

TOBIAS HARKS¹

On the Efficiency of Equilibria in Sequential Nonatomic Routing Games

¹ This work has been supported by the German research funding agency 'Deutsche Forschungsgemeinschaft' under the graduate program 'Graduiertenkolleg 621 (MAGSI/Berlin)'

On the Efficiency of Equilibria in Sequential Nonatomic Routing Games

Tobias Harks*

Konrad-Zuse-Zentrum für Informationstechnik Berlin,
Takustr. 7, 14195 Berlin, Germany.
{harks}@zib.de

Abstract. In this paper, we study the efficiency of Nash equilibria for a sequence of nonatomic routing games. We assume that the games are played consecutively in time in an online fashion: by the time of playing game i , future games $i+1, \dots, n$ are not known, and, once players of game i are in equilibrium, their corresponding strategies and costs remain fixed. Given a sequence of games, the cost for the sequence of Nash equilibria is defined as the sum of the cost of each game. We analyze the efficiency of a sequence of Nash equilibria in terms of competitive analysis arising in the online optimization field. Our main result states that the online algorithm SEQNASH consisting of the sequence of Nash equilibria is $\frac{4n}{2+n}$ -competitive for affine linear latency functions. For $n = 1$, this result contains the bound on the price of anarchy of $\frac{4}{3}$ for affine linear latency functions of Roughgarden and Tardos [1] as a special case. Furthermore, we analyze a problem variant with a modified cost function that reflects the total congestion cost, when all games have been played. In this case, we prove an upper bound of $\frac{4n}{2+n}$ on the competitive ratio of SEQNASH. We further prove a lower bound of $\frac{3n-2}{n}$ of SEQNASH showing that for $n = 2$ our upper bound is tight.

1 Introduction

Future telecommunication networks are expected to support applications with various Quality of Service requirements. The timely delivery of digital audio-visual information for example becomes a major challenge in designing future networks. One of the key issues to accommodate these QoS requirements in the Internet is QoS routing, see Chen and Nahrstedt [2] and references therein for a survey on this rich topic. The challenge in QoS routing is to route the data along paths that meet the QoS requirements of the corresponding application. Most of the work in this field assumes a source routing model, where routing decisions are made at the source of a communication request. The links in the network only advertise their current status (price) that is based on the current congestion situation. Knowing these prices, sources calculate the optimal paths

* Supported by the German research funding agency 'DFG' under the graduate program 'Graduiertenkolleg 621 (MAGSI/Berlin)'

to route their demand. If the link prices correspond to the expected delay on that link, minimum cost routing is a natural goal for time critical real-time applications.

Minimum cost routing also arises in an inter domain QoS market, where multiple service providers offer network resources (capacity) to enable Internet traffic with specific QoS constraints, cf. Yahaya and Suda [3, 4] and Yahaya, Harks, and Suda [5]. Presently, any Internet service provider can offer QoS traffic such as Internet Telephony (VoIP) or video on demand within the domains that the provider controls. Since no single service provider controls all domains in the Internet, deploying end-to-end inter domain QoS traffic requires trading and negotiating for resources between different service providers. In such a market, each service provider advertises prices for resources that he wants to sell. Buying providers choose the cheapest available paths to route demand (coming from own customers) from source to destination.

Recently, several works drew connections between the source routing model and selfish routing models coming from non-cooperative game theory, see among others Roughgarden [6] and Altman, Basar, Jimenez, and Shimkin [7], and references therein. The main focus of this line of research is to quantify the efficiency loss of Nash equilibria compared to the system optimum. Here, two assumptions are crucial: if the traffic matrix changes, (i) all sources may possibly change their routes and form a new equilibrium (ii) the cost of routing demand in the new equilibrium may change. These two assumptions, however, do not hold in the afore mentioned scenarios.

In the source routing model, the routing decision is usually made at the beginning of connection setup. A rerouting attempt during transmission would result in a transient transmission stop and would possibly lower the data rate due to the restart of the transmission control protocol (TCP)[8]. This makes assumption (i) unrealistic for the source routing model even if the traffic matrix changes.

In the inter-provider QoS model both assumptions do not hold. First, buying providers reserve QoS capacity along routes for their communication request. This involves a binding contract between selling providers and the buying provider. Since the risk of congestion externalities is covered by the selling providers, rerouting is dispensable for the buying provider. Secondly, once such a reservation request has been established, the cost for this reservation remains unchanged.

The purpose of this paper is to introduce a new model that does not rely on the afore mentioned two assumptions. It turns out, that a combination of the online optimization field with algorithmic game theory provides a fruitful way to analyze the efficiency of routing strategies in this new model.

1.1 Contribution

In this paper, we introduce the concept of *sequential routing games*. In this concept, we assume a sequence of (nonatomic) routing games $\sigma = (1 \dots n)$ that are played consecutively in time in an online fashion. By the time of playing game

i , future games $i + 1, \dots, n$ are not known. We further assume that once players of game i are in equilibrium, their corresponding strategies remain fixed, that is, the strategies are irrevocable. We analyze the efficiency of a sequence of Nash equilibria in terms of competitive analysis coming from the online optimization field. An online algorithm ALG is called *c-competitive* if the cost of ALG is never larger than c times the cost of an optimal offline solution. The optimal offline solution in our model is derived by minimizing the total routing cost for all games. Note that for deriving the optimal offline solution, the sequence σ is known a priori.

Our main result states that the online algorithm SEQNASH consisting of the sequence of Nash equilibria is $\frac{4n}{2+n}$ -competitive for affine linear latency functions. This result contains the bound on the price of anarchy of $\frac{4}{3}$ for affine linear latency functions of Roughgarden and Tardos [1] as a special case of our model, where $n = 1$. Furthermore, we investigate a problem variant with a different cost function that captures the total congestion cost provided all games are played. We prove an upper bound of $\frac{4n}{2+n}$ on the competitive ratio of SEQNASH and prove a lower bound of $\frac{3n-2}{n}$ of SEQNASH showing that for $n = 2$, the upper bound is tight.

1.2 Related Work

In the last years there has been an exciting development in algorithmic game theory trying to quantify the efficiency loss of Nash equilibria (user equilibria) in non-cooperative games. The fact that there exists an efficiency loss of the user equilibrium compared to a system optimum is well known in the transportation literature, see Braess [9] and Dubey [10]. A first attempt to exactly quantify the so called “price of anarchy” is given by Papadimitriou and Koutsoupias [11] in the context of a load balancing game in communication networks. Subsequently, Roughgarden and Tardos [1] applied this approach to quantify the price of anarchy in nonatomic selfish routing games. In nonatomic games, a large number of players is assumed, each consuming an infinitesimal part of the resources. In particular, Roughgarden and Tardos [1] proved for a set of separable affine cost functions a bound of $\frac{4}{3}$ on the price of anarchy. A series of several other papers analyzed the price of anarchy for more general cost functions and model features; see for example Correa Schulz, and Stier-Moses [12], Jahn, Möhring, Schulz and Stier-Mose [13], Perakis [14], and Roughgarden [6].

In the online routing field, mainly call admission control problems have been considered. An overview article about these problems is given by Leonardi in [15]. In the paper by Awerbuch, Azar, and Plotkin [16], online routing algorithms are presented to maximize throughput under the assumption that routings are irrevocable. They, however, restrict the analysis to single path routing and present competitive bounds that depend on the number of nodes in the network. Our work is motivated by the paper by Harks, Heinz, and Pfetsch [17], where online multicommodity routing problems are considered. They present a greedy online algorithm for a different convex cost function that is $\frac{4K}{2+K}$ competitive, where

K is the number of commodities. In their framework, only single demands are released consecutively.

2 Problem Description

An instance of the *Online Sequential Routing Game* (ONLINESRG) consists of a directed network $D = (V, A)$ and nondecreasing continuous price or latency functions $p_a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for each link $a \in A$. Furthermore, a sequence $\sigma = 1, \dots, n$ of routing games are given. We denote for each game i the set of different types of players by $[K_i] = \{(i, 1), \dots, (i, n_i)\}$. For each $(i, j) \in [K_i]$, a flow of rate $d_{(i,j)} > 0$ must be routed from the origin $s_{(i,j)}$ to the destination $t_{(i,j)}$. We assume infinitely many agents carrying the flow, where each agent controls only an infinitesimal fraction of the flow. For ease of notation, we express in the following the strategy of player (i, j) in terms of a routing assignment. A routing assignment, or *flow*, for player $(i, j) \in [K_i]$ is a nonnegative vector $\mathbf{f}^{(i,j)} \in \mathbb{R}_+^A$. This flow is *feasible* if for all $v \in V$

$$\sum_{a \in \delta^+(v)} f_a^{(i,j)} - \sum_{a \in \delta^-(v)} f_a^{(i,j)} = \gamma(v), \quad (1)$$

where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering v , respectively; furthermore, $\gamma(v) = d_{(i,j)}$ if $v = s_{(i,j)}$, $\gamma(v) = -d_{(i,j)}$ if $v = t_{(i,j)}$, and $\gamma(v) = 0$ otherwise.

Alternatively, one can consider a *path flow* for a player $(i, j) \in [K_i]$. Let $\mathcal{P}_{(i,j)}$ be the set of all paths from $s_{(i,j)}$ to $t_{(i,j)}$ in D . A path flow is a nonnegative vector $(f_P^{(i,j)})_{P \in \mathcal{P}_{(i,j)}}$. The corresponding flow on link $a \in A$ for player $(i, j) \in [K_i]$ is then

$$f_a^{(i,j)} := \sum_{P \ni a} f_P^{(i,j)}.$$

We define \mathcal{F}_i with $0 \leq i \leq n$ to be the set of vectors $(\mathbf{f}^1, \dots, \mathbf{f}^i)$ such that \mathbf{f}^j is a feasible flow for games j , $j = 1, \dots, i$. The entire flow for a sequence of games $\sigma = (1, \dots, n)$ is denoted by $\mathbf{f} = (\mathbf{f}^1, \dots, \mathbf{f}^n)$. We denote by \mathbf{f}_a^i the aggregate flow of game i on link a . The cost of a flow on link $a \in A$ of game i is defined by

$$C_a^i(\mathbf{f}_a^i; \mathbf{f}_a^1, \dots, \mathbf{f}_a^{i-1}) = p_a \left(\sum_{j=1}^i \mathbf{f}_a^j \right) \mathbf{f}_a^i \quad (2)$$

This expression can be obtained as the routing cost on arc a for a feasible flow for game i , given the flows $(\mathbf{f}^1, \dots, \mathbf{f}^{i-1})$ of previous games $1, \dots, i-1$. In the sequel of this paper we assume that $p_a(x)$ is a convex function. Hence $C_a^i(\cdot)$ is also a convex function. The cost for game i is given by the sum of arc costs

$$C^i(\mathbf{f}^i; \mathbf{f}^1, \dots, \mathbf{f}^{i-1}) = \sum_{a \in A} C_a^i(\mathbf{f}_a^i; \mathbf{f}_a^1, \dots, \mathbf{f}_a^{i-1}).$$

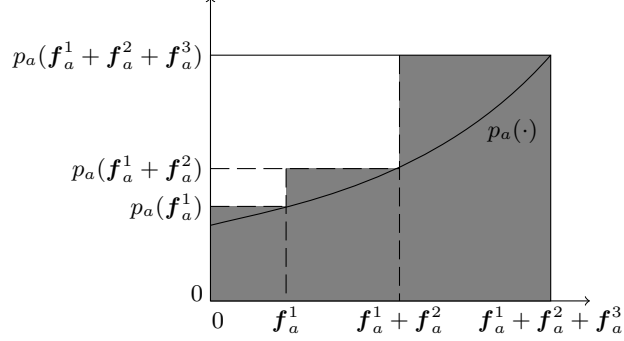


Fig. 1. Illustration of the cost function $C(\mathbf{f})$ for $n = 3$.

If we assume that the flows of games $i + 1, \dots, n$ do not affect the cost of the flows of the first $1, \dots, i$ games, then the total cost of all sequentially played games is given by:

$$C(\mathbf{f}) = \sum_{i=1}^n C^i(\mathbf{f}^i; \mathbf{f}^1, \dots, \mathbf{f}^{i-1}). \quad (3)$$

Figure 1 illustrates the cost function for a feasible flow. The cost on an arc a is actually a right-hand Riemann sum of the price function $p_a(x)$ on the interval $[0, \sum_{i=1}^n \mathbf{f}_a^i]$.

For a sequence of games $\sigma = (1, \dots, n)$, we investigate in this paper the online algorithm SEQNASH, that consists of the sequence of Nash equilibria for the corresponding games $(1, \dots, n)$. We focus on the *efficiency* of SEQNASH compared to the offline optimum OPT.

2.1 Characterizing Sequential Nash Equilibria

A strategy distribution or flow for game i is at Nash equilibrium when no player has an incentive to unilaterally change his strategy.

It is easy to check that a Nash flow \mathbf{f}^i is the optimum of the following convex optimization problem, see for example Roughgarden and Tardos [1].

$$\begin{aligned} \min \quad & \sum_{a \in A_0} \int_0^{\mathbf{f}_a^i} p_a\left(\sum_{j=1}^{i-1} \mathbf{f}_a^j + z\right) dz \\ \text{s.t.} \quad & \sum_{a \in \delta^+(v)} \mathbf{f}_a^{(i,j)} - \sum_{a \in \delta^-(v)} \mathbf{f}_a^{(i,j)} = \gamma(v) \quad \forall v \in V, (i,j) \in [K_i] \quad (4) \\ & \mathbf{f}_a^{(i,j)} \geq 0 \quad \forall a \in A, (i,j) \in [K_i], \end{aligned}$$

where $\gamma(v)$ is defined as in (1). The following conditions are necessary and sufficient to characterize a Nash equilibrium for game i .

Lemma 1. *A feasible flow \mathbf{f}^i for the game i is a Nash equilibrium if and only if it satisfies:*

$$\sum_{a \in A} p_a \left(\sum_{j=1}^i \mathbf{f}_a^j \right) (\mathbf{f}_a^i - \mathbf{x}_a^i) \leq 0 \text{ for all feasible flows } \mathbf{x}^i \text{ for game } i. \quad (5)$$

The proof is based on the first order optimality conditions and the convexity of $C^i(\cdot)$, see Dafermos and Sparrow [18]. For a given sequence σ of games, we denote in the following the deterministic online algorithm that consists of the sequence of Nash equilibria by SEQNASH.

2.2 Total Offline Optimum

Finally, the *total offline optimum* is characterized by:

$$\begin{aligned} \min \quad & C(\mathbf{f}) \\ \text{s.t.} \quad & \sum_{a \in \delta^+(v)} f_a^{(i,j)} - \sum_{a \in \delta^-(v)} f_a^{(i,j)} = \gamma(v) \quad \forall v \in V, (i,j) \in [K_i], i \in [n] \\ & f_a^{(i,j)} \geq 0 \quad \forall a \in A, (i,j) \in [K_i], i \in [n] \end{aligned} \quad (6)$$

where $\gamma(v)$ is defined as in (1).

Lemma 2. *A feasible flow \mathbf{f} for the parallel game $\sigma = (1, \dots, n)$ is total system optimal if and only if it satisfies:*

$$\sum_{a \in A} \frac{\partial C}{\partial \mathbf{f}_a}(\mathbf{f}_a) (\mathbf{f}_a - \mathbf{x}_a) \leq 0 \text{ for all feasible flows } \mathbf{x} \text{ for game } \sigma. \quad (7)$$

The optimal offline algorithm that generates for a given game sequence σ an optimal flow \mathbf{f} for the above problem is called OPT. We denote its value for σ by $\text{OPT}(\sigma) = C(\mathbf{f})$.

3 Bounding the Inefficiency for SEQNASH

For a given sequence of games $\sigma = 1, \dots, n$ we denote the corresponding sequences set of players by $([K_1], \dots, [K_n])$. A solution \mathbf{f} produced by an online algorithm ALG for σ we denote by $\text{ALG}(\sigma) = C(\mathbf{f})$ its cost. The online algorithm ALG is called *c-competitive* if the cost of ALG is never larger than c times the cost of an optimal offline solution. The *competitive ratio* of ALG is the infimum over all $c \geq 1$ such that ALG is c -competitive, see Borodin and El-Yaniv [19].

We can classify the presented algorithms SEQNASH and OPT as follows: SEQNASH is a *deterministic online* algorithms for ONLINE SRG; OPT is an *offline* solution for ONLINE SRG since here, the entire game sequence is known in advance.

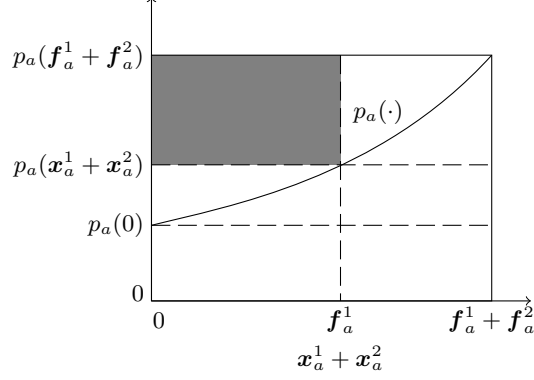


Fig. 2. Illustration of the value $\phi_n(\mathcal{C})$ in Theorem 1 with $n = 2$.

3.1 Competitive Analysis of SEQNASH

Before we state the main result, we need some useful prerequisites. We define for every $a \in A$, for any nonnegative vectors $\mathbf{v}_a, \mathbf{w}_a \in \mathbb{R}^n$ the following values (we assume by convention $0/0 = 0$):

$$\alpha(p_a, n) := \max_{\mathbf{v}_a \geq 0} \frac{p_a\left(\sum_{k=1}^n v_a^k\right) \left(\sum_{k=1}^n v_a^k\right)}{\sum_{k=1}^n p_a\left(\sum_{i=1}^k v_a^i\right) v_a^k} \quad (8)$$

$$\phi(p_a, \mathbf{v}_a, n) := \max_{\mathbf{w}_a \geq 0} \frac{\left(p_a\left(\sum_{k=1}^n v_a^k\right) - p_a\left(\sum_{k=1}^n w_a^k\right)\right) \left(\sum_{k=1}^n w_a^k\right)}{\sum_{k=1}^n p_a\left(\sum_{i=1}^k v_a^i\right) v_a^k}. \quad (9)$$

Figure 2 illustrates the value $\phi(p_a, \mathbf{v}_a, n)$ as the ratio of the area of the shaded rectangle and the white area within the entire rectangle.

For a given class \mathcal{C} of nondecreasing price functions, we further define

$$\begin{aligned} \alpha_n(\mathcal{C}) &:= \sup_{p_a \in \mathcal{C}, \mathbf{v}_a \geq 0} \alpha(p_a, n) \\ \phi_n(\mathcal{C}) &:= \sup_{p_a \in \mathcal{C}, \mathbf{v}_a \geq 0} \phi(p_a, \mathbf{v}_a, n) \end{aligned}$$

Theorem 1. *If the values $\alpha_n(\mathcal{C})$, and $\phi_n(\mathcal{C})$ are finite and $1 - \phi_n(\mathcal{C}) > 0$ holds, then SEQNASH is $\frac{\alpha_n(\mathcal{C})}{1 - \phi_n(\mathcal{C})}$ -competitive.*

Proof. Let \mathbf{f} be the flow generated by SEQNASH, and \mathbf{x} be any feasible flow for a given sequence of games $\sigma = (1, \dots, n)$.

$$\begin{aligned} C(\mathbf{f}) &= \sum_{a \in A} \sum_{k=1}^n p_a \left(\sum_{i=1}^k \mathbf{f}_a^i \right) \mathbf{f}_a^k \\ &\leq \sum_{a \in A} \sum_{k=1}^n p_a \left(\sum_{i=1}^k \mathbf{f}_a^i \right) \mathbf{x}_a^k \end{aligned} \quad (10)$$

$$\leq \sum_{a \in A} p_a \left(\sum_{i=1}^n \mathbf{f}_a^i \right) \left(\sum_{k=1}^n \mathbf{x}_a^k \right) \quad (11)$$

$$\begin{aligned} &= \sum_{a \in A} p_a \left(\sum_{i=1}^n \mathbf{x}_a^i \right) \left(\sum_{k=1}^n \mathbf{x}_a^k \right) + \left(p_a \left(\sum_{i=1}^n \mathbf{f}_a^i \right) - p_a \left(\sum_{i=1}^n \mathbf{x}_a^i \right) \right) \left(\sum_{k=1}^n \mathbf{x}_a^k \right) \\ &\leq \alpha_n(\mathcal{C}) C(\mathbf{x}) + \phi_n(\mathcal{C}) C(\mathbf{f}), \end{aligned} \quad (12)$$

where (10) follows from the variational inequality (4), (11) is valid since price functions are nondecreasing, and (12) follows by the definition of $\alpha_n(\mathcal{C})$ and $\phi_n(\mathcal{C})$, respectively. \square

In the following, we explicitly determine bounds on the values $\alpha_n(\mathcal{C})$ and $\phi_n(\mathcal{C})$ for affine linear price functions.

Lemma 3. For $x_1, \dots, x_n \in \mathbb{R}$ we have:

$$\sum_{i=1}^n x_i^2 \geq \frac{1}{n} \left(\sum_{i=1}^n x_i \right)^2. \quad (13)$$

Proof. Consider the vector $\mathbf{x} = (x_1, \dots, x_n)^\top$ and $\mathbf{1}$, the vector of all ones. By the inequality of Cauchy-Schwarz, we obtain:

$$\left(\sum_{i=1}^n x_i \right)^2 = (\mathbf{1}^\top \mathbf{x})^2 \leq \|\mathbf{1}\|_2^2 \|\mathbf{x}\|_2^2 = n \sum_{i=1}^n x_i^2,$$

which yields the result. \square

Remark 1. The following equation is useful:

$$\sum_{i=1}^n \sum_{j=1}^n f_a^j f_a^i = 2 \sum_{i=1}^n \sum_{j=1}^i f_a^j f_a^i - \sum_{i=1}^n f_a^i f_a^i. \quad (14)$$

Lemma 4. For affine linear price functions $p_a(z) = q_a z + r_a$, $q_a \geq 0, r_a \geq 0$, the value $\alpha_n(\mathcal{C})$ is less than or equal to $\frac{2n}{n+1}$.

Proof. If $q_a = 0$, then we have $\alpha_n(\mathcal{C}) = 1$. For $q_a > 0$, we define:

$$\nu := \frac{r_a}{q_a \sum_{i=1}^n v_a^i}.$$

Then,

$$\begin{aligned}
& \frac{q_a \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k + r_a \sum_{i=1}^n v_a^i}{q_a \sum_{k=1}^n \sum_{i=1}^k v_a^i v_a^k + r_a \sum_{i=1}^n v_a^i} = \frac{2 \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k + 2\nu \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k}{2 \sum_{k=1}^n \sum_{i=1}^k v_a^i v_a^k + 2\nu \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k} \\
& \leq \frac{2 \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k + 2\nu \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k}{\sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k + \frac{1}{n} \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k + 2\nu \sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k} \\
& = \frac{2 + 2\nu}{1 + \frac{1}{n} + 2\nu},
\end{aligned} \tag{15}$$

where (15) follows from Lemma 3. Hence,

$$\alpha_n(p_a, \mathbf{v}_a) \leq \frac{2 + 2\nu}{1 + \frac{1}{n} + 2\nu}.$$

Maximizing $F(\nu) := \frac{2+2\nu}{1+\frac{1}{n}+2\nu}$ with respect to ν yields:

$$\max_{\nu \geq 0} F(\nu) = F(0) = \frac{2n}{n+1}.$$

□

Lemma 5. For parameters $a, b > 0$ and any numbers $x, y \geq 0$ the following inequality is valid:

$$xy \leq \frac{a}{2b}x^2 + \frac{b}{2a}y^2. \tag{16}$$

Lemma 6. For affine linear price functions $p_a(z) = q_a z + r_a$, $q_a \geq 0, r_a \geq 0$, the value $\phi_n(\mathcal{C})$ is less than or equal to $\frac{n}{2(n+1)}$.

Proof.

$$\begin{aligned}
& \frac{q_a \left(\sum_{k=1}^n v_a^i - \sum_{k=1}^n w_a^k \right) \sum_{k=1}^n w_a^k}{q_a \sum_{k=1}^n \sum_{i=1}^k v_a^i v_a^k + r_a \sum_{i=1}^n v_a^i} \leq \frac{\left(\sum_{k=1}^n v_a^i - \sum_{k=1}^n w_a^k \right) \sum_{k=1}^n w_a^k}{\sum_{k=1}^n \sum_{i=1}^k v_a^i v_a^k} \\
& = \frac{\left(\sum_{k=1}^n v_a^i - \sum_{k=1}^n w_a^k \right) \sum_{k=1}^n w_a^k}{\frac{1}{2} \left(\sum_{k=1}^n \sum_{i=1}^n v_a^i v_a^k + \sum_{k=1}^n v_a^k v_a^k \right)} \\
& \leq \frac{\frac{1}{4} \left(\sum_{k=1}^n v_a^i \right)^2}{\frac{1}{2} \left(\sum_{k=1}^n v_a^k \right)^2 + \frac{1}{2n} \left(\sum_{k=1}^n v_a^k \right)^2} \\
& = \frac{n}{2(n+1)},
\end{aligned} \tag{17}$$

$$\tag{18}$$

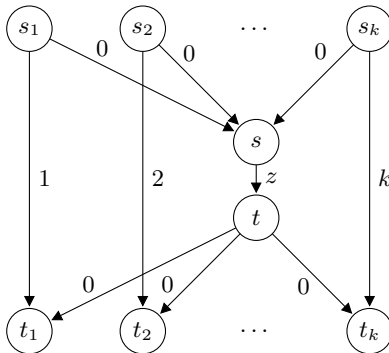


Fig. 3. Graph construction for the proof of Theorem 2.

where (17) is valid since $r_a \geq 0$. Inequality (18) follows from Lemma 5, where we set $x = \sum_{k=1}^n v_a^k, y = \sum_{k=1}^n w_a^k, a = 1$, and $b = 2$. Hence, $\phi_n(\mathcal{C}) \leq \frac{n}{2(n+1)}$. \square

Now, we state our main result.

Corollary 1. *If the price functions of the ONLINESRG are affine, the online algorithm SEQNASH is $\frac{4n}{n+2}$ -competitive, where n is the number of games.*

Proof. Replacing $\alpha_n(\mathcal{C}), \phi_n(\mathcal{C})$ with $\frac{2n}{n+1}, \frac{n}{2(n+1)}$, respectively, and applying Theorem 1 yields the desired result. \square

Note 1. For $n = 1$, that is, only a single game is considered, the competitive ratio is equal to the price of anarchy of $\frac{4}{3}$ for nonatomic routing games with affine linear price functions.

We do not know, whether the above result is tight for $n \geq 2$. In the following, we present lower bounds on the competitive ratio for SEQNASH.

3.2 Lower Bounds for SEQNASH

Theorem 2. *In case of affinely linear cost functions, the online algorithm SEQNASH for ONLINESRG has a competitive ratio greater than or equal to $\frac{5n-3}{3n-1}$, where n is the number of games.*

Proof. Consider the network presented in Figure 3 with the following price functions: $p_{(s_i, s)}(z) = 0, p_{(t, t_i)}(z) = 0, p_{(s_i, t_i)}(z) = i, i = 1, \dots, k$, and $p_{(s, t)}(z) = z$. We consecutively release a sequence of games $(1, \dots, k)$, where in each game i , there is a single player type $(i, 1)$. The demand of player type $(i, 1)$ is of size 1 that has to be routed from s_i to t_i , for $i = 1, \dots, k$. Due to the choice of the

affine terms i , SEQNASH routes for every game the corresponding demand over the arc from s to t . The cost for these k demands is:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}.$$

Then we release the $(k+1)$ -th game with demand d from s to t , which generates the following cost:

$$(k+d)d = kd + d^2.$$

Thus, the total cost for the sequence $\sigma = (1, \dots, k+1)$ for SEQNASH is given by:

$$\text{SEQNASH}(\sigma) = \frac{k(k+1)}{2} + kd + d^2.$$

The optimal offline algorithm OPT routes the demands of the first k games along the direct arcs from s_i to t_i incurring cost of:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}.$$

The last demand in game $k+1$ is routed from s to t with cost d^2 . The total cost for the sequence $\sigma = (1, \dots, k+1)$ for OPT is given by:

$$\text{OPT}(\sigma) = \frac{k(k+1)}{2} + d^2.$$

Setting $d = k$ and replacing $k = n - 1$ yields

$$\frac{\text{SEQNASH}(\sigma)}{\text{OPT}(\sigma)} = \frac{k(k+1) + 2kd + 2d^2}{k(k+1) + 2d^2} = \frac{5n-4}{3n-2}, \quad (19)$$

which proves the theorem. \square

The asymptotic bound for n tending to infinity is $\frac{5}{3}$. However, for large k we can numerically optimize (19) over the parameter d .

Corollary 2. *In case of affinely linear cost functions, the online algorithm SEQNASH for ONLINESRG has a competitive ratio greater than or equal to 1.7.*

Proof. We set $k = 100$. Optimizing over d yields $d = 71$. Evaluating (19) with these values proves the corollary.

4 Total Congestion Cost

In the previous sections we assumed that the cost of a given sequence of games σ is the sum of the cost of the individual games. In this section, we define the total cost as:

$$C(\mathbf{f}) = \sum_{a \in A} p_a \left(\sum_{i=1}^n f_a^i \right) \sum_{i=1}^n f_a^i. \quad (20)$$

This cost function reflects the congestion cost provided the entire sequence of games has been played. To this end, we assume the same basic setting as in the previous sections except that our performance measure for an online algorithm is given by (20). We define for every $a \in A$, for any nonnegative vectors $\mathbf{v}_a, \mathbf{w}_a \in \mathbb{R}^n$ the following value (we assume by convention $0/0 = 0$):

$$\gamma(p_a, \mathbf{v}_a) := \max_{\mathbf{w}_a \geq 0} \frac{(p_a(\sum_{k=1}^n v_a^k) - p_a(\sum_{k=1}^n w_a^k)) \left(\sum_{k=1}^n w_a^k \right)}{p_a(\sum_{k=1}^n v_a^k) \sum_{k=1}^n v_a^k}. \quad (21)$$

We further define for a given class of price functions \mathcal{C} :

$$\gamma(\mathcal{C}) := \sup_{p_a \in \mathcal{C}, \mathbf{v}_a \geq 0} \gamma(p_a, \mathbf{v}_a) \quad (22)$$

$$\omega_n(\mathcal{C}) := \sup_{p_a \in \mathcal{C}, \mathbf{v}_a \geq 0} \alpha(p_a; n) \cdot \gamma(p_a, \mathbf{v}_a). \quad (23)$$

Note that the value $\gamma(\mathcal{C})$ was first defined in Correa, Schulz, and Stier-Moses [12] in the context of selfish routing in nonatomic routing games.

Theorem 3. *If the values $\alpha_n(\mathcal{C})$, and $\omega_n(\mathcal{C})$ are finite and $1 - \omega_n(\mathcal{C}) > 0$ holds, then SEQNASH is $\frac{\alpha_n(\mathcal{C})}{1 - \omega_n(\mathcal{C})}$ -competitive for ONLINESRG.*

Proof. Let \mathbf{f} be the flow generated by SEQNASH, and \mathbf{x} be any feasible flow for a given sequence of games $\sigma = (1, \dots, n)$.

$$\begin{aligned} C(\mathbf{f}) &= \sum_{a \in A} p_a \left(\sum_{i=1}^n \mathbf{f}_a^i \right) \sum_{k=1}^n \mathbf{f}_a^k \\ &\leq \alpha_n(\mathcal{C}) \sum_{a \in A} \sum_{k=1}^n p_a \left(\sum_{i=1}^k \mathbf{f}_a^i \right) \mathbf{f}_a^k \end{aligned} \quad (24)$$

$$\leq \alpha_n(\mathcal{C}) \sum_{a \in A} \sum_{k=1}^n p_a \left(\sum_{i=1}^k \mathbf{f}_a^i \right) \mathbf{x}_a^k \quad (25)$$

$$\leq \alpha_n(\mathcal{C}) \sum_{a \in A} p_a \left(\sum_{i=1}^n \mathbf{f}_a^i \right) \left(\sum_{k=1}^n \mathbf{x}_a^k \right) \quad (26)$$

$$\begin{aligned} &= \sum_{a \in A} \alpha_n(\mathcal{C}) p_a \left(\sum_{i=1}^n \mathbf{x}_a^i \right) \left(\sum_{k=1}^n \mathbf{x}_a^k \right) + \alpha_n(\mathcal{C}) \left(p_a \left(\sum_{i=1}^n \mathbf{f}_a^i \right) - p_a \left(\sum_{i=1}^n \mathbf{x}_a^i \right) \right) \left(\sum_{k=1}^n \mathbf{x}_a^k \right) \\ &\leq \alpha_n(\mathcal{C}) C(\mathbf{x}) + \omega_n(\mathcal{C}) C(\mathbf{f}), \end{aligned} \quad (27)$$

where (24) follows by the definition of $\alpha_n(\mathcal{C})$, (25) follows from the variational inequality (4), (26) is valid since price functions are nondecreasing, and (27) follows by the definition of $\omega_n(\mathcal{C})$. \square

Corollary 3. *If the price functions of the ONLINESRG are affine, the online algorithm SEQNASH is $\frac{4n}{n+2}$ -competitive, where n is the number of games.*

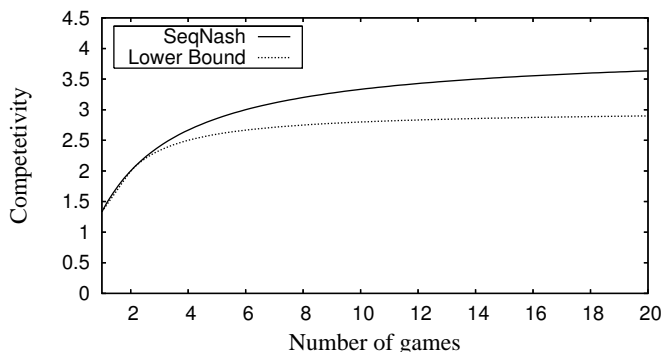


Fig. 4. Competitive bound of $\frac{4n^2}{n+2}$ on the competitive ratio of SEQNASH for affine linear price functions compared to the lower bound of $\frac{3n-2}{n}$.

Proof. Replacing $\alpha_n(\mathcal{C}), \omega_n(\mathcal{C})$ with $\frac{2n}{n+1}, \frac{n}{2(n+1)}$, respectively, and applying Theorem 3 yields the desired result. \square

Lower Bounds for SEQNASH.

Theorem 4. *In case of affinely linear cost functions, the online algorithm SEQNASH for ONLINESRG has a competitive ratio greater than or equal to $\frac{3n-2}{n}$, where n is the number of games.*

Proof. We consider the network presented in Figure 3 with the price functions: $p_{(s_i, s)}(z) = 0$, $p_{(t, t_i)}(z) = 0$, $p_{(s_i, t_i)}(z) = i$, $i = 1, \dots, k$, and $p_{(s, t)}(z) = z$. We consecutively release a sequence of games $(1, \dots, k)$, where in each game i , there is a single player type $(i, 1)$. The demand of player type $(i, 1)$ is 1 that has to be routed from s_i to t_i , for $i = 1, \dots, k$. Due to the choice of the affine terms i , SEQNASH routes for every game the corresponding demand over the arc from s to t . Then we release the $(k+1)$ -th game with demand d from s to t . Thus, the total cost for the sequence $\sigma = (1, \dots, k+1)$ for SEQNASH with the new cost function is given by:

$$\text{SEQNASH}(\sigma) = (k+d)^2.$$

The optimal offline algorithm OPT routes the demands of the first k games along the direct arcs from s_i to t_i incurring cost of:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}.$$

The last demand in game $k+1$ is routed from s to t with cost d^2 . The total cost for the sequence $\sigma = (1, \dots, k+1)$ for OPT is given by:

$$\text{OPT}(\sigma) = \frac{k(k+1)}{2} + d^2.$$

Replacing $k = n - 1$ and setting $d = \frac{n}{2}$ yields

$$\frac{\text{SEQNASH}(\sigma)}{\text{OPT}(\sigma)} = \frac{2(k+d)^2}{k(k+1) + 2d^2} = \frac{3n-2}{n}, \quad (28)$$

which proves the theorem. \square

Remark 2. For $n = 2$, the upper bound given in Corollary 3 is tight.

5 Concluding Remarks

This paper constitutes a first attempt to combine recent advances in the algorithmic game theory field with the online optimization field. Our initial results show that despite the fact that a Nash equilibrium is usually inefficient compared to the system optimum, the online algorithms SEQNASH consisting of a sequence of Nash equilibria is asymptotically 4-competitive.

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