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generalized processor sharing**

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# Waiting times for $M/M$ systems under generalized processor sharing<sup>1</sup>

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## Abstract

We consider a system where the arrivals form a Poisson process and the required service times of the requests are exponentially distributed. According to the generalized processor sharing discipline, each request in the system receives a fraction of the capacity of one processor which depends on the actual number of requests in the system. We derive systems of ordinary differential equations for the LST and for the moments of the conditional waiting time of a request with given required service time as well as a stable and fast recursive algorithm for the LST of the second moment of the conditional waiting time, which in particular yields the second moment of the unconditional waiting time. Moreover, asymptotically tight upper bounds for the moments of the conditional waiting time are given. The presented numerical results for the first two moments of the sojourn times in the  $M/M/m - PS$  system show that the proposed algorithms work well.

**Mathematics Subject Classification (MSC 2000):** 60K25, 68M20, 60G10.

**Keywords:** Poisson arrivals; exponential service times; generalized processor sharing; many-server; permanent customers;  $M/M/m - PS$ ; waiting time; sojourn time; response time; LST; moments; recursions.

## 1 Introduction

Processor Sharing (PS) systems have been widely used in the last decades for modeling and analyzing computer and communication systems. Early

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<sup>1</sup>This work was supported by a grant from the Siemens AG.

work on PS systems was motivated in the analysis of time sharing systems, like multi-programmed computer systems, cf. e.g. [Kle], [CMT], [YK]. In the last years PS systems became popular for studying bandwidth sharing among flows in links of telecommunication networks, cf. e.g. [MR], and job schedulers in web servers. Generalized processor sharing disciplines – combined with leaky bucket admission control – for packet schedulers in IP routers and ATM switches are found to have good properties. By choosing appropriate weights for the generalized processor sharing discipline, a certain quality of service (QoS) can be guaranteed for different packet streams with different bandwidth requirements, cf. e.g. [PG], [ZTK], [EM], [BBJ]. Generalized processor sharing disciplines are appropriate for modeling very complex scheduling strategies, including multi-class and multi-buffer systems as well as multi-processor systems. During the last two decades a huge number of models and various kinds of processor and generalized processor sharing disciplines have been investigated. We mention here only a few of them: [AH], [AAB], [BMU], [BP], [BB1]–[BB4], [GRZ], [Mo2], [Nun], [RS], [SGB], [Ya1], [Ya2] and the references therein.

In this paper we deal with waiting times of requests in a node, where the requests are served according to the following Generalized Processor Sharing (GPS) discipline, cf. [Coh]. If there are  $n \in \mathbb{N} := \{1, 2, \dots\}$  requests in the node then each of them receives a positive fraction  $\varphi(n)$  of the capacity of one processor, i.e., each of the  $n$  requests receives during an interval of length  $\Delta\tau$  the amount  $\varphi(n)\Delta\tau$  of service. Concerning the fractions  $\varphi(n)$  we assume that  $0 < \varphi(n) \leq 1$ ,  $n \in \mathbb{N}$ , and that there exists a  $n \in \mathbb{N}$  such that  $\varphi(n) < 1$ , ensuring that the waiting times are non-negative<sup>1</sup>. In case of  $\varphi_1(n) = 1/n$ ,  $n \in \mathbb{N}$ , we obtain the dynamics of the well-known single server PS system, also called single server system with Egalitarian Processor Sharing (EPS) discipline, cf. [CMT], [Ya2], in case of  $\varphi_{1,k}(n) = 1/(n+k)$ ,  $n \in \mathbb{N}$ , we have a single server PS system with  $k \in \mathbb{N}$  permanent requests in the system, cf. [YY], [BB], in case of  $\varphi_m(n) = \min(m/n, 1)$ ,  $n \in \mathbb{N}$ , a  $m$ -server PS system, where all requests are served in a PS mode, but each request receives at most the capacity of one processor, cf. [Coh] p. 283, [Br1], [Br2], [GRZ], in case of  $\varphi_{m,k}(n) = \min(m/(n+k), 1)$ ,  $n \in \mathbb{N}$ , a  $m$ -server PS system with  $k \in \mathbb{N}$  permanent requests.

A system working under the GPS discipline and where the requests arrive according to a Poisson process of intensity  $\lambda$ , the required service times are i.i.d. with df.  $B(x) := P(S \leq x)$ , where  $S$  denotes a generic service

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<sup>1</sup>In case of  $\varphi(n) = 1$ ,  $n \in \mathbb{N}$ , the system corresponds to a  $M/GI/\infty$  system, where no waiting occurs.

time, and finite mean  $ES$  and independent of the arrival process is denoted by  $M/GI/GPS$ , the corresponding  $m$ -server PS system is denoted by  $M/GI/m - PS$ .

Networks with nodes working under the GPS discipline are investigated in [Coh], [BP]. In particular, for the  $M/GI/GPS$  system the following basic results are known, cf. [Coh] (7.18) and (7.19). If the stability condition

$$\sum_{n=0}^{\infty} \prod_{j=1}^n \frac{\varrho \chi(j)}{j} < \infty, \quad (1.1)$$

where

$$\chi(n) := \frac{1}{\varphi(n)}, \quad n \in \mathbb{N}, \quad \chi(0) := 0 \quad (1.2)$$

and  $\varrho := \lambda ES$  denotes the offered load, is satisfied, then the distribution  $p(n) := P(N = n)$ ,  $n \in \mathbb{Z}_+$ , of the stationary number  $N$  of requests in the system exists and is given by

$$p(n) = \left( \sum_{m=0}^{\infty} \prod_{j=1}^m \frac{\varrho \chi(j)}{j} \right)^{-1} \prod_{j=1}^n \frac{\varrho \chi(j)}{j}, \quad n \in \mathbb{Z}_+. \quad (1.3)$$

Moreover, the stationary distribution of the Markov process of the vector of the number of requests in the system and their attained as well as residual service times is also of a product form, cf. [Coh] p. 279.

Let  $V$  the sojourn time of an arbitrary arriving request with required service time  $S$ . Then  $W := V - S$  is its waiting time, and Little's law provides

$$EV = \frac{1}{\lambda} EN, \quad EW = EV - ES. \quad (1.4)$$

From (1.3) it follows that (1.4) is equivalent to

$$EV = ES E[\chi(N+1)], \quad EW = ES E[\chi(N+1) - 1]. \quad (1.5)$$

Further, for the conditional sojourn time  $V(\tau)$  and conditional waiting time  $W(\tau) := V(\tau) - \tau$  of a request with required service time  $\tau \in \mathbb{R}_+$  it is stated that

$$EV(\tau) = \frac{\tau}{ES} EV, \quad EW(\tau) = \frac{\tau}{ES} EW, \quad (1.6)$$

cf. [Coh] (7.27). It seems that in case of the general  $M/GI/GPS$  system for  $V$ ,  $W$  and  $V(\tau)$ ,  $W(\tau)$  besides (1.4)–(1.6) there are known only asymptotic

results for heavy tailed service times, cf. [GRZ]. However, for special cases several results are well-known. For the  $M/M/1 - PS$  system in [CMT], [Mo1], [SJ] there are given analytical results and numerical algorithms for conditional sojourn time characteristics, in particular for the LST, the distribution function and the variance of  $V(\tau)$ , cf. also [Ya2] (4.1)–(4.4).

For the  $M/GI/1 - PS$  system the LST of  $V(\tau)$  was firstly derived in [KY], cf. also [Ya1], leading in particular to expressions for the variance of  $V(\tau)$ . Corresponding results for the  $M/GI/1 - PS$  system with  $k$  permanent requests are given in [BB], [YY]. For the  $M/PH/1 - PS$  system recently in [SGB] it is presented a numerical algorithm – basing on a uniformization procedure – for computing sojourn time characteristics. For the  $M/M/2 - PS$  system in [Tol] the second moment of  $V(\tau)$  is given. The general  $M/M/m - PS$  system is treated in [Br1], [Br2]. By using the approach of [CMT], for the LST's of the conditional waiting times  $W_n(\tau)$  of a request with required service time  $\tau$  and which finds  $n$  requests at its arrival in the system, a system of differential equations is derived. Further, numerical algorithms for computing the first two moments of the conditional waiting times  $W_n$  of a tagged request which finds at its arrival  $n$  requests in the system are presented and illustrated, where numerical problems in case of light traffic are reported.

The aim of this paper is to derive stable and efficient numerical algorithms for the variances  $var(W)$  and  $var(W(\tau))$  in the general  $M/M/GPS$  system. The paper is organized as follows. In Section 2 for the  $M/M/GPS$  system first we derive a linear system of differential equations for the LST's of  $W_n(\tau)$  by using the approach from [CMT] and [Br1], which provides asymptotically tight upper bounds for the moments of  $W(\tau)$  as well as corresponding linear systems of differential equations for the moments of  $W_n(\tau)$  and linear systems of algebraic equations for the LST's of the moments of  $W_n(\tau)$ . In Section 3 we derive a stable and fast recursive algorithm for computing the LST of the second moment of  $W(\tau)$  in the right half plane, which in particular yields the second moment of  $W$ . The first two moments of the sojourn times  $V$  and  $V(\tau)$  in the  $M/M/GPS$  system as well as numerical results for the  $M/M/m - PS$  system are given in Section 4.

## 2 Moments of the conditional waiting times

Consider a  $M/M/GPS$  system with arrival intensity  $\lambda$  and exponential service times with parameter  $\mu := 1/ES$  in steady state, i.e., let the stability

condition (1.1) be satisfied. For complex  $s$  with  $\Re s > 0$  let

$$w_n(s, \tau) := E[e^{-sW_n(\tau)}], \quad n \in \mathbb{Z}_+, \tau \in \mathbb{R}_+,$$

be the LST of the waiting time of a request with required service time  $\tau$  and finding at its arrival  $n$  requests in the system. Extending the ideas given in [CMT] and [Br1] for the  $M/M/1 - PS$  and  $M/M/m - PS$  system, respectively, we obtain the following system of ordinary differential equations:

**Lemma 2.1** *For fixed complex  $s$  with  $\Re s > 0$ , the LST's  $w_n(s, \tau)$  of the conditional waiting time distributions in the  $M/M/GPS$  system satisfy the linear system of ordinary differential equations*

$$\begin{aligned} \frac{\partial}{\partial \tau} w_n(s, \tau) &= n\mu w_{(n-1)_+}(s, \tau) \\ &\quad - (n\mu + \lambda\chi(n+1) + s(\chi(n+1) - 1))w_n(s, \tau) \\ &\quad + \lambda\chi(n+1)w_{n+1}(s, \tau), \quad n \in \mathbb{Z}_+, \tau \in (0, \infty), \end{aligned} \quad (2.1)$$

with the initial condition

$$w_n(s, 0) = 1, \quad n \in \mathbb{Z}_+. \quad (2.2)$$

**Proof** Let  $\tau > 0$ . Consider a tagged request with required residual service time  $\tau + \Delta\tau$ , where  $n$  further requests are in the system. Then during the time interval  $I$  of length  $\chi(n+1)\Delta\tau$  the tagged request receives an amount  $\Delta\tau$  of service time and the waiting time during  $I$  is  $(\chi(n+1) - 1)\Delta\tau$ , provided that there is whether an arrival nor a departure during  $I$ . Neglecting terms of order  $o(\Delta\tau)$ , the probability of an arrival during  $I$  is  $\lambda\chi(n+1)\Delta\tau$ , and the probability of a departure is  $n\mu\chi(n+1)\Delta\tau = n\mu\Delta\tau$  as the tagged request with required residual service time  $\tau + \Delta\tau$  cannot depart during  $I$  if  $\Delta\tau$  is sufficiently small. In view of the Markov property, thus we obtain

$$\begin{aligned} E[e^{-sW_n(\tau+\Delta\tau)}] &= \lambda\chi(n+1)\Delta\tau E[e^{-sW_{n+1}(\tau)}] \\ &\quad + (1 - (\lambda\chi(n+1) + n\mu)\Delta\tau) E[e^{-s((\chi(n+1)-1)\Delta\tau + W_n(\tau))}] \\ &\quad + n\mu\Delta\tau E[e^{-sW_{(n-1)_+}(\tau)}] + o(\Delta\tau). \end{aligned} \quad (2.3)$$

Rearranging the terms in (2.3) appropriately and taking the limit  $\Delta\tau \downarrow 0$ , it follows (2.1). The initial condition (2.2) is an immediate consequence from  $W_n(0) \equiv 0$ ,  $n \in \mathbb{Z}_+$ , in view of the GPS discipline and  $\varphi(n) > 0$ ,  $n \in \mathbb{N}$ .

□

## 2.1 Moments of $W(\tau)$

Let

$$M_n^{(k)}(\tau) := EW_n^k(\tau), \quad k \in \mathbb{Z}_+, \quad (2.4)$$

be the  $k$ -th moments of the waiting times  $W_n(\tau)$ ,  $n \in \mathbb{Z}_+$ ,  $\tau \in \mathbb{R}_+$ . Note that  $M_n^{(0)}(\tau) = 1$  for  $n \in \mathbb{Z}_+$ ,  $\tau \in (0, \infty)$  and  $M_n^{(k)}(0) = 0$  for  $k \in \mathbb{Z}_+$ ,  $n \in \mathbb{Z}_+$ . Further, let

$$M^{(k)}(\tau) := EW^k(\tau) = \sum_{n=0}^{\infty} p(n)M_n^{(k)}(\tau), \quad k \in \mathbb{Z}_+, \tau \in \mathbb{R}_+, \quad (2.5)$$

be the  $k$ -th moment of the conditional waiting time  $W(\tau)$  of an arriving request with required service time  $\tau$ .

**Lemma 2.2** *For  $k \in \mathbb{N}$ ,  $\tau \in \mathbb{R}_+$  it holds*

$$M^{(k)}(\tau) \leq \tau^k \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n), \quad (2.6)$$

$$\sum_{n=0}^{\infty} (\chi(n+1)-1)p(n)M_n^{(k-1)}(\tau) \leq \tau^{k-1} \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n). \quad (2.7)$$

**Proof** For  $s \in (0, \infty)$ ,  $\tau \in \mathbb{R}_+$ ,  $k \in \mathbb{Z}_+$ ,  $n \in \mathbb{Z}_+$  let

$$M_n^{(k)}(s, \tau) := E[e^{-sW_n(\tau)}W_n^k(\tau)] = (-1)^k \frac{\partial^k}{\partial s^k} w_n(s, \tau), \quad (2.8)$$

$$M^{(k)}(s, \tau) := E[e^{-sW(\tau)}W^k(\tau)] = \sum_{n=0}^{\infty} p(n)M_n^{(k)}(s, \tau). \quad (2.9)$$

Note that these expectations are bounded by  $k!s^{-k}$  because of  $e^x \geq x^k/k!$  for  $x \in \mathbb{R}_+$ . For fixed  $s \in (0, \infty)$ ,  $k \in \mathbb{N}$ , (2.1) yields the system of differential equations

$$\begin{aligned} \frac{\partial}{\partial \tau} M_n^{(k)}(s, \tau) &= n\mu M_{(n-1)_+}^{(k)}(s, \tau) \\ &\quad - (n\mu + \lambda\chi(n+1) + s(\chi(n+1)-1))M_n^{(k)}(s, \tau) \\ &\quad + \lambda\chi(n+1)M_{n+1}^{(k)}(s, \tau) + k(\chi(n+1)-1)M_n^{(k-1)}(s, \tau), \\ &\quad n \in \mathbb{Z}_+, \tau \in (0, \infty). \end{aligned} \quad (2.10)$$

Multiplying (2.10) by  $p(n)$  and using (1.3) and (1.2), it follows that (2.10) is equivalent to

$$\begin{aligned} \frac{\partial}{\partial \tau} (p(n)M_n^{(k)}(s, \tau)) &= \lambda\chi(n)p((n-1)_+)M_{(n-1)_+}^{(k)}(s, \tau) \\ &\quad - (n\mu + \lambda\chi(n+1) + s(\chi(n+1)-1))p(n)M_n^{(k)}(s, \tau) \\ &\quad + (n+1)\mu p(n+1)M_{n+1}^{(k)}(s, \tau) + k(\chi(n+1)-1)p(n)M_n^{(k-1)}(s, \tau), \\ &\quad n \in \mathbb{Z}_+, \tau \in (0, \infty). \end{aligned} \quad (2.11)$$

For  $k \in \mathbb{N} \setminus \{1\}$ ,  $\tau \in (0, \infty)$  by means of Hölder's inequality we find

$$\begin{aligned} M_n^{(k-1)}(s, \tau) &= E[e^{-sW_n(\tau)}W_n^{k-1}(\tau)] \\ &\leq (E[e^{-sW_n(\tau)}])^{1/k}(E[e^{-sW_n(\tau)}W_n^k(\tau)])^{(k-1)/k} \\ &\leq (M_n^{(k)}(s, \tau))^{(k-1)/k}. \end{aligned}$$

Using the above inequality, applying Hölder's inequality to the series and taking into account (2.9) provides

$$\begin{aligned} &\sum_{n=0}^{\infty} (\chi(n+1)-1)p(n)M_n^{(k-1)}(s, \tau) \\ &\leq \sum_{n=0}^{\infty} (\chi(n+1)-1)p(n)(M_n^{(k)}(s, \tau))^{(k-1)/k} \\ &\leq \left( \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n) \right)^{1/k} \left( \sum_{n=0}^{\infty} p(n)M_n^{(k)}(s, \tau) \right)^{(k-1)/k} \\ &= \left( \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n) \right)^{1/k} (M^{(k)}(s, \tau))^{(k-1)/k}. \end{aligned} \quad (2.12)$$

Note that (2.12) also holds for  $k = 1$  in view of (2.8). Assume now that

$$\sum_{n=0}^{\infty} (\chi(n+1)-1)p(n) < \infty.$$

Since the sequence  $(M_n^{(k)}(s, \tau))_{n \in \mathbb{Z}_+}$  is bounded, summing up (2.11) over  $n \in \mathbb{Z}_+$  and taking into account (2.9) provides that for  $s \in (0, \infty)$ ,  $k \in \mathbb{N}$

$$\begin{aligned} \frac{\partial}{\partial \tau} M^{(k)}(s, \tau) &= \sum_{n=0}^{\infty} (\chi(n+1)-1)p(n)(kM_n^{(k-1)}(s, \tau) - sM_n^{(k)}(s, \tau)), \\ &\quad \tau \in (0, \infty). \end{aligned} \quad (2.13)$$



For  $k \in \mathbb{N}$ ,  $\tau \in (0, \infty)$  from (2.13) and (2.12) it follows

$$\frac{\partial}{\partial \tau} M^{(k)}(s, \tau) \leq k \left( \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n) \right)^{1/k} (M^{(k)}(s, \tau))^{(k-1)/k},$$

which is equivalent to

$$\frac{\partial}{\partial \tau} (M^{(k)}(s, \tau))^{1/k} \leq \left( \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n) \right)^{1/k}. \quad (2.14)$$

Because of  $M^{(k)}(s, 0) = 0$ , integrating and taking the  $k$ -th power yields that

$$M^{(k)}(s, \tau) \leq \tau^k \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n), \quad k \in \mathbb{N}, \tau \in \mathbb{R}_+. \quad (2.15)$$

Taking into account (2.9), (2.5), the limit  $s \downarrow 0$  provides (2.6). For  $k \in \mathbb{N}$ ,  $\tau \in (0, \infty)$  from (2.12) and (2.15) we find (2.7) after taking the limit  $s \downarrow 0$ .  $\square$

**Theorem 2.1** *The  $k$ -th moments  $M_n^{(k)}(\tau)$  and  $M^{(k)}(\tau)$  are finite if*

$$E[(\chi(N+1)-1)^k] < \infty. \quad (2.16)$$

For  $k \in \mathbb{N}$  it holds

$$M^{(k)}(\tau) \leq \tau^k E[(\chi(N+1)-1)^k], \quad \tau \in \mathbb{R}_+, \quad (2.17)$$

$$\lim_{\tau \downarrow 0} \frac{M^{(k)}(\tau)}{\tau^k} = E[(\chi(N+1)-1)^k]. \quad (2.18)$$

**Proof** Obviously, (2.17) is equivalent to (2.6). Because of (2.17) and (2.5),  $M^{(k)}(\tau)$  and the  $M_n^{(k)}(\tau)$  are finite if (2.16) is fulfilled. For  $s \in (0, \infty)$ ,  $\tau \in (0, \infty)$ ,  $k \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$  from (2.10) it follows

$$\begin{aligned} & \frac{\partial}{\partial \tau} (e^{(n\mu + \lambda\chi(n+1) + s(\chi(n+1)-1))\tau} M_n^{(k)}(s, \tau)) \\ & \geq k(\chi(n+1)-1) e^{(n\mu + \lambda\chi(n+1) + s(\chi(n+1)-1))\tau} M_n^{(k-1)}(s, \tau). \end{aligned}$$

Because of  $M_n^{(k)}(s, 0) = 0$ , integrating and taking the limit  $s \downarrow 0$  yields

$$e^{(n\mu + \lambda\chi(n+1))\tau} M_n^{(k)}(\tau) \geq k(\chi(n+1)-1) \int_0^\tau e^{(n\mu + \lambda\chi(n+1))\xi} M_n^{(k-1)}(\xi) d\xi,$$

and because of  $M_n^{(0)}(\tau) = 1$  for  $n \in \mathbb{Z}_+$ ,  $\tau \in (0, \infty)$ , by induction over  $k \in \mathbb{N}$  we find

$$e^{(n\mu + \lambda\chi(n+1))\tau} M_n^{(k)}(\tau) \geq \tau^k (\chi(n+1) - 1)^k.$$

Multiplying by  $p(n)e^{-(n\mu + \lambda\chi(n+1))\tau}$ , summing up over  $n \in \mathbb{Z}_+$  and taking into account (2.5) implies

$$M^{(k)}(\tau) \geq \tau^k \sum_{n=0}^{\infty} (\chi(n+1) - 1)^k p(n) e^{-(n\mu + \lambda\chi(n+1))\tau}. \quad (2.19)$$

The estimates (2.6) and (2.19) provide (2.18). □

For the variance of  $W(\tau)$  from (2.5), (1.6), (1.5), (2.17) and (2.18) we find

**Corollary 2.1** *Let (2.16) be fulfilled for  $k = 2$ . Then it holds*

$$\text{var}(W(\tau)) \leq \tau^2 \text{var}(\chi(N+1)), \quad \tau \in \mathbb{R}_+, \quad (2.20)$$

$$\lim_{\tau \downarrow 0} \frac{\text{var}(W(\tau))}{\tau^2} = \text{var}(\chi(N+1)). \quad (2.21)$$

Let  $k \in \mathbb{N}$  be fixed such that (2.16) is fulfilled, i.e., we assume that the  $k$ -th moment of  $\chi(N+1) - 1$  is finite. In view of Theorem 2.1, taking the limit  $s \downarrow 0$  in (2.10) provides

$$\begin{aligned} \frac{d}{d\tau} M_n^{(k)}(\tau) &= n\mu M_{(n-1)_+}^{(k)}(\tau) - (n\mu + \lambda\chi(n+1))M_n^{(k)}(\tau) \\ &\quad + \lambda\chi(n+1)M_{n+1}^{(k)}(\tau) + k(\chi(n+1) - 1)M_n^{(k-1)}(\tau), \\ &\quad n \in \mathbb{Z}_+, \tau \in (0, \infty), \end{aligned} \quad (2.22)$$

and because of Lemma 2.2,

$$\begin{aligned} sM_n^{(k)}(s, \tau) &= E[(e^{-sW_n(\tau)} sW_n(\tau))W_n^{k-1}(\tau)] \leq M_n^{(k-1)}(\tau), \\ &\quad n \in \mathbb{Z}_+, \tau \in (0, \infty), \end{aligned}$$

and Lebesgue's theorem, taking the limit  $s \downarrow 0$  in (2.13) yields

$$\frac{d}{d\tau} M^{(k)}(\tau) = k \sum_{n=0}^{\infty} (\chi(n+1) - 1) p(n) M_n^{(k-1)}(\tau), \quad \tau \in (0, \infty). \quad (2.23)$$

Therefore,  $M^{(k)}(\tau)$  is a monotonically increasing and convex function of  $\tau$ . Integrating (2.23) and taking into account  $M^{(k)}(0) = 0$ , it follows

$$M^{(k)}(\tau) = k \sum_{n=0}^{\infty} (\chi(n+1) - 1) p(n) H_n^{(k-1)}(\tau), \quad \tau \in \mathbb{R}_+, \quad (2.24)$$

where

$$H_n^{(\ell)}(\tau) := \int_0^\tau M_n^{(\ell)}(\xi) d\xi, \quad \ell \in \{0, 1, \dots, k\}, \quad n \in \mathbb{Z}_+, \quad \tau \in \mathbb{R}_+, \quad (2.25)$$

and

$$H_n^{(0)}(\tau) = \tau, \quad n \in \mathbb{Z}_+, \quad \tau \in \mathbb{R}_+, \quad (2.26)$$

in view of  $M_n^{(0)}(\tau) = 1$  for  $n \in \mathbb{Z}_+$ ,  $\tau \in (0, \infty)$ . Integrating (2.22), for fixed  $\ell \in \{1, \dots, k\}$  for the integrals  $H_n^{(\ell)}(\tau)$ ,  $n \in \mathbb{Z}_+$ , we obtain the system of ordinary differential equations

$$\begin{aligned} \frac{d}{d\tau} H_n^{(\ell)}(\tau) &= n\mu H_{(n-1)_+}^{(\ell)}(\tau) - (n\mu + \lambda\chi(n+1))H_n^{(\ell)}(\tau) \\ &\quad + \lambda\chi(n+1)H_{n+1}^{(\ell)}(\tau) + \ell(\chi(n+1) - 1)H_n^{(\ell-1)}(\tau), \\ &\quad n \in \mathbb{Z}_+, \quad \tau \in \mathbb{R}_+, \end{aligned} \quad (2.27)$$

with the initial condition

$$H_n^{(\ell)}(0) = 0, \quad n \in \mathbb{Z}_+. \quad (2.28)$$

**Remark 2.1** Let (2.16) be fulfilled for  $k = 1$ . From (2.5), (2.24) for  $k = 1$  and (2.26) it follows

$$EW(\tau) = \tau E[\chi(N+1) - 1], \quad \tau \in \mathbb{R}_+.$$

In view of (1.5), we have a simple proof of (1.6) in case of a  $M/M/GPS$  system.

Consider the case of  $k = 2$  in more detail. For computing the second moment  $M^{(2)}(\tau)$  via (2.24), we need the  $H_n^{(1)}(\tau)$ ,  $n \in \mathbb{Z}_+$ , which are given by (2.26)–(2.28) for  $\ell = 1$ . This system of differential equations can be solved by numerical integration in an efficient manner, cf. Section 4.1. Another approach for computing the second moment  $M^{(2)}(\tau)$  – not outlined in this paper – is based on a numerical computation of its LST in the right half plane and an application of the inversion formula for LST's. This approach would make use of results given in the following, too.

## 2.2 LST's of the moments of $W(\tau)$ and the moments of $W$

Let  $k \in \mathbb{N}$  be fixed such that (2.16) is fulfilled. For complex  $s$  with  $\Re s > 0$  and  $\ell \in \{0, 1, \dots, k\}$  let

$$L_n^{(\ell)}(s) := \int_{\mathbb{R}_+} e^{-s\tau} dM_n^{(\ell)}(\tau), \quad n \in \mathbb{Z}_+, \quad (2.29)$$

$$L^{(\ell)}(s) := \int_{\mathbb{R}_+} e^{-s\tau} dM^{(\ell)}(\tau) = \sum_{n=0}^{\infty} p(n) L_n^{(\ell)}(s), \quad (2.30)$$

be the LST's of  $M_n^{(\ell)}(\tau)$  and  $M^{(\ell)}(\tau)$ , respectively, where

$$L_n^{(0)}(s) = L^{(0)}(s) = 1, \quad n \in \mathbb{Z}_+,$$

in view of  $M_n^{(0)}(\tau) = M^{(0)}(\tau) = \mathbb{I}\{\tau > 0\}$ ,  $n \in \mathbb{Z}_+$ ,  $\tau \in \mathbb{R}_+$ .

**Remark 2.2** *Using integration by parts, for  $\sigma \in (0, \infty)$  from (2.30), (2.17) we find*

$$L^{(k)}(\sigma) = \sigma \int_{\mathbb{R}_+} e^{-\sigma\tau} M^{(k)}(\tau) d\tau \leq k! \sigma^{-k} E[(\chi(N+1) - 1)^k], \quad (2.31)$$

which is tight for  $k = 1$  and always asymptotically tight for  $\sigma \rightarrow \infty$ , cf. Theorem 2.1. Because of (2.31) and (2.16),  $L^{(k)}(\sigma)$  exists for  $\sigma \in (0, \infty)$ . Note that the existence of  $L^{(k)}(\sigma)$  for fixed  $\sigma \in (0, \infty)$  implies the existence of  $L_n^{(k)}(\sigma)$  for  $n \in \mathbb{Z}_+$  in view of (2.30), and hence the existence of the LST's  $L_n^{(k)}(s)$  and  $L^{(k)}(s)$  for complex  $s$  with  $\Re s \geq \sigma$ .

As the service times are exponentially distributed with parameter  $\mu$ , the moments  $M_n^{(\ell)} := EW_n^\ell$  and  $M^{(\ell)} := EW^\ell$  are just given by

$$M_n^{(\ell)} = L_n^{(\ell)}(\mu), \quad \ell \in \{0, 1, \dots, k\}, \quad n \in \mathbb{Z}_+, \quad (2.32)$$

$$M^{(\ell)} = L^{(\ell)}(\mu), \quad \ell \in \{0, 1, \dots, k\}, \quad (2.33)$$

where

$$M_n^{(0)} = M^{(0)} = 1, \quad n \in \mathbb{Z}_+.$$

Multiplying (2.22) by  $se^{-s\tau}$ , integrating over  $\mathbb{R}_+$  with respect to  $\tau$ , using integration by parts and that  $M_n^{(k)}(0) = M_n^{(k-1)}(0) = 0$  for  $n \in \mathbb{Z}_+$ , for fixed  $s$  with  $\Re s > 0$  we obtain the linear system of algebraic equations

$$\begin{aligned} n\mu L_{(n-1)_+}^{(k)}(s) - (n\mu + \lambda\chi(n+1) + s)L_n^{(k)}(s) + \lambda\chi(n+1)L_{n+1}^{(k)}(s) \\ = -k(\chi(n+1) - 1)L_n^{(k-1)}(s), \quad n \in \mathbb{Z}_+. \end{aligned} \quad (2.34)$$

**Remark 2.3** *In view of  $L_n^{(0)}(s) = 1$ ,  $n \in \mathbb{Z}_+$ , for fixed  $s$  with  $\Re s > 0$ , in principle the sequences  $(L_n^{(k)}(s))_{n \in \mathbb{Z}_+}$  can be computed by solving a suitable finite version of the linear system of equations (2.34) iteratively, recursively with respect to  $k$ , cf. (3.11), (3.28), (3.2) in Section 3 for the case of  $k = 1$ .*

Multiplying (2.34) by  $p(n)$  and using (1.3) and (1.2), it follows that (2.34) is equivalent to

$$\begin{aligned} & \lambda \chi(n) p((n-1)_+) L_{(n-1)_+}^{(k)}(s) - (n\mu + \lambda \chi(n+1) + s) p(n) L_n^{(k)}(s) \\ & \qquad \qquad \qquad + (n+1)\mu p(n+1) L_{n+1}^{(k)}(s) \\ & = -k(\chi(n+1) - 1) p(n) L_n^{(k-1)}(s), \quad n \in \mathbb{Z}_+. \end{aligned} \quad (2.35)$$

Multiplying (2.23) by  $se^{-s\tau}$ , integrating over  $\mathbb{R}_+$  with respect to  $\tau$ , using integration by parts and that  $M_n^{(k-1)}(0) = 0$  for  $n \in \mathbb{Z}_+$ , for fixed  $s$  with  $\Re s > 0$  we obtain

$$L^{(k)}(s) = \frac{k}{s} \sum_{n=0}^{\infty} (\chi(n+1) - 1) p(n) L_n^{(k-1)}(s). \quad (2.36)$$

Because of  $L_n^{(0)}(s) = 1$  for  $n \in \mathbb{Z}_+$ , from (2.36) for  $k = 1$  we find

$$L^{(1)}(s) = \frac{1}{s} \sum_{n=0}^{\infty} (\chi(n+1) - 1) p(n), \quad (2.37)$$

which is the LST translation of (1.6), and for  $k = 2$  we obtain

$$L^{(2)}(s) = \frac{2}{s} \sum_{n=0}^{\infty} (\chi(n+1) - 1) p(n) L_n^{(1)}(s). \quad (2.38)$$

For  $s = \mu$  from (2.36), (2.32), (2.33) it follows

$$M^{(k)} = \frac{k}{\mu} \sum_{n=0}^{\infty} (\chi(n+1) - 1) p(n) M_n^{(k-1)}. \quad (2.39)$$

The second moment of the waiting time  $W$  is given by  $L^{(2)}(\mu)$ , cf. (2.33). In Section 3 we present a stable recursive numerical algorithm for computing  $L^{(2)}(s)$  for fixed complex  $s$  with  $\Re s > 0$  via (2.35) for  $k = 1$  and (2.38).

### 3 A recursive algorithm for computing $L^{(2)}(s)$

In this section we derive a stable and fast recursive algorithm for computing  $L^{(2)}(s)$ . We assume that

$$E[(\chi(N+1)-1)^2] < \infty \quad (3.1)$$

is fulfilled. Let  $\Re s > 0$  in the following, and let

$$z_n(s) := p(n)L_n^{(1)}(s), \quad n \in \mathbb{Z}_+. \quad (3.2)$$

From (2.29) for  $\ell = 1$  it follows

$$|z_n(s)| = \left| p(n) \int_{\mathbb{R}_+} e^{-s\tau} dM_n^{(1)}(\tau) \right| \leq p(n) \int_{\mathbb{R}_+} |e^{-s\tau}| dM_n^{(1)}(\tau) \leq z_n(\sigma),$$

$$\Re s \geq \sigma > 0, \quad n \in \mathbb{Z}_+. \quad (3.3)$$

Because of (3.1), (2.30) for  $\ell = 1$  and (2.38), we have

$$L^{(1)}(s) = \sum_{n=0}^{\infty} z_n(s), \quad (3.4)$$

$$L^{(2)}(s) = \frac{2}{s} \sum_{n=0}^{\infty} (\chi(n+1)-1)z_n(s). \quad (3.5)$$

The representations (3.4) and (3.5) imply that

$$\lim_{n \rightarrow \infty} \chi(n+1)z_n(s) = 0. \quad (3.6)$$

In view of (2.35) for  $k = 1$  and  $L_n^{(0)}(s) = 1$ , the  $z_n(s)$ ,  $n \in \mathbb{Z}_+$ , satisfy the recursion

$$\begin{aligned} & \lambda\chi(n)z_{(n-1)_+}(s) - (n\mu + \lambda\chi(n+1) + s)z_n(s) + (n+1)\mu z_{n+1}(s) \\ & = -(\chi(n+1)-1)p(n), \quad n \in \mathbb{Z}_+, \end{aligned} \quad (3.7)$$

and because of (2.34) for  $k = 1$  and  $L_n^{(0)}(s) = 1$ , for fixed  $\ell \in \mathbb{N}$  we have the linear system of equations

$$\begin{aligned} \frac{z_n(s)}{p(n)} &= (n\mu + \lambda\chi(n+1) + s)^{-1} \\ & \left( n\mu \frac{z_{(n-1)_+}(s)}{p((n-1)_+)} + \lambda\chi(n+1) \frac{z_{n+1}(s)}{p(n+1)} + (\chi(n+1)-1) \right), \\ & n = 0, \dots, \ell-1, \end{aligned} \quad (3.8)$$

for  $(z_n(s)/p(n))_{n \in \{0, \dots, \ell-1\}}$  if  $z_\ell(s)/p(\ell)$  is given. For fixed  $\ell \in \mathbb{Z}_+$  let  $(a_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  the solution of the homogeneous version of the recursion (3.7)

$$\lambda\chi(n)\tilde{z}_{(n-1)_+}(s) - (n\mu + \lambda\chi(n+1) + s)\tilde{z}_n(s) + (n+1)\mu\tilde{z}_{n+1}(s) = 0, \quad n \in \mathbb{Z}_+, \quad (3.9)$$

with the inhomogeneous side condition  $a_\ell^{(\ell)}(s) := 1$ , and for fixed  $\ell \in \mathbb{Z}_+$  let  $(b_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  the solution of the inhomogeneous recursion (3.7) with the homogeneous side condition  $b_\ell^{(\ell)}(s) := 0$ . Dividing (3.9) by  $p(n)$  and taking into account (1.3), for fixed  $\ell \in \mathbb{N}$  it follows that  $(a_n^{(\ell)}(s)/p(n))_{n \in \{0, \dots, \ell-1\}}$  is given by the uniquely determined solution of the linear system of equations

$$\begin{aligned} \frac{a_n^{(\ell)}(s)}{p(n)} &= (n\mu + \lambda\chi(n+1) + s)^{-1} \\ &\left( n\mu \frac{a_{(n-1)_+}^{(\ell)}(s)}{p((n-1)_+)} + \lambda\chi(n+1) \frac{a_{n+1}^{(\ell)}(s)}{p(n+1)} \right), \\ &n = 0, \dots, \ell-1, \end{aligned} \quad (3.10)$$

where  $a_\ell^{(\ell)}(s)/p(\ell) = 1/p(\ell)$ . Analogously, dividing (3.7) by  $p(n)$  and taking into account (1.3), for fixed  $\ell \in \mathbb{N}$  it follows that  $(b_n^{(\ell)}(s)/p(n))_{n \in \{0, \dots, \ell-1\}}$  is given by the uniquely determined solution of the linear system of equations

$$\begin{aligned} \frac{b_n^{(\ell)}(s)}{p(n)} &= (n\mu + \lambda\chi(n+1) + s)^{-1} \\ &\left( n\mu \frac{b_{(n-1)_+}^{(\ell)}(s)}{p((n-1)_+)} + \lambda\chi(n+1) \frac{b_{n+1}^{(\ell)}(s)}{p(n+1)} + (\chi(n+1) - 1) \right), \\ &n = 0, \dots, \ell-1, \end{aligned} \quad (3.11)$$

where  $b_\ell^{(\ell)}(s)/p(\ell) = 0$ , cf. (3.8). Moreover, from (3.8) and (3.11) we obtain

$$\begin{aligned} \frac{z_n(s) - b_n^{(\ell)}(s)}{p(n)} &= (n\mu + \lambda\chi(n+1) + s)^{-1} \\ &\left( n\mu \frac{z_{(n-1)_+}(s) - b_{(n-1)_+}^{(\ell)}(s)}{p((n-1)_+)} + \lambda\chi(n+1) \frac{z_{n+1}(s) - b_{n+1}^{(\ell)}(s)}{p(n+1)} \right), \\ &n = 0, \dots, \ell-1, \end{aligned} \quad (3.12)$$

where  $(z_\ell(s) - b_\ell^{(\ell)}(s))/p(\ell) = z_\ell(s)/p(\ell)$ . The linear systems of equations (3.10)–(3.12) can be solved iteratively due to the row sum criterion. In view

of  $a_\ell^{(\ell)}(s) = a_\ell^{(\ell)}(\sigma) = 1$ ,  $b_\ell^{(\ell)}(s) = b_\ell^{(\ell)}(\sigma) = 0$  and (3.3) for  $n = \ell$ , by starting the iterations from zero, by induction over the iteration steps and taking the limit thus it follows that for  $\Re s \geq \sigma > 0$  and  $n \in \{0, \dots, \ell\}$  it holds

$$|a_n^{(\ell)}(s)| \leq a_n^{(\ell)}(\sigma), \quad (3.13)$$

$$|b_n^{(\ell)}(s)| \leq b_n^{(\ell)}(\sigma), \quad (3.14)$$

$$|z_n(s) - b_n^{(\ell)}(s)| \leq z_n(\sigma) - b_n^{(\ell)}(\sigma). \quad (3.15)$$

Assume that there exists  $j \in \mathbb{Z}_+$  such that  $a_j^{(\ell)}(s) = 0$ . Then the sequence  $(a_n^{(j)}(s) + a_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  satisfies (3.9) and the side condition of the sequence  $(a_n^{(j)}(s))_{n \in \mathbb{Z}_+}$ , and thus it follows  $a_n^{(j)}(s) + a_n^{(\ell)}(s) = a_n^{(j)}(s)$ ,  $n \in \mathbb{Z}_+$ , in contradiction to  $a_\ell^{(\ell)}(s) = 1$ . Hence it holds

$$a_n^{(\ell)}(s) \neq 0, \quad n \in \mathbb{Z}_+. \quad (3.16)$$

Since the sequences  $(a_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  and  $(b_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  are uniquely determined, further we obtain

$$a_n^{(\ell+1)}(s) = \frac{a_n^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)}, \quad n \in \mathbb{Z}_+, \quad (3.17)$$

as the r.h.s. satisfies (3.9) and takes the value 1 for  $n = \ell + 1$ , as well as

$$b_n^{(\ell+1)}(s) = b_n^{(\ell)}(s) - \frac{b_{\ell+1}^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)} a_n^{(\ell)}(s), \quad n \in \mathbb{Z}_+, \quad (3.18)$$

as the r.h.s. satisfies (3.7) and takes the value 0 for  $n = \ell + 1$ .

Summation of (3.7) over  $n = 0, \dots, j$  and interchanging  $n$  and  $j$ , respectively, yield that the  $z_n(s)$ ,  $n \in \mathbb{Z}_+$ , satisfy also the recursion

$$\begin{aligned} z_{n+1}(s) &= \frac{\lambda \chi(n+1)}{(n+1)\mu} z_n(s) + \frac{s}{(n+1)\mu} \sum_{j=0}^n z_j(s) \\ &\quad - \frac{1}{(n+1)\mu} \sum_{j=0}^n (\chi(j+1) - 1) p(j), \quad n \in \mathbb{Z}_+. \end{aligned} \quad (3.19)$$

Multiplying (3.19) by  $(n+1)$  and taking the limit  $n \rightarrow \infty$ , because of (3.6), (3.4) and (2.37) we find the boundary condition

$$\lim_{n \rightarrow \infty} n z_n(s) = 0. \quad (3.20)$$



Analogously, for fixed  $\ell \in \mathbb{Z}_+$  the sequence  $(a_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  satisfies the homogeneous recursion

$$a_{n+1}^{(\ell)}(s) = \frac{\lambda\chi(n+1)}{(n+1)\mu} a_n^{(\ell)}(s) + \frac{s}{(n+1)\mu} \sum_{j=0}^n a_j^{(\ell)}(s), \quad n \in \mathbb{Z}_+, \quad (3.21)$$

and the sequence  $(b_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  satisfies the inhomogeneous recursion

$$\begin{aligned} b_{n+1}^{(\ell)}(s) &= \frac{\lambda\chi(n+1)}{(n+1)\mu} b_n^{(\ell)}(s) + \frac{s}{(n+1)\mu} \sum_{j=0}^n b_j^{(\ell)}(s) \\ &\quad - \frac{1}{(n+1)\mu} \sum_{j=0}^n (\chi(j+1)-1)p(j), \quad n \in \mathbb{Z}_+. \end{aligned} \quad (3.22)$$

**Theorem 3.1** *For  $\ell \in \mathbb{Z}_+$  let  $(b_n^{(\ell)}(s))_{n \in \mathbb{Z}_+}$  the uniquely determined sequence satisfying (3.22) and the side condition  $b_\ell^{(\ell)}(s) = 0$ . Then for  $\Re s > 0$  it holds*

$$L^{(1)}(s) = \lim_{\ell \rightarrow \infty} \sum_{n=0}^{\ell} b_n^{(\ell)}(s), \quad (3.23)$$

$$L^{(2)}(s) = \lim_{\ell \rightarrow \infty} \frac{2}{s} \sum_{n=0}^{\ell} (\chi(n+1)-1)b_n^{(\ell)}(s). \quad (3.24)$$

**Proof** Let  $\sigma \in (0, \infty)$  be fixed. Because of  $a_0^{(0)}(\sigma) = 1$ , from (3.21) for  $\ell = 0$  and  $s = \sigma$  by induction we find

$$a_n^{(0)}(\sigma) > \frac{\sigma}{n\mu}, \quad n \in \mathbb{N}. \quad (3.25)$$

Let  $\ell \in \mathbb{N}$  be given. As  $(z_n(\sigma))_{n \in \mathbb{Z}_+}$  and  $(b_n^{(\ell)}(\sigma))_{n \in \mathbb{Z}_+}$  satisfy (3.22) for  $s = \sigma$ , cf. (3.19), and  $(a_n^{(\ell)}(\sigma))_{n \in \mathbb{Z}_+}$  satisfies the corresponding homogeneous recursion (3.21) for  $s = \sigma$ , the sequence

$$(z_n(\sigma) - b_n^{(\ell)}(\sigma) - (z_\ell(\sigma)/a_\ell^{(0)}(\sigma))a_n^{(0)}(\sigma))_{n \in \mathbb{Z}_+}$$

satisfies the homogeneous recursion (3.21) for  $s = \sigma$ , too. All terms of this sequence are zero as obviously the term with index  $n = \ell$  is zero and the sequence  $(a_n^{(\ell)}(\sigma))_{n \in \mathbb{Z}_+}$  is uniquely determined by the homogeneous recursion (3.21) for  $s = \sigma$  and the side condition  $a_\ell^{(\ell)}(\sigma) = 1$ . Thus it holds

$$z_n(\sigma) - b_n^{(\ell)}(\sigma) = (z_\ell(\sigma)/a_\ell^{(0)}(\sigma))a_n^{(0)}(\sigma), \quad n \in \mathbb{Z}_+. \quad (3.26)$$

Hence from (3.15) and (3.25) for  $n = \ell$ , for  $\Re s \geq \sigma > 0$  we obtain

$$|z_n(s) - b_n^{(\ell)}(s)| \leq z_n(\sigma) - b_n^{(\ell)}(\sigma) < \frac{\mu}{\sigma} a_n^{(0)}(\sigma) \ell z_\ell(\sigma), \quad n \in \mathbb{Z}_+, \ell \in \mathbb{N}. \quad (3.27)$$

Because of (3.20), therefore we conclude that for fixed  $n \in \mathbb{Z}_+$

$$z_n(s) = \lim_{\ell \rightarrow \infty} b_n^{(\ell)}(s) \quad (3.28)$$

locally uniformly for  $\Re s > 0$ . For fixed  $n \in \mathbb{Z}_+$  and  $\Re s \geq \sigma > 0$  thus the sequence  $(\mathbb{I}\{n \leq \ell\} b_n^{(\ell)}(s))_{\ell \in \mathbb{Z}_+}$  converges to  $z_n(s)$  and is bounded by  $z_n(\sigma)$  because of (3.14), (3.15). Hence (3.4) and (3.5) imply (3.23) and (3.24), respectively, due to Lebesgue's theorem. □

Now we are in a position to present a stable recursive algorithm for computing  $L^{(2)}(s)$ . For fixed complex  $s$  with  $\Re s > 0$  and  $\ell \in \mathbb{Z}_+$  let

$$x_1^{(\ell)} := \sum_{n=0}^{\ell} s b_n^{(\ell)}(s), \quad (3.29)$$

$$x_2^{(\ell)} := \sum_{n=0}^{\ell} (\chi(n+1) - 1) b_n^{(\ell)}(s), \quad (3.30)$$

$$x_3^{(\ell)} := \sum_{n=0}^{\ell} (\chi(n+1) - 1) p(n), \quad (3.31)$$

$$x_4^{(\ell)} := \sum_{n=0}^{\ell} s a_n^{(\ell)}(s), \quad (3.32)$$

$$x_5^{(\ell)} := \sum_{n=0}^{\ell} (\chi(n+1) - 1) a_n^{(\ell)}(s). \quad (3.33)$$

In view of  $a_0^{(0)}(s) = 1$  and  $b_0^{(0)}(s) = 0$ , we have the initial values

$$\begin{aligned} x_1^{(0)} &= 0, & x_2^{(0)} &= 0, & x_3^{(0)} &= (\chi(1) - 1)p(0), & x_4^{(0)} &= s, \\ x_5^{(0)} &= \chi(1) - 1. \end{aligned} \quad (3.34)$$

The recursion runs as follows: Let we are given  $x_1^{(\ell)}, \dots, x_5^{(\ell)}$  for a fixed  $\ell \in \mathbb{Z}_+$ . Because of  $a_\ell^{(\ell)}(s) = 1$ ,  $b_\ell^{(\ell)}(s) = 0$ , (3.21) and (3.22), then we have

$$a_{\ell+1}^{(\ell)}(s) = \frac{1}{(\ell+1)\mu} (\lambda\chi(\ell+1) + x_4^{(\ell)}), \quad (3.35)$$

$$b_{\ell+1}^{(\ell)}(s) = \frac{1}{(\ell+1)\mu} (x_1^{(\ell)} - x_3^{(\ell)}), \quad (3.36)$$

and because of  $a_{\ell+1}^{(\ell+1)}(s) = 1$ ,  $b_{\ell+1}^{(\ell+1)}(s) = 0$ , (3.17) and (3.18), we obtain

$$x_1^{(\ell+1)} = \sum_{n=0}^{\ell} s \left( b_n^{(\ell)}(s) - \frac{b_{\ell+1}^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)} a_n^{(\ell)}(s) \right) = x_1^{(\ell)} - \frac{b_{\ell+1}^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)} x_4^{(\ell)}, \quad (3.37)$$

$$\begin{aligned} x_2^{(\ell+1)} &= \sum_{n=0}^{\ell} (\chi(n+1) - 1) \left( b_n^{(\ell)}(s) - \frac{b_{\ell+1}^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)} a_n^{(\ell)}(s) \right) \\ &= x_2^{(\ell)} - \frac{b_{\ell+1}^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)} x_5^{(\ell)}, \end{aligned} \quad (3.38)$$

$$x_3^{(\ell+1)} = x_3^{(\ell)} + (\chi(\ell+2) - 1)p(\ell+1), \quad (3.39)$$

$$x_4^{(\ell+1)} = \sum_{n=0}^{\ell} s \frac{a_n^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)} + s = \frac{1}{a_{\ell+1}^{(\ell)}(s)} x_4^{(\ell)} + s, \quad (3.40)$$

$$\begin{aligned} x_5^{(\ell+1)} &= \sum_{n=0}^{\ell} (\chi(n+1) - 1) \frac{a_n^{(\ell)}(s)}{a_{\ell+1}^{(\ell)}(s)} + (\chi(\ell+2) - 1) \\ &= \frac{1}{a_{\ell+1}^{(\ell)}(s)} x_5^{(\ell)} + (\chi(\ell+2) - 1). \end{aligned} \quad (3.41)$$

Theorem 3.1 now reads

$$L^{(1)}(s) = \frac{1}{s} \lim_{\ell \rightarrow \infty} x_1^{(\ell)}, \quad L^{(2)}(s) = \frac{2}{s} \lim_{\ell \rightarrow \infty} x_2^{(\ell)}. \quad (3.42)$$

Moreover, (2.37) implies

$$L^{(1)}(s) = \frac{1}{s} \lim_{\ell \rightarrow \infty} x_3^{(\ell)}, \quad (3.43)$$

which can be used for controlling the numerical accuracy.

**Remark 3.1** Note that the recursive computation of  $x_2^{(\ell)}$  is of complexity  $O(\ell)$ . Our non-linear recursion avoids the numerical instability of a direct recursive computation of  $(b_n^{(\ell)}(s))_{n \in \{0, \dots, \ell-1\}}$  via (3.10) and (3.11), cf. [Br1] p. 69 for the case of a  $M/M/m - PS$  system and  $s = \mu$ , as well as the solution of the  $\ell$ -dimensional linear system of equations (3.11) by iteration, being of much higher complexity.

## 4 Mean and variance of the sojourn times

We assume that

$$E[(\chi(N+1)-1)^2] < \infty \quad (4.1)$$

is fulfilled. Because of

$$V = W + S, \quad V(\tau) = W(\tau) + \tau, \quad (4.2)$$

the means  $EV$  and  $EW$  of the sojourn time  $V$  and waiting time  $W$  are related by  $EV = EW + ES$ , and the conditional means  $EV(\tau)$  and  $EW(\tau)$  by  $EV(\tau) = EW(\tau) + \tau$ . In view of (4.2), for the conditional variances obviously it holds

$$\text{var}(V(\tau)) = \text{var}(W(\tau)). \quad (4.3)$$

Because of (1.6) and (4.3), for the second moments  $EV^2$ ,  $EW^2$  we obtain

$$\begin{aligned} EV^2 - \frac{ES^2}{(ES)^2} (EV)^2 &= \int_{\mathbb{R}_+} EV^2(\tau) dB(\tau) - \int_{\mathbb{R}_+} \left( \frac{\tau}{ES} EV \right)^2 dB(\tau) \\ &= \int_{\mathbb{R}_+} (EV^2(\tau) - (EV(\tau))^2) dB(\tau) = \int_{\mathbb{R}_+} \text{var}(V(\tau)) dB(\tau) \\ &= \int_{\mathbb{R}_+} \text{var}(W(\tau)) dB(\tau) = EW^2 - \frac{ES^2}{(ES)^2} (EW)^2. \end{aligned} \quad (4.4)$$

In view of  $ES^2 = 2(ES)^2$ , from (4.4) and (2.20) for the variances it follows

$$0 \leq \text{var}(V) - (EV)^2 = \text{var}(W) - (EW)^2 \leq 2(ES)^2 \text{var}(\chi(N+1)). \quad (4.5)$$

**Remark 4.1** (i) Note that in (4.5) the sojourn and waiting time occur symmetrically. This symmetry reflects the fact that statements for the sojourn times hold, which are analogous to the results for the waiting times given in the preceding sections and which can be derived analogously, too.

(ii) Note that (4.5) implies

$$\text{var}(V) = \text{var}(W) + 2EWES + \text{var}(S). \quad (4.6)$$

The quantity  $2EWES$  is a measure for the dependence of waiting and service times under generalized processor sharing. Note that they are independent in FCFS systems, which yields  $\text{var}(V) = \text{var}(W) + \text{var}(S)$  there.

#### 4.1 Numerical results for the sojourn times in $M/M/m - PS$

Table 4.1 and Figure 4.1 have been drawn up using the derived algorithms. The variance  $\text{var}(W(\tau))$  has been computed by solving an appropriate finite version of the system of differential equations (2.26)–(2.28) for  $\ell = 1$  numerically via Euler integration and using a convergence acceleration procedure. The algorithm for  $\text{var}(W(\tau))$  is numerically stable up to  $\rho/m \approx 0.99$  if  $\mu\tau$  is not too large, the algorithm for  $\text{var}(W)$  given in Section 3 is numerically stable up to  $\rho/m \approx 0.999$  and very fast.

Having in mind a possible time scaling, the mean service time is set to  $ES = 1$  without loss of generality. In Table 4.1 there are given the mean  $EV$  and variance  $\text{var}(V)$  of the sojourn time  $V$  as well as the variance  $\text{var}(V(\tau))$  of the conditional sojourn time  $V(\tau)$  of a request with required service time  $\tau = 0.5, 1, 2, 4, 8$  in the  $M/M/m - PS$  system. Remember that for the mean  $EV(\tau)$  of the conditional sojourn time  $V(\tau)$  it holds (1.6). For fixed  $\rho/m < 1$ , the mean  $EV$  and variance  $\text{var}(V)$  of the sojourn time  $V$  as well as the mean  $EV(\tau)$  and the variance  $\text{var}(V(\tau))$  of the conditional sojourn time  $V(\tau)$  for fixed  $\tau$  seem to be decreasing with respect to the number  $m$  of processors (economy of scale). The squared coefficient of variation  $c_V^2 := \text{var}(V)/(EV)^2$  of the sojourn time  $V$  seems to be increasing with respect to the offered load  $\rho$ , but  $c_V^2$  seems to be decreasing with respect to the number  $m$  of processors for fixed  $\rho/m < 1$ .

In Figure 4.1 there is given the variance  $\text{var}(V(\tau))$  of the conditional sojourn time  $V(\tau)$  in the  $M/M/m - PS$  system as a function of the required service time  $\tau$ . It can be seen that  $\text{var}(V(\tau))$  is well approximated by an affine function for large values of  $\tau$ . However, due to (4.3) and Corollary 2.1,  $\text{var}(V(\tau))$  is well approximated by  $\tau^2 \text{var}(\chi(N+1))$  for small values of  $\tau$  and always bounded by  $\tau^2 \text{var}(\chi(N+1))$ .

Table 4.1: *The mean EV and variance  $var(V)$  of the sojourn time  $V$  and the variance  $var(V(\tau))$  of the conditional sojourn time  $V(\tau)$  in  $M/M/m - PS$  in case of  $ES = 1$ . Remember that  $EV(\tau) = (\tau/ES) EV$ .*

$\rho/m$	$m$	$EV$	$var(V)$	$var(V(\tau))$				
				$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 4.0$	$\tau = 8.0$
0.30	4	1.0132	1.0315	0.0012	0.0038	0.0110	0.0279	0.0628
0.35	4	1.0232	1.0571	0.0023	0.0077	0.0231	0.0598	0.1370
0.40	4	1.0378	1.0967	0.0043	0.0145	0.0445	0.1184	0.2763
0.45	4	1.0584	1.1562	0.0074	0.0258	0.0811	0.2217	0.5279
0.50	4	1.0870	1.2446	0.0125	0.0441	0.1419	0.3994	0.9727
0.55	4	1.1260	1.3761	0.0205	0.0735	0.2421	0.7020	1.7550
0.60	4	1.1794	1.5742	0.0332	0.1208	0.4072	1.2188	3.1419
0.65	4	1.2532	1.8808	0.0537	0.1983	0.6843	2.1169	5.6552
0.70	4	1.3572	2.3754	0.0881	0.3297	1.1645	3.7299	10.3852
0.75	4	1.5094	3.2253	0.1493	0.5660	2.0444	6.7894	19.8237
0.80	4	1.7455	4.8327	0.2686	1.0306	3.8065	13.1226	40.4322
0.85	4	2.1489	8.3749	0.5388	2.0918	7.8948	28.2732	92.4788
0.90	4	2.9694	18.6358	1.3429	5.2704	20.3086	75.5606	263.7134
0.95	4	5.4571	74.5394	5.8448	23.1705	91.0474	351.6349	1313.7818
0.30	8	1.0006	1.0013	0.0000	0.0001	0.0002	0.0004	0.0009
0.35	8	1.0017	1.0037	0.0001	0.0002	0.0006	0.0015	0.0032
0.40	8	1.0039	1.0085	0.0002	0.0006	0.0018	0.0044	0.0097
0.45	8	1.0079	1.0178	0.0005	0.0016	0.0046	0.0116	0.0260
0.50	8	1.0148	1.0345	0.0011	0.0036	0.0107	0.0278	0.0638
0.55	8	1.0260	1.0629	0.0022	0.0075	0.0232	0.0625	0.1469
0.60	8	1.0436	1.1104	0.0043	0.0150	0.0480	0.1340	0.3245
0.65	8	1.0708	1.1896	0.0081	0.0291	0.0960	0.2793	0.7006
0.70	8	1.1128	1.3238	0.0152	0.0557	0.1896	0.5756	1.5079
0.75	8	1.1785	1.5612	0.0288	0.1074	0.3771	1.1985	3.3089
0.80	8	1.2860	2.0150	0.0567	0.2151	0.7785	2.5954	7.6272
0.85	8	1.4771	3.0096	0.1222	0.4709	1.7541	6.1412	19.3919
0.90	8	1.8769	5.8330	0.3209	1.2549	4.8031	17.6519	60.3355
0.95	8	3.1104	20.6793	1.4440	5.7183	22.4245	86.2739	320.0697
0.30	16	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.35	16	1.0000	1.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.40	16	1.0001	1.0002	0.0000	0.0000	0.0000	0.0000	0.0001
0.45	16	1.0004	1.0008	0.0000	0.0000	0.0001	0.0002	0.0004
0.50	16	1.0011	1.0024	0.0000	0.0001	0.0003	0.0007	0.0016
0.55	16	1.0029	1.0062	0.0001	0.0004	0.0010	0.0025	0.0055
0.60	16	1.0065	1.0145	0.0003	0.0010	0.0030	0.0078	0.0176
0.65	16	1.0137	1.0313	0.0008	0.0027	0.0083	0.0223	0.0521
0.70	16	1.0270	1.0643	0.0019	0.0067	0.0215	0.0604	0.1471
0.75	16	1.0511	1.1290	0.0044	0.0160	0.0536	0.1593	0.4084
0.80	16	1.0953	1.2605	0.0103	0.0382	0.1334	0.4214	1.1563
0.85	16	1.1805	1.5564	0.0253	0.0961	0.3497	1.1782	3.5181
0.90	16	1.3696	2.3888	0.0733	0.2847	1.0761	3.8680	12.7512
0.95	16	1.9752	6.5602	0.3522	1.3917	5.4360	20.7575	75.9753

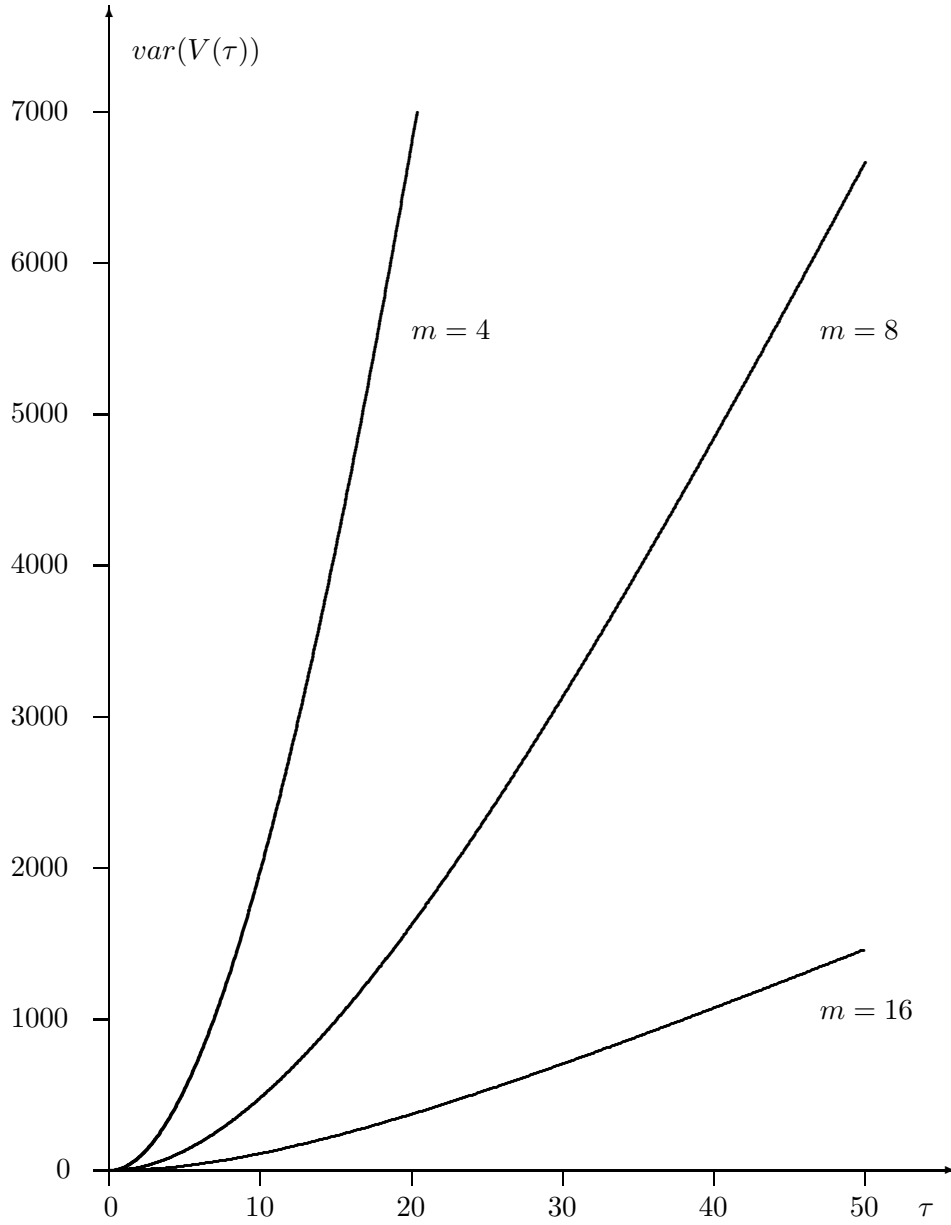


Figure 4.1: The variance  $\text{var}(V(\tau))$  of the conditional sojourn time  $V(\tau)$  of a request with required service time  $\tau$  in the  $M/M/m - PS$  system in case of  $\rho/m = 0.95$  and  $ES = 1$ .

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