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Graph Coloring Manifolds

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A . We introduce a new and rich class of *graph coloring manifolds* via the Hom complex construction of Lovász. The class comprises examples of Stiefel manifolds, series of spheres and products of spheres, cubical surfaces, as well as examples of Seifert manifolds. Asymptotically, graph coloring manifolds provide examples of highly connected, highly symmetric manifolds.

1. Introduction

In the topological approach to graph coloring, initiated by Lovász' proof [**19**] of the Kneser Conjecture [**17**], lower bounds on the *chromatic number* $\chi(H)$ of a graph H are obtained by exploiting topological invariants of a simplicial or cell complex $K(H)$ that is associated with H.

There are several standard constructions that associate a topological space $K(H)$ with a graph H, e.g., the (simplicial) neighborhood complex $\mathcal{N}(H)$ of Lovász [**19**], the (simplicial) box complex $B(H)$ of Matoušek and Ziegler [**26**], and, with respect to a reference graph G, the (cellular) Hom complex $\mathrm{Hom}(G, H)$ of Lovász (cf. [**2**], [**18**]).

From an algorithmic point of view, the topological approach seems, up to now, not suitable to produce "good" lower bounds on $\chi(H)$ for general input graphs H: For example, the historically first topological lower bound by Lovász requires the computation of the connectivity of the neighborhood complex $\mathcal{N}(H)$.

T 1. (Lovász [**19**]) *Let H be a graph. If $\mathcal{N}(H)$ is k-connected, then $\chi(H) \geq k+3$.*

However, neighborhood complexes of graphs can be of "arbitrary" homotopy type [**9**], and for general complexes it is *not decidable* whether they are 1-connected or not! Moreover, there are cases for which the connectivity could be determined, but for which the corresponding lower bounds are far from tight [**36**].

It is therefore most surprising that for highly structured, highly symmetric graphs such as Kneser graphs and generalization [**1**] the topological approach provides sharp lower bounds while other approaches fail badly; [**26**] discusses this issue and gives further references.

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In order to get away from connectivity, lower bounds have been formulated in terms of topological invariants that *are* computable [**3**, Remark 2.7], or the topological tools have been replaced by purely combinatorial ones; see Matoušek [**25**]. Still, the size of the associated complexes causes problems, since for the standard constructions the number of cells of the complexes $K(H)$ grows exponentially.

In a recent series of papers, Babson and Kozlov [**2**], [**3**] and Kozlov [**18**] intensively studied properties of Hom complexes $\mathrm{Hom}(G, H)$ and proved new topological lower bounds (see as well Čukić and Kozlov [**11**], [**10**], Schultz [**32**], and Živaljević [**35**]). For example, it turned out that $\mathrm{Hom}(K_2, K_n)$ is a PL sphere of dimension $n-2$, for $n \geq 2$, and by spectral sequence calculations that the Hom complexes $\mathrm{Hom}(C_5, K_n)$ have the (co)homology of Stiefel manifolds. This was the starting point for the first author to formulate Conjecture 12 (see Section 5) that the Hom complexes $\mathrm{Hom}(C_5, K_n)$ *are* (PL) *homeomorphic* to Stiefel manifolds.

In this paper, we will show that the Hom complexes $\mathrm{Hom}(C_5, K_n)$ indeed are PL manifolds. More generally, we will characterize in Theorem 6 (Section 3) those graphs G for which the Hom complexes $\mathrm{Hom}(G, K_n)$ are PL manifolds for all $n \geq \chi(G)$. Such manifolds we call *graph coloring manifolds*.

In Section 2, we give a short account on Hom complexes. Section 3 introduces graph coloring manifolds. Various examples and series of examples of graph coloring manifolds are discussed in Sections 4–6.

2. Basic Definitions, Notations, and Examples

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs with node sets $V(G)$ and $V(H)$ and edge sets $E(G) \subseteq \binom{V(G)}{2}$ and $E(H) \subseteq \binom{V(H)}{2}$, respectively. We assume that the graphs are *simple graphs*, i.e., graphs without loops and parallel edges.

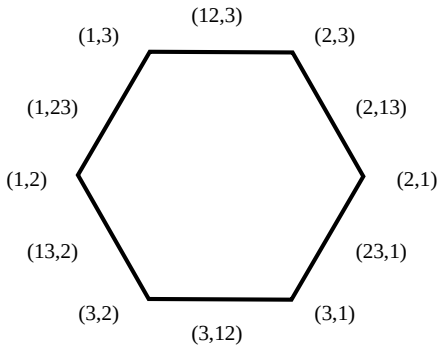
A *graph homomorphism* is a map $\phi : V(G) \to V(H)$, such that if $\{i, j\} \in E(G)$, then $\{\phi(i), \phi(j)\} \in E(H)$, that is, the image of every edge of the graph G is an edge of the graph H. Let the set of all graph homomorphisms from G to H be denoted by $\mathcal{H}om(G, H)$. For two disjoint sets of vertices $A, B \subseteq V(G)$ we define $G[A, B]$ as the subgraph of G with $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{\{a, b\} \in E(G) \colon a \in A, b \in B\}$.

Let $\Delta^{V(H)}$ be the (abstract) simplex whose set of vertices is $V(H)$. Furthermore, let $C(G, H)$ denote the direct product $\prod_{x \in V(G)} \Delta^{V(H)}$, i.e., the copies of $\Delta^{V(H)}$ are indexed by vertices of G. A cell of $C(G, H)$ is a direct product of simplices $\prod_{x \in V(G)} \sigma_x$.

D 2. *For any pair of graphs G and H let the* Hom complex $\mathrm{Hom}(G, H)$ *be a subcomplex of $C(G, H)$ defined by the following condition: $c = \prod_{x \in V(G)} \sigma_x \in \mathrm{Hom}(G, H)$ if and only if for any $u, v \in V(G)$ if $\{u, v\} \in E(G)$, then $H[\sigma_u, \sigma_v]$ is complete bipartite.*

The topology of $\mathrm{Hom}(G, H)$ is inherited from the product topology of $C(G, H)$. Thus, $\mathrm{Hom}(G, H)$ is a polyhedral complex whose (non-empty) cells are products of simplices and are indexed by functions (multi-homomorphisms) $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that if $\{\imath, \jmath\} \in E(G)$, then for every $\tilde{\imath} \in \eta(\imath)$ and $\tilde{\jmath} \in \eta(\jmath)$ it follows that $\{\tilde{\imath}, \tilde{\jmath}\} \in E(H)$.

Let $V(G) = \{1, \ldots, m\}$. We encode the functions η by vectors $(\eta(1), \ldots, \eta(m))$ of non-empty sets $\eta(i) \subseteq V(H)$ with the above properties. A cell $(A_1, \ldots, A_m)$ of $\mathrm{Hom}(G, H)$ is a *face* of a cell $(B_1, \ldots, B_m)$ of $\mathrm{Hom}(G, H)$ if $A_i \subseteq B_i$ for all $1 \leq i \leq m$. In particular, $\mathrm{Hom}(G, H)$ has $\mathcal{H}om(G, H)$ as its set of vertices. Moreover, every cell $(A_1, \ldots, A_m)$ of $\mathrm{Hom}(G, H)$ is a product of m simplices of dimension $|A_i| - 1$ for $1 \leq i \leq m$. For brevity, we write sets $A = \{a_1, \ldots, a_k\} \subseteq V(H)$ in compressed form as strings, i.e., $A = a_1 \ldots a_k$.



F 1. The Hom complex Hom(K_2, K_3).

A cell of Hom(G, H) is a *maximal face* or *facet* if it is not contained in any higher-dimensional cell of Hom(G, H).

Example 1: The cells of the Hom complex Hom(K_2, K_3) are given by the vectors $(1, 2)$, $(1, 3)$, $(2, 3)$, $(2, 1)$, $(3, 1)$, $(3, 2)$, $(12, 3)$, $(13, 2)$, $(23, 1)$, $(3, 12)$, $(2, 13)$, and $(1, 23)$. Therefore, Hom(K_2, K_3) is a cycle with six edges; see Figure 1.

Example 2: The Hom complex Hom(K_2, K_n) is a PL sphere of dimension $n-2$ for $n \geq 2$. In fact, Hom(K_2, K_n) is the boundary complex of a polytope [**2**, Sect. 4.2]: it can be described as the boundary of the Minkowski sum of an $(n-1)$-dimensional simplex σ_{n-1} and its negative $-\sigma_{n-1}$, as stated in [**24**, p. 107, Ex. 3 (c)].

3. Vertex-Links and Flag Simplicial Spheres

Babson and Kozlov asked in [**2**] for what graphs the Hom complex construction provides a connection to polytopes. In this section, we will characterize those graphs G for which Hom(G, K_n) is a piecewise linear (PL) manifold for all $n \geq \chi(G)$.

A (finite) simplicial complex is a *PL d-manifold* if and only if every vertex-link is a PL $(d-1)$-sphere, i.e., every vertex-link is PL homeomorphic to the boundary of the standard d-simplex σ_d.

There are several ways to define the *link* of a vertex v for polyhedral complexes. For Hom complexes Hom(G, H) we will use the following. Let the face poset of Hom(G, H) be denoted by $\mathcal{F}(\text{Hom}(G, H))$ and let the link of v in Hom(G, H) be the cell complex whose face poset is given by $\mathcal{F}_{>v}(\text{Hom}(G, H))$. This link then is a simplicial complex since Hom(G, H) is a prodsimplicial complex (cf. [**18**, 2.4.3]).

For a graph G we say that $X \subseteq V(G)$ is an *independent set* if there is no edge between any two vertices of X. The *independent set complex* Ind(G) of a graph G is the simplicial complex with vertex set $V(G)$ and $X \subseteq V(G)$ forming a simplex if and only if X is an independent set in G, i.e., Ind$(G) = \{X \subseteq V(G) \mid X \text{ is independent in } G\}$.

Every cell of Hom(G, K_n) corresponds to a multi-coloring $f\colon V(G) \to 2^{\{1,\ldots,n\}} \setminus \{\emptyset\}$, where the map f assigns $f(v)$ distinct colors to every vertex $v \in V(G)$, such that the set of vertices colored by any color $i \in \{1, \ldots, n\}$ forms an independent set in G. We denote these sets by $\Delta_i(f) = \{v \in V(G) \mid i \in f(v)\}$ and consider them as simplices of Ind(G).

L 3. *Let ϕ be a vertex of* $\mathrm{Hom}(G, K_n)$*, i.e., a proper coloring of G which we regard as a multi-coloring $\phi\colon V(G) \to 2^{\{1,\dots,n\}}\setminus\{\emptyset\}$ with $|\phi(v)| = 1$ for all $v \in V(G)$. Then*

$$\mathrm{link}_{\mathrm{Hom}(G,K_n)}(\phi)$$

is isomorphic to the join product

$$\mathrm{link}_{\mathrm{Ind}(G)}(\Delta_1(\phi)) * \cdots * \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_n(\phi)).$$

Proof. A simplex of the first complex $\mathrm{link}_{\mathrm{Hom}(G,K_n)}(\phi)$ corresponds to a multi-coloring $f\colon V(G) \to 2^{\{1,\dots,n\}}\setminus\{\emptyset\}$ which extends ϕ. We can consider such an extension color-wise. For color $i \in \{1,\dots,n\}$ we have that $\Delta_i(\phi) \subseteq \Delta_i(f) \in \mathrm{Ind}(G)$ and therefore $\Delta_i(f)\setminus\Delta_i(\phi) \in \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_i(\phi))$. Thus we can identify f with $(\Delta_1(f)\setminus\Delta_1(\phi),\dots,\Delta_n(f)\setminus\Delta_n(\phi))$ and therefore can regard f as an element of $\mathrm{link}_{\mathrm{Ind}(G)}(\Delta_1(\phi)) * \cdots * \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_n(\phi))$. Conversely, every simplex of $\mathrm{link}_{\mathrm{Ind}(G)}(\Delta_1(\phi)) * \cdots * \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_n(\phi))$ gives rise to a unique extension f of ϕ. □

L 4.

(1) *If* $\mathrm{Ind}(G)$ *is a PL sphere, then* $\mathrm{Hom}(G, K_n)$ *is a PL manifold for any $n \geq \chi(G)$.*
(2) *If* $\mathrm{Hom}(G, K_n)$ *is a PL manifold and $n > \chi(G)$, then* $\mathrm{Ind}(G)$ *is a PL sphere.*

Proof. 1. Let $\mathrm{Ind}(G)$ be a PL sphere. Since the link of any simplex of a PL sphere is a PL sphere (of lower dimension) and since the join product of PL spheres is again a PL sphere, it follows by the previous lemma that the link of any vertex of $\mathrm{Hom}(G, K_n)$ is a PL sphere. Thus, $\mathrm{Hom}(G, K_n)$ is a PL manifold.

2. Let $\mathrm{Hom}(G, K_n)$ be a PL manifold. Since $n > \chi(G)$, there is a vertex ϕ of $\mathrm{Hom}(G, K_n)$ that does not use the color n. Hence, $\mathrm{link}_{\mathrm{Ind}(G)}(\Delta_n(\phi)) = \mathrm{Ind}(G)$. Since $\mathrm{Hom}(G, K_n)$ is a PL manifold, $\mathrm{link}_{\mathrm{Hom}(G,K_n)}(\phi) \cong \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_1(\phi)) * \cdots * \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_n(\phi))$ is a PL sphere. Now, the join product of simplicial complexes is a PL sphere if and only if every factor is a PL sphere (see [**30**, 2.24(5)]). It follows that the last factor, $\mathrm{link}_{\mathrm{Ind}(G)}(\Delta_n(\phi)) = \mathrm{Ind}(G)$, is a PL sphere. □

We can formulate this result in terms of G using the following definition.

D 5. *Let K be a (finite) simplicial complex. If K has no "empty simplices", i.e., if every set of vertices of K which form a clique in the* 1*-skeleton* $\mathrm{Skel}_1(K)$ *actually spans a simplex, then K is a* flag simplicial complex *(cf.* [**8**]*). A* flag simplicial sphere *is a flag simplicial complex which triangulates a sphere.*

The *clique complex* $\mathrm{Cliq}(G) = \{X \subseteq V(G) \mid X \text{ is a clique in } G\}$ of any graph G is a flag simplicial complex in a natural way with $G = \mathrm{Skel}_1(\mathrm{Cliq}(G))$.

T 6. *Let G be a graph. Then the Hom complex* $\mathrm{Hom}(G, K_n)$ *is a PL manifold for all $n \geq \chi(G)$ if and only if G is the complement of the* 1*-skeleton of a flag simplicial PL sphere.*

Proof. Let $\mathrm{Hom}(G, K_n)$ be a PL manifold for all $n \geq \chi(G)$. Then, in particular, $\mathrm{Hom}(G, K_{\chi(G)+1})$ is a PL manifold, and thus, by Lemma 4, $\mathrm{Ind}(G) = \mathrm{Cliq}(\overline{G})$ is a PL sphere. Hence, G is the complement of the 1-skeleton of the flag simplicial PL sphere $\mathrm{Cliq}(\overline{G})$.

Conversely, if G is the complement of the 1-skeleton of a flag simplicial PL sphere K, i.e., $G = \overline{\mathrm{Skel}_1(K)}$, then $\mathrm{Ind}(G) = \mathrm{Cliq}(\mathrm{Skel}_1(K)) = K$ is a flag simplicial PL sphere and therefore $\mathrm{Hom}(G, K_n)$ a PL manifold by Lemma 4. □

Remark 1: If $n < \chi(G)$, then $\mathrm{Hom}(G, K_n) = \emptyset$. If $n = \chi(G)$, then every vertex ϕ of $\mathrm{Hom}(G, K_{\chi(G)})$ uses all colors $1,\dots,\chi(G)$. If $\mathrm{Hom}(G, K_{\chi(G)})$ is a PL manifold, then $\mathrm{Ind}(G)$

need not be a PL sphere. It is only required, that the links of vertices (or of higher-dimensional faces if every color is used more than once in every vertex of $\mathrm{Hom}(G, K_{\chi(G)})$) of $\mathrm{Ind}(G)$ are flag simplicial PL spheres. In particular, if G is the complement of the 1-skeleton of a flag combinatorial manifold, then $\mathrm{Hom}(G, K_{\chi(G)})$ is a PL manifold. As another example, if G is a connected bipartite graph, then $\mathrm{Hom}(G, K_2) = S^0$.

Remark 2: It is possible for $\mathrm{Hom}(G, K_n)$ to be a (non-PL) manifold, even without $\mathrm{Ind}(G)$ being a sphere. (See [**5**] for a discussion of non-PL spheres and non-PL manifolds.) For example, if $\mathrm{Ind}(G)$ is a flag combinatorial homology sphere (i.e., a combinatorial manifold with the homology of a sphere, but not homeomorphic to the standard sphere) and $n > \chi(G)$, then for every vertex ϕ of $\mathrm{Hom}(G, K_n)$ the join product $\mathrm{link}_{\mathrm{Hom}(G,K_n)}(\phi) \cong \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_1(\phi)) * \cdots * \mathrm{link}_{\mathrm{Ind}(G)}(\Delta_n(\phi))$ is a simplicial sphere by the double suspension theorem of Edwards [**13**] and Cannon [**7**]. Also, if G is the complement of the 1-skeleton of a flag simplicial non-PL sphere, then $\mathrm{Hom}(G, K_n)$ is a non-PL manifold for $n > \chi(G)$.

D 7. *A Hom complex* $\mathrm{Hom}(G, K_n)$ *is a* graph coloring manifold *if* G *is the complement of the* 1*-skeleton of a flag simplicial PL sphere.*

Remark 3: By Definition 7 and Theorem 6 graph coloring manifold are PL manifolds.

Remark 4: Graph coloring manifolds are highly symmetric: relabeling the colors of K_n defines an action of the symmetric group S_n on $\mathrm{Hom}(G, K_n)$.

Babson and Kozlov [**2**, 2.4] stated as a basic property of Hom complexes that

$$\mathrm{Hom}(G_1 \dot\cup\, G_2, H) = \mathrm{Hom}(G_1, H) \times \mathrm{Hom}(G_2, H), \tag{1}$$

from which it follows that if $G = \dot{\bigcup}_{i=1,\ldots,k} K_2$ is the complement of the 1-skeleton of the boundary of the k-dimensional crosspolytope ∂C_k^{Δ}, then

$$\mathrm{Hom}\,(\dot{\bigcup}_{i=1,\ldots,k} K_2, K_n) = \bigtimes_{i=1,\ldots,k} S^{n-2}. \tag{2}$$

D 8. *A flag simplicial PL sphere is* prime *if the complement of its* 1*-skeleton is connected. A Hom complex* $\mathrm{Hom}(G, K_n)$ *is a* graph coloring manifold of sphere dimension d *if* G *is the complement of the* 1*-skeleton of a prime flag simplicial PL sphere of dimension* d*.*

Since every coloring of a graph G can be regarded as a covering of G by independent sets, the following lower bound holds for the chromatic number $\chi(G)$ of G:

$$\chi(G) \geq \left\lceil \frac{|V|}{\alpha(G)} \right\rceil = \left\lceil \frac{|V|}{\omega(\overline{G})} \right\rceil, \tag{3}$$

where $\alpha(G)$ is the *independence number* or *stable set number* of G (i.e., the maximum size of an independent set in G) and $\omega(G)$ is the *clique number* of G (i.e., the maximum size of a clique in G). If G is the complement of the 1-skeleton of a prime flag simplicial *PL* d-sphere on m vertices, then $\alpha(G) = \omega(\overline{G}) = d + 1$. Thus

$$\chi(G) \geq \left\lceil \frac{m}{d+1} \right\rceil \tag{4}$$

and

$$\dim(\mathrm{Hom}(G, K_{\chi(G)+k})) = (\chi(G) + k)(d+1) - m \tag{5}$$

for all $k \geq 0$.

The lower bound (4) can be arbitrarily bad: If G is the complement of the 1-skeleton of the suspension $S^0 * C_{2r+1}$ of an odd cycle C_{2r+1}, $r \geq 2$, then $\chi(G) = 2r + 1 > \lceil \frac{2r+3}{3} \rceil$.

From the following theorem it follows that graph coloring manifolds provide examples of highly connected manifolds.

T 9. (Čukić and Kozlov [**10**]) *Let G be a graph of maximal valency s, then the Hom complex* $\mathrm{Hom}(G, K_n)$ *is at least* $(n - s - 2)$*-connected.*

Let G be the complement of the 1-skeleton of a flag simplicial PL sphere. If G has maximal valency s, then $\mathrm{Hom}(G, K_n)$ is simply connected and thus orientable for $n \geq s+3$. We expect that $\mathrm{Hom}(G, K_n)$ is orientable also for $\chi(G) \leq n < s + 3$.

C 10. *Graph coloring manifolds are orientable.*

4. Graph Coloring Manifolds of Sphere Dimension Zero

Trivially, S^0, consisting of two isolated vertices, is the only zero-dimensional flag simplicial sphere. The complement of its (empty) 1-skeleton is the complete graph K_2. Hence, the graph coloring manifolds of sphere dimension zero are the Hom complexes $\mathrm{Hom}(K_2, K_n) \cong S^{n-2}$, for $n \geq 2$.

5. Graph Coloring Manifolds of Sphere Dimension One

The one-dimensional flag simplicial spheres are the cycles C_m of length $m \geq 4$. For $m = 4$ we have that (the 1-skeleton) $\mathrm{SK}_1(\overline{C}_4) = \overline{C}_4 = K_2 \dot\cup K_2$ with

$$\mathrm{Hom}(K_2 \dot\cup K_2, K_n) = \mathrm{Hom}(K_2, K_n) \times \mathrm{Hom}(K_2, K_n) \cong S^{n-2} \times S^{n-2}.$$

If $m \geq 5$, then $\mathrm{SK}_1(\overline{C}_m) = \overline{C}_m$ is connected. In the following, we treat odd and even cycles separately.

5.1. Hom Complexes of Complements of Odd Cycles. Babson and Kozlov [**3**] used topological information on the Hom complexes $\mathrm{Hom}(C_5, K_n)$ (with $\overline{C}_5 \cong C_5$ for $m = 5$) and, more generally, on the Hom complexes $\mathrm{Hom}(C_{2r+1}, K_n)$, for $r \geq 2$ and $n \geq r + 1$, to prove the Lovász Conjecture:

T 11. (Babson and Kozlov [**3**]) *If for a graph H the complex* $\mathrm{Hom}(C_{2r+1}, H)$ *is k-connected, for some $r \geq 1$ and $k \geq -1$, then $\chi(H) \geq k + 4$.*

Babson and Kozlov computed various cohomology groups of the Hom complexes $\mathrm{Hom}(C_m, K_n)$. For $m = 5$, the respective cohomology groups are those of Stiefel manifolds.

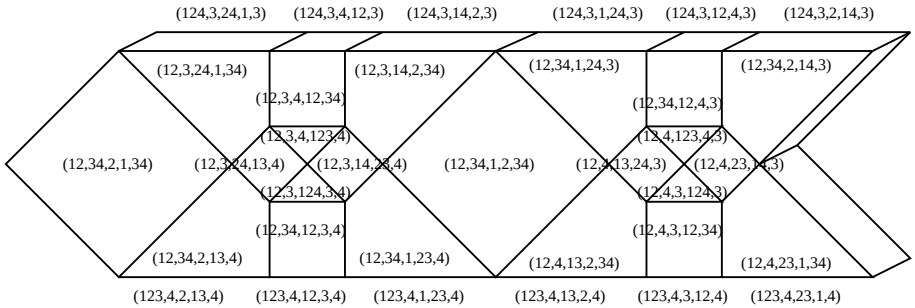
C 12. (Csorba) *The Hom complex* $\mathrm{Hom}(C_5, K_{n+2})$ *is PL homeomorphic to the Stiefel manifold $V_{n+1,2}$.*

It is elementary to verify that $\mathrm{Hom}(C_5, K_3)$ consists of two cycles with 15 vertices and 15 edges each.

Example 5: $\mathrm{Hom}(C_5, K_3) \cong V_{2,2} \cong S^0 \times S^1$.

For $n = 2$, the complex $\mathrm{Hom}(C_5, K_4)$ has 240 vertices and 300 maximal cells that are either cubes or prisms over triangles.

Since every cell of a Hom complex is a product of simplices, triangulations of graph coloring manifolds (*without additional vertices*) can easily be obtained by the product triangulation construction as described in [**22**]. For small examples, the homology of the resulting triangulations can then be computed with one of the programs [**12**] or [**15**].



F 2. The solid torus $(12,*,*,*,*)$ in $\mathrm{Hom}(C_5,K_4)$.

The product triangulation of $\mathrm{Hom}(C_5,K_4)$ has f-vector $f=(240,1680,2880,1440)$. As homology we obtained $H_*(\mathrm{Hom}(C_5,K_4))=(\mathbb{Z},\mathbb{Z}_2,0,\mathbb{Z})$, which coincides with the spectral sequence computations of Babson and Kozlov in [**3**]. We also computed the homology of $\mathrm{Hom}(C_7,K_4)$ and obtained that $H_*(\mathrm{Hom}(C_7,K_4))=(\mathbb{Z},\mathbb{Z}_2,0,\mathbb{Z},0,0)$ which was conjectural in [**3**].

We next used the bistellar flip heuristic BISTELLAR [**21**] to determine that the complex $\mathrm{Hom}(C_5,K_4)$ is homeomorphic to $\mathbb{R}\mathbf{P}^3$. (See [**5**] for a discussion of the heuristic; for large complexes the `bistellar` client (due to N. Witte) of the TOPAZ module of the `polymake` system [**15**] provides a fast implementation of BISTELLAR.)

T 13. $\mathrm{Hom}(C_5,K_4)\cong V_{3,2}\cong\mathbb{R}\mathbf{P}^3$.

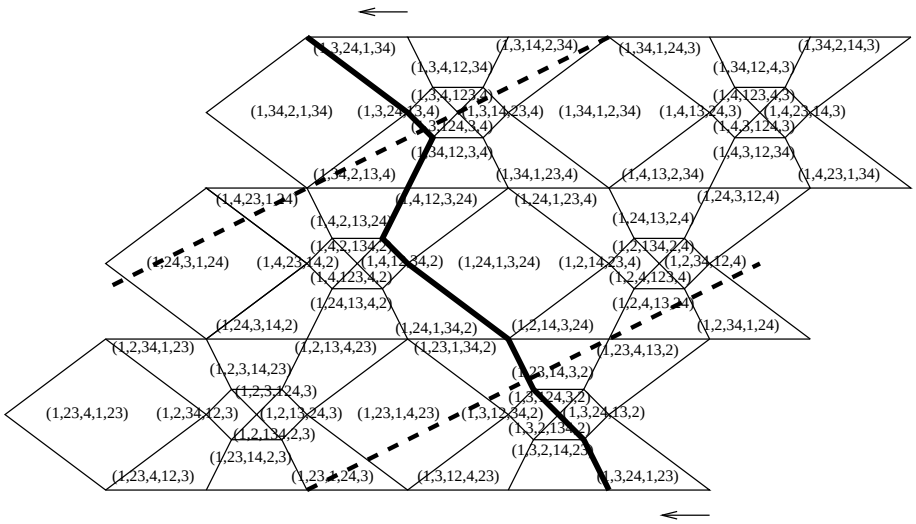
Proof. In addition to the above computer proof, we give an explicit Heegaard decomposition of $\mathrm{Hom}(C_5,K_4)$ from which one can see that this Hom complex is homeomorphic to $\mathbb{R}\mathbf{P}^3$ (and thus homeomorphic to the Stiefel manifold $V_{3,2}$).

First we show that the collection of cells of the form $(ijk,*,*,*,*)$ forms a solid torus. By symmetry it is enough to consider the collection of cells $(123,*,*,*,*)$. Since the numbers 1, 2, and 3 can not occur at positions 2 and 5, it immediately follows that the cells of this collection are of the form $(123,4,*,*,4)$. The middle $(*,*)$-part is the six-gon corresponding to $\mathrm{Hom}(K_2,K_3)$; see Figure 1. So $(ijk,*,*,*,*)$ is the product of a triangle and a circle.

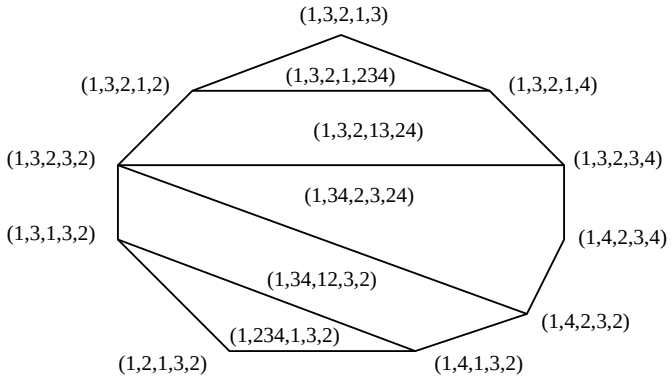
In Figure 2 we display the collections of cells of the form $(12,*,*,*,*)$ (with the cell on the left glued to the cells on the right). Clearly, this collection of cells forms a torus, and therefore, by symmetry, also every collection $(ij,*,*,*,*)$. Finally, the cells of the form $(i,*,*,*,*)$ form a solid torus as well. The boundary torus of $(1,*,*,*,*)$ can be seen in Figure 3. Again, the left side is glued to the right side of Figure 3. The gluing of the top and bottom is indicated by the arrows.

In order to understand how these solid tori are glued together we have to identify meridian disks. For the collections $(ijk,*,*,*,*)$ and $(ij,*,*,*,*)$ this is clear. A meridian disk of the collection $(1,*,*,*,*)$ is given in Figure 4; its boundary corresponds to the thick line in Figure 3. The complement of $(1,*,*,*,*)$ in $\mathrm{Hom}(C_5,K_4)$ is a solid torus composed of the collections $(12,*,*,*,*)$, $(13,*,*,*,*)$, …, $(2,*,*,*,*)$, …, $(234,*,*,*,*)$, which we abbreviate by 12, 13, …, 2, …, 234 in Figure 5. In fact, Figure 5 gives the base sphere of the S^1-fibered space $\mathrm{Hom}(C_5,K_4)$ and makes clear how the different tori are glued together.

A meridian curve of the complement of $(1,*,*,*,*)$ is drawn as a dashed curve in Figure 3. Since this curve is a $(2,1)$-curve, it follows that $\mathrm{Hom}(C_5,K_4)$ is homeomorphic



F 3. The boundary of the solid torus $(1, *, *, *, *)$ in $\mathrm{Hom}(C_5, K_4)$.



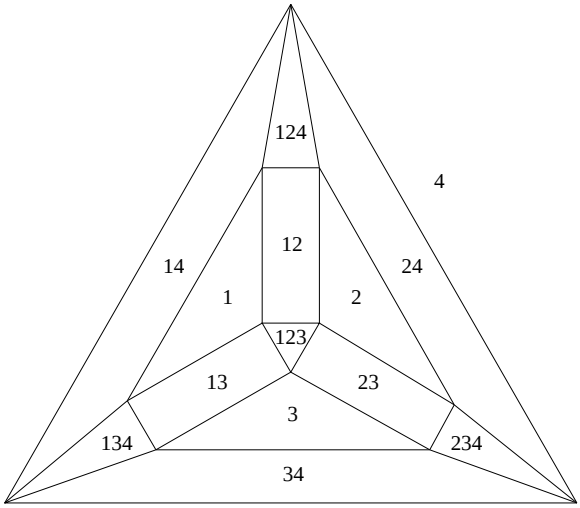
F 4. A meridian disk of the solid torus $(1, *, *, *, *)$ in $\mathrm{Hom}(C_5, K_4)$.

to the 3-dimensional real projective space. The latter space is homeomorphic to the Stiefel manifold $V_{3,2}$. □

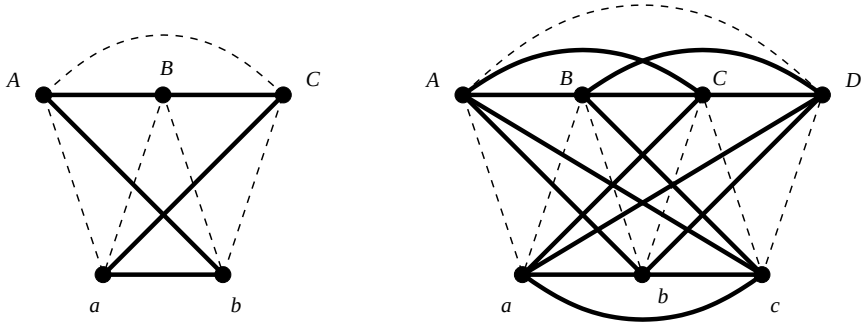
The 5-dimensional Hom complex $\mathrm{Hom}(C_5, K_5)$ consists of 2070 maximal cells and has 1020 vertices. The corresponding product triangulation has f-vector (1020, 25770, 143900, 307950, 283200, 94400). With the `bistellar` client it took less than a week to reduce this triangulation to a triangulation with $f = (12, 66, 220, 390, 336, 112)$. The latter triangulation is 3-neighborly, i.e., it has a complete 2-skeleton, and thus is simply connected. Its homology is $(\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z})$. Moreover, its second Stiefel-Whitney class is trivial, as we computed with `polymake`. By the classification of simply connected 5-manifolds of Barden [**4**], the unique simply connected 5-manifold with homology $(\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z})$ and trivial second Stiefel-Whitney class is $S^3 \times S^2$.

T 14. $\mathrm{Hom}(C_5, K_5) \cong V_{4,2} \cong S^3 \times S^2$.

In the following, we discuss a particular representation of odd cycles that gives some insight into *all* Hom complexes $\mathrm{Hom}(\overline{C}_{2r+1}, K_n)$ of complements of odd cycles $\overline{C}_{2r+1}$, $r \geq 2$.



F 5. Cell decomposition of the base sphere S^2 in $\mathrm{Hom}(C_5, K_4)$.

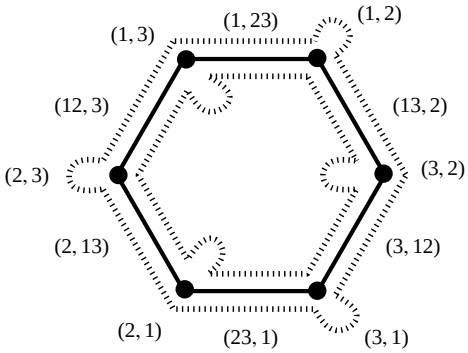


F 6. The (dashed) cycles C_5 and C_7 and their complements.

(With a similar approach we will analyze the Hom complexes $\mathrm{Hom}(\overline{C}_{2r}, K_n)$ of complements of even cycles $\overline{C}_{2r}$, $r \geq 2$, in the next section.)

We display the cycles C_{2r+1}, $r \geq 2$, in form of a crown that is turned upside down; see Figure 6 for the *crown representations* of the (dashed) cycles C_5 and C_7. Clearly, the bottom vertices of a crown representation form a clique, i.e., a complete graph K_r, in the complement $\overline{C}_{2r+1}$.

Let us have a look at the crown representation of C_5. Every cell (a, b, A, B, C) of $\mathrm{Hom}(\overline{C}_5, K_n)$ contains every number $x \in \{1, \ldots, n\}$ at exactly two positions. Since the sets a and b are associated with the bottom vertices that form a clique K_2 in $\overline{C}_5$, the number x can appear in at most one of the sets a and b. If it is contained in, say, a, then the second copy of x can only be placed in the sets A and B that are connected with a by a dashed edge of C_5. The top vertices of $\overline{C}_5$ form a clique minus the (dashed) edge between the leftmost vertex and the rightmost vertex. Hence, if x is contained in neither a nor b, then it is contained in the leftmost top set A and in the rightmost top set C.



F 7. Hom(K_2, K_3) and Hom($\overline{C}_5, K_3$).

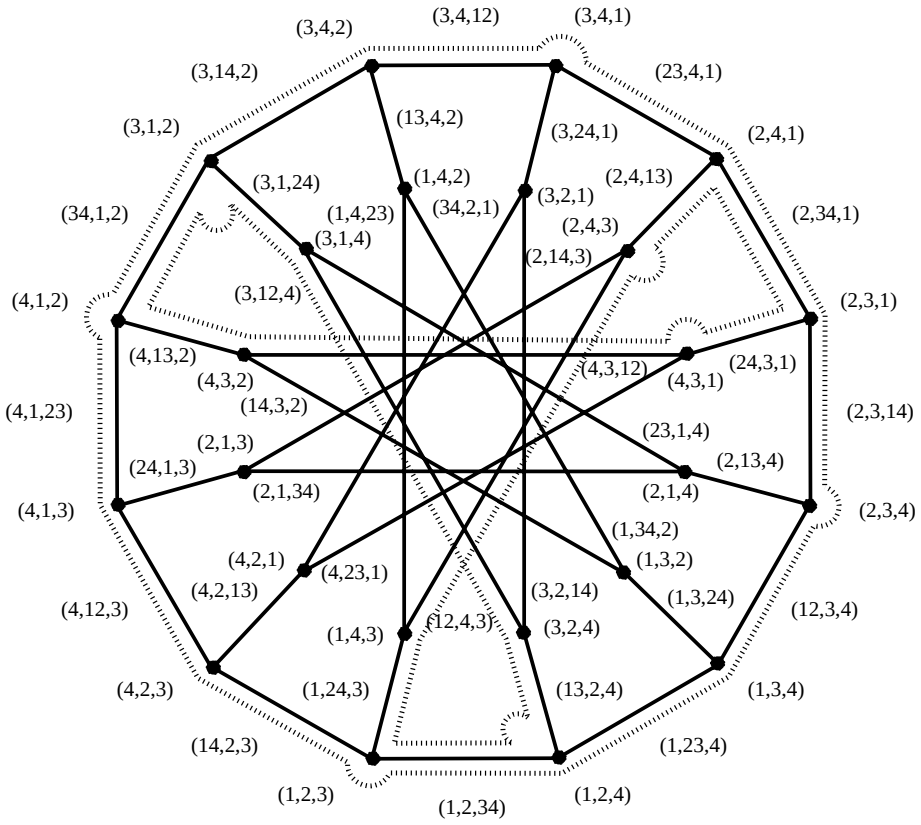
If we restrict us further to $n = 3$ colors, then Hom(K_2, K_3) is a six-gon as displayed in solid in Figure 7. The cell $(a, b) = (1, 23)$ of Hom(K_2, K_3) can be extended to a cell $(a, b; A, B, C)$ of Hom($\overline{C}_5, K_3$) in precisely two ways, either to $(1, 23; 1, 2, 3)$ or to $(1, 23; 1, 3, 2)$. We depict these edges of Hom($\overline{C}_5, K_3$) as dashed edges in Figure 7, parallel to the edge $(1, 23)$ of Hom(K_2, K_3). Let $(1, 23; 1, 2, 3)$ be the upper dashed edge. If we move the number 3 from the second to the third position, then we obtain the cell $(1, 2; 13, 2, 3)$ from which we move on to $(1, 2; 3, 12, 3)$, and from there to $(1, 2; 3, 1, 23)$. These three cells of Hom($\overline{C}_5, K_3$) correspond to the vertex $(1, 2)$ of Hom(K_2, K_3) and are displayed together by a dashed half-cycle at the vertex $(1, 2)$ in Figure 7. If we move on further, then we get to the dashed edge $(13, 2; 3, 1, 2)$, from there to the dashed edge $(3, 12; 3, 1, 2)$, before we again start a half-cycle $(3, 1; 23, 1, 2)$, $(3, 1; 2, 13, 2)$, $(3, 1; 2, 3, 12)$, this time at the vertex $(3, 1)$ of Hom(K_2, K_3). We can then continue on the outer dashed cycle until we reach our starting edge $(1, 23; 1, 2, 3)$ of Hom($\overline{C}_5, K_3$). Similarly, we can move around the inner dashed cycle when we start with $(1, 23; 1, 3, 2)$.

P 15. *The Hom complex* Hom($\overline{C}_{2r+1}, K_{r+1}$) *is the disjoint union of* $r!$ *cycles with* $(2r^2 + 3r + 1)$ *vertices each.*

Proof. We first count the number of vertices of Hom($\overline{C}_{2r+1}, K_{r+1}$), i.e., the number of distinct colorings with $r+1$ colors of $\overline{C}_{2r+1}$. To color the bottom K_r in the crown representation of $\overline{C}_{2r+1}$ we choose r of the $r + 1$ colors and then have $r!$ choices to place these r colors. For one such coloring, say $(1, 2, \ldots, r)$, there are $(2r + 1)$ ways to extend it to a coloring of $\overline{C}_{2r+1}$: If we use the color $r + 1$ just once, then we have $r + 1$ choices to place it in the top row of the crown; the remaining positions for the colors in the top row are then completely determined by the position of the color $(r + 1)$ and by our choice of the colors in the bottom row. If we use the color $r + 1$ twice, then we have to put it at the positions 1 and $r + 1$ of the top row. We further choose one of the colors $1, \ldots, r$ not to be used in the top row; this again determines all the positions for the colors in the top row. Thus we have $(r + 1)$ choices if color $r + 1$ appears once in the top row and r choices if color $r + 1$ appears twice in the top row. Altogether we have

$$\binom{r+1}{r} r!(r + 1 + r) = (2r^2 + 3r + 1)r!$$

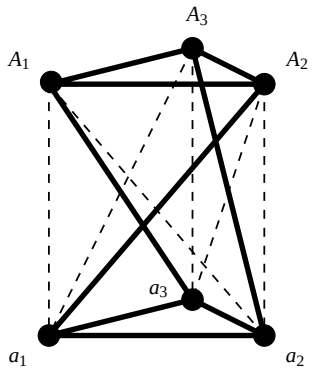
different colorings of $\overline{C}_{2r+1}$ with $r + 1$ colors.



F 8. The Hom complex Hom(K_3, K_4).

Since every number $1, \ldots, r+1$ appears exactly twice in a cell of Hom($\overline{C}_{2r+1}, K_{r+1}$), the dimension of Hom($\overline{C}_{2r+1}, K_{r+1}$) is $2(r+1) - (2r+1) = 1$. If we move for the edge $(1, 2, \ldots, r-1, r(r+1); 1, 2, \ldots, r, r+1)$ of Hom($\overline{C}_{2r+1}, K_{r+1}$) the number $r+1$ from the last position of the bottom row to the first position of the top row and then continue until we reach the edge $(r+1, 1, 2, \ldots, (r-1)r; r+1, 1, 2, \ldots, r-1, r)$, this takes $r+1+r = 2r+1$ steps. After $r+1$ rounds we return to the starting edge $(1, 2, \ldots, r-1, r(r+1); 1, 2, \ldots, r-1, r, r+1)$. Thus, by symmetry, every cycle of Hom($\overline{C}_{2r+1}, K_{r+1}$) has length $(2r+1)(r+1) = 2r^2+3r+1$. Since Hom($\overline{C}_{2r+1}, K_{r+1}$) has $(2r^2 + 3r + 1)r!$ vertices, it follows that Hom($\overline{C}_{2r+1}, K_{r+1}$) consists of $r!$ cycles with $(2r^2 + 3r + 1)$ vertices each. □

As before in the case of Hom($\overline{C}_5, K_3$), every edge of Hom(K_r, K_{r+1}) can be extended in exactly two ways to an edge of Hom($\overline{C}_{2r+1}, K_{r+1}$). This can be interpreted geometrically by thickening every edge of the 1-dimensional manifold Hom(K_r, K_{r+1}) to a 2-dimensional strip and then gluing these strips together at the vertices of Hom(K_r, K_{r+1}). In this way, we get a two-dimensional manifold with boundary, with the boundary being homeomorphic to Hom($\overline{C}_{2r+1}, K_{r+1}$). In Figure 8 we display the Hom complex Hom(K_3, K_4), consisting of 24 vertices and 36 edges, together with two of the $3! = 6$ (dotted) cycles of Hom($\overline{C}_7, K_4$). Every vertex of Hom(K_r, K_{r+1}) can be extended in $r+1$ ways to an edge of Hom($\overline{C}_{2r+1}, K_{r+1}$). These $r+1$ edges form a path that we display as dotted half-cycles in the Figures 7 and 8.



F 9. The cycle C_6 (dashed) and its complement $\overline{C}_6$.

C 16. *The* 3*-dimensional graph coloring manifold* $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+2})$, $r \geq 2$, *is homeomorphic to the orientable Seifert manifold* $\{\, Oo, r! - 2 \,|\, r! \,\}$ *with homology* $(\mathbb{Z}, \mathbb{Z}^{2(r!-2)} \oplus \mathbb{Z}_{r!}, \mathbb{Z}^{2(r!-2)}, \mathbb{Z})$.

The conjecture holds for $r = 2$ and $r = 3$. (For an introduction to Seifert manifolds see Seifert [**34**] as well as [**22**] and [**29**].)

For $r = 2$, Theorem 13 yields $\{\, Oo, 0, |\, 2 \,\} \cong \mathbb{R}\mathbf{P}^3 \cong \mathrm{Hom}(\overline{C}_5, K_4)$. For $r = 3$, the product triangulation of $\mathrm{Hom}(\overline{C}_7, K_5)$ has f-vector $f = (2520, 20160, 35280, 17640)$ and homology $(\mathbb{Z}, \mathbb{Z}^8 \oplus \mathbb{Z}_6, \mathbb{Z}^8, \mathbb{Z})$. It took ten minutes on a Pentium R 2.8 GHz processor to reduce the triangulation with the `bistellar` client system of [**15**] to a triangulation with $f = (27, 289, 524, 262)$. In a second step, the topological type of the resulting triangulation was recognized within seconds with the program `Three-manifold Recognizer` [**28**] (see also [**27**]). Many thanks to S. V. Matveev, E. Pervova, and V. Tarkaev for their help with the recognition!

T 17. $\mathrm{Hom}(\overline{C}_7, K_5) \cong \{\, Oo, 4, |\, 6 \,\}$.

We will describe further graph coloring manifolds of similar size in Section 6, for which their topological type was recognized in the same manner.

Recognition heuristic for Seifert and graph manifolds:

1. Reduce the size of a given triangulation with the `bistellar` client of the TOPAZ module of the `polymake` system [**15**].
2. Use the program `Three-manifold Recognizer` [**28**] for the recognition.

If the (Matveev) complexity of a given triangulation is not too large, there is a good chance to recognize the topological type, even when the triangulation is huge.

5.2. Hom Complexes of Complements of Even Cycles. Similar to the crown representation of (complements of) odd cycles, we split the vertices of even cycles C_{2r} into a lower and an upper part, corresponding to the bipartition of C_{2r}. The lower and also the upper part form a complete graph K_r in $\overline{C}_{2r}$, i.e., every maximal cell of $\mathrm{Hom}(\overline{C}_{2r}, K_r)$ contains each color $1, \ldots, r$ exactly twice, once in the lower part and once in the upper part. (Figure 9 displays C_6 and its complement $\overline{C}_6$ together with a cell $(a_1, a_2, a_3; A_1, A_2, A_3)$ of $\mathrm{Hom}(\overline{C}_6, K_r)$.)

We will employ the following two propositions to describe the 2-dimensional Hom complexes $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$.

P 18. (Babson and Kozlov [**2**]) *The Hom complex* $\mathrm{Hom}(K_r, K_s)$ *is homotopy equivalent to a wedge of* $f(r,s)$ *spheres of dimension* $s-r$*, where the numbers* $f(r,s)$ *satisfy the recurrence relation*

$$f(r,s) = r f(r-1, s-1) + (r-1) f(r, s-1), \tag{6}$$

for $s > r \geq 2$*; with the boundary values* $f(r,r) = r! - 1$*,* $f(1,s) = 0$ *for* $s \geq 1$*, and* $f(r,s) = 0$ *for* $r > s$*.*

P 19. (Čukić and Kozlov [**10**]) $f(r, r+1) = r!\,\frac{r^2-r-2}{2} + 1$.

T 20. *The Hom complex* $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$*,* $r \geq 2$*, is an orientable cubical surface of genus*

$$g(r) = f(r, r+1) = r!\,\tfrac{r^2-r-2}{2} + 1 \tag{7}$$

with $n(r) = (2+r^2)\cdot(r+1)!$ *vertices,* $2(n(r)+2g(r)-2)$ *edges, and* $n(r)+2g(r)-2$ *squares.*

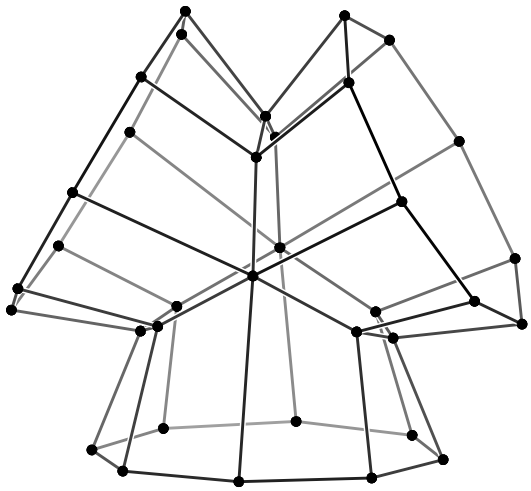
Proof. Let $(a_1, \ldots, a_r; A_1, \ldots, A_r)$ be a maximal cell of $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$. Since every color $1, \ldots, r+1$ appears exactly once in $(a_1, \ldots, a_r)$ and once in $(A_1, \ldots, A_r)$ the cell $(a_1, \ldots, a_r; A_1, \ldots, A_r)$ is the product of the edge $(a_1, \ldots, a_r)$ with the edge $(A_1, \ldots, A_r)$. Hence, $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$ is a cubical surface.

We count the vertices of $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$. For every vertex $(v_1, \ldots, v_r, w_1, \ldots, w_r)$ we have to choose r of the $r+1$ colors for the lower part and then have $r!$ choices to place these r colors. Let $(v_1, \ldots, v_r) = (1, \ldots, r)$ be such a placement. If the left out color $r+1$ does not appear in the upper part, then $(1, \ldots, r)$ can be extended in exactly two ways to a coloring of $\overline{C}_{2r}$, yielding the vertices $(1, \ldots, r; 1, \ldots, r)$ and $(1, \ldots, r; 2, \ldots, r, 1)$ of $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$. If the left out color $r+1$ *is* used in the top part, then there are r choices to place it, and for each such placement every choice to not use one of the colors $1, \ldots, r$ determines a vertex. Therefore, we have altogether $2 + r^2$ choices to extend $(1, \ldots, r)$ to a vertex of $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$; i.e., $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$ has $n(r) := (2+r^2)\cdot(r+1)!$ vertices.

Let M be an orientable cubical surface of genus g with n vertices, e edges, and s squares. Since every square is bounded by four edges and every edge appears in two squares, double counting yields $2e = 4s$. By this equation and by Euler's relation, $s - e + n = \chi(M) = 2 - 2g$, we get that $s = n + 2g - 2$ and $e = 2(n + 2g - 2)$.

It remains to show that $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$ is orientable and has genus $g(r) = f(r, r+1) = r!\,\frac{r^2-r-2}{2} + 1$. For this, let us fix an edge, say $(a_1, \ldots, a_r) = (1, 2, \ldots, r-1, r(r+1))$, of $\mathrm{Hom}(K_r, K_{r+1})$. Then the sequence of $2r$ squares

$$\begin{array}{l}
(1,2,\ldots,r-1,r(r+1);1,2,\ldots,r-2,r-1,r(r+1)),\\
(1,2,\ldots,r-1,r(r+1);1,2,\ldots,r-2,(r-1)r,r+1),\\
(1,2,\ldots,r-1,r(r+1);1,2,\ldots,(r-2)(r-1),r,r+1),\\
\ldots\\
(1,2,\ldots,r-1,r(r+1);1,23,\ldots,r-1,r,r+1),\\
(1,2,\ldots,r-1,r(r+1);12,3,\ldots,r-1,r,r+1),\\
(1,2,\ldots,r-1,r(r+1);2,3,\ldots,r-1,r,1(r+1)),\\
(1,2,\ldots,r-1,r(r+1);2,3,\ldots,r-1,r(r+1),1),\\
(1,2,\ldots,r-1,r(r+1);2,3,\ldots,r-1,r+1,1r),\\
(1,2,\ldots,r-1,r(r+1);12,3,\ldots,r-1,r+1,r),\\
(1,2,\ldots,r-1,r(r+1);1,23,\ldots,r-1,r+1,r),\\
\ldots\\
(1,2,\ldots,r-1,r(r+1);1,2,\ldots,(r-2)(r-1),r+1,r),\\
(1,2,\ldots,r-1,r(r+1);1,2,\ldots,r-2,(r-1)(r+1),r)
\end{array}$$



F 10. Three cylinders forming a trinoid in Hom($\overline{C}_6, K_4$).

forms a cylinder $C_{2r} \times I$. By symmetry, we get such a cylinder for every edge $(a_1, \ldots, a_r)$ of Hom(K_r, K_{r+1}). Since every vertex of the graph Hom(K_r, K_{r+1}) has degree r, we have r cylinders in Hom($\overline{C}_{2r}, K_{r+1}$) meeting "at a vertex" of Hom(K_r, K_{r+1}). (In the case of Hom(K_3, K_4) three cylinders meet at a vertex, which yields a trinoid as depicted in Figure 10.) By inspecting the gluing at the vertices, it is easy to deduce that Hom($\overline{C}_{2r}, K_{r+1}$) is orientable. It moreover follows that Hom($\overline{C}_{2r}, K_{r+1}$) has genus $f(r, r+1)$, which is the number of wedged 1-spheres in the graph Hom(K_r, K_{r+1}). □

As in the case of Hom($\overline{C}_{2r+1}, K_{r+1}$), we can interpret Hom($\overline{C}_{2r}, K_{r+1}$) geometrically in the following way. If we thicken the edges of the 1-dimensional manifold Hom(K_r, K_{r+1}) to solid tubes, then for the resulting 3-manifold with boundary the boundary is homeomorphic to Hom($\overline{C}_{2r}, K_{r+1}$).

C 21. *The Hom complex* Hom($\overline{C}_{2r}, K_s$) *is, for* $s > r \geq 2$, *homeomorphic to the connected sum of* $f(r, s)$ *copies of* $S^{s-r} \times S^{s-r}$.

The 4-dimensional Hom complex Hom($\overline{C}_6, K_5$) consists of 3180 cells and has 1920 vertices. The corresponding product triangulation has $f = (1920, 30780, 104520, 126000, 50400)$. With the `bistellar` client it took half a day to reduce this triangulation to a triangulation with f-vector $(33, 379, 1786, 2300, 920)$. The latter triangulation is simply connected, as we computed with the group algebra package `GAP` [**16**]. The homology of the triangulation is $(\mathbb{Z}, 0, \mathbb{Z}^{58}, 0, \mathbb{Z})$. Moreover, we used `polymake` to compute the intersection form of the example, which turned out to be indefinite, even, and of rank 29. By the classification of Freedman [**14**], this shows:

T 22. Hom($\overline{C}_6, K_5$) $\cong (S^2 \times S^2)^{\#29}$.

Conjecture 21 thus holds in the case $(r, s) = (3, 5)$, and, by Theorem 20, also for the series $s = r + 1 \geq 3$.

T 1. Triangulated surfaces with few vertices.

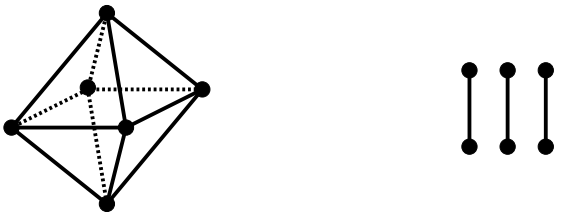
# Vertices	6	7	8	9	10
# Manifolds	3	9	43	655	42426
# Spheres	2	5	14	50	233
# Flag Spheres	1	1	2	4	10

6. Graph Coloring Manifolds of Sphere Dimension Two

Flag simplicial 2-spheres with small numbers n of vertices can be obtained by first enumerating *all* triangulated 2-spheres with n vertices and then testing which of these are flag. Triangulations of two-dimensional spheres with up to 23 vertices have been enumerated with the program `plantri` by Brinkmann and McKay [**6**] (see the manual of `plantri` or the web-page of Royle [**31**] for the numbers of triangulations on $n \leq 23$ vertices). With another approach, triangulations of all two-dimensional manifolds with up to 10 vertices have been enumerated by Lutz (cf. [**23**]); the respective numbers of triangulations are given in Table 1.

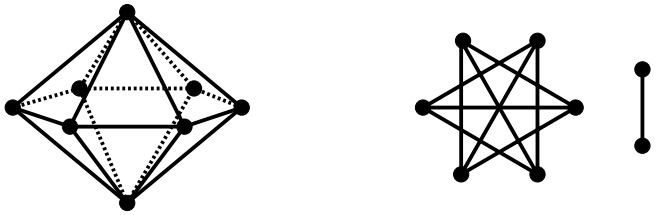
The flag simplicial spheres with up to 9 vertices together with the complements of their 1-skeleta are displayed in Figures 11–18. (The symbol ${}^2n_k$ stands for the kth 2-manifold with n vertices in the catalog [**20**].)

For the flag 2-spheres ${}^26_3 = \partial C_3^\Delta$ (the boundary of the 3-dimensional cross-polytope C_3^Δ), ${}^27_9 = C_5 * S^0$, ${}^28_{41} = C_6 * S^0$, and ${}^29_{630} = C_7 * S^0$, the complements of the respective 1-skeleta are not connected, and therefore, by Equation 1, are direct products.

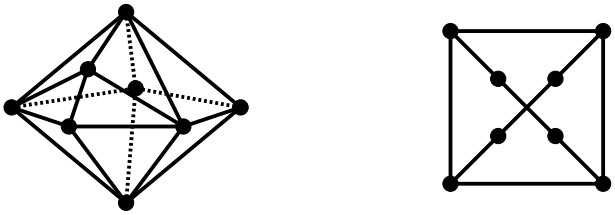


F 11. The flag sphere ${}^26_3 = \partial C_3^\Delta$ and the complement of its 1-skeleton.

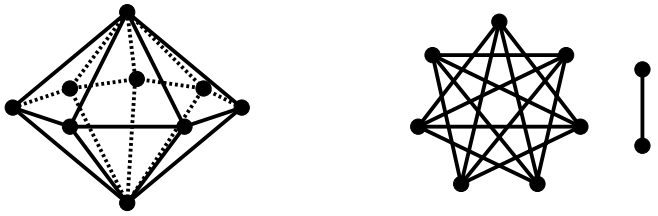
F 12. The flag sphere ${}^27_9 = C_5 * S^0$ and the complement of its 1-skeleton.



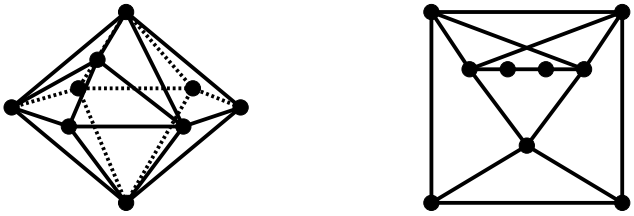
F 13. The flag sphere ${}^2 8_{41} = C_6 * S^0$ and the complement of its 1-skeleton.



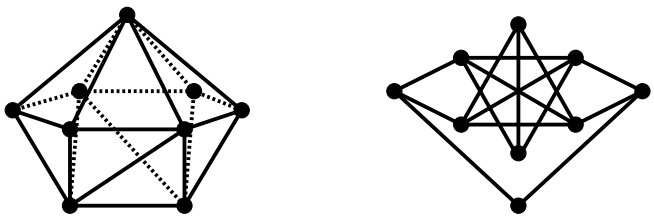
F 14. The flag sphere ${}^2 8_{43}$ and the complement of its 1-skeleton.



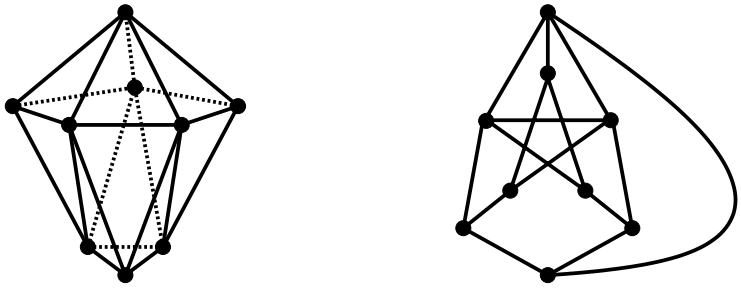
F 15. The flag sphere ${}^2 9_{630} = C_7 * S^0$ and the complement of its 1-skeleton.



F 16. The flag sphere ${}^2 9_{651}$ and the complement of its 1-skeleton.



F 17. The flag sphere ${}^2 9_{652}$ and the complement of its 1-skeleton.



F 18. The flag sphere ${}^2 9_{652}$ and the complement of its 1-skeleton.

T 2. Hom complexes associated with the flag spheres ${}^2 8_{43}$, ${}^2 9_{651}$, ${}^2 9_{652}$, and ${}^2 9_{655}$.

Hom complex	Type	Homology	f-Vector of Product Triangulation
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 8_{43})}, K_3)$	4 cycles		each cycle has 24 vertices
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{651})}, K_3)$	24 vertices		
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{652})}, K_3)$	24 vertices		
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{655})}, K_3)$	12 vertices		
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 8_{43})}, K_4)$	?	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z})$	(3624, 55224, 184656, 221760, 88704)
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{651})}, K_4)$	$(\mathbb{T}^2)^{\#13} \times S^1$	$(\mathbb{Z}, \mathbb{Z}^{27}, \mathbb{Z}^{27}, \mathbb{Z})$	(2928, 21360, 36864, 18432)
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{652})}, K_4)$	$(\mathbb{T}^2)^{\#13} \times S^1$	$(\mathbb{Z}, \mathbb{Z}^{27}, \mathbb{Z}^{27}, \mathbb{Z})$	(3120, 22992, 39744, 19872)
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{655})}, K_4)$	$(S^2 \times S^1)^{\#13}$	$(\mathbb{Z}, \mathbb{Z}^{13}, \mathbb{Z}^{13}, \mathbb{Z})$	(3096, 22104, 38016, 19008)

For those flag 2-spheres with $n \leq 9$ vertices, for which the complements of their 1-skeleta are connected, we analyzed the product triangulations of their Hom complexes with few colors. Table 2 gives the results.

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Note added in proof. Conjecture 12 has recently been proved by C. Schultz [**33**].

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