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Facets of the $(s,t)-p$-path polytope

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Abstract

We give a partial description of the $(s,t)-p$-path polytope of a directed graph D which is the convex hull of the incidence vectors of simple directed (s,t)-paths in D of length p. First, we point out how the $(s,t)-p$-path polytope is located in the family of path and cycle polyhedra. Next, we give some classes of valid inequalities which are very similar to inequalities which are valid for the p-cycle polytope, that is, the convex hull of the incidence vectors of simple cycles of length p in D. We give necessary and sufficient conditions for these inequalities to be facet defining. Furthermore, we consider a class of inequalities that has been identified to be valid for (s,t)-paths of cardinality at most p. Finally, we transfer the results to related polytopes, in particular, the undirected counterpart of the $(s,t)-p$-path polytope.

1 Introduction

Given a directed graph $D=(V,A)$, we say that a subset

$$P=\{(i_1,i_2),(i_2,i_3),\ldots,(i_{k-1,i_k})\}$$

of A is a directed simple (s,t)-path if $k\geq 2$, all nodes $i_1,\ldots,i_k$ are distinct, s is the origin, and t is the terminus, that is, $s=i_1$, $t=i_k$. Below a directed simple path will be sometimes denoted by a tuple of nodes. For example, (i_1,i_2,i_3,i_4) denotes the path $\{(i_1,i_2),(i_2,i_3),(i_3,i_4)\}$. In this paper we study the facial structure of the $(s,t)-p$-path polytope $P^p_{s,t-\text{path}}(D)$ which is the convex hull of the incidence vectors of directed (s,t)-paths with exactly p arcs. The corresponding $(s,t)-p$-path problem, that is, the problem of finding a minimum cost $(s,t)-p$-path , is NP-hard, since for $p=n$ and negative arc cost it is equivalent to the Hamiltonian (s,t)-path problem. So for general p we cannot expect to obtain a complete and tractable linear characterization of the $(s,t)-p$-path polytope $P^p_{s,t-\text{path}}(D)$.

A lot of path and cycle polyhedra are well studied. Dahl and Gouveia [6] gave some valid inequalities for polyhedra associated with the directed hop-constrained shortest path problem which is the problem of finding a minimum (s,t)-path with at most p arcs. Dahl and Realfsen [7] studied the same problem on acyclic directed graphs, in particular, on *2-graphs*. The dominant of the directed (s,t)-path polytope which is the Minkowski sum of the convex hull of the incidence vectors of simple (s,t)-paths and the Euclidean space $\mathbb{R}^A$ is determined by nonnegativity constraints $x_{ij}\geq 0$ and cut inequalities $x(C)\geq 1$ for all (s,t)-cuts C (see Schrijver [15], chapter 13). The cycle polytope $P_C(D_n)$ which is the convex hull of the incidence vectors of all simple directed cycles of the complete directed graph D_n has been investigated by Balas and Oosten [1], while the undirected counterpart, the circuit polytope, has been studied by Coullard & Pulleyblank [5] and Bauer [2]. Hartmann and Özlük [10] gave a partial description of the p-cycle polytope $P_C^p(D_n)$ which is the convex hull of the incidence vectors of all simple p-cycles of D_n. Maurras and Nguyen [12, 13] studied the facial structure of the undirected analog. Finally, Bauer et al. [3] also studied the cardinality constrained circuit polytope, which is the convex hull of the incidence vectors of all undirected simple cycles with at most p edges on the complete graph K_n.

The present paper is motivated by the observation that the $(s,t)-p$-path polytope is closely related to the polyhedra mentioned in the last paragraph and it has an exposed position among them. Indeed, valid inequalities for the $(s,t)-p$-path polytope can easily be transformed into valid inequalities for some related polytopes, for example, by lifting. A first overview is given in Figure 1. An arrow there means that facet defining inequalities (or some classes of facet defining inequalities) of the polytope at the tail of the arrow can be transformed into facet defining inequalities for the polytope at the head of the arrow, where G and D are appropriate graphs and digraphs, respectively.

The remainder of the paper is organized as follows: In Section 2 we propose an integer programming formulation of the $(s,t)-p$-path polytope and describe how valid inequalities can be lifted to valid inequalities of the p-cycle polytope. Section 3 contains the study of the facial structure of the $(s,t)-p$-path polytope $P^p_{s,t-\text{path}}(D)$ on an appropriate digraph D. Finally, in Section 4 we transfer the results of Section 3 to the polytopes mentioned that are related to the $(s,t)-p$-path polytope $P^p_{s,t-\text{path}}(D)$.

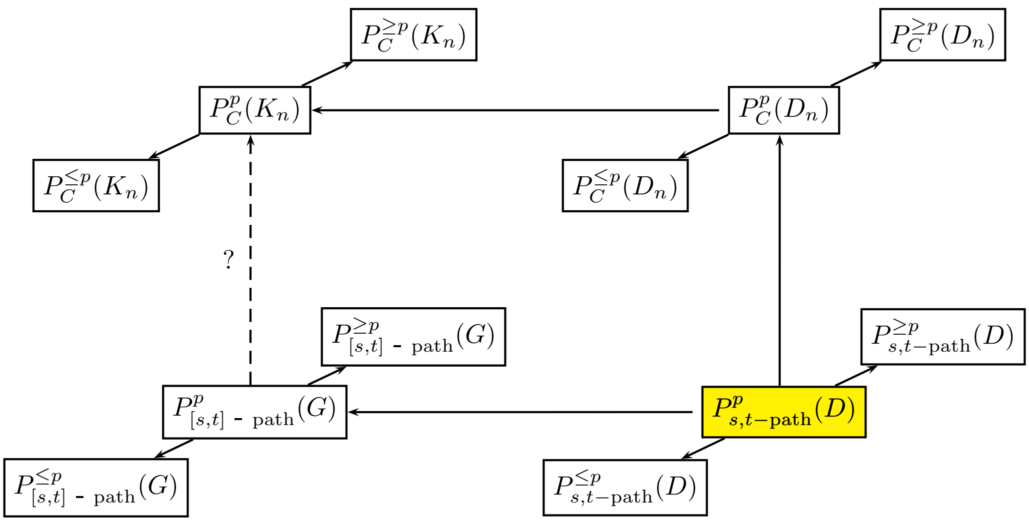

Figure 1. The $(s,t)-p$-path polytope $P^p_{s,t-\text{path}}(D)$ and related polytopes.

2 Basic results

We start the polyhedral analysis of the $(s,t)-p$-path polytope with an integer programming formulation. In the sequel, $D=(V,A)$ is a digraph on node set $V=\{0,\dots,n\}$ whose arc set A contains neither loops nor parallel arcs. The nodes s and t will be identified with the nodes 0 and n, respectively. Consequently, the $(0,n)-p$-path polytope will be denoted by $P^p_{0,n-\text{path}}(D)$. The integer points of $P^p_{0,n-\text{path}}(D)$ are characterized by the system

$$
\begin{aligned}
x(\delta^-(0)) &= 0, && (1)\\
x(\delta^+(n)) &= 0, && (2)\\
x(\delta^+(i))-x(\delta^-(i)) &= \begin{cases} 1 \text{ if } i=0,\\ 0 \text{ if } i\in V\setminus\{0,n\},\\ -1 \text{ if } i=n,\end{cases} && (3)\\
x(A) &= p, && (4)\\
x(\delta^+(i)) &\le 1 & \forall\ i\in V\setminus\{0,n\}, & \quad (5)\\
x((S:V\setminus S)) &\ge x(\delta^+(j)) & \forall S\subset V, 3\le |S|\le n-2, & \quad (6)\\
& & 0,n\in S, j\in V\setminus S, & \\
x_{ij} &\in \{0,1\} & \forall\ (i,j)\in A. & \quad (7)
\end{aligned}
$$

Here, we denote by $\delta^+(k)$ and $\delta^-(k)$ the set of arcs directed out of and into node k, respectively. For an arc set $F\subseteq A$ we set $x(F):=\sum_{(i,j)\in F} x_{ij}$, and for any node sets S,T of V, $(S:T)$ is short for $\{(i,j)\in A|i\in S, j\in T\}$. Furthermore, in the following we denote by $A(S)$ the subset of arcs whose both endnodes are in S, for some $S\subseteq V$.

The incidence vectors of node-disjoint unions of a $(0,n)$-path and cycles on node set $V\setminus\{(0,n)\}$ are described by the equations (1)-(2), the *flow constraints* (3), *degree constraints* (5), and the *integrality constraints* (7). The *one-sided min-cut inequalities* (6) are satisfied by all $(0,n)$-paths but violated by unions of a $(0,n)$-paths and cycles on $V\setminus\{0,n\}$. Finally, the *cardinality constraint* (4) ensures that all $(0,n)$-paths are of length p.

Complete linear descriptions of $P^p_{0,n-\text{path}}(D)$ for $p=1,2,3$ are given in Table 1, where D is the complete digraph on node set $\{0,\dots,n\}$. The results for $p=2$ and $p=3$ follows from the fact that a $(0,n)-2$-path visits exactly one internal node and a $(0,n)-3$-path contains exactly one internal arc. Since the number of internal nodes is $n-1$, the dimension of $P^2_{0,n-\text{path}}(D)$ is $n-2$, and since the number of internal arcs is $(n-1)(n-2)$, the dimension of $P^3_{0,n-\text{path}}(D)$ is equal to $(n-1)(n-2)-1=n-3n+1$. The $(0,n)-1$-path polytope $P^1_{0,n-\text{path}}(D)$ has clearly dimension 0 and is determined by the equations $x_{0n}=1$ and $x_{ij}=0$ for all $(i,j)\in A\setminus\{(0,n)\}$. We suppose in the sequel that A contains all arcs (i,j), where $i\neq j\in V$, except the arcs $(i,0)$, (n,i) for $i=1,\dots,n-1$, $(0,n)$, and $(n,0)$.

Contracting the nodes 0 and n to the single node n we obtain the complete digraph D_n on n nodes, and we see that the set of simple $(0,n)-p$-paths defined on D can be identified with the set of simple p-cycles defined on D_n that contain node n. Hence, the $(0,n)-p$-path polytope $P^p_{0,n-\text{path}}(D)$ and the node constraint cycle polytope $P^p_C(D_n)_{|n}:=\{\mathbf{x}\in P^p_C(D_n)|x(\delta^+(n))=1\}$ are isomorphic. In particular, when $p=n$, $P^p_{0,n-\text{path}}(D)$ is isomorphic to the

Table 1. Polyhedral Analysis of $P^p_{0,n-\text{path}}(D)$, where D is the complete digrah on node set $\{0,\dots,n\}$. For $p=n$, $P^p_{0,n-\text{path}}(D)$ is equivalent to the asymmetric traveling salesman polytope defined on n nodes.

p	Dimension	Complete linear description
1	0	$x_{0n} = 1$ $x_{ij} = 0 \quad \forall\, (i,j) \in A \setminus \{(0,1)\}$
2	$n-2$	$x(\delta^-(0)) = 0$ $x(\delta^+(n)) = 0$ $x_{ij} = 0 \quad \forall\, (i,j) \in A(V \setminus \{0,n\})$ $x(\delta^+(0)) = 1$ $x_{0j} - x_{jn} = 0 \quad \forall\, j \in V \setminus \{0,n\}$ $x_{0j} \geq 0 \quad \forall\, j \in V \setminus \{0,n\}$
3	n^2-3n+1	$x(\delta^-(0)) = 0$ $x(\delta^+(n)) = 0$ $x(A(V \setminus \{0,n\})) = 1$ $x(\delta^+(i)) = x_{0i} + x_{in} \quad \forall\, i \in V \setminus \{0,n\}$ $x(\delta^-(i)) = x_{0i} + x_{in} \quad \forall\, i \in V \setminus \{0,n\}$ $x_{ij} \geq 0 \quad \forall\, (i,j) \in A(V \setminus \{0,n\})$
		Partial linear description
4 $\vdots$ $n-1$	n^2-2n-1	equations (1)-(4) see Section 3

asymmetric traveling salesman polytope which has dimension n^2-3n+1 (see [9]). Furthermore, Hartmann and Özlük [10] showed that $P^p_C(D_n)_{|n}$ is a facet of the p-cycle polytope if $4 \leq p < n$. For $4 \leq p < n$, the p-cycle polytope has dimension n^2-2n and therefore the dimension of $P^p_{0,n-\text{path}}(D)$ is equal to n^2-2n-1. Moreover, this relation leads to the following theorem obtained by standard lifting (see Nemhauser and Wolsey [14]).

Theorem 2.1. *Let $\boldsymbol{ax} \leq a_0$ be a facet defining inequality for the $(0,n)-p$-path polytope $P^p_{0,n-path}(D)$, where $4 \leq p < n$, and let γ be the maximum of $a(C)$ over all p-cycles C in D. Setting $a_{ni} := a_{0i}$ for $i = 1,\dots,n-1$, the inequality*

$$\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_{ij} + (\gamma - a_0)x(\delta^+(n)) \leq \gamma \tag{8}$$

defines a facet of the p-cycle polytope $P^p_C(D_n)$, where D_n is the complete digraph on node set $\{1,\dots,n\}$. □

This easy but fundamental relation between the $(0,n)-p$-path polytope $P^p_{0,n-\text{path}}(D)$ and the p-cycle polytope $P^p_C(D_n)$ also holds between other length restricted path and cycle polytopes (see [16]). This fact implies that it would be profitably to study first the facial structure of a length restricted directed path polytope and afterwards that of the corresponding cycle polytope. In our special case, the p-cycle polytope is already well studied; so we will proceed in the opposite direction, that is, starting from the results for the p-cycle polytope $P_C^p(D_n)$ given by Hartmann and Özlük [10] we will prove in many cases analogous results for the $(0,n)-p$-path polytope $P_{0,n-\text{path}}^p(D)$ and it is not surprising that this can be often done along the lines of the proofs of the authors mentioned above. Lemma 2.2 adapts Lemmas 2 and 6 of Hartmann and Özlük [10] for our purposes. The other statements of this section can be proved in the same manner as the original statements in [10]; so we omit their proofs.

Lemma 2.2 (cf. Lemmas 2 and 6 of Hartmann and Özlük [10])**.** *Let $3 \leq p < n$, $\boldsymbol{c}$ be a row vector, $s,t \in V$, $s \neq t$, and $R \subseteq V \setminus \{s,t,0,n\}$. There are λ, π_s, π_t, and $\{\pi_j | j \in R\}$ with*

$$\begin{array}{rcll} c_{si} & = & \lambda + \pi_s - \pi_i & \forall\, i \in R, \\ c_{it} & = & \lambda + \pi_i - \pi_t & \forall\, i \in R, \\ c_{ij} & = & \lambda + \pi_i - \pi_j & \forall\, (i,j) \in A(R), \end{array}$$

if one of the following conditions holds:

(i) $|R| \geq 5$ and $c_{ik}+c_{kj} = c_{il}+c_{lj}$ for all distinct nodes $i \in R \cup \{s\}$, $j \in R \cup \{t\}$, $k,l \in R$.

(ii) $|R| \geq p \geq 4$ and $c(P) = \gamma$ for all $(s,t) - p$-paths P, whose internal nodes are all in R.

(iii) $|R| = p-1$, $c(P) = \gamma$ for all $(s,t) - p$-paths P, whose internal nodes are all the nodes of R, and $c(P) = \delta$ for all $(s,t) - r$-paths P, all $r-1$ of whose internal are in R, for some $2 \leq r < p$.

(iv) $p = 3$, $|R| \geq 3$, $c(P) = \gamma$ for all $(s,t) - 3$-paths P, whose internal nodes are all in R, and $c(P) = \delta$ for each $(s,t) - 2$-path P whose inner node is in R.

Proof. (i) In particular, $c_{ik} + c_{kj} = c_{il} + c_{lj}$ for all distinct nodes $i,j,k,l \in R$. Using Lemma 2 of Hartmann and Özlük [10], it follows that there are λ and $\{\pi_j | j \in R\}$ with

$$c_{ij} \quad = \quad \lambda + \pi_i - \pi_j \quad \forall\, (i,j) \in A(R).$$

Next, setting $\pi_s := c_{sk} + \pi_k - \lambda$ and $\pi_t := \lambda + \pi_k - c_{kt}$ for some $k \in R$, we derive

$$\begin{array}{rcl} c_{si} & = & c_{sk} + c_{kl} - c_{il} = \lambda + \pi_s - \pi_i, \\ c_{it} & = & c_{kt} + c_{lk} - c_{li} = \lambda + \pi_i - \pi_t \end{array}$$

for all $i \in R$.

(ii) First, let $|R| \geq 5$. Since $|R| \geq p$, for all distinct nodes $i,j,k,l \in R$ there is a $(s,t) - p$-path that contains the arcs (i,k) and (k,j) but does not visit node l. Replacing node k by node l in P yields another $(s,t) - p$-path and thus $c_{ik} + c_{kj} = c_{il} + c_{lj}$ for all distinct nodes $i,j,k,l \in R$. Lemma 2 of Hartmann and Özlük implies that there are λ and $\{\pi_j | j \in R\}$ such that $c_{ij} = \lambda + \pi_i - \pi_j$ for all $(i,j) \in A(R)$. Set $\pi_s := c_{sk} + \pi_k - \lambda$ and $\pi_t := \lambda + \pi_l - c_{lt}$ for some $k \neq l \in R$. Any $(s,t) - p$-path whose internal nodes are in R and that uses the arcs $(s,k),(l,t)$ yields $\gamma = p\lambda + \pi_s - \pi_t$. Further, considering for $i \in R$ a

$(s,t)-p$-path P whose internal nodes are in R and that uses the arcs $(s,i),(l,t)$ yields $c_{si} = \lambda + \pi_s - \pi_i$ for all $i \in R$. Analogous it follows that $c_{jt} = \lambda + \pi_j - \pi_t$ for all $j \in R$.

Next, let $|R| = p = 4$. Without loss of generality, we may assume that $R = \{1,2,3,4\}$. Setting $Q := \{1,2,3\}$ and identifying the nodes s and t, Theorem 23 of Grötschel and Padberg implies that there are α_s, β_t, $\{\alpha_j | j \in Q\}$, and $\{\beta_j | j \in Q\}$ such that

$$\begin{array}{rcll} c_{si} & = & \alpha_s + \beta_i & \forall\, i \in Q, \\ c_{ij} & = & \alpha_i + \beta_j & \forall\, (i,j) \in A(Q), \\ c_{it} & = & \alpha_i + \beta_t & \forall\, i \in Q. \end{array}$$

Considering for any two nodes $i \neq j \in Q$ the $(s,t) - 4$-paths $(s,4,k,i,t)$ and $(s,4,k,j,t)$, where k is the remaining node in Q, we see that $c_{ki} + c_{it} = c_{kj} + c_{jt}$ which implies that $\alpha_i + \beta_i = \alpha_j + \beta_j$ for all $i,j \in Q$. Denoting by λ this common value and setting $\pi_s := \alpha_s$, $\pi_j := \alpha_j$ for $j = 1,2,3$, and $\pi_t := \lambda - \beta_t$, yields $c_{si} = \lambda + \pi_s - \pi_i$, $c_{it} = \lambda + \pi_i - \pi_t$ for $i = 1,2,3$, and $c_{ij} = \lambda + \pi_i - \pi_j$ for all $(i,j) \in A(Q)$. Now setting $\pi_4 := \lambda + \pi_s - c_{s4}$, we see that $c_{4t} = \lambda + \pi_4 - \pi_t$, $c_{i4} = \lambda + \pi_i - \pi_4$, and $c_{4i} = \lambda + \pi_4 - \pi_i$ for $i = 1,2,3$.

(iii) This is Lemma 6 of Hartmann and Özlük [10].

(iv) Without loss of generality, let $1,2 \in R$. Condition (iii) implies that there are λ, π_s, π_1, π_2, and π_t with the required property restricted on $Q := \{1,2\}$. Further, it follows that $\gamma = 3\lambda + \pi_s - \pi_t$ and $\delta = 2\lambda + \pi_s - \pi_t$. Setting $\pi_i := \lambda + \pi_s - c_{si}$ for all $i \in R \setminus Q$, we see immediately that $c_{it} = \lambda + \pi_i \pi_t$ for all $i \in R \setminus Q$. Thus we also obtain $c_{ij} = \lambda + \pi - \pi_j$ for all $(i,j) \in A(R)$. □

Equivalence of inequalities is an important matter when studying polyhedra. Two valid inequalities for the $(0,n) - p$-path polytope $P^p_{0,n-\text{path}}(D)$ are equivalent if one can be obtained from the other by multiplication with a positive scalar and adding appropriate multiples of the flow conservation constraints (3) and the cardinality constraint (4). Clearly, two valid inequalities define the same facet of $P^p_{0,n-\text{path}}(D)$ if and only if they are equivalent. For the next theorem that establishes a relationship between a linear basis of equality system (3), (4) and the arcs defining it we introduce the following two definitions: a *balanced cycle* is a (not necessarily directed) simple cycle that contains the same number of forward and backward arcs and an *unbalanced 1-tree* is a subgraph of D consisting of a spanning tree T plus an arc (k,l) whose fundamental cycle $C(k,l)$ is not balanced.

Theorem 2.3 (cf. Theorem 3 of Hartmann and Özlük [10])**.** *Let $n \geq 2$ and let H be a subgraph of D. The variables corresponding to the arcs of H form a basis for the linear equality system* (3), (4) *if and only if H is an unbalanced 1-tree.* □

Corollary 2.4 (cf. Corollary 4 of Hartmann and Özlük [10])**.** *Let $\boldsymbol{c}\boldsymbol{x} \leq c_0$ be a valid inequality for $P^p_{0,n-path}(D)$, and let values b_{ij} be specified for the arcs (i,j) in an unbalanced 1-tree H. Then there is an equivalent inequality $\boldsymbol{c}'\boldsymbol{x} \leq c'_0$ for which $c'_{ij} = b_{ij}$ for all arcs $(i,j) \in H$.* □

Corollary 2.5 (cf. Corollary 5 of Hartmann and Özlük [10])**.** *Let $3 \leq p < n$, $\boldsymbol{c}$ be a row vector, $s \in V \setminus \{n\}$, $t \in V \setminus \{0\}$, $s \neq t$, $R \subseteq V \setminus \{s,t,0,n\}$ with*

$|R| \geq 2$, let either of the conditions of Lemma 2.2 be satisfied, and suppose that $c_{ij} = \beta$ holds for all (i,j) in an unbalanced 1-tree H on R. Then $c_{ij} = \beta$ for all $i,j \in R$. Moreover, there are σ and τ with $c_{si} = \sigma$ and $c_{it} = \tau$ for all $i \in R$.

Proof. In either case, Lemma 2.2 implies that there are λ, π_s, π_t, and $\{\pi_j | j \in R\}$ with

$$\begin{array}{rcll} c_{si} & = & \lambda + \pi_s - \pi_i & \forall\, i \in R, \\ c_{it} & = & \lambda + \pi_i - \pi_t & \forall\, i \in R, \\ c_{ij} & = & \lambda + \pi_i - \pi_j & \forall\, (i,j) \in A(R), \end{array}$$

Without loss of generality, let $\pi_k = 0$ for some $k \in R$. Theorem 2.3 then implies that $\lambda = \beta$ and $\pi_j = 0$ for all $j \in R$. Thus, $c_{si} = \beta + \pi_s$ and $c_{it} = \beta - \pi_t$ for all $i \in R$. □

The next theorem can be used to lift facet defining inequalities for the $(0,n)-$ p-path polytope $P^p_{0,n-\text{path}}(D)$ into facet defining inequalities for $P^p_{0,n-\text{path}}(D')$, where $D' = D_{n+k+1} - (\delta^-(0) \cup \delta^+(n))$. Before stating it we need some definitions. A subset $B \subseteq A$ of cardinality p is called a *p-bowtie* if it is the union of a $(0,n)$-path P and a simple cycle C connected at exactly one node. The p-bowtie B is said to be *tied* at node k if $V(P) \cap V(C) = \{k\}$. A facet F of $P^p_{0,n-\text{path}}(D)$ is called *regular* if it is defined by an inequality $\mathbf{cx} \leq c_0$ that is not equivalent to a nonnegativity constraint $x_{ij} \geq 0$ or a *broom inequality*

$$x((\delta^+(i)) \geq x_{ji} + x_{ik} \tag{9}$$

for some internal node i, where $j = k$ is an internal node or $j = 0$ and $k = n$. Note that F is already regular if for each internal node k, there is a $(0,n)-p$-path P with $c(P) < c_0$ that does not visit node k (see [10]).

Theorem 2.6 (cf. Theorem 8 of Hartmann and Özlük [10])**.** *Suppose that $\boldsymbol{cx} \leq c_0$ induces a regular facet of $P^p_{0,n-path}(D)$, where $3 < p < n$. Let k be an internal node such that $c(B) \leq c_0$ for all p-bowties B tied at node k and let δ_k be the maximum of $c(\Gamma)$ over all $0,n$-paths Γ of length $p-1$ that visit node k. Then*

$$\mathbf{cx} + \sum_{\substack{i=0 \\ i \neq k}}^{n-1} c_{ik} x_{i,n+1} + \sum_{\substack{j=1 \\ j \neq k}}^{n} c_{kj} x_{n+1,j} + (c_0 - \delta_k)[x_{k,n+1} + x_{n+1,k}] \leq c_0 \tag{10}$$

defines a regular facet of $P^p_{0,n-path}(D')$, where D' is the digraph obtained by subtracting from the complete digraph on node set $\{0, \ldots, n+1\}$ the arc sets $(\delta^-(0)$ and $\delta^+(n))$. □

Since inequality (10) is obtained by copying the coefficient structure of node k, one refers to this process as "lifting by cloning node k". In order to show that a class $\mathcal{K}$ of regular inequalities define facets of the $(0,n)-p$-path polytope it suffices to show it for a subclass $\mathcal{K}' \subset \mathcal{K}$ from which the remaining inequalities in $\mathcal{K} \setminus \mathcal{K}'$ can be obtained by cloning internal nodes. The members of a minimal subclass $\mathcal{K}'$ (minimal with respect to set inclusion) are said to be *primitive.*

Before stating the last theorem of this section we also need some definitions. Let F be a subset of A, the *auxiliary graph* G_F is an undirected bipartite graph on $2n$ nodes $v_0, \ldots, v_{n-1}$, $w_1, \ldots, w_n$, with the property that $(i,j) \in F$ if and only if G_F contains the arc (v_i, w_j). Given a valid inequality $\mathbf{cx} \le c_0$, a $(0,n)-p$-path P is said to be *tight* if $c(P) = c_0$. Moreover, we define the following equivalence relation on the arc set A: two arcs (i,j) and (k,l) are related with respect to $\mathbf{cx} \le c_0$, if there is an arc $(f,g) \in A$ with $a_{ij} = a_{fg} = a_{kl}$ and two tight $(0,n)-p$-paths P_{ij}, P_{kl} such that $(i,j),(f,g) \in P_{ij}$ and $(k,l),(f,g) \in P_{kl}$.

Theorem 2.7 (cf. Theorem 9 of Hartmann and Özlük [10]). *Let $\boldsymbol{a} \ge \boldsymbol{0}$ and $\boldsymbol{ax} \le a_0$ be a facet defining inequality for $P^p_{0,n-path}(D)$, where $3 < p < n$. Suppose that the auxiliary graph G_Z for the arc set $Z := \{(i,j) \in A | a_{ij} = 0\}$ is connected, every tight $(0,n)-p$-path with respect to $\boldsymbol{ax} \le a_0$ contains at least one arc $(i,j) \in Z$, and every arc (i,j) belongs to the same equivalence class with respect to $\boldsymbol{ax} \le a_0$. Let R be a set of nodes, set $q := p + |R|$, and let t be the smallest number such that*

$$\boldsymbol{ax} + t \sum_{j \in R} x(\delta^+(j)) \le a_0 + |R|t \tag{11}$$

is valid for all $(0,n)-q$-paths on $V \cup R$, and if $|R| \ge 2$ suppose that at least one tight $(0,n)-q$-path with respect to (11) visits r nodes in R with $0 < r < |R|$. Then (11) is facet defining for the $(0,n)-q$-path polytope on $V \cup R$. □

3 Facets and valid inequalities

In the sequel we will show that the inequalities given in the IP-formulation, the *nonnegativity constraints* $x_{ij} \ge 0$, as well as some more inequalities are in general facet defining for $P^p_{0,n-\text{path}}(D)$. Throughout, we assume that $4 \le p \le n-1$. The inequalities considered in Theorems 3.1 - 3.5 were shown to be valid for the p-cycle polytope in Hartmann and Özlük [10]. So they are also valid for $P^p_{0,n-\text{path}}(D)$, since the $(0,n)-p$-path polytope on D can be interpreted as the restriction of the p-cycle polytope on D_n to the hyperplane defined by $x(\delta^+(n)) = 1$.

3.1 Trivial inequalities

Theorem 3.1 (cf. Theorem 10 of Hartmann and Özlük [10]). *The nonnegativity constraint*

$$x_{ij} \ge 0 \tag{12}$$

is valid for $P^p_{0,n-path}(D)$ and induces a facet of $P^p_{0,n-path}(D)$ whenever $4 \le p \le n-1$.

Proof. When $n \le 6$ and $p = 4$ or $p = 5$, (12) can be proved to induce a facet by application of a convex hull code (e.g. Polymake [8]), so we assume that $n \ge 7$. Suppose that $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ with $x_{ij} = 0$. At least one of the two nodes i and j is an internal node, because $(0,n) \notin A$. Without loss of generality, we may assume that $j \in \{1,\dots,n-1\}$ and set $R := V \setminus \{0,n,j\}$. By Corollary 2.4, we may assume that $c_{jw} = c_{0w} = c_{wn} = 0$ for some $w \in R$ and $c_{kl} = 0$ for all arcs (k,l) in some unbalanced 1-tree on R.

Let $q \in R \cup \{0\}$, $r,s \in R$, $t \in R \cup \{n\}$ be distinct nodes and let P be a $(0,n)-p$-path that contains the arcs (q,r) and (r,t) but does not visit node s or use the arc (i,j). Substituting node r by node s in P we obtain another

$(0, n) - p$-path that does not use (i, j). Hence condition (2.2) of Lemmma 2.2 holds and Corollary 2.5 implies that $c_{kl} = 0$ for all $(k, l) \in A(V \setminus \{j\})$ which also implies that $c_0 = 0$.

Each $(0, n) - p$-path that uses the arc (j, w) but does not use the arc (i, j) also satisfies (12) with equality, so $c_{kj} = 0$ for all $k \in V \setminus \{i, n, w\}$. Similar considerations yield $c_{jk} = 0$ for all $k \in V \setminus \{0\}$ and $c_{wj} = 0$ if $w \neq i$. Thus, $c_{kl} = 0$ for all arcs $(k, l) \neq (i, j)$ and therefore $\mathbf{cx} = c_0$ is simply $c_{ij}x_{ij} = 0$. □

Theorem 3.2 (cf. Theorem 11 of Hartmann and Özlük [10]). *Let j be an internal node. The degree constraint*

$$x(\delta^+(j)) \leq 1 \tag{13}$$

is valid for $P^p_{0,n-path}(D)$ and induces a facet of $P^p_{0,n-path}(D)$ whenever $4 \leq p \leq n - 1$.

Proof. Without loss of generality, we will show that $x(\delta^+(1)) \leq 1$ defines a facet of $P^p_{0,n-\text{path}}(D)$. First we will show that Theorem 3.2 holds when $p = 4$. If $n = 5$, $x(\delta^+(1)) \leq 1$ can be proved to define a facet using a convex hull code. Theorem 2.6 applied to node 2 yields then the result when $n \geq 6$.

Secondly, we will investigate the case $p \geq 5$. Suppose that $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ with $x(\delta^+(1)) = 1$. By Corollary 2.4, we may assume that $c_{21} = c_{02} = c_{2n} = 0$ and $c_{ij} = 0$ in some unbalanced 1-tree on $R := \{2, 3, \ldots, n - 1\}$. Since $|R| \geq p - 1 \geq 4$ and $c(P) = c_0 - c_{01}$ for all $(1, n)$-paths P of length $p - 1$ whose internal nodes are all in R, condition (2.2) of Lemma 2.2 holds. Thus, $c_{ij} = 0$ for all $(i, j) \in A(R \cup \{n\})$ and $c_{1j} = 1$ for all $j \in R$ using Corollary 2.5. Now it is easy to see that $\mathbf{cx} = c_0$ is simply $x(\delta^+(1)) = 1$. □

3.2 Cut inequalities

Theorem 3.3 (cf. Theorem 12 of Hartmann and Özlük [10]). *Let $S \subset V$ and $0, n \in S$. The min-cut inequality*

$$x((S : V \setminus S)) \geq 1 \tag{14}$$

is valid for $P^p_{0,n-path}(D)$ if and only if $|S| \leq p$ and facet defining for $P^p_{0,n-path}(D)$ if and only if $3 \leq |S| \leq p$ and $|V \setminus S| \geq 2$.

Proof. The min-cut inequality (14) is valid for $P^p_{0,n-\text{path}}(D)$ if and only if $|S| \leq p$, since a $(0, n) - p$-path can be obtained in S if and only if $|S| \geq p + 1$. When $|S| = 2$, (14) is an implicit equation. When $|V \setminus S| = 1$, $n \leq p$. So we suppose that $3 \leq |S| \leq p$ and $|V \setminus S| \geq 2$.

First let $|S| = 3$. When $|V \setminus S| \leq 4$, (14) can be shown to be facet defining by means of a convex hull code, so let $|V \setminus S| \geq 5$. Let w.l.o.g. $S = \{0, 1, n\}$ and suppose that $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ that satisfies (14) with equality. Using Corollary 2.4, we may assume that $c_{01} = 0$, $c_{0w} = c_0$ and $c_{wn} = 0$ for some $w \in V \setminus S$, as well as $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree H on $V \setminus S$.

Let $i \in (V \setminus S) \cup \{0\}$, $j \in (V \setminus S) \cup \{n\}$, $k, l \in V \setminus S$ be distinct nodes and let P be a tight $(0, n) - p$-path that contains the arcs $(i, k), (k, j)$ but does not visit node l. Such a path P exists even when $p = 4$. Replacing node k by node

l yields another tight $(0,n)-p$-path , and hence condition (2.2) of Lemma 2.2 holds. Corollary 2.5 implies that $c_{ij} = 0$ for all $(i,j) \in A(V \setminus S)$, $c_{0i} = c_0$, and $c_{in} = 0$ for all $i \in V \setminus S$. Now it is easy to see that $c_{1i} = c_0$ and $c_{i1} + c_{1n} = 0$ for all $i \in V \setminus S$. Subtracting c_{1n} times the equation $x(\delta^-(n)) = 1$ and adding c_{1n} times the equation $x((V \setminus S : S)) - x((S : V \setminus S)) = 0$, we see that $\mathbf{cx} = c_0$ is equivalent to $(c_0 - c_{1n})x(S : V \setminus S) = c_0 - c_{1n}$.

Secondly, let $|S| \geq 4$. Let w.l.o.g. $S = \{0, 1, 2, \ldots, q, n\}$ for some $q < p$ and suppose that $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ that satisfies (14) with equality. Using Corollary 2.4, we may assume that $c_{01} = c_{1n} = 0$, $c_{1i} = c_0$ for all $i \in (V \setminus S)$, and $c_{ij} = 0$ for all arcs (i,j) in some unbalanced 1-tree on $R := S \setminus \{0, n\}$.

Let P be the path $(q+1, \ldots, p-1, n)$ and Q be the path $(q+1, \ldots, p, n)$ Then $c(\Gamma) = c_0 - c(P)$ for all $(0, q+1)$-paths Γ, whose internal nodes are all the nodes of R. Further, $c(\Delta) = c_0 - c(Q)$ for all $(0, q+1)$-paths Δ, all q of whose internal nodes are in R. Therefore, condition (2.2) of Lemma 2.2 holds and Corollary 2.5 implies that $c_{ij} = 0$ for all $(i,j) \in A(R \cup \{0\})$ and $c_{i,q+1} = c_0$ for all $i \in R$. Replacing node $q+1$ by any other node in $V \setminus S$ (in the above argumentation), we obtain $c_{ij} = c_0$ for all $(i,j) \in (R : V \setminus S)$.

Next, consider for any arc $(i,j) \in A(V \setminus S)$ a tight $(0,n)-p$-path P that uses the arcs $(0,1), (1,2), (2,j)$ and skips node i. Then the $(0,n)-p$-path $P' := (P \setminus \{(0,1), (1,2), (2,j)\}) \cup \{(0,2), (2,i), (i,j)\}$ is also tight. Thus, we derive that $c_{ij} = 0$ for all $(i,j) \in A(V \setminus S)$. Further, from the tight $(0,n)-p$-paths that starts with the arc $(0,1)$ and use some arc (i,n) with $i \in V \setminus S$ we deduce $c_{in} = 0$ for all those arcs (i,j). Moreover, from the tight $(0,n)-p$-paths starting with the arc $(0,2)$ and ending with the arcs $(i,1), (1,n)$ for some $i \in V \setminus S$ we obtain $c_{i1} = 0$ for $i \in V \setminus S$. It is now easy to see that $c_{0i} = c_0$ for all $i \in V \setminus S$, $c_{jn} = 0$ for all $j \in R$, and $c_{ij} = 0$ for all $(i,j) \in (V \setminus S : R)$ (distinguish the cases $p = 4$ and $p \geq 5$). Therefore $\mathbf{cx} = c_0$ is simply $c_0 x((S : V \setminus S)) = c_0$. $\square$

Theorem 3.4 (cf. Theorem 13 of Hartmann and Özlük [10])**.** *Let $S \subset V$ and $0, n \in S$. The one-sided min-cut inequality*

$$x((S : V \setminus S)) \geq x(\delta^+(l)) \tag{15}$$

is valid for $P^p_{0,n-path}(D)$ for all $l \in V \setminus S$, and facet defining for $P^p_{0,n-path}(D)$ if and only if $|S| \geq p+1$ and $|V \setminus S| \geq 2$.

Proof. The one-sided min-cut inequality (15) is valid, because all $(0,n)-p$-paths that visits some node $l \in V \setminus S$ use at least one arc in $(S : V \setminus S)$. If $|V \setminus S| = 1$, then (15) is the flow constraint $x(\delta^-(l)) - x(\delta^+(l)) = 0$. If indeed $|V \setminus S| \geq 2$ but $|S| \leq p$, then (15) can be obtained by summing the min-cut inequality (14) and the degree constraint $-x(\delta^+(l)) \geq -1$.

So suppose that $|S| \geq p+1$ and $|V \setminus S| \geq 2$. Let w.l.o.g. $l = 1$ and set $R := S \cup \{1\}$. By adding to (15) the flow constraint $x(\delta^+(1)) - x(\delta^-(1)) = 0$, it can be easily seen that (15) is equivalent to

$$x((S : V \setminus R)) - \sum_{i \in V \setminus R} x_{i1} \geq 0. \tag{16}$$

Suppose that $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ that satisfies (16) with equality. By Corollary 2.4, we may assume that $c_{in} = 0$ for all $i \in V \setminus R$

and $c_{ij}=0$ for all arcs (i,j) in some unbalanced 1-tree on R. Condition (2.2) of Lemma 2.2 is satisfied; hence, from Corollary 2.5 follows that $c_{ij}=0$ for all $(i,j)\in A(R)$ which also implies that $c_0=0$.

Any $(0,n)-p$-path that contains the arcs $(1,i),(i,n)$ for some $i\in V\setminus R$ and whose remaining arcs are in $A(R)$ satisfies (16) with equality. Since $c_{in}=0$ and $c_a=0$ for all $a\in A(R)$, it follows that $c_{1i}=0$ for all $i\in V\setminus R$. Now considering tight $(0,n)-p$-paths that contain the arcs $(1,i),(i,j),(j,n)$ for some $(i,j)\in A(V\setminus R)$ and whose remaining arcs are in $A(R)$, we see that $c_{ij}=0$ for all $(i,j)\in A(V\setminus R)$. Further, the $(0,n)-p$-paths that use the arcs $(1,i),(i,j)$ for $i\in V\setminus R, j\in S\setminus\{n\}$ and whose remaining arcs are in $A(R)$ yield $c_{ij}=0$ for all $(i,j)\in(V\setminus R:S\setminus\{n\})$. Finally, considering for each $(i,j)\in(S:V\setminus R)$ and $k\in V\setminus R$ a tight $(0,n)-p$-path that contains the arcs $(i,j),(j,1)$ and a tight $(0,n)-p$-path that contains the arcs $(i,j),(j,k),(k,1)$, we see that $c_{j1}=c_{k1}$ for all $j,k\in V\setminus R$, $c_{ij}=c_{kl}$ for all $(i,j),(k,l)\in(S:V\setminus R)$, and $c_{ij}+c_{k1}=0$ for all $(i,j)\in(S:V\setminus R)$, $k\in V\setminus R$. Thus $\mathbf{cx}=c_0$ is simply $c_{jk}x((S:V\setminus R))-c_{jk}\sum_{i\in V\setminus R}x_{i1}=0$ for some $(j,k)\in(S:V\setminus R)$. □

Theorem 3.5 (cf. Theorem 15 of Hartmann and Özlük [10])**.** *Let $\langle R,S,T\rangle$ be a partition of V and let $0,n\in S$. The generalized max-cut inequality*

$$x((S:T))+\sum_{i\in R}x(\delta^+(i))\le\lfloor(p+|R|)/2\rfloor \tag{17}$$

is valid for the $(0,n)-p$-path polytope $P^p_{0,n-path}(D)$ for $p\ge 4$ and facet defining for $P^p_{0,n-path}(D)$ if and only if $p+|R|$ is odd, $|S\setminus\{n\}|>(p-|R|)/2$, $|T|>(p-|R|)/2$, and

(i) $p=|R|+3$, $|R|\ge 2$, and $|S|=3$, or

(ii) $p\ge|R|+5$.

Proof. Necessity. From $x(A)=p$ and $x((S:T))\le x((T:S))+x((T:R))$ we derive the inequality $2x((S:T))+\sum_{i\in R}x(\delta^+(i))\le p$. Adding the inequality $\sum_{i\in R}x(\delta^+(i))\le|R|$, dividing by two, and rounding down, we obtain (17). When $p+|R|$ is even, then (17) is obtained with no rounding, and hence it is not facet defining. When $|S\setminus\{n\}|\le(p-|R|)/2$ or $|T|\le(p-|R|)/2$, then (17) is implied by degree constraints $x(\delta^(i))\le 1$.

Let P be any $(0,n)-p$-path and denote by r the number of nodes in R visited by P. Then $|v(P)\cap(S\setminus\{n\}\cup T)|=p-r$ and hence $\chi^P((S:T))\le(p-r)/2$. This in turn implies that there is no tight $(0,n)-p$-path if $r\le|R|-2$, where $|R|\ge 2$. Now, when $p=|R|+3$ and $|S|\ge 4$, (17) is dominated by nonnegativity constraints $x_{ij}\ge 0$ for $(i,j)\in A(S\setminus\{0,n\})$. Further, when $p=|R|+3$, $|S|=3$, and $|R|=1$, (17) is dominated by the inequality (23). Finally, when $p\le|R|+1$, (17) is dominated by some nonnegativity constraints, for example, $c_{in}=0$ for some $i\in T$.

Suffiency. First we will show that (17) is facet defining if $R=\emptyset$. In this the case, the resulting inequality

$$x((S:T))\le\lfloor p/2\rfloor=q \tag{18}$$

where $p=2q+1$, is called *max-cut inequality*. First, we show that (18) is facet defining for $P^p_{0,n-\mathrm{path}}(D)$. If $p=5$ and $|S\setminus\{n\}|=3$ or $|T|=3$, we will show that (18) defines a facet using Theorem 2.6. The only primitive inequalities are those with $n = 6$ and by application of a convex hull code, we see that in this case (18) is facet defining for $P^p_{0,n-\text{path}}(D)$. Moreover, (18) is regular, since for each inner node k there is a non-tight $(0, n) - p$-path that does not visit k. Without loss of generality, let $T = \{1, 2, \ldots, t\}$ and $S = \{t + 1, \ldots, n, 0\}$ for some $4 \leq t \leq n - 4$.

Suppose that $\mathbf{cx} = c_0$ holds for all $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ satisfying (18) with equality. By Corollary 2.4, we may assume that $c_{02} = 1$, $c_{t+1,n} = 0$, $c_{j1} = 1$ for all $j \in S \setminus \{n\}$, and $c_{1i} = 0$ for all $i \in T$.

First, consider any $(0, n)-2q$-path P that alternates between nodes in S and nodes in T, but does not visit node 1. Replacing any arc $(i, j) \in P$ with $i \in S$, $j \in T$ by the arcs $(i, 1), (1, j)$ we obtain a tight $(0, n) - p$-path , and therefore $c(P) - c_{ij} = c_0 - 1$ holds for all $(i, j) \in P \cap (S : T)$. This in turn implies that $c_{ij} = 1$ for all $(i, j) \in (S : T)$, since we have $3 \leq t \leq n - 3$ and $c_{02} = 1$. Next, consider any tight $(0, n) - p$-path that uses arcs $(i, k), (k, j)$ for $i, j \in S \setminus \{0, n\}$, $k \in T$ but does not visit node $l \in T$. Replacing node k by node l yields another tight path which implies immediately $c_{ik} + c_{kj} = c_{il} + c_{lj}$. Similarly we obtain $c_{ki} + c_{il} = c_{kj} + c_{jl}$ and thus $c_{ik} + c_{ki} = c_{jl} + c_{lj}$ for all $i, j \in S \setminus \{0, n\}$ and $k, l \in T$. Since $t \geq 3$ and $c_{ik} = c_{jl} = 1$, we see that there is some σ with $c_{ki} = \sigma$ for all $k \in T$, $i \in S \setminus \{0, n\}$. Now consider any tight path that contains the arcs $(1, t+1), (t+1, n)$ and does not visit some node $l \in T$. Replacing node $t+1$ by node l yields another tight $(0, n) - p$-path and hence $c_{1,t+1} + c_{t+1,n} = c_{1l} + c_{ln}$. Since $c_{1,t+1} = \sigma$ and $c_{t+1,n} = c_{1l} = 0$, this implies $c_{ln} = \sigma$ for all $l \in T$, $l \neq 1$. Of course, it follows also that $c_{1n} = \sigma$.

Finally, any tight $(0, n)-p$-path contains exactly one arc $(i, j) \in A(S) \cup A(T)$, so $c_{ij} = c_0 - q(1 + \sigma)$ for all $(i, j) \in A(S) \cup A(T)$. Due to $c_{t+1,n} = 0$, this implies that $c_{ij} = 0$ for all $(i, j) \in A(S) \cup A(T)$. Adding σ times the equation $x((S : T)) - x((T : S)) = 0$, we see that $\mathbf{cx} = c_0$ is equivalent $x((S : T)) = q$. This proves that (refoddmaxcut) is also facet defining when $0, n \in T$.

When $R \neq \emptyset$, we prove the claim by showing that the conditions of Theorem 2.7 hold for (18). Since $w = p - |R|$ is odd and $w \geq 5$, $x(S : T) \leq \lfloor w/2 \rfloor$ induces a facet of the $(0, n) - w$-path polytope defined on the digraph $D^+ = (V \setminus R, A(V \setminus R))$. Let us denote this inequality by $\mathbf{ax} \leq a_0$. It is easy to see that the auxiliary graph G_Z for the arc set $Z = \{(i, j)|a_{ij} = 0\}$ is connected (cf. [10]). Further, each tight $(0, n) - w$-path contains two arcs (i, j) and (k, l) which are not adjacent and hence all arcs in Z are in the same equivalency class with respect to $\mathbf{ax} \leq a_0$. Since there are tight $(0, n) - p$-paths with respect to (17) that visit $|R| - 1$ of the nodes in R, Theorem 2.7 implies that (17) induces a facet of $P^p_{0,n-\text{path}}(D)$ unless $(p = |R| + 3, |R| \geq 2, and |S| = 3)$.

Finally, suppose that $p = |R| + 3$,$|R| \geq 2$, and $|S| = 3$. Without loss of generality, we may assume that $S = \{0, 1, n\}$, $2, 3 \in R$, and $4, 5 \in T$. Suppose that $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ that satisfies (17) with equality. By Corollary 2.4, we may assume that $c_{2j} = 1$ for all $j \in R$, $c_{i2} = 0$ for all $i \in T$, $c_{32} = 1$, $c_{21} = 1$, $c_{1n} = 0$, and $c_{04} = 1$. There are tight $(0, n) - p$-paths that visits a node $l \in T$ followed by all $|R|$ (or any $|r| - 1$) nodes in R and a node 1. Applying Lemma 2.2, we see that

$$\begin{array}{rcll} c_{lj} & = & \lambda + \pi_l - \pi_j & (j \in R) \\ c_{ij} & = & \lambda + \pi_i - \pi_j & (i, j \in R) \\ c_{im} & = & \lambda + \pi_i - \pi_m & (i \in R) \end{array}$$

for some λ, $\{\pi_j | j \in R\}$, π_l, and π_1. Let w.l.o.g. $\pi_2 = 0$. Theorem 2.3 then implies that $\lambda = 1$ and $\pi_j = 0$ for all $j \in R$, $c_{i2} = 0$ implies that $\pi_l = -1$, and $c_{21} = 1$ implies that $\pi_1 = 0$. Thus, $c_{ij} = 1$ for all $(i,j) \in A(R)$, $c_{ij} = 0$ for all $i \in T, j \in R$, and $c_{i1} = 1$ for all $i \in R$. Next, considering any (tight) $(0,n) - p$-path P that uses the arcs $(0,4), (2,1), (1,n)$ and visits all $|R|$ nodes in R yields $c_0 = |R| + 1$. Replacing node 4 by another node $j \in T$ yields $c_{0j} = 1$ for all $j \in T$. Next, consider any tight $(0,n) - p$-path P that uses the arcs $(0,i), (i,j), (j,1)$ for some $i, j \in R$. Then the $(0,n) - p$-path $P' := (P \setminus \{(0,i),(i,j),(j,1)\}) \cup \{(0,j),(j,i),(i,1)\}$ is also tight, and hence, $c_{0i} = c_{0j}$ for all $i, j \in R$. Denote this common value by σ. From the tight $(0,n) - p$-paths that visits the nodes 1 and t for some $t \in T$ and all nodes in R, we derive $c_{ij} = 1 - \sigma$ for all $i \in R, j \in T$. Now it is easy to see that $c_{in} = 1 + \sigma$ for all $i \in T$. Considering any tight $(0,n) - p$-path that uses the arcs $(0,2), (2,1), (1,4), (4,3)$, and (m,n) for an appropriate $m \in R$ yields $\sigma = 0$. Thus, $c_{0i} = 0$ and $c_{in} = 1$ for all $i \in R$, $c_{1j} = 1$ for all $j \in T$, and $c_{ij} = 1$ for all $i \in R, j \in T$. Determining the coefficients of the remaining arcs is an easy task. So we see that $\mathbf{cx} = c_0$ is simply (17). □

Theorem 3.6. *Let $\langle R, S, T\rangle$ be a partition of V and let $0, n \in T$. The generalized max-cut inequality*

$$x((S:T)) + \sum_{i \in R} x(\delta^+(i)) \leq \lfloor (p + |R|)/2 \rfloor \tag{19}$$

is valid for the $(0,n) - p$-path polytope $P^p_{0,n-path}(D)$ for $p \geq 4$ and facet defining for $P^p_{0,n-path}(D)$ if and only if $p + |R|$ is odd, $|S| > (p - |R|)/2$, $|T \setminus 0| > (p - |R|)/2$, and

(i) $p = |R| + 3$, $|R| \geq 2$, *and* $|T| = 3$, *or*

(ii) $p \geq |R| + 5$.

□

Theorem 3.7. *Let $\langle R, S, T\rangle$ be a partition of V, let $0 \in S$, and let $n \in T$. The generalized max-cut inequality*

$$x((S:T)) + \sum_{i \in R} x(\delta^+(i)) \leq \lfloor (p + |R| + 1)/2 \rfloor \tag{20}$$

is valid for the $(0,n) - p$-path polytope $P^p_{0,n-path}(D)$ for $p \geq 4$ and facet defining for $P^p_{0,n-path}(D)$ if and only if $p + |R|$ is even, $p \geq |R| + 4$, $|S| > (p - |R|)/2$, and $|T| > (p - |R|)/2$.

Proof. From the equation $x(A) = p$ and the inequality $x((S:T)) \leq x((T:S)) + x((T:R)) + 1$ we derive the inequality $2x((S:T)) + \sum_{i \in R} x(\delta^+(i)) \leq p + 1$. Adding the inequality $\sum_{i \in R} x(\delta^+(i)) \leq |R|$, dividing by two, and rounding down yields (20). If $p + |R|$ is odd we obtain (20) without rounding and hence it is not facet defining for $P^p_{0,n-\mathrm{path}}(D)$. When $|S| \leq (p - |R|)/2$ or $|T| \leq (p - |R|)/2$, (20) is dominated by degree constraints $x(\delta^+(j)) \leq 1$. Furthermore, we have to show that (20) is not facet defining if $p \leq |R| + 2$. When $R = \emptyset$, it is clear. Otherwise consider any $(0,n) - p$-path P and denote the number of nodes in

R visited by P by r. It is easy to see that P is tight only if $r \geq |R|-1$. For the sake of contradiction, assume that $p \leq |R|$ and P is tight. Then we have $r=|R|-1$ and thus $p=|R|$ which implies $\lfloor(p+|R|+1)/2\rfloor=|R|$. But $\chi^P((S:T))+\sum_{i\in R} chi^P(\delta^+(i))=|R|-1$, so P is not tight, a contradiction. Hence, the only possibility is that $p=|R|+2$. Now, $p=|R|+2$ implies that $|S|,|T|\geq 2$ and $\lfloor(p+|R|+1)/2\rfloor=|R|+1$. But then (20) is dominated by the nonnegativity constraints $x_{ij}\geq 0$ for all $(i,j)\in A(S)\cup A(T)$.

First, we show that (20) is facet defining when $R=\emptyset$. In this case, p is even and (20) is the max-cut inequality

$$x((S:V\setminus S))\leq\lfloor(p+1)/2\rfloor=p/2. \tag{21}$$

If $p=4$ and $|S|=3$ or $|V\setminus S|=3$, we will show that (21) defines a facet of $P^p_{0,n-\text{path}}(D)$ using Theorem 2.6. The only primitive members of family (21) with $p=4$ are those with $|S|=|V\setminus S|=3$. Inequality (21) is obviously regular and using a convex hull code, we see that (21) defines a facet of $P^p_{0,n-\text{path}}(D)$. Moreover, all p-bowties tied at an inner node satisfy (21).

If $p\geq 6$ suppose that the equation $\mathbf{cx}=c_0$ is satisfied by every $\mathbf{x}\in P^p_{0,n-\text{path}}(D)$ that satisfies ((21)) with equality. Let w.l.o.g. $1,2\in V\setminus S$. By Corollary 2.4, we may assume that $c_{02}=1$, $c_{i1}=1$ for all $i\in S$, and $c_{1j}=0$ for all $j\in V\setminus S$, $j\neq 1$. Since $|S|,|V\setminus S|\geq 4$, we can apply the same argumentation as in the proof to Theorem 3.5. Thus $c_{ij}=1$ for all $(i,j)\in(S:V\setminus S)$, $c_{ij}=\sigma$ for all $(i,j)\in(V\setminus(S\cup\{n\}):S\setminus\{0\})$, for some σ, and $c_{ij}=0$ for all $(i,j)\in A(S)\cup A(T)$. Evaluating the cost of tight $(0,n)-p$-paths yields $c_0=\frac{p}{2}+(\frac{p}{2}-1)\sigma$ which implies that $\mathbf{cx}=c_0$ is the equation $x((S:V\setminus S))+\sigma x((V\setminus S:S))=\frac{p}{2}+\sigma(\frac{p}{2}-1)$. Adding σ times the equation $x((S:V\setminus S))-x((V\setminus S:S))=1$, we see that (21) is equivalent to $x((S:V\setminus S))=p/2$.

Applying Theorem 2.7 to the $(0,n)-w$-path polytope defined on the digraph $D^*=(V\setminus R, A(V\setminus R))$, where $w=p-|R|$, proves that (20) is facet defining for $P^p_{0,n-\text{path}}(D)$ even for $R\neq\emptyset$. □

Theorem 3.8. *Let $\langle R,S,T\rangle$ be a partition of V, let $0\in T$, and let $n\in S$. The generalized max-cut inequality*

$$x((S:T))+\sum_{i\in R}x(\delta^+(i))\leq\lfloor(p+|R|-1)/2\rfloor \tag{22}$$

is valid for the $(0,n)-p$-path polytope $P^p_{0,n-path}(D)$ for $p\geq 4$ and facet defining for $P^p_{0,n-path}(D)$ if and only if $p+|R|$ is even, $p\geq|R|+4$, $|S|>(p-|R|)/2$, and $|T|>(p-|R|)/2$. □

Remark 3.9. *If $R=\emptyset$, inequality (22) is equivalent to the inequality*

$$x((T:S))\leq\lfloor(p+1)/2\rfloor,$$

since in this case holds the equation $x((S:T))=x((T:S))-1$.

Theorem 3.10. *Let $\emptyset\neq T=V\setminus\{0,1,2,3,n\}$. The inequality*

$$\begin{array}{rl} x_{03}-x_{3n}+3x_{12}-x_{21}+2x_{13}-2x_{31}-2x_{2n}+2x((T:\{3\})) & \\ +x(A(T))+x((\{1\}:T))-x((T:\{1\}))+x((T:\{2\}))-x((\{2\}:T) & \geq\ 0 \end{array} \tag{23}$$

is facet defining for $P^4_{(s,t)-path}(D)$.

Proof. When $|T|=1$, the claim can be verified with a convex hull code. For $|T|\geq 2$ we apply Theorem 2.6. $\square$

3.3 Jump inequalities

Dahl and Gouveia [6] introduced a class of valid inequalities for the directed hop-constrained shortest path problem (the problem of finding a minimum $(0,n)$-path with at most p arcs) they called *jump and lifted jump inequalities*. Given a partition $\langle S_0, S_1, S_2, \ldots, S_p, S_{p+1}\rangle$ of V into $p+2$ node sets, where $S_0=\{0\}$ and $S_{p+1}=\{n\}$, these inequalities encode the fact that a $(0,n)$-path P of length at most p must make at least one "jump" from a node set S_i to a node set S_j, with $j-i\geq 2$. Transferring them to the $(0,n)-p$-path polytope and lifting them (see [6]) we can give a sufficient condition for them to be facet defining for $P^p_{0,n-\text{path}}(D)$. But it seems to be hard to give a complete classification of the jump inequalities.

Theorem 3.11. *Let $\langle S_0, S_1, S_2, \ldots, S_p, S_{p+1}\rangle$ be a partition of V, where $S_0 = \{0\}$ and $S_{p+1}=\{n\}$. The jump inequality*

$$\sum_{i=0}^{p-1}\sum_{j=i+2}^{p+1} x((S_i:S_j)) - x((S_{p-1}\cup S_p : S_1\cup S_2)) \geq 1 \tag{24}$$

is facet defining for the $(0,n)-p$-path polytope $P^p_{0,n-path}(D)$ if $|S_i|\geq 2$ for $i=1,\ldots,p$.

Proof. We refer to an arc (i,j) as forward arc if $(i,j)\in(S_k:S_l)$ for some $k<l$ and as backward arc if $(i,j)\in(S_q:S_r)$ for some $q>r$. We say, the $(0,n)-p$-path P makes a "jump" with respect to (24)if P uses an arc $(i,j)\in(S_k:S_l)$ for some $0\leq k<l\leq p+1$ with $l\geq k+2$.

The jump inequality (24) is valid for $P^p_{0,n-\text{path}}(D)$, since it is valid for the path polytope $P^{\leq p}_{0,n-\text{path}}(D)$ which is the convex hull of all incidence vectors of simple $(0,n)$-paths with at most p arcs (see [6]).

To show that (24) is facet defining for $P^p_{0,n-\text{path}}(D)$, we apply Theorem 2.6. So we have to verify that the conditions of Theorem 2.6 hold for (24), when $|S_i|=2$ for $i=1,\ldots,p$, that is, when $n=2p+1$. In the sequel, let $\mathbf{dx}\geq 1$ be such an jump inequality.

Let $B=P\cup C$ be any p-bowtie , where C is a simple cycle and P is a simple $(0,n)$-path. Since $|P|\leq p$, $d(P)\geq 1$. When $d(C)\geq 0$, it follows $d(B)\geq 1$, too. Otherwise $d(C)=-1$ and C is a cycle in

$$\left(\bigcup_{j=2}^{p-2}(S_j:S_{j+1})\right)\cup(S_{p-1}:S_2),$$

since $|C|\leq p-2$. Thus, the cardinality of C is equal to $p-2$ and P is a $(0,n)-2$-path that makes two "jumps". Therefore, the jump inequality $\mathbf{dx}\geq 1$ is satisfied by all p-bowties .

Further, $\mathbf{dx}\geq 1$ is regular, since to each internal node k there exists a non-tight $(0,n)-p$-path that does not visit node k.

It remains to be shown that $\mathbf{dx}\geq 1$ is facet defining for $P^p_{0,n-\text{path}}(D)$. Without loss of generality, let $S_i=\{i,p+i\}$ for $i=1,\ldots,p$. When $p=4$ or

$p = 5$, the inequality $\mathbf{dx} \geq 1$ can be seen facet defining using a convex hull code. So let $p \geq 6$. Suppose that $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ that satisfies (24) with equality. Denoting by P the $(0, 2p+1)$-path $(0, \ldots, p, 2p+1)$, we may assume by Corollary 2.5 that $c(P) = 0$, $c_{0,p+1} = 0$, and $c_{i,p+i} = 0$ for $i = 1, \ldots, p$. Substituting two connected arcs $(i, j), (j, k) \in P$ by the arc (i, k), we see that $c_{m-1,m+1} = c_0$ for $m = 1, \ldots, p-1$, and $c_{p-1,2p+1} = c_0$. Next, replacing three connected arcs $(i, j), (j, k), (k, l) \in P$ with $i > 0$ by the arcs $(i, p+i), (p+i, l)$, we see that $c_{2p-2,2p+1} = c_0$ and $c_{p+i,i+3} = c_0$ for $i = 1, \ldots, p-3$. Further, replacing in these $(0, n) - p$-paths node i by node $p+i-1$ (for $i \geq 2$) yields $c_{m,m+1} = 0$ for $m = p+1, \ldots, 2p-3$ and considering successively the $(0, n) - p$-paths

$$(0, p+1, 4, \ldots, q, p+q, \ldots, 2p+1)$$

for $q = p, \ldots, 4$, we see that even $c_{m,m+1} = 0$ for $m = p+1, \ldots, 2p$, since $p \geq 6$. We can now easily deduce that $c_{i,p+i+1} = c_{p+i,i+1} = c_{p+i,i} = 0$ for $i = 1, \ldots, p$, $c_a = c_0$ for all $a \in (S_i : S_{i+2})$ $(i = 0, \ldots, p-1)$, and $c_a = c_0$ for all $a \in (S_i : S_{i+3})$ $(i = 0, \ldots, p-2)$. Furthermore, for each arc $a \in (S_i : S_{i+4})$, $i = 0, \ldots, p-3$, there is a tight $(0, n) - p$-path containing a that does not use any backward arc, which implies that $c_a = c_0$ for all those arcs a. Moreover, for each arc $a \in (S_m : S_{m-1})$ there is a tight $(0, n) - p$-path that uses a, makes a jump from S_i to S_{i+4} for some i, and does not use any further backward arcs. Hence, $c_a = 0$ for all $a \in (S_m : S_{m-1})$, $m = 2, \ldots, m$. It is now easy to see that the remaining coefficients can be determined as required, and therefore, $\mathbf{cx} = c_0$ is simply $c_0\mathbf{dx} = c_0$. □

3.4 Cardinality-path inequalities

The cardinality-path inequalities were originally formulated for the cardinality constrained circuit polytope. They say that a (undirected) simple cycle of cardinality at most p never uses more edges of a (undirected) simple path P of cardinality p than internal nodes of P. This idea can be transferred to the $(0, n) - p$-path polytope. Before stating the next theorem we introduce two notations. For any simple path P we denote its internal nodes by $\dot{P}$. Furthermore, we define $\text{bid}(P) := P \cup \{(i, j) | (j, i) \in P\}$.

Theorem 3.12. *Let s, t be internal nodes and P be a (s, t)-path of length $p-1$. The cardinality path inequality*

$$\sum_{i \in \dot{P}} x(\delta^-(i)) - x(\text{bid}(P)) \geq 0 \tag{25}$$

is valid for the $(0, n) - p$-path polytope $P^p_{0,n-path}(D)$ and induces a facet of $P^p_{0,n-path}(D)$ if and only if $p \in \{4, 5\}$ and $n \geq p+2$ or $p \geq 6$ and $n \geq 2p-3$.

Proof. Without loss of generality, let $P = (1, 2, \ldots, p)$.

Necessity. When $p \in \{4, 5\}$ and $n = p+1$, (25) can be seen not to induce a facet using a convex hull code. When $p \geq 6$ and $p+1 \leq n \leq 2p-4$, (25) is dominated by the nonnegativity constraints $x_{2,p-1} \geq 0$ and $x_{p-1,2} \geq 0$.

Suffiency. When the conditions in Theorem 3.12 are satisfied and the cardinality of the node set $S := \{1, p, p+1, \ldots, n-1\}$ is at most 4, (25) can be seen to induce a facet using a convex hull code. So suppose that $|S| \geq 5$ and $\mathbf{cx} = c_0$ is satisfied by every $\mathbf{x} \in P^p_{0,n-\text{path}}(D)$ that satisfies (25) with equality. By Corollary 2.5 we may assume that $c_{j,j+1} = 0$ for $j = 1, \ldots, p-2$, $c_{0,n-1} = c_{n-1,n} = 0$, and $c_{ij} = 0$ for all arcs (i,j) in some unbalanced 1-tree on S.

For any four distinct nodes $i \in S \cup \{0\}$, $j,k \in S$, and $l \in S \cup \{n\}$ there is a tight $(0,n)-p$-path that uses the arcs $(i,k),(k,j)$ and skips node l. Replacing node k by node l yields another tight $(0,n)-p$-path and thus $c_{ik}+c_{kj} = c_{il}+c_{lj}$. Using Corollary 2.5 we obtain $c_{ij} = 0$ for all $(i,j) \in A(S \cup \{0,n\})$ and therefore also $c_0 = 0$.

In the following we distinguish the three cases $p = 4$, $p = 5$, and $p \geq 6$.

CASE 1: $p = 4$

From the $(0,n)-4$-paths $(0,5,1,2,n)$ and $(0,1,2,3,n)$ we derive $c_{2n} = c_{3n} = 0$ and from the $(0,n)-4$-paths $(0,1,2,i,n)$ for $i = p, \ldots, n-1$ we derive $c_{2i} = 0$.

Next, considering the $(0,n)-4$-paths $(0,5,4,3,n)$ and $(0,4,3,2,n)$ yields $c_{43} = c_{32} = 0$. Hence, we can also deduce that $c_{3j} = 0$ for all $j \in S \setminus \{4\}$.

Further, from all tight $(0,n)-4$-paths that use the arc $(3,4)$ we deduce that $c_{ij} + c_{34} = 0$ for $(i,j) \in \{(0,2),(0,3),(1,3)\} \cup (S : \{3\})$. It follows analogously that $c_{kl} + c_{21} = 0$ for all $(k,l) \in \{(0,2),(0,3),(4,2)\} \cup (S : \{2\})$. In particular, $c_{02} + c_{21} = c_{02} + c_{34} = 0$ which implies that $c_{21} = c_{34}$ and hence, $c_{ij} + c_{kl} = 0$ for all $(i,j) \in \{(1,3),(4,2)\} \cup (S \cup \{0\} : \{2,3\})$ and $(k.l) \in \{(2,1),(3,4)\}$. So $\mathbf{cx} = c_0$ is obviously equivalent to (25).

CASE 2: $p = 5$

This case can be carry out similar as the case $p = 4$; so we omit this part of the proof.

CASE 3: $p \geq 6$

From the $(0,n)-p$-path $(0,\ldots,p-1,n)$ we derive that $c_{p-1,n} = 0$. Further, setting $T := \{3,\ldots,p-2\}$, it can be easily seen that $c_{ij} = 0$ for all $i \in T, j \in (S \setminus \{1\}) \cup \{n\}$. Next, for any arc $(i,j) \in (\dot{P} \setminus \{p-1\} : S \cup \{n\} \cup \{(p-1,n)\})$ there is a tight $(0,n)-p$-path that uses the arcs (i,j) and $(k,k+1)$ for $k = 1,\ldots,i-1$ and whose remaining arcs are in $A(S \cup \{0,n\})$. Hence, $c_{ij} = 0$ for all those arcs (i,j). Further, from the $(0,n)-p$-path $(0,\ldots,p-3,p,p-1,n)$ we derive that $c_{p,p-1} = 0$. Moreover, for any node $i \in S \setminus \{1\}$ there is a tight $(0,n)-p$-path that uses the arcs $(0,1),(1,2),(2,i),(p,p-1),(p-1,n)$ and whose remaining arcs are in $A(S)$. Thus, $c_{2i} = 0$ for all $i \in S \setminus \{1\}$. Considering further tight $(0,n)-p$-paths on node set $S \cup \{0,2,p-1,n\}$, we see that also $c_{p-1,i} = 0$ for all $i \in S \setminus \{p\}$ and $c_{2n} = 0$. Finally, considering successively the $(0,n)-p$-paths $(0,\ldots,i-2,p,p-1,\ldots,i,n)$ for $i = p-2,\ldots,2$, we find that $c_{i+1,i} = 0$ for $i = 2,\ldots,p-2$.

It remains to be shown that $c_{21} = c_{p-1,p} = \sigma$ and $c_{ij} = -\sigma$ for all arcs (i,j) in $\bigcup_{k=2}^{p-1} \delta^-(k) \setminus \text{bid}(P)$ for some σ. From the two tight $(0,n)-p$-paths $(0,4,5,\ldots,p+2,n)$ and $(0,4,3,2,1,p+1,p+2,\ldots,n)$ we derive that $c_{21} = c_{p-1,p}$. Denote this common value by σ. Since to each arc $(i,j) \in \bigcup_{k=2}^{p-1} \delta^-(k) \setminus \text{bid}(P)$ there is a tight $(0,n)-p$-path that uses either the arc $(2,1)$ or $(p-1,p)$

and therefore, $c_{ij} = -\sigma$ for all those arcs (i,j). Thus, $\mathbf{cx} = c_0$ is simply

$$\sigma x(bid(P)) - \sigma \sum_{i \in V(\dot{P})} x(\delta^-(i)) = 0.$$

$\square$

4 Facets of related polytopes

In this section, we derive facet defining inequalities for related polytopes from facet defining inequalities for the $(0,n)-p$-path polytope. We exploit three tools to do this; the first is Theorem 2.1 which can be applied to derive facets for the p-cycle polytope. The two other tools were already mentioned in Hartmann and Özlük [10]. They showed that the undirected counterpart $\bar{\mathbf{c}}\mathbf{y} \leq c_0$ of a symmetric inequality $\mathbf{cx} \leq c_0$ is facet inducing for the (undirected) p-circuit polytope $P_C^p(K_n)$ if $\mathbf{cx} \leq c_0$ is facet inducing for $P_C^p(D_n)$. Here, $\mathbf{cx} \leq c_0$ is called *symmetric* if $c_{ij} = c_{ji}$ for all $i < j$ and the induced inequality $\bar{\mathbf{c}}\mathbf{y} \leq c_0$ for $P_C^p(K_n)$ is defined by $\bar{c}_{ij} = c_{ij} = c_{ji}$ for all $i < j$. This concept can be adapted to the directed and undirected path polytopes in a modified version. We refer to 4.2. The third tool can be applied to the undirected/directed $(0,n)-p$-path or p-cycle polytopes (basic polytopes), when relaxing the cardinality constraint $x(B) = p$ to $x(B) \geq p$ or $x(B) \leq p$, where B is the ground set (the arc set or edge set). The resulting upper and lower polytopes have one dimension more than their basis polytopes, respectively, and this fact can be exploited to lift facets of the basis polytope into facets of the related upper and lower polytopes (see 4.3).

We illustrate the three tools by examples in the next subsections. In 4.2, we apply not only the second tool, but also give a short polyhedral analysis of the undirected counterpart of the $(0,n)-p$-path polytope.

4.1 New facets of the directed p-cycle polytope

Applying Theorem 2.1 to Theorems 3.7 and 3.8 we obtain some new facet defining inequalities for the directed p-cycle polytope $P_C^p(D_n)$.

Corollary 4.1. *Let $\langle \{j\}, R, S, T \rangle$ be a partition of V. The inequality*

$$x((S:\{j\})) + x((\{j\}:T)) + x((S:T)) + \sum_{i \in R} x(\delta^+(i)) \leq \lfloor (p+|R|+1)/2 \rfloor \quad (26)$$

defines a facet of the p-cycle polytope $P_C^p(D_n)$ if $p+|R|$ is even, $p \geq |R|+4$, $|S| > (p-|R|)/2-1$, and $|T| > (p-|R|)/2-1$. $\square$

Corollary 4.2. *Let $\langle \{j\}, R, S, T \rangle$ be a partition of V. The inequality*

$$x(\delta^+(r)) + x((S:T)) + \sum_{i \in R} x(\delta^+(i)) \leq \lfloor (p+|R|+1)/2 \rfloor \quad (27)$$

defines a facet of the p-cycle polytope $P_C^p(D_n)$ if $p+|R|$ is even, $p \geq |R|+4$, $|S| > (p-|R|)/2-1$, and $|T| > (p-|R|)/2-1$. $\square$

4.2 Facets of the undirected $(0,n)-p$-path polytope

The undirected $(0,n)-p$-path polytope $P^p_{[0,n]\text{ - path}}(K_{n+1})$ is the symmetric counterpart of the directed $(0,n)-p$-path polytope $P^p_{0,n-\text{path}}(D)$. Here, $K_{n+1}=(V,E)$ denotes the complete graph on node set $V=\{0,\dots,n\}$. Table 2 gives linear descriptions of $P^1_{[0,n]\text{-path}}(K_{n+1})$ and $P^2_{[0,n]\text{-path}}(K_{n+1})$. The complete polyhedral analysis of the $[0,n]-p$ - path polytope $P^3_{[0,n]\text{-path}}(K_{n+1})$ begins with the next theorem and afterwards we will turn to the $[0,n]-p$ - path polytopes $P^p_{[0,n]\text{ - path}}(K_{n+1})$ with $4\le p\le n-1$.

Theorem 4.3. *Let $K_{n+1}=(V,E)$ be the complete graph on node set $V=\{0,\dots,n\}$. Then*

$$\dim P^3_{[0,n]\text{-}path}(K_{n+1}) = |E|-n-2.$$

Proof. First note that each internal edge $e=[i,j]$ corresponds to two incidence vectors $P^{(i,j)}$ and $P^{j,i}$ of $[0,n]-3$-paths as follows: $P^{(i,j)}=\chi^{[0,i],[i,j],[j,n]}$ and $P^{(j,i)}=\chi^{[0,j],[j,i],[i,n]}$. Consider the points $P^{(k,n-1)}$, $P^{(n-1,k)}$ for $k=1,\dots,n-2$ and $P^{(i,j)}$ for $1\le i<j\le n-2$. It is easy to see that these $|E|-n-1$ points are linearly independent and thus, $\dim P^3_{[0,n]\text{-path}}(K_{n+1})\ge |E|-n-2$.

Next, all incidence vectors of $[0,n]-3$-paths satisfy the following system of linearly independent equations:

$$\begin{aligned} y_{0n} &= 0, && (28)\\ y(\delta(0)) &= 1, && (29)\\ y(\delta(n)) &= 1, && (30)\\ y(\delta(i))-2(y_{0i}+y_{in}) &= 0, \qquad i=1,\dots,n-1, && (31)\end{aligned}$$

where $\delta(j)$ denotes the set of edges which are incident with node j and $y(F)=\sum_{e\in F} y_e$ for any $F\subseteq E$. This implies that $\dim P^3_{[0,n]\text{-path}}(K_{n+1})\le |E|-n-2$, which completes the proof. □

Remark 4.4. *Adding the equations (29)-(31), subtracting two times* (28)*, and dividing by two, yields the equation*

$$\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} y_{ij} = 1. \qquad (32)$$

In the next theorem, $\delta_{in}\in\{0,1\}$ for $i=1,\dots,n-1$.

Theorem 4.5. *A complete and nonredundant linear description of the $[0,n]-3$-path polytope $P^3_{[0,n]\text{-}path}(K_{n+1})$ is given by the equations* (28)-(31)*, the nonnegativity constraints $y_{ij}\ge 0$ for $1\le i<j\le n$, and the inequalities*

$$\sum_{i=1}^{n-1}\delta_{in}y_{in}+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\left\lfloor\frac{2-\delta_{in}-\delta_{jn}}{2}\right\rfloor y_{ij}\le 1 \qquad (33)$$

for all $(n-1)$-tupels $(\delta_{1n},\dots,\delta_{n-1,n})$ satisfying $1\le\sum_{i=1}^{n-1}\delta_{in}\le n-2$.

Table 2. Polyhedral analysis of $P^1_{[0,n]\text{-path}}(K_{n+1})$ and $P^2_{[0,n]\text{-path}}(K_{n+1})$.

p	Dimension	Complete linear description
1	0	$y_{0n} = 1$ $y_{ij} = 0 \quad \forall\, [i,j] \in E \setminus \{[0,n]\}$
2	$n-2$	$y_{0n} = 0$ $y(\delta(0)) = 1$ $y_{0i} - y_{in} = 0 \quad i = 1,\dots,n-1$ $y_{0i} \geq 0 \quad i = 1,\dots,n-1$ $y_{ij} = 0 \quad 1 \leq i < j \leq n-1$

Proof. Validity. Let $\mathbf{cy} \leq 1$ be some inequality of family (33). The edge set of the support graph $G = (V,F)$, defined by $F := \{e \in E | c_e = 1\}$, decomposes into two disconnected subsets $F^n := \{[i,n] \in F | \delta_{in} = 1\}$ and $F^{\neg n} := F \setminus F^n$, and as is easily seen, each $[0,n] - 3$-path P uses at most one edge of F in the subgraph $G \subset K_{n+1}$. Hence, $\mathbf{cy} \leq 1$ is valid for $P^p_{[0,n]\text{ - path}}(K_{n+1})$.

Nonredundancy. Since the equations (28)-(31) are linearly independent, they induce a nonredundant description of the lineality space of $P^3_{[0,n]\text{-path}}(K_{n+1})$.

Next, we prove that the inequalities given in Theorem 4.5 are nonredundant by showing that the set of induced faces is an anti-chain. Let F_1 and F_2 be from two different inequalities induced faces of $P^3_{[0,n]\text{-path}}(K_{n+1})$. When F_1 and F_2 are induced by nonnegativity constraints, they are clearly not contained into each other. If only one of them is induced by a nonnegativity constraint $y_{ij} \geq 0$ $(1 \leq i < j \leq n)$, say F_1, it follows immediately that $F_2 \not\subset F_1$. Since $|V(F^{\neg n})| \geq 2$, there is also a point $P^{(k,l)}$ in F_1 that is not in F_2 and thus, $F_1 \not\subset F_2$.

Finally, let both faces not induced by nonnegativity constraints. Denote the edge sets of the support graphs corresponding to F_1 and F_2 by E_1 and E_2, respectively. Since $E_1 \not\subset E_2$ and $E_2 \not\subset E_1$, it follows also that $F_1 \not\subset F_2$ and $F_2 \not\subset F_1$.

Completeness. We will show that each facet defining inequality $\mathbf{cx} \leq c_0$ for $P^3_{[0,n]\text{-path}}(K_{n+1})$ is equivalent to a nonnegativity constraint $y_{ij} \geq 0$ $(1 \leq i < j \leq n)$ or an inequality of family (33).

Adding appropriate multiples of the equations (28)-(31), we see that $\mathbf{cy} \leq c_0$ is equivalent to an inequality $\mathbf{dy} \leq d_0$ with

(i) $d_{0i} = 0$ for $i = 1,\dots,n$,

(ii) $d_{zn} = 0$ for some internal node z,

(iii) $d_{uw} = 0$ for some internal edge $[u,w]$, and

(iv) $d_{ij} \geq 0$ for $1 \leq i < j \leq n$.

This immediately implies that $d_0 > 0$ and $0 \leq d_e \leq d_0$ for all $e \in E$.

Next, we will show that $d_e \in \{0, d_0\}$ for all $e \in E$. Suppose, for the sake of contradiction, that $M := \{[i,j] \in E | 0 < d_{ij} < d_0\} \neq \emptyset$. Assuming that there is some internal edge $[k,l] \in M$ with $[k,n],[l,n] \notin M$, we see that $d_{kn} = d_{ln} = 0$, since $d_{kl} + d_{ln} \leq d_0$ and $d_{kl} + d_{kn} \leq d_0$. Thus, $\mathbf{dy} \leq d_0$ is dominated by the inequality $\tilde{\mathbf{d}}\mathbf{y} \leq d_0$, where $\tilde{d}_{kl} = d_0$ and $\tilde{d}_e = d_e$ for all $e \in E \setminus \{[k,l]\}$. Assuming that there is some edge $[m,n]$ such that $[i,m] \notin M$ for all internal nodes $i \neq m$, yields $d_{im} = 0$ for all internal nodes $i \neq m$. Therefore $\mathbf{dy} \leq d_0$ is dominated by the inequality $\mathbf{d}'y \leq d_0$, where $d'_{mn} = d_0$ and $d'_e = d_e$ for all $e \in E \setminus \{[m,n]\}$. So we may assume in the sequel:

(a) $[i,n] \in M$ or $[j,n] \in M$ for each internal edge $[i,j] \in M$;

(b) for each edge $[k,n] \in M$ there is an internal edge $[i,k] \in M$.

In particular, we deduce that $M \cap \{[i,j] | 1 \leq i < j \leq n-1\} \neq \emptyset$ and $M \cap \{[i,n] | 1 \leq i \leq n-1\} \neq \emptyset$.

Let d_{rs} be the minimum over all edges in $M \cap \{[i,j] | 1 \leq i < j \leq n-1\}$ and d_{vn} be the minimum over all edges in $M \cap \{[i,n] | 1 \leq i \leq n-1\}$. We now construct two different inequalities $\mathbf{ay} \leq a_0$ and $\mathbf{by} \leq b_0$ that together imply $\mathbf{dy} \leq d_0$. The coefficients of the both inequalities we set as follows:

$$\begin{array}{ccccll} a_0 & = & b_0 & = & d_0, & \\ a_{ij} & = & b_{ij} & = & d_{ij} & \forall\, [i,j] \in E \setminus M, \\ a_{ij} & = & & & d_{ij} - d_{rs} & \text{for } 1 \leq i < j \leq n-1, \\ a_{kn} & = & & & d_{kn} + d_{rs} & \text{for } 1 \leq k \leq n-1, \\ & & b_{ij} & = & d_{ij} + d_{vn} & \text{for } 1 \leq i < j \leq n-1, \\ & & b_{kn} & = & d_{kn} - d_{vn} & \text{for } 1 \leq k \leq n-1. \end{array}$$

It can be easily seen that $\mathbf{dy} \leq d_0$ is a convex combination of $\mathbf{ay} \leq a_0$ and $\mathbf{by} \leq b_0$:

$$(\mathbf{d}, d_0) = \frac{d_{vn}}{d_{rs} + d_{vn}}(\mathbf{a}, a_0) + \frac{d_{rs}}{d_{rs} + d_{vn}}(\mathbf{b}, b_0).$$

Further, all three inequalities are pairwise nonequivalent; so it remains to be shown that the inequalities $\mathbf{ay} \leq a_0$ and $\mathbf{by} \leq b_0$ are valid for $P^3_{[0,n]\text{-path}}(K_{n+1})$. This can be done by checking $a_{ij} + a_{jn} \leq a_0$ and $b_{ij} + b_{jn} \leq b_0$ for all $1 \leq i,j \leq n-1$ with $i \neq j$.

Let i and j be distinct nodes in $\{1, \ldots, n-1\}$.

CASE 1: $[i,j], [j,n] \notin M$.

We have $a_{ij} = b_{ij} = d_{ij}$ and $a_{jn} = b_{jn} = d_{jn}$. Thus, $a_{ij} + a_{jn} \leq a_0$ and $b_{ij} + b_{jn} \leq b_0$, since $d_{ij} + d_{jn} \leq d_0$.

CASE 2: $[i,j] \in M$, $[j,n] \notin M$.

Since $0 < d_{ij} < d_0$, $d_{jn} \in \{0, d_0\}$, and $d_{ij} + d_{jn} \leq d_0$, we deduce that $d_{jn} = 0$. Hence, also $a_{jn} = b_{jn} = 0$. Since $a_{ij} = d_{ij} - d_{rs} < d_{ij}$, it follows that $a_{ij} + a_{jn} \leq a_0$. Due to (a), $[i,n] \in M$, and since $d_{in} \geq d_{vn}$, we deduce that $d_{ij} \leq d_0 - d_{vn}$. Thus, $b_{ij} + b_{jn} = d_{ij} + d_{vn} \leq d_0 = b_0$.

CASE 3: $[i,j] \notin M$, $[j,n] \in M$.

This implies that $a_{ij} = b_{ij} = d_{ij} = 0$ and thus, $b_{ij} + b_{jn} \leq b_0$. Due to (b), there is some internal node l such that $[l,j] \in M$. Since $d_{lj} \geq d_{rs}$, we deduce that $d_{jn} \leq d_0 - d_{rs}$ and hence, $a_{ij} + a_{jn} = d_{jn} + d_{rs} \leq d_0 = a_0$.

CASE 4: $[i,j], [j,n] \in M$.

Clear.

Thus, in all four cases, the inequalities $\mathbf{ay} \leq a_0$ and $\mathbf{by} \leq b_0$ are valid for $P^3_{[0,n]\text{-path}}(K_{n+1})$. So we have shown that $d_e \in \{0, d_0\}$ for all $e \in E$ and without loss of generality, we may assume that $d_0 = 1$.

We resume: the facet defining inequality $\mathbf{dy} \leq d_0$ satisfies (i)-(iii), $d_0 = 1$, and $d_e \in \{0,1\}$ for all $e \in E$. Note that $d_{ln} = 1$ for some internal node l implies that $d_{il} = 0$ for all internal nodes $i \neq l$.

When $d_{in} = 0$ for $1 \leq i \leq n-1$, we deduce that $d_e = 1$ for all internal edges $e \neq [u,w]$, i.e., $\mathbf{dy} \leq d_0$ is equivalent to the nonnegativity constraint $y_{uw} \geq 0$.

When $d_{in} = 1$ for all internal nodes $i \neq z$, we see that $d_e = 0$ for all internal edges e. Then, $\mathbf{dy} \leq d_0$ is equivalent to the nonnegativity constraint $y_{zn} \geq 0$.

In all other cases, i.e., for $1 \leq \sum_{i=1}^{n-1} d_{in} \leq n-2$, the inequality $\mathbf{dy} \leq d_0$ is not equivalent to a nonnegativity constraint which implies that for each edge e there is a tight $[0,n]-3$-path containing e. Thus, $d_{ij} = 1$ for all internal edges $[i,j]$ for which $d_{in} = d_{jn} = 0$. Therefore, $\mathbf{dy} \leq d_0$ is a member of family (33). □

Next, we turn to the polytopes $P^p_{[0,n]\text{ - path}}(K_{n+1})$ when $4 \leq p \leq n-1$. The integer points in $P^p_{[0,n]\text{ - path}}(K_{n+1})$ are characterized by the following model:

$$
\begin{aligned}
y_{0n} &= 0 && && (34)\\
y(\delta(0)) &= 1 && && (35)\\
y(\delta(n)) &= 1 && && (36)\\
y(\delta(j)) &\leq 2 && \forall\ j \in V \setminus \{0,n\} && (37)\\
y(\delta(j) \setminus \{e\}) - y_e &\geq 0 && \forall\ j \in V \setminus \{0,n\}, e \in \delta(j), && (38)\\
y((S : V \setminus S)) &\geq y(\delta(j)) && \forall\ S \subset V, 3 \leq |S| \leq n-2, && (39)\\
& && \quad 0, n \in S, j \in V \setminus S &&\\
y(E) &= p && && (40)\\
x_e &\in \{0,1\} && \forall\ e \in E. && (41)
\end{aligned}
$$

Here, for any node sets S, T of V, $y((S : T))$ is short for $\sum_{i \in S} \sum_{j \in T} y_{ij}$, where the summation does not extend over loops (i,i) for $i \in S \cap T$.

The *parity* constraints (38) together with the degree (37) and the integrality constraints (41) ensure that every internal node has degree 0 or 2. Hence, constraints (34) - (38) and the integrality constraint (41) are satisfied by the incidence vector of the node disjoint union of a simple $[0,n]$-path and simple cycles on the set of internal nodes. The one-sided min-cut inequality (39) is satisfied by the incidence vectors of simple $[0,n]$-paths but violated by the incidence vectors of the union of a simple $[0,n]$-path and simple cycles. Finally, the cardinality constraint (40) excludes all incidence vectors of $[0,n]$-paths which have a length that is not equal to p.

Lemma 4.6. *Let $4 \leq p \leq n-1$ and $n \geq 6$. If the equation*

$$\mathbf{cy} = c_0$$

is satisfied by all $[0,n]-p$-paths, then there are α, β, γ, such that $c_{0i} = \alpha$, $c_{in} = \beta$ for all $i \in \{1, \dots, n-1\}$ and $c_{ij} = \gamma$ for all $i, j \in \{1, \dots, n-1\}$.

Proof. Set $S := \{1, \dots, n-1\}$ and let i, j, k, l be any distinct nodes in S and consider any $[0,n]-p$-path P that uses the edges $[i,j], [j,k]$ but does not visit node l. Replacing node j by node l yields $c_{ij}+c_{jk}=c_{il}+c_{kl}$. Next, consider any $[0,n]-p$-path P' that uses the edges $[j,i],[i,l]$ but does not visit the node k. Replacing node i by node k yields $c_{ij}+c_{il}=c_{jk}+c_{kl}$. We deduce that $c_{ij}=_{kl}$ and since $|S|\geq 5$, we see that $c_{ij}=c_{kl}$ for all distinct nodes $i,j,k,l\in S$. Denoting this common value by γ, it follows immediately that there are α,β with $c_{0i}=\alpha$ and $c_{in}=\beta$ for all $i\in S$. □

We are now well prepared to determine the dimension of $P^p_{[0,n]\text{ - path}}(K_{n+1})$ depending on n and p. For the sake of completeness we determine also the dimension of $P^p_{[0,n]\text{ - path}}(K_{n+1})$ when $p=n$.

Theorem 4.7. *Let $n\geq p\geq 4$. Then*

$$\dim P^p_{[0,n]\text{ - }path}(K_{n+1})=\begin{cases}|E|-4 & \text{if } p\leq n-1,\\ |E|-n-2 & \text{if } p=n\geq 4.\end{cases}$$

Proof. Using a convex hull code we see that $\dim P^4_{[0,6]\text{-path}}(K_6)=11$. Next, suppose that $n\geq 6$ and $4\leq p\leq n-1$. We will show that (34)-(36) and (40) is a minimal equality subsystem for $P^p_{[0,n]\text{ - path}}(K_{n+1})$. Since the equations (34)-(36) and (40) are linearly independent, $\dim P^p_{[0,n]\text{ - path}}(K_{n+1})\leq \frac{(n+1)n}{2}-4$. It remains to be shown that any equation that is satisfied by all $\mathbf{y}\in P^p_{[0,n]\text{ - path}}(K_{n+1})$ is a linear combination of (34)-(36) and (40). Let $\mathbf{cy}=c_0$ be such an equation. By Lemma 4.6, there are α,β,γ with $c_{0i}=\alpha$, $c_{in}=\beta$ for all internal nodes i and $c_{ij}=\gamma$ for all internal nodes $i\neq j$. Thus,

$$\begin{aligned}(\mathbf{cy},c_0) &= \gamma(y(E),p)\\ &\quad+(\alpha-\gamma)(y(\delta(0)),1)\\ &\quad+(\beta-\gamma)(y(\delta(n)),1)\\ &\quad+(c_{0n}+\gamma-\alpha-\beta)(y_{0n},0).\end{aligned}$$

Finally, let $p=n\geq 4$. Theorem 7 of Grötschel and Padberg [11] implies that the dimension of the traveling salesman polytope Q^{n+1}_T defined on the complete graph on node set V is equal to $|E|-n-1$ for $n\geq 2$ and Theorem 8 of the same authors [11] says that the inequalities $x_e\leq 1$ induce facets F_e of Q^{n+1}_T for $n\geq 3$. Since F_{0n} is isomorphic to $P^n_{[0,n]\text{-path}}(K_{n+1})$, we obtain the required result. □

A valid inequality $\mathbf{cx}\leq c_0$ for the $(0,n)-p$-path polytope $P^p_{0,n-\text{path}}(D)$ is said to be pseudo-symmetric if $c_{ij}=c_{ji}$ for all $1\leq i<j\leq n-1$. It is easy to see that the undirected counterpart $\bar{\mathbf{c}}\mathbf{y}\leq c_0$ of a pseudo-symmetric inequality $\mathbf{cx}\leq c_0$ (obtained by setting $\bar{c}_{0i}=c_{0i},\bar{c}_{in}=c_{in}$ for all internal nodes i and $\bar{c}_{ij}=c_{ij}=c_{ij}$ for all $1\leq i<j\leq n-1$) is facet defining for $P^p_{[0,n]\text{ - path}}(K_{n+1})$ if $\mathbf{cx}\leq c_0$ is facet defining for $P^p_{0,n-\text{path}}(D)$ (cf. [10]). The argument that can be used to prove the statement is the following: assuming that $\bar{\mathbf{c}}\mathbf{y}\leq c_0$ does not induce a facet of $P^p_{[0,n]\text{ - path}}(K_{n+1})$, then there is a facet inducing inequality $\bar{\mathbf{d}}\mathbf{y}\leq d_0$ for $P^p_{[0,n]\text{ - path}}(K_{n+1})$ such that $\{\mathbf{y}\in P^p_{[0,n]\text{ - path}}(K_{n+1})|\bar{\mathbf{c}}\mathbf{y}=c_0\}\subsetneq\{\mathbf{y}\in P^p_{[0,n]\text{ - path}}(K_{n+1})|\bar{\mathbf{d}}\mathbf{y}=d_0\}$. But then $\{\mathbf{x}\in P^p_{0,n-\text{path}}(D)|\mathbf{cy}=c_0\}\subsetneq\{\mathbf{x}\in P^p_{0,n-\text{path}}(D)|\mathbf{dy}=d_0\}$, where $\mathbf{dx}\leq d_0$ is the directed counterpart of $\bar{\mathbf{d}}\mathbf{y}\leq d_0$ (obtained by setting $d_{0i}=\bar{d}_{0i}$, $d_{in}=\bar{d}_{in}$ for all $i\in\{1,\ldots,n-1\}$ and $d_{ij}=dji=\bar{d}_{ij}$ for all $1\leq i<j\leq n-1$).

Since the degree constraint (13) and the cut inequalities (14), (15), (18), and (21) are pseudo-symmetric , their undirected counterparts are facet defining for $P^p_{[0,n]\text{ - path}}(K_{n+1})$.

Corollary 4.8. *Let $4 \leq p < n$.*

(i) The degree constraint $y(\delta(j)) \leq 2$ induces a facet of $P^p_{[0,n]\text{ - }path}(K_{n+1})$ for every internal node j of G.

(ii) Let $S \subset V$ and $0, n \in S$. The min-cut inequality $y((S : V \setminus S)) \geq 2$ induces a facet of $P^p_{[0,n]\text{ - }path}(K_{n+1})$ if $3 \leq |S| \leq p$.

(iii) Let $S \subset V$ and $0, n \in S$. The one-sided min-cut inequality $y((S : V \setminus S)) \geq y(\delta(j))$ defines a facet of $P^p_{[0,n]\text{ - }path}(K_{n+1})$ for every node $j \in V \setminus S$.

(iv) Let $S \subset V$ and $0, n \in S$. The max-cut inequality $y((S : T)) \leq p-1$ defines a facet of $P^p_{[0,n]\text{ - }path}(K_{n+1})$ if p is odd, $S \setminus \{n\} > p/2$, and $T > p/2$.

(v) Let $S \subset V$ and $0 \in S$ and $n \in T$. The max-cut inequality $y((S : T)) \leq p/2$ induces a facet of $P^p_{[0,n]\text{ - }path}(K_{n+1})$ if p is even, $|S| > p/2$, and $|T| > p/2$.

□

Finally, we show that the nonnegativity constraints $x_e \geq 0$ define facets of the $[0, n] - p$-path polytope $P^p_{[0,n]\text{ - path}}(K_{n+1})$.

Theorem 4.9. *Let $4 \leq p < n$. The nonnegativity constraint*

$$y_e \geq 0 \tag{42}$$

defines a facet of the $[0, n] - p$-path polytope $P^p_{[0,n]\text{ - }path}(K_{n+1})$ for all edges $e \neq [0, n]$ of K_{n+1}.

Proof. When $n \leq 5$, (42) can be seen to be facet defining using a convex hull code; so assume that $n \geq 6$. Let $\mathbf{cy} = c_0$ be an equation that is satisfied by every $\mathbf{y} \in P^p_{[0,n]\text{ - path}}(K_{n+1})$ with $y_e = 0$. Since the lineality space of $P^p_{[0,n]\text{ - path}}(K_{n+1})$ is determined by the equations (34)-(36) and (40), we may assume that $c_{0n} = 0$, $c_{0m} = c_{mn} = 0$ for some internal node m with $[0, m] \neq e \neq [m, n]$, and $c_f = 0$ for some internal edge $f \neq e$.

Let $g = [i, j], h = [k, l] \in E \setminus \{e\}$ be not adjacent edges. Without loss of generality, we may assume that the nodes j and l are not incident with edge e. Let P be any tight $[0, n] - p$-path that uses the edges $[i, j], [j, k]$ but does not visit node l. Replacing node j by node l yields another tight path and hence, $c_{ij} + c_{jk} = c_{il} + c_{lk}$. Next, consider any tight $[0, n] - p$-path P' that uses the edges $[j, i], [i, l]$ and does not visit node k. Replacing node i by node k yields another tight path and thus, $c_{ij} + c_{jk} = c_{il} + c_{lk}$. Adding both equations, we obtain $c_g = c_h$, and since $|V \setminus \{0, n\}| \geq 5$, this implies $c_g = c_h$ for all internal edges g, h that are not equal to e. Now it is easy to see that also $c_{0i} = c_{0j}$ and $c_{kn} = c_{ln}$ for all edges $[0i], [0j], [kn], [ln]$ not equal to e. Since $c_{0m} = c_{mn} = 0$ and $c_f = 0$, it follows that $c_g = 0$ for all edges $g \neq e$ which implies also $c_0 = 0$. Hence, $\mathbf{cx} = c_0$ is simply $c_e y_e = 0$. □

4.3 Facets of the lower and upper directed $(0,n) - p$-path polytopes

Theorem 4.10 (cf. Theorem 18 of Hartmann and Özlük [10])**.** *Let* $\mathbf{cx} \le c_0$ *induce a facet of the* $(0,n) - p$*-path polytope* $P^p_{0,n-path}(D)$*, where* $4 \le p < n$*. If* μ *is the smallest (largest) value such that*

$$\mu x(A) + \mathbf{cx} \le \mu p + c_0 \tag{43}$$

is valid for the lower (upper) $(0,n) - p$*-path polytope, then* (43) *is facet inducing for the lower (upper)* $(0,n) - p$*-path polytope.* □

Corollary 4.11. *Let* $4 \le p < n$*. The nonnegativity constraints* (12)*, degree constraints* (13)*, one-sided min-cut inequalities* (15)*, max-cut inequalities* (17) *-* (22)*, jump inequalities* (24)*, and cardinality-path inequalities* (25) *are facet defining for the lower* $(0,n) - p$*-path polytope, if the accordant conditions hold.* □

Corollary 4.12. *Let* $4 \le p < n$*,* $S \subset V$*, and* $0, n \in S$*. The inequality*

$$x(A) - x((S : V \setminus S)) \le p - 1 \tag{44}$$

induces a facet of the lower $(0,n) - p$*-path polytope if and only if* $|S| \le p$ *and* $|V \setminus S| \ge 2$*.*

Proof. The inequality (44) is derived from the min-cut inequality (14) with parameter $\mu = -1$. Hence it is facet defining, if $3 \le |S| \le p$ and $|V \setminus S| \ge 2$. When $S = \{0, n\}$, (44) is equivalent to the cardinality constraint $x(A) \le p$ and hence facet defining for the lower $(0,n) - p$-path polytope.

Conversely, when $|S| \ge p + 1$, (44) is no longer valid, and when $|V \setminus S| = 1$, $n \le p$, a contradiction. □

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