$ and $\beta_e = \beta$ for all $e \in E_1 \backslash E_2$.

Plugging in all coefficients in (23) we arrive at:

$$\begin{aligned} \beta x(E_1 \backslash E_2) \ &+ \ \tfrac{\beta}{r} f^Q(\bar{A}_1^+ \backslash A_2^+) \\ &+ \ \tfrac{\beta}{r}(c-r)x(\bar{E}_1 \cap E_2) + \tfrac{\beta}{r} f^Q(\bar{A}_1^+ \cap A_2^+) - \tfrac{\beta}{r} f^Q(\bar{A}_1^- \cap A_2^-) \\ &+ \ c\tfrac{\beta}{r} x(E_1 \cap E_2) - \tfrac{\beta}{r} f^Q(A_1^- \cap A_2^-) \ = \ \beta\eta \end{aligned}$$

which is equivalent to

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + rx(E_1) + (c-r)x(E_2) = r\eta$$

We have shown that the hyperplane (23) is a multiple of (11) plus a linear combination of flow conservation constraints. It follows that $F = \tilde{F}$. This concludes the proof. □

A.2 Proof of Theorem 4.13

Proof. We proceed as in the proof of Theorem 4.12 and apply the definitions of the faces F, $\tilde{F}$ with $F \subseteq \tilde{F}$ and the vectors b_e, $g^k_{e^+}, g^k_{e^-}$. We set $d := d_S^{K_S^+}$, $d^- := |d_S^{K_S^-}|$, $\eta := \eta^{K_S^+}$, $r := r^{K_S^+} < c$, $\epsilon > 0$ small enough and $\mathcal{M}$ a large integral number.

In contrast to the proof of Theorem 4.12 the point p to start from is defined as follows. Choose $l \in E_1 \backslash E_2$. All demand is routed on l with capacity exactly $c\eta$, more precisely all flow for positive

commodities is routed on l^+ and all flow for negative commodities is routed on l^-. Notice that we assume $d \geq d^-$ and $r < c$. The point p can be written as:

$$p := \eta b_l + \sum_{k \in K_S^+} d_S^k g_{l^+}^k + \sum_{k \in K_S^-} d_S^k g_{l^-}^k.$$

By considering p and $p + b_l$ we conclude that $\emptyset \neq F \neq \mathcal{CS}_S^{bi}$. It is missing to prove that $F = \tilde{F}$. We will show that (23) is (11) up to a scalar multiple and a linear combination of flow conservation constraints. We can assume that $\gamma_{l^+}^k = 0 \ \forall k \in K_S^+$ and $\gamma_{l^-}^k = 0 \ \forall k \in K_S^-$ w.l.o.g. by adding multiples of the flow conservation constraints to (23).

Set $\beta := \beta_l$. Since p lies on the hyperplane, we conclude that

$$\beta\eta = \pi \tag{33}$$

Modifying p by simultaneously increasing flow on l^+ and l^- by ϵ for every commodity gives new points on the face and thus $\gamma_{l^+}^k, \gamma_{l^-}^k = 0 \ \forall k \in K$.

Now consider the disjoint partition $E_S = (\bar{E}_1 \cap E_2) \cup (E_1 \cap E_2) \cup (E_1 \backslash E_2)$. (Note that $\bar{E}_1 \backslash E_2 = \emptyset$). We calculate the coefficients $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k$ for e in each of the three sets by constructing new points on the face F. Note that all the points to be defined for edges in $\bar{E}_1 \cap E_2$ and $E_1 \backslash E_2$ additionally satisfy the Undirected capacity constraints when no negative commodities are given. Hence with $E_1 \cap E_2 = \emptyset$ and $K_S^- = \emptyset$ the theorem holds for $\mathcal{CS}_S^{un}$.

i) $\mathbf{\bar{E}_1 \cap E_2}$: For $e \in \bar{E}_1 \cap E_2$ and $k \in K_S^+$ we define the points (25), (26) and (27) and as in the proof of Theorem 4.12 we conclude that

$$-r\gamma_{e^-}^k = \beta, \ r\gamma_{e^+}^k = \beta \ \text{ and } \ \beta_e = \tfrac{\beta}{r}(c - r) \quad \forall e \in \bar{E}_1 \cap E_2, k \in K_S^+.$$

To see that $\gamma_{e^+}^k = \gamma_{e^-}^k = 0 \ \forall k \in K_S^-$ modify the point in (25) by first increasing flows on l^-, e^+ and then increasing flows on e^+, e^-.

ii) $\mathbf{E_1 \cap E_2}$: Let $e \in E_1 \cap E_2$ and $k \in K_S^+$. By defining v_e^k as in (28) and with the same arguments it can be shown that

$$\gamma_{e^+}^k = 0 \ \forall k \in K_S^+, K_S^- \ \text{ and } \ \gamma_{e^-} := \gamma_{e^-}^{k_1} = \gamma_{e^-}^{k_2} \ \forall k_1, k_2 \in K_S^+.$$

Modifying p by deleting one unit of capacity on l and installing one unit of capacity on $e \in E_1 \cap E_2$ gives a point p_e on the face F as in the proof of Theorem 4.12. If $K_S^- = \emptyset$ we can conclude $-r\gamma_{e^-}^k = \beta \ \forall k \in K_S^+$. and $\beta_e = c\frac{\beta}{r} \ \forall e \in E_1 \cap E_2$ in a similar way. Else if $K_S^- \neq \emptyset$ we also have to reroute flow on l^- and e^- for commodities in K_S^-. We can assume $d > d^-$ and $d > c$ by assumption. It follows that a rerouting can be done in such a way that flow for $k \in K_S^-$ is still positive on l^- and that the capacity for e^- is not saturated. If in this case $\varphi_{l^+}^k, \varphi_{l^-}^k, \varphi_{e^+}^k, \varphi_{e^-}^k > 0$ denote the rerouted flows on l^+, l^-, e^+, e^- for $k \in K$, then p_e can be written as

$$p_e := p - b_l + b_e - \sum_{k \in K_S^+} \varphi_{l^+}^k g_{l^+}^k - \sum_{k \in K_S^-} \varphi_{l^-}^k g_{l^-}^k + \sum_{k \in K_S^+} \varphi_{e^+}^k g_{e^+}^k + \sum_{k \in K_S^+} \varphi_{e^-}^k g_{e^-}^k + \sum_{k \in K_S^-} \varphi_{e^-}^k g_{e^-}^k$$

with

$$\sum_{k \in K_S^+} \varphi_{l^+}^k = r, \ \sum_{k \in K_S^+} \varphi_{e^+}^k = c, \ \sum_{k \in K_S^+} \varphi_{e^-}^k = c - r \ \text{ and } \ \sum_{k \in K_S^-} \varphi_{e^-}^k = \sum_{k \in K_S^-} \varphi_{l^-}^k < r$$

implying

$$\beta\eta - \beta + \beta_e + (c - r)\gamma_{e^-} + \sum_{k \in K_S^-} \varphi_{e^-}^k \gamma_{e^-}^k = \pi$$

We can decrease flow on l^- and increase it on e^- which gives $\gamma_{e^-}^k = 0 \ \forall k \in K_S^-$ and thus $-r\gamma_{e^-}^k = \beta \ \forall k \in K_S^+$ and $\beta_e = c\frac{\beta}{r} \ \forall e \in E_1 \cap E_2$ as above.

iii) $\mathbf{E_1 \backslash E_2}$: For $e \in E_1 \backslash E_2$ we construct the point q_e as in the proof Theorem 4.12 but we also have to reroute flow for K_S^-:

$$q_e := p - b_l + b_e - \sum_{k \in K_S^+} \varphi^k g_{l^+}^k - \sum_{k \in K_S^-} \varphi^k g_{l^-}^k + \sum_{k \in K_S^+} \varphi^k g_{e^+}^k + \sum_{k \in K_S^-} \varphi^k g_{e^-}^k$$

with $\sum_{k \in K_S^+} \varphi^k = r$ and $\sum_{k \in K_S^-} \varphi^k \leq r$. We conclude

$$\beta\eta - \beta + \beta_e + \sum_{k \in K_S^+} \varphi^k \gamma_{e^+}^k + \sum_{k \in K_S^-} \varphi^k \gamma_{e^-}^k = \pi \tag{34}$$

For $k \in K$ add an ϵ-flow to e^+ and e^- to conclude that $\gamma_{e^+}^k = -\gamma_{e^-}^k \ \forall k \in K$. If for all $k \in K$ we can either show $\gamma_{e^+}^k = 0$ or $\gamma_{e^-}^k = 0$ we conclude $\beta_e = \bar{\beta} \ \forall e \in E_1 \backslash E_2$ by using (34).

By assumption $E_2 \neq \emptyset$. First suppose that there is $\bar{e}$ in $E_1 \cap E_2$. Modify q_e by installing one unit of capacity on $\bar{e}$ and sending a flow of c on $\bar{e}^+$ and $\bar{e}^-$ for a commodity $k \in K_S^+$, which again gives a point on F. Now decrease flow on $\bar{e}^+$ and increase it on e^+ by ϵ. Hence $\gamma_{e^+}^k = 0 \ \forall k \in K_S^+$. Having done so simultaneously increasing flow on $\bar{e}^+$ and e^- gives $\gamma_{e^-}^k = 0 \ \forall k \in K_S^-$. Finally suppose that there is $\bar{e}$ in $\bar{E}_1 \cap E_2$. For $k \in K_S^+$ consider the vector

$$p + (c - r) g_{l^+}^k + b_{\bar{e}} + b_e + c g_{\bar{e}^-}^k + r g_{e^+}^k$$

Simultaneously increasing flow on l^- and on e^+ for any commodity gives $\gamma_{e^+}^k = 0 \ \forall k \in K$.

Plugging in all coefficients in (23) gives a multiple of (11) as in the proof of Theorem 4.12. □

A.3 Proof of Theorem 4.15

Proof. Necessity has been proven in Lemma 4.10. We show sufficiency here. If $E_1 = \emptyset$ and $d_S^{K_S^+} < c$, then (12) reduces to the cutset inequality $x(E_S) \geq 1$ which is facet-defining for $\mathcal{CS}_S^{bi}$ ($\mathcal{CS}_S^{un}$) if $|E_S| = 1$ (see Theorem 4.11). If $\bar{E}_1 = \emptyset$, then (12) reduces to $x(E_S) \geq \eta^{K_S^+}$, which is facet-defining if $|E_S| = 1$ or $d_S^{K_S^+} > c$ (see Theorem 4.11). For the rest of the proof we can assume that $E_1, \bar{E}_1 \neq \emptyset$. For $Q = K_S^+$ we define the face

$$F := \Big\{ (f, x) \in \mathcal{CS}_S^{bi} \ (\mathcal{CS}_S^{un}) : (f, x) \text{ satisfies (12) with equality} \Big\}$$

Given $e \in E_S$ let b_e denote the incidence vector of the design variable of e and let $g_{e^+}^k, g_{e^-}^k$ be the unit vectors for the flow variables for commodity $k \in K$ of e in both directions. Suppose $r^{K_S^+} < c$ and set $d := d_S^{K_S^+}$, $\eta := \eta^{K_S^+}$, $r := r^{K_S^+}$ and $\epsilon > 0$ small enough. Choose $l \in E_1$ and $\bar{l} \in \bar{E}_1$. We construct a point p on the face F by installing η capacity units on the link l and by using this link to satisfy all demands. The point p is given by

$$p := \eta b_l + \sum_{k \in K_S^+} d_S^k g_{l^+}^k + \sum_{k \in K_S^-} d_S^k g_{l^-}^k .$$

Use $c\eta - d = c - r$ to verify that p is on the face. Considering $p + b_l$ we conclude $\emptyset \neq F \neq \mathcal{CS}_S^{bi}$. It remains to show that F is inclusion-wise maximal. Choose a facet $\tilde{F}$ of $\mathcal{CS}_S^{bi}$ with $F \subseteq \tilde{F}$ and let $\tilde{F}$ be defined by (23). We may add multiples of the $|K|$ flow conservation constraint to (23). Therefore we assume that $\gamma_{\bar{l}^-}^k = 0$ for all $k \in K$ w. l. o. g.. Set $\beta := \beta_l$ and $\bar{\beta} := \beta_{\bar{l}}$. The point p lies on $F \subseteq \tilde{F}$, hence

$$\beta\eta + \sum_{k \in K_S^+} d_S^k \gamma_{l^+}^k + \sum_{k \in K_S^-} d_S^k \gamma_{l^-}^k = \pi \tag{35}$$

Now we define a point p_e for all $e \in E_S$ the following way. We modify p by deleting one unit of capacity on l and installing one unit of capacity on $e \in E_S$. We decrease flow for K_S^+ on l^+ by a

total of r and increase it by the same amount on e^+. Some flow for K_S^- is also rerouted now using e^-. This can be done in such a way that flow is positive on e^+ for all $k \in K_S^+$, that flow is positive on e^- for all $k \in K_S^-$, and that the capacity on e is not saturated. Note that $p_e \in F$ for $e \in E_1$ and also for $e \in \bar{E}_1$. If $\varphi^k > 0$ denotes the rerouted flow for commodity k, then p_e can be written as

$$p_e := p - b_l + b_e - \sum_{k \in K_S^+} \varphi^k g_{l^+}^k - \sum_{k \in K_S^-} \varphi^k g_{l^-}^k + \sum_{k \in K_S^+} \varphi^k g_{e^+}^k + \sum_{k \in K_S^-} \varphi^k g_{e^-}^k,$$

with $\sum_{k \in K_S^+} \varphi^k g_{l^+}^k = \sum_{k \in K_S^+} \varphi^k g_{e^+}^k = r$ and $\sum_{k \in K_S^-} \varphi^k g_{l^-}^k = \sum_{k \in K_S^-} \varphi^k g_{e^-}^k \leq r$. From $p_e \in F \subseteq \tilde{F}$ follows

$$\beta\eta - \beta + \beta_e + \sum_{k \in K_S^+} (d_S^k - \varphi^k)\gamma_{l^+}^k + \sum_{k \in K_S^-} (d_S^k - \varphi^k)\gamma_{l^-}^k + \sum_{k \in K_S^+} \varphi^k \gamma_{e^+}^k + \sum_{k \in K_S^-} \varphi^k \gamma_{e^-}^k = \pi. \tag{36}$$

Modifying p_e by simultaneously increasing flow on e^+ and e^- by ϵ for every commodity gives

$$\gamma_{e^+}^k = -\gamma_{e^-}^k \ \ \forall e \in E_S, k \in K.$$

Now consider the disjoint partition $E_S := E_1 \cup \bar{E}_1$. We calculate the coefficients $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k$ for e in each of the two sets by constructing new points from p. All these points are on the face F for $\mathcal{CS}_S^{bi}$. If $K_S^- = \emptyset$, then all the points additionally satisfy Undirected capacity constraints. Thus with $K_S^- = \emptyset$ the theorem holds for $\mathcal{CS}_S^{un}$.

i) $\mathbf{E_1}$: For $e \in E_1$ and $k \in K_S^+$ consider the point

$$v_e^k := p_e + b_{\bar{l}} + (c - r)g_{e^+}^k + (c - r)g_{\bar{l}^-}^k.$$

Since v_e^k as well as p_e satisfy (23) and because $\gamma_{\bar{l}^-}^k = 0$ we conclude $\bar{\beta} + (c - r)\gamma_{e^+}^k = 0$, and thus

$$\gamma_{e^-}^k = -\gamma_{e^+}^k = \frac{\bar{\beta}}{c - r} \ \ \forall e \in E_1, k \in K_S^+. \tag{37}$$

We modify v_e^k by increasing flow on e^- and $\bar{l}^+$ for commodities in K_S^-. It follows $\gamma_{e^-}^k = -\gamma_{e^+}^k = 0 \ \forall e \in E_1, k \in K_S^-$. The equations (35) and (36) (with $e \in E_1$) now reduce to

$$\beta\eta - d\frac{\bar{\beta}}{c - r} = \pi \quad \text{and} \quad \beta\eta - \beta - d\frac{\bar{\beta}}{c - r} + \beta_e = \pi$$

which implies $\beta_e = \beta \ \forall e \in E_1$.

ii) $\bar{\mathbf{E}}_\mathbf{1}$: For $e \in \bar{E}_1$ and $k \in K_S^+$ define the point

$$w_e^k := p + b_e + (c - r)g_{l^+}^k + (c - r)g_{e^-}^k$$

on the face F. Since w_e^k satisfies (23) and because of (37), we get

$$\beta\eta + \beta_e - (d + c - r)\frac{\bar{\beta}}{c - r} - (c - r)\gamma_{e^+}^k = \pi \tag{38}$$

For commodities in K_S^- increasing flow on l^- and e^+ gives

$$\gamma_{e^-}^k = -\gamma_{e^+}^k = \gamma_{l^-}^k = 0 \ \forall e \in \bar{E}_1, k \in K_S^-.$$

For a fixed commodity $k \in K_S^+$ modify w_e^k by decreasing flow for k on l^+, e^- and simultaneously increasing flow on l^+, e^- for an arbitrary commodity $k^* \in K_S^+$. Hence

$$\gamma_e := \gamma_{e^-}^{k^*} = \gamma_{e^-}^k = -\gamma_{e^+}^{k^*} = -\gamma_{e^+}^k \ \ \forall e \in \bar{E}_1, k \in K.$$

The equation (36) with $e \in \bar{E}_1$ now reduces to

$$\beta\eta - \beta - (d - r)\frac{\bar{\beta}}{c - r} + \beta_e - r\gamma_e = \pi \tag{39}$$

Evaluating (39) for $e = \bar{l}$ and comparing with (35) gives

$$\beta = \frac{c\bar{\beta}}{c-r}$$

since $\beta_{\bar{l}} = \bar{\beta}$ and $-\gamma_{\bar{l}+} = \gamma^k_{\bar{l}+} = 0$ for all $k \in K$. Then from (38) and (39) follows that $r\gamma_e = (r-c)\gamma_e$ But $c > r > 0$ and thus

$$\gamma_e = \gamma^k_{e^-} = -\gamma^k_{e^+} = 0 \ \forall e \in \bar{E}_1, k \in K.$$

Now comparing (38) with (35) results in

$$\beta_e = \bar{\beta} \ \forall e \in \bar{E}_1.$$

Plugging in all coefficients in (23) we arrive at:

$$\frac{c\bar{\beta}}{c-r}x(E_1) + \bar{\beta}x(\bar{E}_1) + \frac{\bar{\beta}}{c-r}f^{K_S^+}(E_1^-) - \frac{\bar{\beta}}{c-r}f^{K_S^+}(E_1^+) = \bar{\beta}$$

which by multiplying with $\frac{c-r}{\bar{\beta}}$ reduces to (12) (with $Q = K_S^+$). We have shown that the hyperplane (23) is a multiple of (12) plus a linear combination of flow conservation constraints. It follows that $F = \tilde{F}$. This concludes the proof. □