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The extreme points of QSTAB(G) and its implications

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The extreme points of QSTAB(G) and its implications

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Abstract

Perfect graphs constitute a well-studied graph class with a rich structure, reflected by many characterizations w.r.t different concepts. Perfect graphs are, e.g., characterized as precisely those graphs G where the stable set polytope STAB(G) coincides with the clique constraint stable set polytope QSTAB(G). For all imperfect graphs STAB(G) $\subset$ QSTAB(G) holds and, therefore, it is natural to measure imperfection in terms of the difference between STAB(G) and QSTAB(G). Several concepts have been developed in this direction, for instance the dilation ratio of STAB(G) and QSTAB(G) which is equivalent to the imperfection ratio imp(G) of G. To determine imp(G), both knowledge on the facets of STAB(G) and the extreme points of QSTAB(G) is required.

The anti-blocking theory of polyhedra yields all *dominating* extreme points of the polytope QSTAB(G), provided a complete description of the facets of STAB($\overline{G}$) is known. As this is typically not the case, we extend the result on anti-blocking polyhedra to a *complete* characterization of the extreme points of QSTAB(G) by establishing a 1-1 correspondence to the facet-defining subgraphs of $\overline{G}$. We discuss several consequences, in particular, we give alternative proofs of several famous results.

1 Motivation

A graph G is *perfect* if, in each induced subgraph $G' \subseteq G$, the clique number $\omega(G')$ equals the chromatic number $\chi(G')$. Perfect graphs turned out to be an interesting class with a rich structure; in particular, both parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [12]. The latter result relies on the characterization of the stable set polytope of perfect graphs by means of facet-inducing inequalities. The *stable set polytope* STAB(G) is the convex hull of the incidence vectors of all stable sets of G. A canonical relaxation is the clique constraint stable set polytope QSTAB(G), given by *nonnegativity constraints* $x_i \geq 0$ for all nodes i of G and *clique constraints* $\sum_{i \in Q} x_i \leq 1$ for all cliques $Q \subseteq G$. We have STAB(G) $\subseteq$ QSTAB(G) in general, but equality for *perfect* graphs only [5]. Hence, if G is imperfect it follows that STAB(G) $\subset$ QSTAB(G) and it is natural to measure imperfection in terms of the difference between STAB(G) and QSTAB(G).

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Several concepts have been developed in this direction, like the description of STAB(G) by additional sets of inequalities [19], the disjunctive index of QSTAB(G) [1] (called imperfection index, see below), or the dilation ratio of STAB(G) and QSTAB(G) [10, 11]. The latter is equivalent to the *imperfection ratio* imp(G) of the graph, introduced in [10, 11] as the maximum ratio of the fractional chromatic number $\chi_f(G, c)$ and the clique number $\omega(G, c)$ in their weighted versions, taken over all positive integral vectors c. It turned out that the imperfection ratio is an appropriate measure for imperfection [10, 11, 13, 20] and we are interested in characterizing it in different ways. Gerke and McDiarmid [10, 11] showed that imp(G) is the dilation ratio

$$\text{imp}(G) = \min\{t : \text{QSTAB}(G) \subseteq t\ \text{STAB}(G)\}$$

of STAB(G) and QSTAB(G). To express imp(G) in a different way, let $\mathcal{F}(G) = \{a \in [0,1]^{|V|} : a^T x \leq 1$ facet of STAB$(G)\}$ denote the set of all normal vectors of nontrivial facets of STAB(G) (scaled to have right hand side equal to one). Hence, t STAB(G) equals $\{x \in \mathbb{R}_+^{|V|} : a^T x \leq t\ \forall a \in \mathcal{F}(G)\}$. As QSTAB$(G)$ fits in t STAB(G) if, for all $y \in$ QSTAB(G), $a^T y \leq t$ holds, we have

$$\text{imp}(G) = \max\{a^T y : a \in \mathcal{F}(G), y \in \text{QSTAB}(G)\}$$

as any smaller t would violate $a^T y \leq t$ for some $a \in \mathcal{F}(G)$ and $y \in$ QSTAB(G). It clearly suffices to consider nontrivial facets of STAB(G) and (fractional) extreme points of QSTAB(G) only. This suggests that both knowledge about the facet-defining system of STAB(G) and the extreme points of QSTAB(G) helps to determine imp(G). It is well-known that STAB$(\overline{G})$ and QSTAB(G) form an anti-blocking pair [8, 9], in fact, QSTAB$(G) = A(\text{STAB}(\overline{G})) = \{z \in \mathbb{R}_+^{|G|} : z^T x \leq 1\ \forall x \in \text{STAB}(\overline{G})\}$ where $\overline{G}$ is the complement of G. Hence, every facet of STAB$(\overline{G})$ is an extreme point of QSTAB(G), but not all extreme points of QSTAB(G) are conversely of importance for the facet-defining system of $A(\text{STAB}(\overline{G}))$: it suffices to consider the *dominating* extreme points (i.e., all $x \in \text{STAB}(\overline{G})$ where $y \geq x$ implies that $y \notin \text{STAB}(\overline{G})$). Applying this knowledge, we can express imp(G) in two more ways, namely, as

$$\text{imp}(G) = \max\{a^T y : a \in \mathcal{F}(G), y \in \mathcal{F}(\overline{G})\},$$

or, alternatively, as

$$\text{imp}(G) = \max\{a^T y : a \in \text{QSTAB}(\overline{G}), y \in \text{QSTAB}(G)\}$$

where we can restrict to dominating extreme points.

However, a complete description of STAB(G) (or STAB$(\overline{G})$) is typical not available, and hence we do not know all dominating extreme points of the corresponding clique relaxations either. In fact, we do not necessarily need to know the dominating extreme points: Let $(\hat{a}, \hat{y})$ be a pair of dominating extreme points that maximizes $a^T y$ and let $X := \{v \in V : \hat{a}_v = 0$ or $\hat{y}_v = 0\}$. Then all pairs $(\tilde{a}, \tilde{y})$ with $\tilde{a} \in \{a \in \text{QSTAB}(\overline{G}) : a_v = \hat{a}_v \forall v \notin X, 0 \leq a_v < \hat{a}_v \forall v \in X\}$ and $\tilde{y} \in \{y \in \text{QSTAB}(G) : y_v = \hat{y}_v \forall v \notin X, 0 \leq y_v < \hat{y}_v \forall v \in X\}$ maximize $a^T y$ as well. If such pairs exist, *dominated* extreme pairs are among them (by setting a_v or y_v to zero for one or more nodes in X).

Knowledge about the extreme points of QSTAB(G) is also of interest for determining the imperfection index [1] of a graph. The imperfection index is given by

$$\mathrm{imp_I}(G) = \min\{|J| : P_J(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G), J \subseteq V\}$$

where $P_J(\mathrm{QSTAB}(G)) = \mathrm{conv}\{x \in \mathrm{QSTAB}(G) : x_j \in \{0,1\}, j \in J\}$ describes the disjunctive procedure [2] for a subset $J \subseteq V$ of nodes (note that $P_V(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G)$ for all G). We have $\mathrm{imp_I}(G) = 0$ iff G is perfect. Every fractional extreme point a indicates that at least one node from G_a has to be included in the subset J, where G_a is the subgraph of G induced by all $v \in V$ with $a_v > 0$.

The above described relations of imperfection ratio and imperfection index with the extreme points of QSTAB(G) have motivated us to characterize the extreme points of QSTAB(G) completely; our main result is the following theorem. For this, let supp(a) be the vector a restricted to the non-zero components only.

Theorem 1 *A vector $a \neq 0$ is an extreme point of* QSTAB(G) *if and only if for the subgraph $\overline{G}_a$ of $\overline{G}$,* supp(a) *belongs to $\mathcal{F}(\overline{G}_a)$.*

Thus, we establish a 1-1 correspondence between the extreme points of QSTAB(G) and facet-inducing *sub*graphs of $\overline{G}$.

Example 1 Let G be a 5-wheel with center c. Its complement $\overline{G}$ is a 5-hole with an isolated node c. Obviously, $\overline{G}$ has this 5-hole $\overline{C}_5$ as only facet-inducing subgraph different from a clique, thus QSTAB(G) has exactly one fractional extreme point, namely $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$. Conversely, G has two facet-inducing subgraphs different from a clique, the 5-hole C_5 and G itself, producing the constraint $x(C_5) \leq 2$ and the constraint $x(C_5) + 2x_c \leq 2$. Accordingly, QSTAB($\overline{G}$) has the two fractional extreme points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$. (Note that $x(C_5) \leq 2$ is not a facet of STAB(G) but only of STAB(C_5).)

In the above example all coefficients of the facet-defining inequalities are known. In those cases where the dominating extreme points cannot be determined due to unknown lifting coefficients, Theorem 1 allows to find extreme points by facet-inducing subgraphs.

The rest of this paper is structured as follows. Theorem 1 is proved in Section 2, whereas we discuss in Section 3 some implications of this theorem by showing how several famous graph theoretical results can be reproved by it. We conclude with some remarks on subgraphs that determine the imperfection ratio and index for particular classes of graphs.

2 Proof of the main theorem

Suppose that $\overline{G}$ contains a subgraph $\overline{G}'$ such that $a^T x \leq 1$ is a facet of STAB($\overline{G}'$) with $0 < a_i \leq 1$ for $i \in \overline{G}'$. Then there exist $n' = |\overline{G}'|$ stable sets $S'_1, \ldots, S'_{n'}$ of $\overline{G}'$ such that $a^T \chi^{S'_i} = 1$ for $1 \leq i \leq n'$ and $\chi^{S'_1}, \ldots, \chi^{S'_{n'}}$ are linearly independent (here χ^S denotes the characteristic vector defined by S).

These stable sets clearly correspond to n' cliques $Q'_1, \dots, Q'_{n'}$ of G'. For any such clique Q'_i, choose a maximal clique $Q_i \subseteq G$ with $Q_i \supseteq Q'_i$. Let (x, a) denote the element-wise product of two vectors. The vector $x' = (\chi^{G'}, a)$ satisfies the n' clique constraints associated with the maximal cliques $Q_1, \dots, Q_{n'}$ at equality, as

$$x'(Q_i) = \sum_{j \in Q'_i \subseteq Q_i} a_j = a^T \chi^{Q'_i} = a^T \chi^{S'_i} = 1$$

holds by the choice of G'. Furthermore, x' satisfies the $n - n' = |G \setminus G'|$ nonnegativity constraints $-x'_j \leq 0 \; \forall j \notin G'$ with equality. Hence, x' belongs to $n = |G|$ facets of QSTAB(G). In order to show that x' is an extreme point it remains to ensure that these facets are linearly independent. For that, construct an $(n \times n)$-matrix A as follows: Let the first n' columns of A correspond to nodes in G' and the last $n - n'$ columns to nodes in $G \setminus G'$. Choose further the incidence vectors of the cliques $Q_1, \dots, Q_{n'}$ as first n' rows and the incidence vectors of the nonnegativity constraints $-x'_j \leq 0 \; \forall j \notin G'$ as last $n - n'$ rows, see Figure 1 (with Id representing an identity matrix of appropriate size).

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & -\mathrm{Id} \end{array} \right)$$

Figure 1: The $(n \times n)$-matrix A

As the submatrix A_1 corresponds to the independent cliques $Q'_1, \dots, Q'_{n'}$ of G', the whole matrix A is invertible due to its block structure. Thus, x' is indeed an extreme point of QSTAB(G).

Conversely, suppose that $x' = (\chi^{G'}, a)$ with $0 < a_i \leq 1$ for $i \in \overline{G}'$ and $a_i = 0$ otherwise is an extreme point of QSTAB(G). Then x' satisfies n linearly independent facets of QSTAB(G) with equality. Among them are clearly the $n - n'$ nonnegativity constraints $-x'_j \leq 0 \; \forall j \notin G'$ and none of the remaining n' nonnegativity constraints. As QSTAB(G) has only two types of facets, x' satisfies also n' maximal clique facets with equality, say the clique constraints associated with the maximal cliques $Q_1, \dots, Q_{n'}$ of G. Let $Q'_i = Q_i \cap G'$, then

$$1 = x'(Q_i) = \sum_{j \in Q_i} a_j = \sum_{j \in Q'_i \subseteq Q_i} a_j = a^T \chi^{Q'_i}$$

follows by the choice of the vector a. Clearly, the cliques $Q'_1, \dots, Q'_{n'}$ of G' correspond to stable sets $S'_1, \dots, S'_{n'}$ of $\overline{G}'$ and $a^T \chi^{S'_i} = 1$ holds for $1 \leq i \leq n'$. In order to show that $a^T x \leq 1$ is a facet of STAB$(\overline{G}')$, it remains to verify that $\chi^{S'_1}, \dots, \chi^{S'_{n'}}$ are linearly independent. For that, construct an $(n \times n)$-matrix A as above, choosing the nodes in G' and in $G \setminus G'$ as first n' and last $n - n'$ columns, respectively, the incidence vectors of the cliques $Q_1, \dots, Q_{n'}$ as first n' and the unit vectors corresponding to $-x'_j = 0 \; \forall j \notin G'$ as last $n - n'$ rows, see again Figure 1. As x' is an extreme point, the matrix A is invertible. In order to show invertibilty for the submatrix A_1, we add, for each 1-entry in A_2, the corresponding unit vector in $(0, -\mathrm{Id})$. That way, we turn A_2 into a matrix with 0-entries only but maintain all entries in A_1. This shows that the rows of A_1 are linearly independent and, therefore, the incidence vectors of the cliques $Q'_1, \dots, Q'_{n'}$ of G' respectively of the corresponding stable sets $S'_1, \dots, S'_{n'}$ in $\overline{G}'$. Hence, $a^T x \leq 1$ is indeed a facet of $\mathrm{STAB}(\overline{G}')$.

3 Some implications

With the help of Theorem 1 we are able to present alternative proofs for several famous results in the field of perfect graphs and stable sets: the Perfect Graph Theorem, the characterization of minimally imperfect graphs, and the half-integrality of $\mathrm{QSTAB}(G)$ for certain graph classes.

3.1 Characterizing perfect graphs

The Perfect Graph Theorem [15] states that a graph is perfect iff its complement is perfect. For perfect graphs, the assertion of Theorem 1 follows directly from the Perfect Graph Theorem and $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$: If G is perfect, then $\mathrm{QSTAB}(G)$ has integral extreme points only, namely $\chi^{G'}$ where $G' \subseteq G$ is an arbitrary stable set; as $\overline{G}$ is perfect as well, its only facet-inducing subgraphs are all cliques $\overline{G}' \subseteq \overline{G}$.

Conversely, we obtain both the Perfect Graph Theorem and the polyhedral characterization of perfect graphs with the help of Theorem 1 as follows.

First, consider an arbitrary graph G, an integral node weighting $c \geq 0$, and the following chain of inequalities and equations, obtained by dropping or adding integrality constraints and linear programming duality (with $\alpha(G, c)$ the weighted stability number and $\overline{\chi}(G, c)$ the weighted clique cover number):

$$
\begin{aligned}
\alpha(G,c) &= \max\{c^T x : x \in \mathrm{STAB}(G)\} \\
&= \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \in \{0,1\}^{|G|}\} \\
&\leq \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \geq 0\} \\
&= \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i \ \forall i \in G,\ y_Q \geq 0 \ \ \forall \text{cliques } Q \subseteq G\} \\
&\leq \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i \ \forall i \in G,\ y_Q \in \mathbb{Z}_+ \ \forall \text{cliques } Q \subseteq G\} \\
&= \overline{\chi}(G,c)
\end{aligned}
$$

The last program is an integer programming formulation of the weighted clique cover problem, the intermediate steps yield the fractional stability and clique cover numbers, $\alpha_f(G, c)$ and $\overline{\chi}_f(G, c)$, which are equal by linear programming duality.

If $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ then obviously $\alpha(G, c) = \alpha_f(G, c)$ follows; in particular $\alpha_f(G, c)$ is integer valued for all $c \geq 0$ and comes from an integer solution. By duality, $\overline{\chi}_f(G, c)$ is integer valued for all $c \geq 0$ as well. Hence, also the optimal solution of the fractional clique cover problem is integral for all $c \geq 0$, and consequently there always exists an integral optimal solution (totally dual integrality). Turning to the complementary graph yields $\omega(\overline{G}, c) = \chi(\overline{G}, c)$ for all $c \geq 0$. This is particularly true for all 0/1-weightings, implying equality for the unweighted case $\omega(\overline{G}', \mathbb{1}) = \chi(\overline{G}', \mathbb{1})$ for all induced subgraphs $\overline{G}'$ of $\overline{G}$.

Hence, $\overline{G}$ cannot contain any minimal imperfect graph as induced subgraph (as clique and chromatic number would differ for such subgraphs) and is, therefore, perfect.

Conversely, if $\overline{G}$ contains a minimal imperfect subgraph $\overline{G}'$, then we have $\omega(\overline{G}', \mathbb{1}) < \chi(\overline{G}', \mathbb{1})$, implying $\alpha(G', c) < \overline{\chi}(G', c)$. We obtain $\mathrm{STAB}(G) \subset \mathrm{QSTAB}(G)$ since otherwise we would have equality through the whole chain, in particular $\alpha(G', c) = \alpha_f(G', c) = \overline{\chi}_f(G', c) = \overline{\chi}(G', c)$, a contradiction.

This implies that $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ *if and only if* $\overline{G}$ *is perfect.* With the help of this fact and Theorem 1 we easily obtain the following:

Corollary 2 *For any graph G, the following assertions are equivalent:*

(1) *G is perfect;*
(2) $\mathrm{STAB}(\overline{G}) = \mathrm{QSTAB}(\overline{G})$*;*
(3) $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$*;*
(4) *$\overline{G}$ is perfect.*

Proof. We have seen that the graph G is perfect if and only if $\mathrm{STAB}(\overline{G}) = \mathrm{QSTAB}(\overline{G})$ holds. This is equivalent to having $\mathrm{QSTAB}(\overline{G})$ 0/1-valued extreme points only, namely the extreme points associated with stable sets in $\overline{G}$. Due to Theorem 1, this is the case if and only if G has cliques as only facet-inducing subgraphs or, equivalently, that $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$. Finally, this is true if and only if $\overline{G}$ is perfect. □

3.2 Near-perfect graphs

A graph G is said to be *minimal imperfect* if G is not perfect but every proper induced subgraph is perfect (by the Strong Perfect Graph Theorem these are exactly the odd holes and the odd antiholes [4]). For minimally imperfect graphs, Theorem 1 corresponds to the well-known characterization of Padberg [16], stating that a graph G is minimally imperfect if and only if $\mathrm{STAB}(G)$ has the full rank facet as only nontrivial, nonclique facet and $\mathrm{QSTAB}(G)$ has exactly one fractional extreme point, namely $\frac{1}{\omega(G)}\chi^G$ (a rank facet is defined by the inequality $x(G') \leq \alpha(G')$ for some $G' \subseteq G$; it is full if $G' = G$).

By the Perfect Graph Theorem, $\overline{G}$ is minimally imperfect as well, and its only facet-inducing subgraph different from a clique is $\overline{G}$ itself, producing the full rank constraint $x(G) \leq \alpha(\overline{G}) = \omega(G)$. Therefore, $\frac{1}{\omega(G)}\chi^G$ is the only fractional extreme point of $\mathrm{QSTAB}(G)$, and conversely.

A graph G is said to be *near-perfect* if the full rank facet is the only nontrivial, nonclique facet of $\mathrm{STAB}(G)$. Clearly, every perfect or minimally imperfect graph is near-perfect; we call a near-perfect graph *proper* if it is neither perfect nor minimally imperfect. For such graphs, we obtain:

Lemma 3 *If G is properly near-perfect, then both* $\mathrm{QSTAB}(G)$ *and* $\mathrm{QSTAB}(\overline{G})$ *have at least two fractional extreme points.*

Proof. As G is properly near-perfect, there is a proper induced minimally imperfect subgraph $G' \subset G$. Hence G has at least two facet-defining subgraphs (namely G' and G itself). By Theorem 1, QSTAB($\overline{G}$) has, therefore, at least two fractional extreme points (corresponding to the rank constraints associated with G' and G).

According to Shepherd [17], $\overline{G}$ is not near-perfect. Thus STAB($\overline{G}$) has at least two non-trivial, non-clique facets, say $a^t x \leq 1$ and $b^t x \leq 1$. Theorem 1 implies that QSTAB(G) has, therefore, at least two fractional extreme points (namely a and b). □

Even worse, the following example exhibits a sequence of near-perfect graphs G where the number of fractional extreme points of QSTAB(G) and QSTAB($\overline{G}$) tends to infinity.

Example 2 *Webs* are graphs W_n^k with n nodes $1, \ldots, n$ where ij is an edge if $|i - j| \leq n \bmod k$ and $i \neq j$, and *antiwebs* are their complements. According to [17, 19], all webs with stability number two are near-perfect, that are the webs W_n^k with $n < 3(k+1)$. With the help of Trotter's formula [14], it is easy to check that the odd antihole $W^l_{2(l+1)+1}$ is an induced subgraph of $W^k_{2(k+1)+2}$ for all $k \geq 2$ if $l \leq \frac{k}{2}$. In particular, we have

$$W^l_{2(l+1)+1} \subset W^{2l}_{2(2l+1)+2} \text{ for all } l \geq 1.$$

Since, by this choice, the number of nodes $2(l+1)+1$ of the odd antihole does not divide the number of nodes $2(2l+1)+2$ of the whole web, we obtain that there are $2(2l+1)+2$ *different* odd antiholes in $W^{2l}_{2(2l+1)+2}$, namely,

$$C(i) = \{i, i+1, (i+1)+2, \ldots, (i+1)+2l, i+2(l+1), \ldots, i+2(2l+1)\}$$

for every node i (thus, we choose i, the next node $i+1$, then l times the next but one node, once more the next node, and finally $l+1$ times the next but one node again). Thus, the web $W^{2l}_{2(2l+1)+2}$ contains as many different odd antiholes as nodes (resp. the antiweb $\overline{W}^{2l}_{2(2l+1)+2}$ as many different odd holes as nodes). According to Theorem 1, each of them yields a fractional extreme point of QSTAB($\overline{W}^{2l}_{2(2l+1)+2}$) (resp. QSTAB($W^{2l}_{2(2l+1)+2}$)). Thus, the number of fractional extreme points is at least $2(2l+1)+2$ for both and tends to infinity if l does. In particular, the odd antiholes in $W^{2l}_{2(2l+1)+2}$ correspond to fractional extreme points of QSTAB($\overline{W}^{2l}_{2(2l+1)+2}$) which are *not* dominating (as $W^{2l}_{2(2l+1)+2}$ does not have odd antihole facets).

3.3 Half-integral fractional stable set polytopes

We say that an inequality $a^T x \leq b$ is given in its integer form if all entries in a and the rhs b are integers with greatest common divisor 1 (i.e., cannot be scaled down to smaller integer values).

An immediate consequence of Theorem 1 is the following.

Corollary 4 QSTAB(G) *is half-integral if and only if any facet-producing subgraph $\overline{G}'$ of $\overline{G}$ induces a facet having rhs ≤ 2 in its integer form.*

This is clearly true for all graphs G such that $\alpha(\overline{G}) \leq 2$ holds. This implies, for any graph G with $\omega(G) \leq 2$, that QSTAB(G) is half-integral. As for such graphs QSTAB(G) obviously coincides with the edge constraint stable set polytope ESTAB(G) (given by nonnegativity and edge constraints only), the above corollary yields the well-known result that ESTAB(G) has half-integral extreme points only.

Further examples are line graphs. A *line graph* is obtained by taking the edges of an original graph as nodes and connecting two nodes iff the original edges are incident.

Shepherd [18] gave a complete description of the stable set polytopes of their complements by showing that the only nontrivial facets of stable set polytopes of complements of line graphs are constraints

$$\sum_{i \leq k} x(A_i) + 2x(Q) \leq 2$$

associated with complete joins of odd antiholes $A_1, \ldots, A_k$ and a clique Q (a *complete join* of two graphs G_1 and G_2 is the graph obtained by adding all edges joining a node of G_1 with a node of G_2). This implies, that the fractional stable set polytopes of line graphs are half-integral.

As the stable set polytopes of line graphs correspond to the matching polytope introduced and described by Edmonds [7], we obtain an alternative proof that the fractional matching polytope has half-integral extreme points only.

4 Concluding remarks

We established a 1-1 correspondence between the extreme points of QSTAB(G) and the facet-inducing subgraphs of $\overline{G}$. We discussed how to use this result to reprove several famous results. Returning to the original motivation, we finally discuss consequences of Theorem 1 for determining the imperfection ratio and the imperfection index of a graph. Our characterization is of interest for this purpose since it helps to indentify both

- minimum node subsets $J \subset V$ such that $P_J(\text{QSTAB}(G)) = \text{STAB}(G)$ holds (here it even suffices to consider "minimal" fractional extreme points which do not dominate any other fractional extreme point of QSTAB(G)),
- facet-defining subgraphs of G and $\overline{G}$ such that the associate extreme point y of QSTAB(G) and normal vector $a \in \mathcal{F}(G)$ satisfy $\text{supp}(y) = \text{supp}(a)$ and $y^T a = \text{imp}(G)$.

In both cases, it is thus worth to consider not only the known facet-defining vectors of STAB(G) and STAB($\overline{G}$), but also those for their subgraphs without lifting. Of particular interest are those subgraphs G' of G for which *both* G' and $\overline{G}'$ induce facets (we call a graph H facet-inducing if there is an $a \in \mathcal{F}(H)$ with $a_v > 0$ for all $v \in V(H)$).

In particular, the following lower bound for the imperfection ratio

$$\text{imp}(G) \quad \geq \quad \max\{\tfrac{|G'|}{\alpha(G')\omega(G')} : G' \subseteq G\} \tag{1}$$

obtained by combining two bounds from [10, 11], was discussed in [6]. The question is to find the crucial subgraphs where the bound is indeed attained. Theorem 1 and the invariance of the imperfection ratio under taking complements imply that such crucial subgraphs G' are those where both G' and $\overline{G}'$ induce facets. Indeed, we have the following examples:

- *Minimally imperfect graphs* I, as both I and $\overline{I}$ induce the full rank facet [16]. For graphs G with $\text{imp}(G) = \max\{\frac{|I|}{\alpha(I)\omega(I)} : I \subseteq G\}$, we have $\text{imp}(G) \leq \frac{5}{4}$ (since the 5-hole is the smallest such graph). This is the case for, e.g., minimally imperfect graphs itself and all line graphs [10, 11].
- *Partitionable graphs*, that are graphs P where for any node x, $P - x$ partitions into $\omega(P)$ maximum stable sets or into $\alpha(P)$ maximum cliques. Both P and $\overline{P}$ induce the full rank facet [3]. For graphs G with $\text{imp}(G) = \max\{\frac{|P|}{\alpha(P)\omega(P)} : P \subseteq G\}$, we have $\text{imp}(G) \leq \frac{5}{4}$ [13] (since the 5-hole is again the smallest such graph).
- *Webs W_n^k and antiwebs $\overline{W}_n^k$* produce both the full rank facet iff $k+1$ and n are relatively prime [14]. For graphs G with $\text{imp}(G) = \max\{\frac{n}{\alpha\omega} : W_n^k, \overline{W}_n^k \subseteq G\}$, $\alpha = \alpha(W_n^k)$, and $\omega = \alpha(\overline{W}_n^k)$, we have $\text{imp}(G) < \frac{3}{2}$ [6]. This is true for webs and antiwebs itself and for so-called quasi-line and near-bipartite graphs [6].

In order to find further subgraphs which are crucial for the imperfection ratio, the task is to find such graphs G where G and $\overline{G}$ are both facet-producing, but induce not necessarily the full rank facet.

Finally, it should be noted that Theorem 1 can be generalized to general anti-blocking pairs. Possible applications of this theorem within other contexts is a direction for further research.

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