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Abstract. We study online multicommodity minimum cost routing problems in networks, where commodities have to be routed sequentially. Arcs are equipped with load dependent price functions defining the routing weights. We discuss an online algorithm that routes each commodity by minimizing a convex cost function that depends on the demands that are previously routed. We present a competitive analysis of this algorithm showing that for affine linear price functions this algorithm is $\frac{4K}{2+K}$-competitive, where K is the number of commodities. For the parallel arc case this algorithm is optimal. Without restrictions on the price functions and network, no algorithm is competitive. Finally, we investigate a variant in which the demands have to be routed unsplittably.

1 Introduction

In this work we study the fundamental problem of sequentially routing demands in a network. We consider a dynamic load dependent weight setting on links. In realistic scenarios the online aspect arises due to the fact that by the time of routing a given demand, future demands are not known. We briefly outline two examples.

Open Shortest Path First (OSPF) is the most commonly used intra-domain internet routing protocol today, see Moy [1]. Traffic is routed along shortest paths from source to destination with respect to weights on the links that are under the control of network operators. A default weight setting strategy is to make the weight inversely proportional to the physical link capacity as suggested by Cisco [2]. If the routing weights are interpreted as prices for reserving capacity on the corresponding link, the OSPF protocol routes demands along the cheapest route.

Minimum cost routing also arises in an inter domain Quality of Service (QoS) market, where multiple service providers offer network resources (capacity) to enable internet traffic with specific QoS constraints, cf. Yahaya and Suda [3, 4]. In such a market, each service provider advertises prices (weights) for resources that

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he wants to sell. Buying providers reserve capacity along the cheapest available path to route demand (coming from own customers) from source to destination.

In this paper we propose a framework, called *Online Multicommodity Routing Problem* (OnlineMCRP), to investigate such online routing problems. We allow arbitrary continuous nondecreasing price functions. As far as we know this approach has not been investigated before.

We investigate an online algorithm for this setting. We show that this algorithm is $\frac{4K}{2+K}$-competitive for affine linear price functions, where K is the number of commodities. For the parallel arc case this algorithm is optimal. The key to derive competitive results is to bound the bundle size for a given demand. That is, demands are allowed to be split into infinitesimal pieces that are sequentially routed over paths and prompt the update of the routing weights along the used path. Without restrictions on the price functions and network, no algorithm is competitive. We also investigate a variant in which the demands have to be routed unsplittably.

Related Work Multicommodity routing problems have been studied in the context of traffic engineering, see Fortz and Thorup [5, 6]. There, the goal is to route given demands subject to capacity constraints in order to minimize a convex load dependent penalty function. In this setting, a central planer has full knowledge of all demands, which is not the case in our approach.

Another related line of research is the investigation of efficient routing in decentralized noncooperative systems. This has been extensively studied using game theoretic concepts, cf. Roughgarden and Tardos [7], Correa, Schulz, and Stier Moses [8], and references therein. In these works the efficiency of Nash equilibria are studied. Hence, rerouting of demands is allowed in this context. In our model, once a routing decisions has been made this routing remains unchanged, i.e., it is irrevocable.

In the online network routing field mainly call admission control problems have been considered. An overview article about these problems is given by Leonardi in [9].

2 Problem description

An instance of the *Online Multicommodity Routing Problem* (OnlineMCRP) consists of a directed network $D = (V, A)$ and nondecreasing continuous price functions $p_a : \mathbb{R}_+ \to \mathbb{R}_+$ for each link $a \in A$. These functions define the price of reserving capacity on a link depending on the current load, see below. Furthermore, a sequence $\sigma = 1, \dots, K$ of commodities must be routed one after the other. We assume that $K \geq 2$ and denote the set of commodities by $[K] := \{1, \dots, K\}$. The routing decision for commodity k is *online*, i.e., it only depends on the routings of commodities $1, \dots, k-1$. Once a commodity has been routed it remains unchanged. Each commodity $k \in [K]$ has a demand $d_k > 0$ that is to be routed from its source $s_k \in V$ to its destination $t_k \in V$ after it arises.

A routing assignment, or *flow*, for commodity $k \in [K]$ is a nonnegative vector $\boldsymbol{f}^k \in \mathbb{R}_+^{|A|}$. This flow is *feasible* if for all $v \in V$

$$\sum_{a \in \delta^+(v)} f_a^k - \sum_{a \in \delta^-(v)} f_a^k = \gamma(v), \tag{1}$$

where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering v, respectively; furthermore, $\gamma(v) = d_k$ if $v = s_k$, $\gamma(v) = -d_k$ if $v = t_k$, and $\gamma(v) = 0$ otherwise. Note that Equation (1) allows to split the demand of a commodity.

Alternatively, one can consider a *path flow* for a commodity $k \in [K]$. Let $\mathcal{P}_k$ be the set of paths from s_k to t_k in D. A path flow is a nonnegative vector $(f_P^k)_{P \in \mathcal{P}_k}$. The corresponding flow on link $a \in A$ for commodity $k \in [K]$ is then

$$f_a^k := \sum_{P \ni a} f_P^k.$$

We define $\mathcal{F}_k$ with $k \in [K]$ to be the set of vectors $(\boldsymbol{f}^1, \dots, \boldsymbol{f}^k)$ such that $\boldsymbol{f}^i$ is feasible for commodity i for $i = 1, \dots, k$. If $(\boldsymbol{f}^1, \dots, \boldsymbol{f}^k) \in \mathcal{F}_k$, we say that it is *feasible* for commodities $1, \dots, k$. The entire flow for a sequence of commodities is denoted by $\boldsymbol{f} = (\boldsymbol{f}^1, \dots, \boldsymbol{f}^K)$. Furthermore, the cost of a flow on link $a \in A$ of commodity k is defined by

$$C_a^k(f_a^1, \dots, f_a^k) = \int_0^{f_a^k} p_a\Big(\sum_{i=1}^{k-1} f_a^i + z\Big)\, dz. \tag{2}$$

This expression can be obtained as the cost of a shortest path routing, where the demand is split into infinitesimal pieces that are routed consecutively. Hence, the integral represents the fact that an infinitesimal amount of flow increases the price for each consecutive piece. Note that C_a^k is a convex function.

The cost for $\boldsymbol{f}^k$ is

$$C^k(\boldsymbol{f}^k) := \sum_{a \in A} C_a^k(f_a^1, \dots, f_a^k), \tag{3}$$

and the total cost is defined by $C(\boldsymbol{f}) := \sum_{k=1}^K C^k(\boldsymbol{f}^k)$.

In this paper we study the online algorithm Seq that sequentially routes the requested demands with minimum cost. Therefore, it solves for every $k \in [K]$ the following convex program

$$\begin{aligned} \min \quad & C^k(\boldsymbol{f}^k) \\ \text{s.t.} \quad & \sum_{a \in \delta^+(v)} f_a^k - \sum_{a \in \delta^-(v)} f_a^k = \gamma(v) && \forall\, v \in V \\ & f_a^k \geq 0 && \forall a \in A, \end{aligned} \tag{4}$$

where the vectors $\boldsymbol{f}^1, \dots, \boldsymbol{f}^{k-1}$ are fixed by solving the first $k-1$ problems. This problem can be efficiently solved within arbitrary precision in polynomial time (see Grötschel, Lovász, and Schrijver [10]).

Using the relation

$$\frac{\partial C^k}{\partial f^k_a}(\boldsymbol{f}^k) = p_a\big(\sum_{i=1}^{k} f^i_a\big),$$

we state necessary and sufficient optimality conditions of the above K problems:

Lemma 1. *A flow $\boldsymbol{f} = (\boldsymbol{f}^1, \dots, \boldsymbol{f}^K)$ is generated by* Seq *if and only if for all $k \in [K]$ the following two equivalent conditions are satisfied. Let $\boldsymbol{x}$ be any feasible flow, and let $P, Q \in \mathcal{P}_k$ where P is flow carrying w.r.t. $\boldsymbol{f}^k$*

$$\sum_{a\in A} p_a\big(\sum_{i=1}^{k} f^i_a\big)\,(f^k_a - x^k_a) \le 0 \tag{5}$$

$$\sum_{a\in P} p_a\big(\sum_{i=1}^{k} f^i_a\big) \le \sum_{a\in Q} p_a\big(\sum_{i=1}^{k} f^i_a\big). \tag{6}$$

The proof is based on the first order optimality conditions and the convexity of C^k, see Dafermos and Sparrow [11].

For the sequence $\sigma = 1, \dots, K$, an *optimal offline flow* is given by the solution $\boldsymbol{f}^\star$ of the following convex optimization problem:

$$\begin{aligned} \min \quad & C(\boldsymbol{f}) & & \\ \text{s.t.} \quad & \sum_{a\in\delta^+(v)} f^k_a - \sum_{a\in\delta^-(v)} f^k_a = \gamma(v) & & \forall\, v \in V,\ k \in K \\ & f^k_a \ge 0 & & \forall\, a \in A,\ k \in K, \end{aligned} \tag{7}$$

where $\gamma(v)$ is defined as in (1). We denote by $\text{Opt}(\sigma) = C(\boldsymbol{f}^\star)$ its value.

Using the relation

$$\frac{\partial C}{\partial f^k_a}(\boldsymbol{f}) = p_a\big(\sum_{i=1}^{K} f^i_a\big),$$

the necessary and sufficient optimality conditions of the above problem are:

Lemma 2. *A flow $\boldsymbol{f} = (\boldsymbol{f}^1, \dots, \boldsymbol{f}^K)$ is offline optimal if and only if for all $k \in [K]$ the following two equivalent conditions are satisfied. Let $\boldsymbol{x}$ be any feasible flow, and let $P, Q \in \mathcal{P}_k$, where P is flow carrying w.r.t. $\boldsymbol{f}^k$*

$$\sum_{a\in A} p_a\big(\sum_{i=1}^{K} f^i_a\big)\,(f^k_a - x^k_a) \le 0 \tag{8}$$

$$\sum_{a\in P} p_a\big(\sum_{i=1}^{K} f^i_a\big) \le \sum_{a\in Q} p_a\big(\sum_{i=1}^{K} f^i_a\big). \tag{9}$$

Note that the only difference to the optimality conditions in Lemma 1 is the summation in the price function up to commodity K instead of k. This reflects the offline aspect since all demands are known.

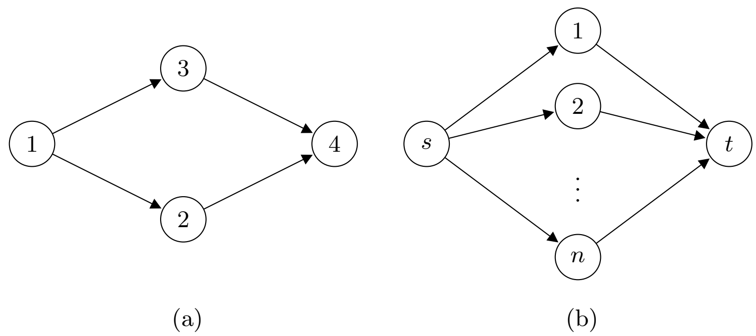

Fig. 1: Construction for the proofs of lower bounds.

For a given sequence of commodities $\sigma = 1, \ldots, K$ and a solution $\boldsymbol{f}$ produced by an online algorithm ALG for σ we denote by ALG$(\sigma) = C(\boldsymbol{f})$ its cost. The online algorithm ALG is called *c-competitive* if the cost of ALG is never larger than c times the cost of an optimal offline solution. The *competitive ratio* of ALG is the infimum over all $c \geq 1$ such that ALG is c-competitive, see Borodin and El-Yaniv [12].

Remark 1. If the price functions are constant, i.e., $p_a(z) = q_a$ for every arc $a \in A$, the algorithm SEQ is optimal for the offline problem. This holds because in this case the routing problems are independent from each other. In fact, each routing decision is just a shortest path problem with respect to the constant costs q_a and the offline problem is a min-cost flow problem without capacity constraints. Hence, both problems can be solved more efficiently than in the general case.

Clearly, also in the case $K = 1$, the competitive ratio of SEQ is 1.

3 Competitive Analysis of SEQ

First, we show that there exists, in general, no competitive deterministic online algorithm.

Proposition 1. *If neither the network nor the price functions are restricted, there exists no competitive deterministic online algorithm for* ONLINEMCRP.

Proof. Consider the network depicted in Figure 1 (a). For all arcs a in the network, the price function is $p_a(z) = m \cdot z^{m-1}$ with $m > 2$. Note that the integral of this function is z^m. Let ALG be an arbitrary deterministic online algorithm. The first commodity has demand $d_1 = 1$, which has to be routed from $s_1 = 1$ to $t_1 = 4$. There are two possible paths for this commodity: path $P_1 = (1, 2, 4)$ and $P_2 = (1, 3, 4)$.

Assume that the algorithm ALG splits this demand evenly. This leads to the cost $4 \cdot (\frac{1}{2})^m$. Now commodity 2 arises with demand $d_2 = 1$, source $s_2 = 1$, and target $t_2 = 2$. This demand has to be routed along $P_3 = (1, 2)$. For this sequence σ we have the total cost $\textsc{Alg}(\sigma) = 4 \cdot (\frac{1}{2})^m + (\frac{1}{2} + 1)^m$. Routing the first commodity completely over path P_2 and the second over path P_3 leads to the total cost $\textsc{Opt}(\sigma) \leq 2 \cdot 1^m + 1^m = 3$. Letting m tend to infinity shows that $\textsc{Alg}$ is not competitive.

Consider now the case where the algorithm $\textsc{Alg}$ does not split commodity 1 evenly. If the algorithm routes the greater portion over path P_1, we reveal the same commodity 2 as in the previous case. Otherwise, the second commodity has a demand $d_2 = 1$, source $s_2 = 1$, and target $t_2 = 3$. It is obviously that in both cases this lead to the cost for the algorithm strictly greater than in the evenly split case. Again for $m \to \infty$, it follows that $\textsc{Alg}$ is not competitive, since still $\textsc{Opt}(\sigma) \leq 3$. □

This shows that to obtain any competitive result, the network or the price functions have to be more restricted. Now we will show that if the price functions are affine, $\textsc{Seq}$ is $\frac{4K}{2+K}$-competitive. For affine price functions $p_a(z) = q_a \cdot z + r_a$ with $q_a > 0$, $r_a \geq 0$ $(a \in A)$, we have for a feasible flow $(\boldsymbol{f}^1, \dots, \boldsymbol{f}^k)$

$$C_a^k(\boldsymbol{f}^1, \dots, \boldsymbol{f}^k) = q_a \, \big(\sum_{i=1}^{k-1} f_a^i + \tfrac{1}{2} f_a^k\big) f_a^k + r_a \, f_a^k.$$

It follows from the optimality conditions (5) that if $(\boldsymbol{f}^1, \dots, \boldsymbol{f}^k)$ is generated by $\textsc{Seq}$, we have

$$\sum_a \big(q_a \sum_{i=1}^{k} f_a^i + r_a\big) \, (f_a^k - x_a^k) \leq 0, \tag{10}$$

for all feasible flows $\boldsymbol{x}^k$.

Theorem 1. *If the price functions are affine,* $\textsc{Seq}$ *is* $\frac{4K}{2+K}$*-competitive.*

Proof. Let $\boldsymbol{f}$ be the flow generated by $\textsc{Seq}$ for a given sequence and let $\boldsymbol{x}$ be any other feasible flow. We start with the following inequality:

$$0 \leq \Big(\tfrac{1}{2} \sum_{k=1}^{K} f_a^k - \sum_{k=1}^{K} x_a^k\Big)^2 = \tfrac{1}{4} \sum_{k=1}^{K} \sum_{i=1}^{K} f_a^i \, f_a^k - \sum_{k=1}^{K} \sum_{i=1}^{K} f_a^i \, x_a^k + \sum_{k=1}^{K} \sum_{i=1}^{K} x_a^i \, x_a^k.$$

We use the following useful relation

$$\sum_{k=1}^{K} \sum_{i=1}^{K} f_a^i \, f_a^k = 2 \sum_{k=1}^{K} \big(\sum_{i=1}^{k-1} f_a^i + \tfrac{1}{2} f_a^k\big) f_a^k, \tag{11}$$

for the first and last sum and obtain:

$$0 \leq \tfrac{1}{2} \sum_{k=1}^{K} \big(\sum_{i=1}^{k-1} f_a^i + \tfrac{1}{2} f_a^k\big) f_a^k - \sum_{k=1}^{K} \sum_{i=1}^{K} f_a^i \, x_a^k + 2 \sum_{k=1}^{K} \big(\sum_{i=1}^{k-1} x_a^i + \tfrac{1}{2} x_a^k\big) x_a^k.$$

Multiplying with q_a and adding over all arcs yields:

$$0 \leq \sum_{a\in A} q_a\Big(\tfrac{1}{2}\sum_{k=1}^{K}\big(\sum_{i=1}^{k-1} f_a^i + \tfrac{1}{2} f_a^k\big) f_a^k - \sum_{k=1}^{K}\sum_{i=1}^{K} f_a^i\, x_a^k + 2\sum_{k=1}^{K}\big(\sum_{i=1}^{k-1} x_a^i + \tfrac{1}{2} x_a^k\big) x_a^k\Big).$$

We now add the inequality

$$0 \leq \sum_{a\in A}\sum_{k=1}^{K}\big(\tfrac{1}{2} r_a\, f_a^k - r_a\, x_a^k + 2 r_a\, x_a^k\big) - \tfrac{1}{K}\sum_{a\in A}\sum_{k=1}^{K} r_a\, f_a^k,$$

which holds because $K \geq 2$, leading to:

$$0 \leq \tfrac{1}{2}\, C(\boldsymbol{f}) - \sum_{a\in A}\sum_{k=1}^{K}\big(q_a \sum_{i=1}^{K} f_a^i + r_a\big) x_a^k + 2\, C(\boldsymbol{x}) - \tfrac{1}{K}\sum_{a\in A}\sum_{k=1}^{K} r_a\, f_a^k.$$

We drop part of the second term and apply (10):

$$\begin{aligned} 0 &\leq \tfrac{1}{2}\, C(\boldsymbol{f}) - \sum_{a\in A}\sum_{k=1}^{K}\big(q_a \sum_{i=1}^{k} f_a^i + r_a\big) f_a^k + 2\, C(\boldsymbol{x}) - \tfrac{1}{K}\sum_{a\in A}\sum_{k=1}^{K} r_a\, f_a^k \\ &= -\tfrac{1}{2}\, C(\boldsymbol{f}) + 2\, C(\boldsymbol{x}) - \tfrac{1}{2}\sum_{a\in A} q_a \sum_{k=1}^{K} f_a^k\, f_a^k - \tfrac{1}{K}\sum_{a\in A}\sum_{k=1}^{K} r_a\, f_a^k. \end{aligned}$$

This yields:

$$\begin{aligned} C(\boldsymbol{f}) &\leq 4\, C(\boldsymbol{x}) - \sum_{a\in A} q_a \sum_{k=1}^{K} f_a^k\, f_a^k - \tfrac{2}{K}\sum_{a\in A}\sum_{k=1}^{K} r_a\, f_a^k \\ &\leq 4\, C(\boldsymbol{x}) - \tfrac{1}{K}\sum_{a\in A} q_a \Big(\sum_{k=1}^{K} f_a^k\Big)^2 - \tfrac{2}{K}\sum_{a\in A}\sum_{k=1}^{K} r_a\, f_a^k \\ &= 4\, C(\boldsymbol{x}) - \tfrac{2}{K}\sum_{a\in A} q_a \sum_{k=1}^{K}\Big(\sum_{i=1}^{k-1} f_a^k + \tfrac{1}{2} f_a^k\Big) f_a^k - \tfrac{2}{K}\sum_{a\in A}\sum_{k=1}^{K} r_a\, f_a^k, \end{aligned}$$

where the second inequality follows from the inequality of Cauchy-Schwarz and the last equation follows by (11). Hence, we get $C(\boldsymbol{f}) \leq 4\, C(\boldsymbol{x}) - \frac{2}{K} C(\boldsymbol{f})$, from which the claim follows. ☐

We do not know whether this result is tight. The best known lower bound is the following.

Theorem 2. *In case of linear cost functions no deterministic online algorithm for* OnlineMCRP *is c-competitive for any $c < 1.309$.*

Proof. The proof is similar to the proof of Proposition 1. Consider the network displayed in Figure 1 (a). Each arc a of the network has the same price function

$p_a(z) = 2\,z$. Let ALG be an arbitrary deterministic online algorithm. We first present ALG commodity 1 with demand 1 that has to be routed from $s_1 = 1$ to $t_1 = 4$.

Assume the algorithm behaves like SEQ. This means that the demand gets evenly divided into two pieces: one half is routed over path $P_1 = (1, 2, 4)$ and the other over path $P_2 = (1, 3, 4)$. In this case we reveal commodity 2 with demand $d \geq 1$ between 1 and 2 and then commodity 3 with demand d between 2 and 4. For these commodities there exists a unique path. Therefore, ALG yields for this sequence σ the cost:

$$\textsc{Alg}(\sigma) = \textsc{Seq}(\sigma) = 4 \cdot (\tfrac{1}{2})^2 + 2 \cdot \big(\tfrac{1}{2} + \tfrac{1}{2}d\big) \cdot d + 2 \cdot \big(\tfrac{1}{2} + \tfrac{1}{2}d\big) \cdot d = 1 + 2d + 2d^2.$$

Since $d \geq 1$, the optimal offline solution is to route commodity 1 only over path P_2 and the other along its unique path. Therefore,

$$\textsc{Opt}(\sigma) = 2 \cdot 1^2 + 2 \cdot \big(\tfrac{1}{2} \cdot d\big) \cdot d + 2 \cdot \big(\tfrac{1}{2} \cdot d\big) \cdot d = 2 + 2d^2.$$

This leads to

$$\frac{\textsc{Alg}(\sigma)}{\textsc{Opt}(\sigma)} = \frac{1 + 2d + 2d^2}{2 + 2d^2}.$$

Hence, ALG cannot have a competitive ratio less than

$$\max_{d \geq 1} \frac{1 + 2d + 2d^2}{2 + 2d^2} \geq 1.309.$$

If ALG does not behave like SEQ for the first commodity, ALG has to route more than one half of the demand over path P_1 or path P_2. If it is path P_1, then we present commodities 2 and 3 as above. Otherwise, we present commodities 2 with demand d between 1 and 3 and commodity 3 with demand d between 3 and 4. W.o.l.g. we assume that ALG routes the greater portion over path P_1. Let α be the demand routed over path P_1; by assumption, $\frac{1}{2} < \alpha \leq 1$. The cost of ALG for the sequence σ is

$$\textsc{Alg}(\sigma) = 2 \cdot \alpha^2 + 2 \cdot (1 - \alpha)^2 + 2 \cdot (2\alpha d + d^2).$$

Since $d \geq 1$, the optimal cost stays the same and we obtain:

$$\frac{\textsc{Alg}(\sigma)}{\textsc{Opt}(\sigma)} = \frac{2\alpha^2 + 2(1-\alpha)^2 + 4\alpha d + 2d^2}{2 + 2d^2} > \frac{1 + 2d + 2d^2}{2 + 2d^2}.$$

Therefore, ALG cannot have a competitive ratio less than 1.309. □

The proof of Theorem 2 also shows that SEQ is not dominated by any other deterministic online algorithm. In fact, as we show next, possible good algorithms for the ONLINEMCRP have to split the demands.

Theorem 3. *An deterministic online algorithm for the* ONLINEMCRP *that routes all demands unsplittable is not competitive.*

Proof. Consider the network shown in Figure 1 (b). This network contains $n+2$ nodes and n paths from node s to node t. The price functions are $p_a(z) = 2\,z$ for all $a \in A$. Let ALG be an arbitrary deterministic online algorithm which does not split demands. We consider a single commodity with demand 1 between s and t. Since ALG does not split, the cost of its routing is independent from the chosen path:

$$\text{ALG}(\sigma) = 2 \cdot (\tfrac{1}{2} \cdot 1) \cdot 1 + 2 \cdot (\tfrac{1}{2} \cdot 1) \cdot 1 = 2.$$

An optimal solution splits the demand into n evenly divided pieces and sends each piece over an different path. This leads to an optimal cost of

$$\text{OPT}(\sigma) = 2 \cdot n \cdot (\tfrac{1}{n})^2 = \tfrac{2}{n}.$$

Therefore, the competitive ratio of ALG is not smaller than n. Since this holds for all $n \in \mathbb{N}$, ALG is not competitive. □

In Section 4 we further investigate the case where we are not allowed to split demand.

We now consider the *parallel arc case*, i.e., D consists of two nodes and parallel arcs only. Recall from Lemma 1 and 2 that a flow $\boldsymbol{x}$ solves the offline problem (7) and $\boldsymbol{f}$ is generated by SEQ if and only if:

$$\text{for all } a \in A \text{ with } \sum_{k=1}^{K} x_a^k > 0: \quad p_a\Big(\sum_{i=1}^{K} x_a^i\Big) \le p_{\hat{a}}\Big(\sum_{i=1}^{K} x_{\hat{a}}^i\Big)$$

$$\text{for all } a \in A,\ k \in [K] \text{ with } f_a^k > 0: \quad p_a\Big(\sum_{i=1}^{k} f_a^i\Big) \le p_{\hat{a}}\Big(\sum_{i=1}^{k} f_{\hat{a}}^i\Big).$$

Lemma 3. *Given a sequence $\sigma = 1, \ldots, K$, let $\boldsymbol{f}$ be the flow generated by* SEQ*. Define $A_k^+ := \{a \in A \,:\, f_a^k > 0\}$ for $k \in [K]$. Then,*

$$p_a\Big(\sum_{i=1}^{k+1} f_a^i\Big) \le p_{\hat{a}}\Big(\sum_{i=1}^{k+1} f_{\hat{a}}^i\Big),\ \forall a \in A_k^+,\ \hat{a} \in A,\ k = 1, \ldots, K-1.$$

Proof. Let $a \in A_k^+$. First assume that $a \in A_{k+1}^+$. Then by the optimality conditions above for $(\boldsymbol{f}^1, \ldots, \boldsymbol{f}^{k+1})$ the claim follows.

Now assume $a \notin A_{k+1}^+$. Then we have for all $\hat{a} \in A$:

$$p_a\Big(\sum_{i=1}^{k+1} f_a^i\Big) = p_a\Big(\sum_{i=1}^{k} f_a^i\Big) \le p_{\hat{a}}\Big(\sum_{i=1}^{k} f_{\hat{a}}^i\Big) \le p_{\hat{a}}\Big(\sum_{k=1}^{k+1} f_{\hat{a}}^i\Big),$$

where the first inequality follows from the optimality condition for the flow $(\boldsymbol{f}^1, \ldots, \boldsymbol{f}^{k+1})$, and the second follows from the assumption that the price functions are nondecreasing. □

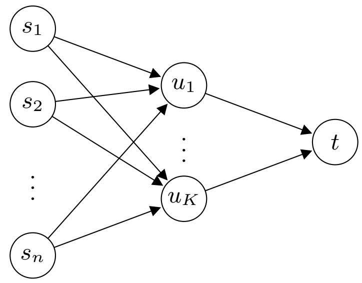

Fig. 2: Construction for the proof of Theorem 5.

Theorem 4. *Given a sequence of commodities, let $\boldsymbol{f}$ be the flow generated by* Seq *for this sequence. Then, $C(\boldsymbol{f}) \leq C(\boldsymbol{x})$ for any feasible $\boldsymbol{x}$, i.e., $\boldsymbol{f}$ is also an offline optimum.*

Proof. For the last commodity K we have the following optimality condition:

$$p_a\Big(\sum_{i=1}^{K} f_a^i\Big) \leq p_{\hat{a}}\Big(\sum_{k=1}^{K} f_{\hat{a}}^i\Big), \ \forall a \in A_K^+. \tag{12}$$

Using Lemma 3 for $k = K-1$ we obtain:

$$p_a\Big(\sum_{i=1}^{K} f_a^i\Big) \leq p_{\hat{a}}\Big(\sum_{k=1}^{K} f_{\hat{a}}^i\Big), \ \forall a \in A_{K-1}^+.$$

Inequality (12) and applying Lemma 3 iteratively $K-1$ times yields the optimality conditions (2) for the offline optimum. □

4 Unsplittable Routings

In this section we study the variant of the OnlineMCRP in which demands are not allowed to be split, i.e., unsplittable routings. Such a restriction often occurs in practice, e.g., in single path routing problems in telecommunication networks. It is possible to formulate a mixed integer convex program for this setting. In contrast to the splittable case, however, the offline problem is NP-hard in this case.

Theorem 5. *The offline problem for the unsplittable* OnlineMCRP *is NP-hard, even when the price functions are linear.*

Proof. Consider an instance of the *minimum sum of squares problem*, which is NP-complete in the strong sense (see Garey and Johnson [13]). Here, one is given nonnegative integers $d_1, \ldots, d_K$ and positive integers $N \leq K$ and J. The question is whether there exists a partition of $[K]$ into N sets $A_1, \ldots, A_N$ such that

$$\sum_{i=1}^{N} \Big(\sum_{k \in A_i} d_k \Big)^2 \leq J?$$

For the reduction to the offline problem, we construct a directed graph D with nodes $\{s_1, \ldots, s_K, u_1, \ldots, u_N, t\}$ and the following arcs: For each $k \in [K]$ and $i \in [N]$ we have an arc (s_k, u_i) with price function 0. For each $i \in [N]$ we add an arc $a = (u_i, t)$ with price function $p_a(z) = 2\,z$; see Figure 2. Furthermore, for $k \in [K]$ there are demands d_k between s_k and t.

We now claim that there exists an unsplittable solution to the offline problem of value at most J if and only if the answer to the minimum sum of squares problem is positive. First assume that $A_1, \ldots, A_N$ is the wanted partition. Then if $k \in A_i$, we route commodity k along u_i to t. Using (11), we obtain the following costs:

$$2 \sum_{i=1}^{N} \sum_{k \in A_i} \big(\sum_{\substack{j \in A_i \\ j<k}} d_j + \tfrac{1}{2} d_k \big) d_k = \sum_{i=1}^{N} \sum_{k \in A_i} \sum_{j \in A_i} d_k\, d_j = \sum_{i=1}^{N} \Big(\sum_{k \in A_i} d_k \Big)^2.$$

This proves the forward direction of the claim. Conversely, assume that there exists an unsplittable flow of value J. For $i = 1, \ldots, N$, let A_i be the set of indices k whose corresponding demands are routed over the arc (u_i, t). Again the cost is given as above, which shows that there exits a solution to the minimum sum of squares problem. □

When the price functions are constant, both the unsplittable variants of (4) and (7) are min cost flow problems and hence polynomial time solvable.

Theorem 6. *In general there exists no competitive deterministic online algorithm for the unsplittable* OnlineMCRP.

Proof. Given the network shown in Figure 1 (a), where each arc a has a price function $p_a(z) = m \cdot z^{m-1}$ with $m > 2$. Let Alg be an arbitrary deterministic online algorithm for the considered problem. First, we reveal a commodity with demand $d_1 = 1$, source $s_1 = 1$, and target $t_1 = 4$. W.l.o.g. we assume that Alg uses path $P = (1, 2, 4)$ to route this demand. Commodity 2 is released with demand $d_2 = 1$, source $s_2 = 1$, and target $t_2 = 2$. For this sequence σ, Alg yields the cost

$$\textsc{Alg}(\sigma) = 2 \cdot 1^m + (1+1)^m = 2 + 2^m.$$

The optimal cost is $\textsc{Opt}(\sigma) = 3$. Therefore, as $m \to \infty$, it follows that Alg is not competitive. □

Theorem 7. *If we consider only linear price functions, no deterministic online algorithm has a competitive ratio less than* 2 *for the unsplittable* OnlineMCRP.

Proof. Consider the network shown in Figure 1 (a), where each link a has the same price function $p_a(z) = 2\,z$. Let $\textsc{Alg}$ be an arbitrary deterministic online algorithm. We first reveal commodity 1 with demand $d_1 = 1$, source $s_1 = 1$, and target $t_1 = 4$. This request can be routed over path $P_1 = (1, 2, 4)$ or over path $P_2 = (1, 3, 4)$. W.l.o.g. assume that $\textsc{Alg}$ chooses path P_1. Now we release two more commodities. Both have a demand of 1. One has to routed from 1 to 2 and the other from 2 to 4. The assignment by $\textsc{Alg}$ for this sequence σ leads to a cost of

$$\textsc{Alg}(\sigma) = 2 \cdot 1^2 + (2 + 1^2) + (2 + 1^2) = 8.$$

Since the optimal cost for σ is $\textsc{Opt}(\sigma) = 4$, the competitive ratio of $\textsc{Alg}$ is at least 2.

5 Final Comments and Future Research

In practice, routings have to consider capacities, which we ignored in our approach. In this case, however, one can easily construct examples in which $\textsc{Seq}$ does not even produce a feasible solution. Further requirements in practice include path length restrictions and survivability issues.

In the future, we plan to investigate the competitiveness of $\textsc{Seq}$ for nonlinear prices functions. It is also an open issue whether the competitiveness bound in Theorem 1 is tight and whether the optimality results in Theorem 4 can be extended.

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