

FRANK H. LUTZ¹

Triangulated Manifolds with Few Vertices: Combinatorial Manifolds

¹*Main address:* Technische Universität Berlin, Fakultät II - Mathematik und Naturwissenschaften, Institut für Mathematik, Sekr. MA 3-2, Straße des 17. Juni 136, D-10623 Berlin, Germany, lutz@math.tu-berlin.de. *Additional address:* Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB), Takustraße 7, D-14195 Berlin, Germany, lutz@zib.de.

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There are essentially two ways to decompose a (compact, connected) d -manifold (without boundary) into d -simplices: as a (standard) simplicial complex or, more generally, as a simplicial cell complex. In the latter case, identifications on the boundaries of the maximal simplices are sometimes allowed.

As an example, we need five tetrahedra to triangulate the 3-sphere as a simplicial complex, but only two tetrahedra to triangulate it as a simplicial cell complex without identifications on the boundaries of the tetrahedra. If we allow for identifications on the boundary, then one tetrahedron suffices.

Triangulations of 3-manifolds as general simplicial cell complexes have been studied intensively in recent years (see for example [103], [104], [105], [106] and the references contained therein): The 3-manifolds S^3 , $\mathbb{R}P^3$, and $L(3, 1)$ can be triangulated with one tetrahedron only. Otherwise the minimal number of tetrahedra for a triangulation of a (closed) orientable irreducible 3-manifold M coincides with the Matveev complexity $c(M)$ of M ; cf. [106, p. 62].

In the following, our focus will be on triangulations of manifolds as proper simplicial complexes and on their combinatorial properties. A triangulated d -manifold M is a *simplicial d -manifold* if it is triangulated as a simplicial complex. If, in addition, every of the vertex-links of M is PL homeomorphic to the boundary of a d -simplex, then M is a *combinatorial (= triangulated PL) d -manifold*.

For combinatorial manifolds, various restrictions are known on the numbers of i -dimensional faces f_i , in particular, on the number of vertices $n = f_0$. In the first two sections of this paper, we give a survey on such restrictions. Explicit examples of small (or otherwise interesting) triangulations of manifolds of dimension up to eight will be discussed in the subsequent sections.

1 Minimal Triangulations

Vertex-minimal triangulations are presently known only for a rather limited number of manifolds due to the following:

- There are, apart from some special cases, no standard constructions to produce triangulations of a given manifold M with *few* vertices.
- If such a small triangulation is obtained in some way with, say, n vertices, then proving that n is minimal is, in general, a difficult task. The only reasonable approach (apart from a complete enumeration) is to provide lower bounds on n in terms of numerical invariants of the given manifold M . Minimality follows if one of these bounds is sharp.

For 2-dimensional manifolds M , Heawood [71] proved a lower bound in terms of the Euler characteristic of M , and Jungerman and Ringel settled the problem of constructing corresponding minimal triangulations.

Let a triangulation on n vertices be *k-neighborly* if every k -subset of the n vertices forms a face of the triangulation. In other words, the triangulation has the $(k - 1)$ -skeleton of the $(n - 1)$ -simplex on n vertices. We underline the k -th entry of the *face vector* $f = (f_0, \dots, f_d)$ of a triangulation, whenever the triangulation is k -neighborly.

Theorem 1 (Jungerman and Ringel [74], [114]) *If M is 2-dimensional and different from the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3, then there is a triangulation of M on n vertices if and only if*

$$\binom{n-3}{2} \geq 3(2 - \chi(M)), \quad (1)$$

with equality if and only if the triangulation is 2-neighborly. (For the three exceptional cases, $\binom{n-3}{2}$ has to be replaced by $\binom{n-4}{2}$, cf. [73].)

Otherwise the outcome is poor: Apart from minimal triangulations of the d -sphere as the boundary of the $(d + 1)$ -simplex and one series of minimal triangulations of (twisted) sphere products by Kühnel [83] (see Theorem 5), there are merely eleven examples of manifolds for which we have minimal triangulations (with an explicit proof for minimality); see Table 2.

Before we have a closer look at these examples in the following sections, we first give an overview over all lower bounds that are currently known.

Theorem 2 (Brehm and Kühnel [38]) *Let M be a combinatorial d -manifold with n vertices.*

- (a) *If M is not a sphere, then*

$$n \geq 3 \left\lceil \frac{d}{2} \right\rceil + 3, \quad (2)$$

where equality can only occur for $d = 2, 4, 8, 16$. In these cases, M is a manifold ‘like a projective plane’ (in the sense of [59]).

(b) If M is $(i - 1)$ -connected but not i -connected and $1 \leq i < d/2$, then

$$n \geq 2d + 4 - i. \quad (3)$$

There is a combinatorially unique triangulation $\mathbb{R}\mathbf{P}_6^2$ of the real projective plane with 6 vertices and a combinatorially unique triangulation $\mathbb{C}\mathbf{P}_9^2$ of the complex projective plane with 9 vertices [87], [88]. Moreover, there are at least six combinatorially different 15-vertex triangulations of an 8-dimensional manifold $\sim \mathbb{H}\mathbf{P}^2$ like the quaternionic projective plane; for details see Sections 3, 5, and 9.

Question 3 (Brehm and Kühnel [38]) *Can the 16-dimensional Cayley projective plane be triangulated with 27 vertices?*

If d is even and M is not $d/2$ -connected, then Part (a) of Theorem 2 yields $n \geq 3d/2 + 3$ for manifolds ‘like a projective plane’ and $n \geq 3d/2 + 4$ for all other manifolds.

Corollary 4 (Brehm and Kühnel [38]) *Let M be a combinatorial d -manifold with n vertices.*

- (a) *If M has the same homology as $S^{d-i} \times S^i$, then $n \geq 2d + 4 - i$.*
- (b) *If M is not simply connected, then $n \geq 6$ for $d = 2$ and $n \geq 2d + 3$ for $d \geq 3$.*

Theorem 2(b) proves vertex-minimality for a series of triangulations of (twisted) sphere products constructed by Kühnel.

Theorem 5 (Kühnel [83]) *There is a series M^d of vertex-minimal combinatorial triangulations with transitive dihedral automorphism group on $2d + 3$ vertices of topological type*

- $M^d \cong S^{d-1} \times S^1$, if d is even, and
- $M^d \cong S^{d-1} \times S^1$, if d is odd.

This series was later generalized by Kühnel and Lassmann [92] to a series of combinatorial d -manifolds $M_k^d(n)$, $1 \leq k \leq d - 1$, with vertex-transitive dihedral symmetry group on $n \geq 2^{d-k}(k+3) - 1$ vertices. In general, $M_k^d(n)$ is a (orientable or non-orientable) k -dimensional sphere bundle over the $(d - k)$ -dimensional torus with $M_{d-1}^d(2d + 3) = M^d$ and $M_{d-1}^d(2d + 4) \cong S^{d-1} \times S^1$ as special cases.

Conjecture 6 *The transitive Kühnel-Lassmann triangulation $M_{d-1}^d(2d + 4)$ of $S^{d-1} \times S^1$ on $2d + 4$ vertices is vertex-minimal for odd d . For even d , $S^{d-1} \times S^1$ can be triangulated minimally on $2d + 4$ vertices.*

The Klein bottle needs at least 8 vertices for a triangulation [61], so the conjecture holds for $d = 2$. For $d = 3$ it follows from Theorem 11 of Walkup below. Moreover, $S^{3-1} \times S^1$ has triangulations with 12 vertices, but minimality is not settled in this case; see Section 5.

Proposition 7 *There are triangulations of $S^{d-1} \times S^1$ with $3d + 3$ vertices.*

Proof. Let I be an interval, triangulated with 4 vertices. If we take the boundary of the d -simplex with $d + 1$ vertices as a triangulation of the sphere S^{d-1} , then the product triangulation (see [98]) of $S^{d-1} \times I$ has $4(d + 1)$ vertices. The boundary of $S^{d-1} \times I$ consists of two disjoint copies of the boundary of the d -simplex. By identifying the two boundary spheres (with taking care of the orientation), we obtain the wanted triangulation of $S^{d-1} \times S^1$ with $3d + 3$ vertices. \square

The Brehm-Kühnel bound of Theorem 2(a) is sharp for $d = 3, 4, 8$: $S^2 \times S^1$ can be triangulated with 9 [128], $\mathbb{C}\mathbf{P}^2$ with 9 [87], and $\sim\mathbb{H}\mathbf{P}^2$ with 15 vertices [39]. In Sections 6 and 7 we give triangulations of $S^3 \times S^2$ with 12 and of $S^3 \times S^3$ with 13 vertices, and therefore show that the Brehm-Kühnel bound is also sharp in dimensions 5 and 6.

Question 8 *Is there a 15-vertex triangulation of $S^4 \times S^3$?*

Such a triangulation would settle the tightness of the Brehm-Kühnel bound for $d = 7$; it would be a minimal triangulation also by Corollary 4(a).

Starting from a product triangulation with $6 \cdot 4 = 24$ vertices and by applying bistellar flips to it (cf. [19] for a discussion of bistellar flips), we found a triangulation of $S^4 \times S^2$ with 16 vertices.

Question 9 *Can $S^4 \times S^2$ be triangulated with 14 vertices?*

For homology spheres, Bagchi and Datta improved Brehm and Kühnel's bounds of Theorem 2 in dimensions $3 \leq d \leq 6$.

Theorem 10 *Let M be a combinatorial d -manifold with n vertices.*

- (a) (Brehm and Kühnel [38]) *If M is a \mathbb{Z} -homology sphere, then $n \geq 2d + 3$ for $d \geq 6$.*
- (b) (Bagchi and Datta [12]) *If M is a \mathbb{Z}_2 -homology sphere, then $n \geq d + 9$ for $3 \leq d \leq 6$. If $n = d + 9$, then M does not admit a (non-trivial) bistellar flip.*

The result of Bagchi and Datta implies that at least 12 vertices are needed to triangulate the lens space $L(3, 1)$. A triangulation with this number of vertices was first found by Brehm [33]; see Section 4.

The minimal number of vertices is known for four further 3-manifolds.

Theorem 11 (Walkup [129]) *With the exception of S^3 , $S^2 \times S^1$, and $S^2 \times S^1$, which have minimal triangulations with 5, 9 and 10 vertices, respectively, every other triangulated 3-manifold has at least 11 vertices. $\mathbb{R}\mathbf{P}^3$ can be triangulated with 11 vertices.*

For real and complex projective spaces there are lower bounds due to Arnoux and Marin.

Theorem 12 (Arnoux and Marin [11]) *Combinatorial triangulations of $\mathbb{R}\mathbf{P}^d$ and $\mathbb{C}\mathbf{P}^r$ have at least $n \geq (d+1)(d+2)/2$ and $n \geq (r+1)^2$ vertices respectively. Equality is possible only for $d=2$ and $r=2$.*

A series of combinatorial triangulations of $\mathbb{R}\mathbf{P}^d$ with $2^{d+1} - 1$ vertices was constructed by Kühnel [84]. We applied the bistellar flip program BISTELLAR [96] to the 4-dimensional example and obtained a 16-vertex triangulation of $\mathbb{R}\mathbf{P}^4$ (see [101]), which is minimal by Theorem 12. Furthermore, we found a triangulation of $\mathbb{R}\mathbf{P}^5$ with 24 vertices.

Question 13 *Is there a 22-vertex triangulation of $\mathbb{R}\mathbf{P}^5$?*

For the complex projective spaces $\mathbb{C}\mathbf{P}^r$ explicit triangulations are not known for $r > 2$.

Question 14 *Is there a 17-vertex triangulation of $\mathbb{C}\mathbf{P}^3$?*

A 4-dimensional analogue of Heawood's bound for surfaces from Theorem 1 was proved by Kühnel.

Theorem 15 (Kühnel [85, 4.1]) *If M is a combinatorial 4-manifold with n vertices, then*

$$\binom{n-4}{3} \geq 10(\chi(M) - 2). \quad (4)$$

Equality holds if and only if M is 3-neighborly.

In generalization of Theorem 1 and Theorem 15, Kühnel and, independently, Kalai conjectured:

Conjecture 16 [85, p. 61] *Let M be a combinatorial $2k$ -manifold with n vertices, then*

$$\binom{n-k-2}{k+1} \geq (-1)^k \binom{2k+1}{k+1} (\chi(M) - 2), \quad (5)$$

with equality if and only if the triangulation is $(k+1)$ -neighborly.

The conjecture holds in several cases (cf. [86, 4.8]), in particular, for $n \leq 3k + 3$ and $n \geq k^2 + 4k + 3$. The latter inequality was improved to $n \geq 4k + 3$ by Novik [112]. Moreover, Novik showed that for both ranges the conjecture holds for arbitrary simplicial triangulations, not just combinatorial ones(!). As the bounds 4 and 5 depend on the Euler characteristic, we get more detailed information on the minimal numbers of vertices n for triangulations of $2k$ -manifolds than by the general bounds of Theorem 2. Since the Euler characteristic is zero for odd-dimensional manifolds, it was proposed by Kalai [18], [75] to replace the factor $(-1)^k(\chi(M) - 2)$ by the sum of the Betti numbers $\sum_{i=1}^{2k-i} \beta_i(M)$ (or possibly by a weighted sum). By Poincaré duality, $(-1)^k(\chi - 2) = \beta_k - (\beta_{k+1} + \beta_{k-1}) + (\beta_{k+2} + \beta_{k-2}) - \dots = \beta_k + 2 \sum_{i=1}^{k-1} (-1)^i \beta_{k-i}$ and $\sum_{i=1}^{2k-1} \beta_i = \beta_k + 2 \sum_{i=1}^{k-1} \beta_{k-i}$, when the Betti numbers β_i are computed with respect to the field \mathbb{F}_2 (M is orientable over \mathbb{F}_2 !).

Theorem 17 (Novik [112]) *Let M be a simplicial d -manifold with n vertices.*

(1) *If $d = 2k$ and $n \leq 3k + 3$ or $n \geq 4k + 3$, then*

$$\binom{n-k-2}{k+1} \geq \binom{2k+1}{k+1} (\beta_k + 2 \sum_{i=0}^{k-2} \tilde{\beta}_i). \quad (6)$$

(2) *If $d = 2k$ and $n \leq 3k + 3$ or $n \geq 7k + 3$, then*

$$\binom{n-k-2}{k+1} \geq \binom{2k+1}{k+1} (\beta_k + 2 \sum_{i=1}^{k-1} \beta_i). \quad (7)$$

(3) *If $d = 2k - 1$ and $n \leq 3k + 2$ or $n \geq 4k + 1$, then*

$$\frac{2n}{n+k+2} \cdot \binom{n-k-2}{k} \geq \binom{2k-1}{k} (2 \sum_{i=1}^{k-1} \beta_i). \quad (8)$$

Kühnel [82] has worked out another elegant (and previously unpublished) conjecture that generalizes Theorem 2(b) (in the case of (twisted) sphere products) and Conjecture 16 (in the case of $(k-1)$ -connected $2k$ -manifolds).

Conjecture 18 (Kühnel) *Let M be a combinatorial d -manifold with n vertices and reduced Betti numbers $\tilde{\beta}_j(M)$, $j = 0, \dots, \lfloor \frac{d}{2} \rfloor$. Then*

$$\binom{n-d+j-2}{j+1} \geq \binom{d+2}{j+1} \tilde{\beta}_j(M) \quad \text{for } j = 0, \dots, \lfloor \frac{d-1}{2} \rfloor. \quad (9)$$

If d is even, then additionally (for $j = \frac{d}{2}$):

$$\binom{n-\frac{d}{2}-2}{\frac{d}{2}+1} \geq \binom{d+2}{\frac{d}{2}+1} \frac{\tilde{\beta}_{\frac{d}{2}}(M)}{2}. \quad (10)$$

If equality holds for one of the bounds of 9 or for 10, say, for $j = s$, then $\tilde{\beta}_j(M) = 0$ for $j \neq s$.

Table 1: The Pascal-like triangle of Kühnel's lower bounds of Conjecture 18.

				$\binom{n-2}{1} \geq 0$
			$\binom{n-3}{1} \geq 0$	$\binom{n-3}{2} \geq \binom{4}{2} \frac{\beta_1}{2}$
		$\binom{n-4}{1} \geq 0$	$\binom{n-4}{2} \geq \binom{5}{2} \beta_1$	$\binom{n-4}{3} \geq \binom{6}{3} \frac{\beta_2}{2}$
$\binom{n-5}{1} \geq 0$	$\binom{n-6}{1} \geq 0$	$\binom{n-5}{2} \geq \binom{6}{2} \beta_1$	$\binom{n-5}{3} \geq \binom{7}{3} \beta_2$	
$\binom{n-6}{2} \geq \binom{7}{2} \beta_1$				
$\binom{n-7}{1} \geq 0$	\dots	\dots	\dots	\dots

The bounds of Kühnel can be arranged in a Pascal-like triangle as displayed in Table 1: There is a row for every dimension $d = 0, 1, \dots$ with the respective bounds for $j = 0, \dots, \lceil \frac{d-1}{2} \rceil$ as entries.

The diagonal in the Kühnel triangle corresponding to $j = 0$ gives the bound $\binom{n-d-2}{1} \geq 0$, which is equivalent to $n \geq d + 2$. In other words, every combinatorial d -manifold trivially has at least as many vertices as the boundary of a $(d + 1)$ -simplex.

For d -manifolds M with $\beta_1(M) = 1$, the second diagonal $j = 1$ in Kühnel's triangle yields (for $d \geq 3$) the bound $\binom{n-d-1}{2} \geq \binom{d+2}{2}$, or, equivalently, $n \geq 2d + 3$. This is precisely the bound for non-simply connected d -manifolds from Corollary 4(b); Kühnel's series from Theorem 5 of vertex-minimal triangulations of (twisted) S^{d-1} -bundles over S^1 with $n = 2d + 3$ vertices shows that this bound is best possible.

In fact, the bounds of Conjecture 18 hold for all sphere products $S^{d-i} \times S^i$, for which the relevant bound $\binom{n-d+i-2}{i+1} \geq \binom{d+2}{i+1}$ gives $n \geq 2d + 4 - i$ in accordance with Corollary 4(a). For $(k - 1)$ -connected $2k$ -manifolds the bounds of Conjecture 18 corresponding to $j = k$ coincide with the generalized Heawood bounds of Conjecture 16.

Let us point out that Kühnel's Conjecture 18 is of particular interest for odd-dimensional manifolds M , where we, up to now, only have the bounds of Theorem 2 (together with the special bounds of Theorem 10(b) and of Theorem 12).

A first case with $\beta_1(M) > 1$, where Kühnel's Conjecture 18 would prove vertex-minimality, is for the manifolds $(S^2 \times S^1)^{\#3}$ and $(S^2 \times S^1)^{\#3}$ for which we know triangulations with 13 vertices. For 4-manifolds M with trivial second homology group $H_2(M) = 0$, Kühnel's bound $\binom{n-5}{2} \geq 15 \cdot \beta_1(M)$ follows from results of Walkup (cf. [86, p. 96], [128, Thm. 5]).

As mentioned above, there are, besides the minimal triangulations by Jungerman and Ringel of surfaces, the d -sphere, which can be triangulated as the boundary of the $(d + 1)$ -simplex, and the series of minimal triangulations of (twisted) sphere products by Kühnel [83], eleven exceptional examples of manifolds for which we explicitly have minimal triangulations; see Table 2.

Table 2: Eleven exceptional cases of vertex-minimal triangulations.

Dimension	Manifold	n	Reference
3	$S^2 \times S^1$	10	[129]
	$\mathbb{R}\mathbf{P}^3 = L(2, 1)$	11	[42, $S_{2.4}$], [129]
4	$L(3, 1)$	12	[12], [33]
	$\mathbb{C}\mathbf{P}^2$	9	[87], [88]
	$S^2 \times S^2$	11	
	$(S^2 \times S^2) \# (S^2 \times S^2)$	12	
	$\mathbb{R}\mathbf{P}^4$	16	
5	K3 surface	16	[48]
	$S^3 \times S^2$	12	
6	$S^3 \times S^3$	13	
8	$\sim \mathbb{H}\mathbf{P}^2$	15	[39]

Minimality of these examples follows

- for $S^2 \times S^1$ and $\mathbb{R}\mathbf{P}^3 = L(2, 1)$ from Theorem 11,
- for $L(3, 1)$ from Theorem 10(b),
- for $\mathbb{C}\mathbf{P}^2$ from Theorem 2(a) as well as from Theorem 15,
- for $S^2 \times S^2$ from [88],
- for $(S^2 \times S^2) \# (S^2 \times S^2)$ and the K3 surface from Theorem 15,
- for $\mathbb{R}\mathbf{P}^4$ from Theorem 12,
- for $S^3 \times S^2$ from Theorem 2(a) as well as from (b), and
- for $S^3 \times S^3$ and $\sim \mathbb{H}\mathbf{P}^2$ from Theorem 2(a).

For various other small triangulations minimality cannot be proven due to a lack of sharper lower bounds.

Conjecture 19 *The minimal number of vertices is*

- 12 for $(S^2 \times S^1) \#^2$ and $(S^2 \times S^1) \#^2$,
- 13 for $(S^2 \times S^1) \#^3$ and $(S^2 \times S^1) \#^3$,
- 14 for the lens spaces $L(4, 1)$, $L(5, 2)$, and $\mathbb{R}\mathbf{P}^2 \times S^1$,
- 15 for $L(5, 1)$, the prism spaces $P(2) = S^3/Q$, $P(3)$, $P(4)$, and the connected sum $\mathbb{R}\mathbf{P}^3 \# \mathbb{R}\mathbf{P}^3$,

(e) 12 for $\mathbb{C}\mathbf{P}^2 \# \mathbb{C}\mathbf{P}^2$ and $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$,

(f) 15 for $(S^3 \times S^1) \# (\mathbb{C}\mathbf{P}^2) \#^5$, and

(g) 13 for $SU(3)/SO(3)$.

Triangulations for these examples with the given numbers of vertices are discussed in the following sections.

Theorem 20 (Kühnel and Lassmann [91]) *The d -dimensional torus T^d can be triangulated combinatorially with $2^{d+1} - 1$ vertices (for $d \geq 2$).*

In fact, the corresponding series is a subseries of the Kühnel and Lassmann series $M_k^d(n)$ with $T^d \cong M_1^d(2^{d+1} - 1)$.

We tried to improve on the respective triangulations of T^3 and T^4 with our bistellar flip program, but with no success.

Conjecture 21 *The Kühnel-Lassmann series of combinatorial d -tori T^d with $2^{d+1} - 1$ vertices is vertex-minimal.*

Besides the Kühnel-Lassmann triangulation of the 4-torus, a non-equivalent triangulation of T^4 with 31 vertices was constructed by Grigis [67]. Dartois and Grigis gave 25 further triangulations of the eight-dimensional torus T^8 with 511 vertices [53], but so far, no additional triangulation of T^3 has been found with 15 vertices.

Conjecture 22 *The Kühnel-Lassmann triangulation [89] of T^3 with 15 vertices is vertex-minimal and unique with $f = (15, \underline{105}, 180, 90)$.*

2 f -Vectors

Of course, it would be desirable to know a complete characterization of the f -vectors of a triangulable manifold, not just the minimal number of vertices.

For various classes of simplicial complexes this rather general question has, in fact, been answered (cf. the surveys by Björner [17] and Billera and Björner [15]). For the class of all simplicial complexes the result is known as the Kruskal-Katona theorem ([77], [81]; see also Schützenberger [120], who gave the first proof). A complete characterization for simplicial polytopes is given by the g -theorem (conjectured by McMullen [108]) of Billera and Lee [16] and Stanley [124]. It is, however, an open problem whether the g -theorem can be extended to general simplicial spheres (' g -conjecture for simplicial spheres' by McMullen [108]). The g -conjecture holds for 3- and 4-dimensional spheres; cf. McMullen [108]: There, the essential condition is that $f_1 \geq (d+1)n - \binom{d+2}{2}$, which is the case $k = 1$ in the lower bound theorem of Kalai and Gromov.

Lower Bound Theorem 23 (Kalai and Gromov, cf. [75] including the note added in proof and [68, Ch. 2.4.10]) *If M is a d -dimensional (pseudo-) manifold, then for every triangulation with n vertices,*

$$f_k \geq \begin{cases} \binom{d+1}{k}n - \binom{d+2}{k+1}k & \text{for } 1 \leq k \leq d-1, \\ d \cdot n - (d-1)(d+2) & \text{for } k = d. \end{cases} \quad (11)$$

This result for polytopal spheres was obtained before by Barnette [13], the case $k = d$ for pseudomanifolds by Klee [78]. The lower bound theorem for 3- and 4-dimensional manifolds was first proved by Walkup [129].

Upper Bound Theorem 24 (Novik [112]) *Let M be a simplicial (homology) d -manifold. If d is odd and also if $d = 2k$ and $\beta_k \leq 2\beta_{k-1} + 2\sum_{i=1}^{k-3} \beta_i$, then M cannot have more i -dimensional faces as the corresponding cyclic polytope with the same number of vertices, that is, $f_k(M) \leq f_k(C_{d+1}(n))$ for $1 \leq k \leq d$.*

The upper bound theorem for polytopal spheres was first proved by McMullen [107] and later generalized to simplicial spheres by Stanley [124].

3 2-Manifolds

The f -vectors of triangulated surfaces M have a particularly simple description. We have Euler's equation, $n - f_1 + f_2 = \chi(M)$, and, by double counting of incidences between edges and triangles, it follows that $2f_1 = 3f_2$. Thus, the number of vertices n determines f_1 and f_2 , that is, a triangulated 2-manifold M of Euler characteristic $\chi(M)$ on n vertices has f -vector

$$f = (n, 3n - 3\chi(M), 2n - 2\chi(M)). \quad (12)$$

The *orientable surface* $M(g, +)$ of genus g has homology

$$H_*(M(g, +)) = (\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}) \quad (13)$$

and Euler characteristic $\chi(M(g, +)) = 2 - 2g$, whereas the *non-orientable surface* $M(g, -)$ of genus g has homology

$$H_*(M(g, -)) = (\mathbb{Z}, \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, 0) \quad (14)$$

and Euler characteristic $\chi(M(g, -)) = 2 - g$. The smallest possible n for a triangulation of a 2-manifold M (with the exception of the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3, where one more vertex is needed) is determined by Heawood's bound

$$n \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \quad (15)$$

of Theorem 1. Corresponding minimal triangulations of non-orientable surfaces were constructed by Ringel [114]. For minimal triangulations of orientable surfaces see Jungerman and Ringel [74] and the references contained therein.

A *map* on a surface M is a decomposition of M into polygonal regions or *countries* such that every vertex has at least degree 3 and each vertex of degree s is incident with s countries. If the decomposition is simplicial, the map is called *simplicial*. For surveys on *polyhedral maps*, i.e., maps that are realized geometrically in \mathbb{R}^3 with straight edges, flat faces, and without self-intersections, see Brehm and Schulte [41] and Brehm and Wills [43].

A map is *regular* if it has a *flag-transitive* automorphism group, that is, if its automorphism group is transitive on the triples $\{\text{vertex} \subset \text{edge} \subset \text{facet}\}$, or *flags* for short. Regular maps can be seen as non-spherical combinatorial analogues of the Platonic solids. Well known examples of flag-transitive simplicial maps with few vertices are – besides the tetrahedron, the octahedron, the icosahedron, and \mathbb{RP}_6^2 – Dyck’s regular map [57], [58] of genus 3 on 12 vertices and Klein’s regular map [79] of genus 3 on 24 vertices. (For polyhedral realizations of Dyck’s regular map see Bokowski [21] and Brehm [35]. A polyhedral realization of Klein’s map has been constructed by Schulte and Wills [119].)

A systematic classification of regular maps on orientable surfaces of genus 1 and 2 was begun by Brahana [31], [32] and completed by Coxeter [50] and Coxeter and Moser [51]). Coxeter and Moser [51] also constructed regular maps on non-orientable surfaces of small genus as well as some interesting infinite families. Regular maps on the orientable surface of genus 3 were classified by Sherk [121] and those on orientable surfaces of genus 4 to 6 by Garbe [62]. Conder and Everitt [49] constructed series of regular maps on non-orientable surfaces and Wilson [131] enumerated all regular maps with up to 100 edges. The regular simplicial maps with up to 31 vertices are listed in [102].

In generalization of regular surfaces, McMullen, Schulz, and Wills [109] called a polyhedral surface *equivelar* of type $\{p, q\}$ if all its 2-faces are p -gons and all its vertices have degree q . (In various older papers, *equivelar polyhedra* are called *regular* as well; cf. [1].) *Equivelar triangulations* of the torus were constructed by Altshuler [1]. Further series were given by McMullen, Schulz, and Wills [109], [110]. Datta and Nilakantan [54] enumerated all *equivelar surfaces* with few vertices, in particular, all simplicial *equivelar polyhedra* with up to 11 vertices. For examples of *face-transitive polyhedra* see Wills [130].

A triangulated surface M with n vertices attains equality in Heawood’s bound, $n = \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$, if and only if the triangulation is *neighborly*, that is, the 1-skeleton of M is the complete graph K_n . In particular, a triangulation is *neighborly* if and only if $n(7 - n)/6 = \chi(M)$ is an integer (smaller or equal to two). The first examples are the tetrahedron, the real projective plane \mathbb{RP}_6^2 with 6 vertices, and Möbius’ torus [111] with 7 vertices.

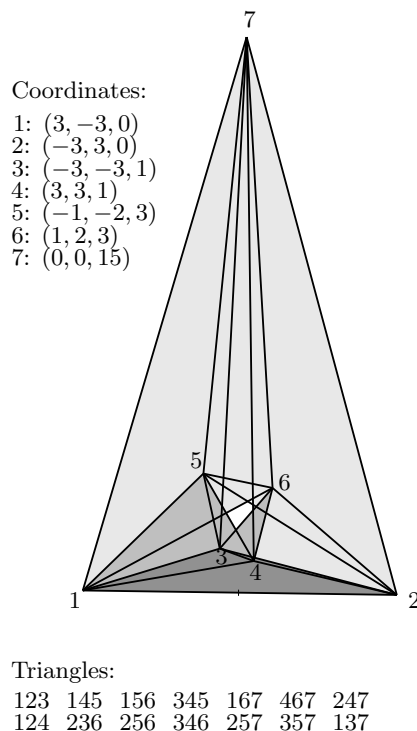


Figure 1: Császár's torus.

Neighborly maps with 9 and 10 vertices were enumerated by Altshuler and Brehm [6]: There are 2 non-orientable neighborly maps of genus 5 on 9 vertices and 14 non-orientable neighborly maps of genus 7 on 10 vertices. Moreover, there are 59 orientable neighborly maps of genus 6 on 12 vertices, as enumerated by Altshuler [4] (see also [3]).

Neighborly maps have attracted attention in various ways. By Theorem 1, they provide examples of minimal triangulations of surfaces. At the same time, every neighborly map M with n vertices is an example of a minimal graph embedding of the complete graph $K_n \hookrightarrow M$.

Theorem 25 (Ringel and Youngs [115]) *For every surface M different from the 2-sphere and with the exception of the Klein bottle, there is an embedding $K_n \hookrightarrow M$ if and only if $n \leq \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$. If equality holds, then the embedding of the complete graph induces a triangulation of M . For the Klein bottle, there exist embeddings for $n \leq 6$.*

Let the *chromatic number* $\chi_{\text{ch}}(M)$ of a surface M be the minimal number of colors that is needed to color any polyhedral map on M . Heawood proved that $\chi_{\text{ch}}(M) \leq \lfloor \frac{7 + \sqrt{49 + 48g}}{2} \rfloor$ for all orientable surfaces of genus $g \geq 1$.

As a direct consequence of the existence of embeddings of complete graphs on surfaces according to Theorem 25, Ringel and Youngs [115] were able to complete in 1968 the *map color theorem*, which first was announced by Heawood [71] in 1890.

Map Color Theorem 26 (Ringel and Youngs [115]) *If M is an orientable surface of genus $g \geq 1$, then*

$$\chi_{\text{ch}}(M) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor. \quad (16)$$

Theorem 25 implies equality $\chi_{\text{ch}}(M) = \lfloor \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \rfloor$ also for all non-orientable surfaces, with the exception of the Klein bottle K , where $\chi_{\text{ch}}(K) = 6$ [61]. For the 2-dimensional sphere, $\chi_{\text{ch}}(S^2) = 4$ is the essence of the famous *Four Color Theorem* of Appel and Haken [10]; cf. also Robertson, Sanders, Seymour, and Thomas [116].

Neighborly maps are of interest also with respect to geometric realizations of surfaces.

Question 27 (Grünbaum [69, Ch. 13.2]) *Can every triangulated orientable 2-manifold be embedded geometrically in \mathbb{R}^3 , i.e., can it be realized with straight edges, flat triangles, and without self intersections?*

By Steinitz' theorem (cf. [132]), every combinatorial 2-sphere is realizable as the boundary complex of a convex 3-dimensional polytope. For the 2-torus of genus 1 the realizability problem is still open.

Conjecture 28 (Duke [56]) *Every triangulated torus can be realized as a polyhedron in \mathbb{R}^3 .*

A first explicit geometric realization of Möbius' minimal 7-vertex triangulation of the 2-torus was given by Császár [52] (see Figure 1 and [95]). Bokowski and Eggert [25] showed that there are altogether 72 different types of realizations of the Möbius torus, and Bokowski and Fendrich [20] verified that triangulated tori with up to 11 vertices are all realizable. Sets of coordinates for triangulated tori with up to 10 vertices can be found in [72].

Brehm and Bokowski [22], [23], [34], [36] constructed geometric realizations for examples of triangulated orientable 2-manifolds of genus $g = 2, 3, 4$ with minimal numbers of vertices $n = 10, 10, 11$, respectively.

Neighborly maps of higher genus, however, were considered as candidates for counter-examples to the Grünbaum realization problem for a while; cf. [29, p. 137]. Neighborly orientable maps have genus $g = (n - 3)(n - 4)/12$ and therefore $n \equiv 0, 3, 4, 7 \pmod{12}$ vertices, with $g = 6$ and $n = 12$ as the first case beyond the tetrahedron and the 7-vertex torus.

Theorem 29 (Bokowski and Guedes de Oliveira [27]) *The triangulated orientable surface N_{54}^{12} of genus 6 with 12 vertices of Alshuler's list [4] is not geometrically embeddable in \mathbb{R}^3 .*

Table 3: Triangulated surfaces with up to 10 vertices.

n	Surface	Types	n	Surface	Types	n	Surface	Types
4	S^2	1	8	S^2	14	10	S^2	233
				T^2	7		T^2	2109
5	S^2	1		$\mathbb{R}\mathbf{P}^2$	16		$M(2, +)$	865
6	S^2	2		K^2	6		$M(3, +)$	20
	$\mathbb{R}\mathbf{P}^2$	1					$\mathbb{R}\mathbf{P}^2$	1210
			9	S^2	50		K^2	4462
				T^2	112		$M(3, -)$	11784
7	S^2	5		$\mathbb{R}\mathbf{P}^2$	134		$M(4, -)$	13657
	T^2	1		K^2	187		$M(5, -)$	7050
	$\mathbb{R}\mathbf{P}^2$	3		$M(3, -)$	133		$M(6, -)$	1022
				$M(4, -)$	37		$M(7, -)$	14
				$M(5, -)$	2			

Datta and Nilakantan [55] determined all triangulated surfaces with 8 vertices. Those with 9 and 10 were enumerated in [100]; see Table 3 for the respective numbers of combinatorial types.

Theorem 30 (Bokowski and Lutz; cf. [100]) *All 865 vertex-minimal 10-vertex triangulations of the orientable surface of genus 2 can be realized geometrically in \mathbb{R}^3 .*

Vertex-transitive triangulations of surfaces with up to 15 vertices are given in [80]. Enumeration results for vertex-transitive neighborly triangulations with up to 22 vertices can be found in [102].

For a given number n of vertices it is way easier to enumerate triangulations of the 2-dimensional sphere only than to enumerate all triangulated 2-manifolds: According to Steinitz' theorem ([125], [126]; cf. [132]), every triangulation of S^2 is polytopal and therefore is, by stereographic projection, equivalent to a planar triangulation (with straight edges). Brückner [45], [46] listed (by hand!) all triangulations of S^2 with up to 12 vertices. His census was later corrected slightly for $n = 11$ by Grace [66] and for $n = 12$ by Bowen and Fisk [30]. Triangulations with up to 23 vertices were enumerated with the program `plantri` by Brinkmann and McKay [44] (see the manual of `plantri` and also Royle [117]). Table 4 gives the respective numbers. Precise formulas for rooted triangulations of the 2-sphere were determined by Tutte [127].

Table 4: Triangulated 2-spheres with $11 \leq n \leq 23$ vertices.

n	Types
11	1249
12	7595
13	49566
14	339722
15	2406841
16	17490241
17	129664753
18	977526957
19	7475907149
20	57896349553
21	453382272049
22	3585853662949
23	28615703421545

4 3-Manifolds

All triangulated 3-manifold with f -vector (n, f_1, f_2, f_3) satisfy the relations $n - f_1 + f_2 - f_3 = 0$ (Euler) and $2f_2 = 4f_3$ (double counting). Thus

$$f = (n, f_1, 2f_1 - 2n, f_1 - n). \quad (17)$$

A complete characterization of the f -vectors of the 3-manifolds S^3 , $S^2 \times S^1$, $S^2 \times S^1$, and $\mathbb{R}P^3$ was given by Walkup.

Theorem 31 (Walkup [129]) *For every 3-manifold M there is an integer $\gamma(M)$ such that*

$$f_1 \geq 4n + \gamma(M) \quad (18)$$

for every triangulation of M with n vertices and f_1 edges. Moreover there is an integer $\gamma^*(M) \geq \gamma(M)$ such that for every pair (n, f_1) with $n \geq 0$ and

$$\binom{n}{2} \geq f_1 \geq 4n + \gamma^*(M) \quad (19)$$

there is a triangulation of M with n vertices and f_1 edges. In particular,

- (a) $\gamma^* = \gamma = -10$ for S^3 ,
- (b) $\gamma^* = \gamma = 0$ for $S^2 \times S^1$,
- (c) $\gamma^* = 1$ and $\gamma = 0$ for $S^2 \times S^1$, where, with the exception $(9, 36)$, all pairs (n, f_1) with $n \geq 0$ and $4n + \gamma(M) \leq f_1 \leq \binom{n}{2}$ occur,
- (d) $\gamma^* = \gamma = 7$ for $\mathbb{R}P^3$, and
- (e) $\gamma^*(M) \geq \gamma(M) \geq 8$ for all other 3-manifolds M .

For an alternative proof of the existence of neighborly triangulations for every 3-manifold see Sarkaria [118].

Let us remark that if a 3-manifold M can be triangulated with n vertices and f_1 edges, then triangulations of M with $n + k$ vertices and $f_1 + 4k$ edges can be obtained for $k \geq 0$ by successive *stacking*. (In every stacking step some tetrahedron of the respective triangulation of M is subdivided: This adds one vertex and four edges each.)

Conjecture 32 *The f -vectors of triangulations of the 3-torus T^3 are characterized by $\gamma^*(T^3) = \gamma(T^3) = 45$.*

Conjecture 32 implies Conjecture 21 for $d = 3$. We used the bistellar flip program BISTELLAR [96] to verify that there are triangulations of \mathbf{T}^3 for all pairs (n, f_1) with $15 \leq n \leq 35$ and $4n + 45 \leq f_1 \leq \binom{n}{2}$.

Also that there are triangulations of the lens space $L(3, 1)$ for all (n, f_1) with $12 \leq n \leq 35$ and $4n + 18 \leq f_1 \leq \binom{n}{2}$.

Conjecture 33 *The f -vectors of triangulations of the lens space $L(3, 1)$ are characterized by $\gamma^*(L(3, 1)) = \gamma(L(3, 1)) = 18$.*

In 1990, Kühnel [85] gave a list of six (pairwise different) 3-manifolds for which he knew triangulations with 15 or less vertices: The 3-sphere S^3 , the twisted S^2 -bundle over S^1 (i.e., the 3-dimensional Klein bottle, which we usually denote by $S^2 \times S^1$), the product $S^2 \times S^1$, the projective 3-space $\mathbb{R}\mathbf{P}^3$, the 3-dimensional torus \mathbf{T}^3 [89], and Cartan's hypersurface S^3/Q ([37]; [47]).

Since then, twenty-one further examples have been found. Brehm and Świątkowski [42] constructed triangulations of $L(3, 1)$ and $L(4, 1)$ with 13 and 15 vertices, respectively; and by local modifications, Brehm [33] found a triangulation of $L(3, 1)$ with 12 vertices. Kühnel and Lassmann [90] listed two combinatorial 3-manifolds with 12 vertices that have homology $H_* = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z})$ and which are triangulations of $(S^2 \times S^1) \# (S^2 \times S^1)$. All other eighteen examples are new. In addition, we improved the number of vertices for $L(4, 1)$ to $n = 14$.

Theorem 34 *There are at least 27 distinct 3-manifolds that can be triangulated with $n \leq 15$ vertices; see Table 5.*

Triangulations of the respective manifolds were constructed as described in [98] and [40]. To these triangulations we applied bistellar flips to reduce the numbers of vertices and edges. The resulting triangulations can be found online at [97].

Conjecture 35 *There are only nine 3-manifolds that can minimally be triangulated with $n \leq 13$ vertices: S^3 with 5, $S^2 \times S^1$ with 9, $S^2 \times S^1$ with 10, $\mathbb{R}\mathbf{P}^3$ with 11, $L(3, 1)$, $(S^2 \times S^1) \#^2$, and $(S^2 \times S^1) \#^2$ with 12, as well as $(S^2 \times S^1) \#^3$ and $(S^2 \times S^1) \#^3$ with 13 vertices. For all other 3-manifolds at least 14 vertices are needed.*

Table 5: Combinatorial 3-manifolds with $n \leq 15$ vertices and smallest known transitive triangulations with n_{vt} vertices (minimal if underlined).

Manifold	Homology	n	n_{vt}	Reference
S^3	$(\mathbb{Z}, 0, 0, \mathbb{Z})$	<u>5</u>	<u>5</u>	
$S^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2, 0)$	<u>9</u>	<u>9</u>	[8], [129]
$S^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$	<u>10</u>	<u>10</u>	[129]
$\mathbb{R}\mathbb{P}^3 = L(2, 1)$	$(\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z})$	<u>11</u>	<u>12</u>	[42], [129]
$L(3, 1)$	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	<u>12</u>	<u>14</u>	[12], [33]; [90]
$(S^2 \times S^1) \# (S^2 \times S^1)$ $= (S^2 \times S^1) \# - (S^2 \times S^1)$	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z})$	12	<u>12</u>	[90]
$(S^2 \times S^1) \# (S^2 \times S^1)$ $= (S^2 \times S^1) \# (S^2 \times S^1)$	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$	12	<u>16</u>	
$(S^2 \times S^1) \#^3$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$	13	<u>16</u>	
$(S^2 \times S^1) \#^3$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$	13	<u>16</u>	
$L(4, 1)$	$(\mathbb{Z}, \mathbb{Z}_4, 0, \mathbb{Z})$	14	<u>16</u>	
$L(5, 2)$	$(\mathbb{Z}, \mathbb{Z}_5, 0, \mathbb{Z})$	14	?	
$(S^2 \times S^1) \# \mathbb{R}\mathbb{P}^3$ $= (S^2 \times S^1) \# - \mathbb{R}\mathbb{P}^3$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z})$	14	?	
$(S^2 \times S^1) \# \mathbb{R}\mathbb{P}^3$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$	14	?	
$\mathbb{R}\mathbb{P}^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$	14	<u>17</u>	[90]
$(S^2 \times S^1) \#^2 \# \mathbb{R}\mathbb{P}^3$	$(\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$	14	?	
S^3/T^*	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	15	<u>16</u>	
$L(5, 1)$	$(\mathbb{Z}, \mathbb{Z}_5, 0, \mathbb{Z})$	15	?	
$P(2) = S^3/Q$	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	15	<u>15</u>	[37]
$P(3)$	$(\mathbb{Z}, \mathbb{Z}_4, 0, \mathbb{Z})$	15	?	
$P(4)$	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	15	?	
$\mathbb{R}\mathbb{P}^3 \# \mathbb{R}\mathbb{P}^3$ $= \mathbb{R}\mathbb{P}^3 \# - \mathbb{R}\mathbb{P}^3$	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	15	?	
$(S^2 \times S^1) \# L(3, 1)$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}, \mathbb{Z})$	15	?	
$(S^2 \times S^1) \# L(3, 1)$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_2, 0)$	15	?	
$(S^2 \times S^1) \#^2 \# \mathbb{R}\mathbb{P}^3$	$(\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_2, \mathbb{Z}^2, \mathbb{Z})$	15	?	
$\mathbf{T}^3 = S^1 \times S^1 \times S^1$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$	15	<u>15</u>	[89]
$(S^2 \times S^1) \#^4$	$(\mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}^4, \mathbb{Z})$	15	?	
$(S^2 \times S^1) \#^4$	$(\mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}^3 \oplus \mathbb{Z}_2, 0)$	15	?	

Table 6: 3-manifolds with isomorphic homology groups on $n \leq 15$ vertices.

Manifold	Homology	π_1	f -vector
$L(3, 1)$	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	\mathbb{Z}_3	(12, <u>66</u> , 108, 54)
S^3/T^*	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	T^*	(15, 102, 174, 87)
$(S^2 \times S^1)^{\#3}$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$	$\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$	(13, 72, 118, 59)
\mathbf{T}^3	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$	\mathbb{Z}^3	(15, <u>105</u> , 180, 90)
$(S^2 \times S^1) \# \mathbb{R}\mathbf{P}^3$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$	$\mathbb{Z} * \mathbb{Z}_2$	(14, 73, 118, 59)
$\mathbb{R}\mathbf{P}^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$	$\mathbb{Z} \oplus \mathbb{Z}_2$	(14, 84, 140, 70)
$L(4, 1)$	$(\mathbb{Z}, \mathbb{Z}_4, 0, \mathbb{Z})$	\mathbb{Z}_4	(14, 84, 140, 70)
$P(3)$	$(\mathbb{Z}, \mathbb{Z}_4, 0, \mathbb{Z})$	D_3^*	(15, 97, 164, 82)
$L(5, 2)$	$(\mathbb{Z}, \mathbb{Z}_5, 0, \mathbb{Z})$	\mathbb{Z}_5	(14, 87, 146, 73)
$L(5, 1)$	$(\mathbb{Z}, \mathbb{Z}_5, 0, \mathbb{Z})$	\mathbb{Z}_5	(15, 97, 164, 82)
$\mathbb{R}\mathbf{P}^3 \# \mathbb{R}\mathbf{P}^3$	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	$\mathbb{Z}_2 * \mathbb{Z}_2$	(15, 86, 142, 71)
$P(2) = S^3/Q$	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	$D_2^* = Q$	(15, 90, 150, 75)
$P(4)$	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	D_4^*	(15, 104, 178, 89)

In view of Walkup's Theorem 31 and Bagchi and Datta's Theorem 10, the open part of the conjecture (see also Conjecture 19(a) and (b)) is to show that $(S^2 \times S^1)^{\#3}$ and $(S^2 \times S^1)^{\#3}$ cannot be triangulated with 12 vertices and that $(S^2 \times S^1)^{\#2}$ and $(S^2 \times S^1)^{\#2}$ cannot be triangulated with 11 vertices, while all other 3-manifolds, different from the nine listed ones, need at least 14 vertices for a triangulation. Kühnel's Conjecture 18 would imply that 13 vertices is best possible for $(S^2 \times S^1)^{\#3}$ and $(S^2 \times S^1)^{\#3}$.

From Conjecture 35 it would follow that the 14-vertex triangulations of $L(4, 1)$, $L(5, 1)$, $(S^2 \times S^1) \# \mathbb{R}\mathbf{P}^3$, $(S^2 \times S^1) \# \mathbb{R}\mathbf{P}^3$, $\mathbb{R}\mathbf{P}^2 \times S^1$, and of the connected sum $(S^2 \times S^1)^{\#2} \# \mathbb{R}\mathbf{P}^3$ from Theorem 34 are vertex-minimal (which comprises Conjecture 19(a)).

Among the triangulations from Table 5 are various examples that have the same homology. We list these examples together with their fundamental groups and their smallest known f -vectors in Table 6.

Suppose, one of the manifolds from Table 6 is given to us as a simplicial complex without further information. Then computing its homology vector, fundamental group, and (as a 'quasi-invariant') the f -vector of the smallest triangulation that we achieve from the given complex by bistellar flips, will allow us to make a quite accurate guess for its topological type. In general, however, many manifolds will share the same minimal f -vector.

The smallest f -vectors that we found for k -fold connected sums of sphere products $(S^2 \times S^1)^{\#k}$ and twisted sphere products $(S^2 \times S^1)^{\#k}$ are identical for $2 \leq k \leq 5$; see Table 7.

Table 7: Smallest known triangulations of connected sums of $S^2 \times S^1$ and $S^2 \times S^1$.

Manifold	f -Vector
$(S^2 \times S^1)^{\#2}, (S^2 \times S^1)^{\#2}$	(12,58,92,46)
$(S^2 \times S^1)^{\#3}, (S^2 \times S^1)^{\#3}$	(13,72,118,59)
$(S^2 \times S^1)^{\#4}, (S^2 \times S^1)^{\#4}$	(15,90,150,75)
$(S^2 \times S^1)^{\#5}, (S^2 \times S^1)^{\#5}$	(16,104,176,88)

Theorem 36 (Brehm and Światkowski [42]) *The number of topologically distinct lens spaces that can be triangulated with n vertices grows exponentially with n .*

Brehm and Światkowski [42] constructed explicit triangulations of all lens spaces $L(p, q)$. In particular, they gave an infinite series of $D_{2(p+2)}$ -symmetric triangulations $S_{2(p+2)}$ of $L(p, 1)$ with $2p + 7$ vertices.

The Brehm and Światkowski example $S_{2.4}$ with D_8 -symmetry on 11-vertices is combinatorially isomorphic to Walkup's minimal triangulation [129] of the lens space $\mathbb{R}\mathbf{P}^3 = L(2, 1)$. The facets of this triangulation $\mathbb{R}\mathbf{P}_{11}^3$ with f -vector (11, 51, 80, 40) are:

1237	123 11	1269	126 11	1279	135 10	135 11	137 10
1479	147 10	1489	148 10	1568	156 11	158 10	1689
2348	234 11	2378	246 10	246 11	248 10	2578	2579
258 10	259 10	269 10	3459	345 11	3489	359 10	3678
367 10	3689	369 10	4567	456 11	4579	467 10	5678.

The full automorphism group of $\mathbb{R}\mathbf{P}_{11}^3$ is larger than D_8 : For every vertex, we computed the Altshuler-Steinberg determinant [7] $\det AA^T$ of the vertex-facet incidence matrix A of the respective vertex-link. Vertices 1–6 yield Altshuler-Steinberg determinant 41616, the determinant for vertices 7–10 is 12096, and vertex 11 gives determinant 0. Thus, the automorphism group of $\mathbb{R}\mathbf{P}_{11}^3$ must be a subgroup of $S_6 \times S_4$. In fact, it is $2S_4$ with generators $(1,2,3,4,5,6)(7,8,9)$, $(1,2)(3,6)(4,5)(7,9)$, and $(3,6)(7,9)(8,10)$, which can easily be verified by a computer.

By applying bistellar flips to the triangulation $S_{2.5}$ with 13 vertices, we obtained a 12-vertex triangulations $L(3, 1)_{12}$ of the lens space $L(3, 1)$ with f -vector (12, 66, 108, 54) and facets

1234	123 10	1249	1256	1259	126 11	12 10 11
1347	1378	138 10	1479	156 12	1579	157 12
16 11 12	178 12	18 10 11	18 11 12	234 12	23 10 12	2489
248 12	2568	2589	2678	267 11	278 12	27 10 11
27 10 12	3456	345 11	3467	34 11 12	3568	3589
359 11	3678	389 10	39 10 12	39 11 12	456 10	45 10 11
4679	469 10	489 10	48 10 11	48 11 12	56 10 12	579 11
57 10 11	57 10 12	679 11	69 10 12	69 11 12.		

Table 8: Triangulated 3-manifolds with up to 10 vertices.

n	Manifold	Types (all)	Types (neighborly)
5	S^3	1	1
6	S^3	2	1
7	S^3	5	1
8	S^3	39	4
9	S^3	1296	50
	$S^2 \times S^1$	1	1
10	S^3	247882	3540
	$S^2 \times S^1$	615	83
	$S^2 \times S^1$	518	54

The symmetry group of $L(3, 1)_{12}$ is S_3 as a subgroup of $S_6 \times S_3 \times S_3$ with generators $(1,2)(3,6)(4,5)(7,8)(10,11)$ and $(1,3,5)(2,4,6)(7,8,9)(10,11,12)$. (The Altshuler-Steinberg determinant is 134784 for the vertices 1–6, 133056 for the vertices 7–9, and 112320 for the vertices 10–12.)

As already mentioned, Brehm [33] previously found a triangulation of $L(3, 1)$ with 12 vertices by modifying $S_{2,5}$.

Conjecture 37 *The minimal triangulation \mathbb{RP}_{11}^3 of \mathbb{RP}^3 is unique with f -vector $(11, 51, 80, 40)$. Also, the minimal triangulation $L(3, 1)_{12}$ of $L(3, 1)$ is unique with f -vector $(12, \underline{66}, 108, 54)$ and is the only triangulation of $L(3, 1)$ with 12 vertices.*

The exact numbers of different combinatorial types of triangulations of S^3 , $S^2 \times S^1$, and $S^2 \times S^1$ with up to 10 vertices were obtained by

Grünbaum and Sreedharan [70] (simplicial 4-polytopes with 8 vertices),
 Barnette [14] (combinatorial 3-spheres with 8 vertices),
 Altshuler and Steinberg [7] (neighborly 4-polytopes with 9 vertices),
 Altshuler and Steinberg [8] (neighborly 3-manifolds with 9 vertices),
 Altshuler and Steinberg [9] (combinatorial 3-manifolds with 9 vertices),
 Altshuler [2] (neighborly 3-manifolds with 10 vertices),
 Lutz [24] (combinatorial 3-manifolds with 10 vertices).

For a discussion of the polytopality of the simplicial 3-spheres with 9 vertices see Altshuler, Bokowski, and Steinberg [5] and Engel [60]. For the polytopality of the neighborly simplicial 3-spheres with 10 vertices see Bokowski and Garms [26] and Bokowski and Sturmfels [28]. The numbers of combinatorial types of 3-manifolds with up to 10 vertices can be found in Table 8.

An upper bound on the numbers of combinatorial types of simplicial 4-polytopes was given by Goodman and Pollack [64], [65]. Many combinatorially different types of triangulated spheres for growing n were constructed by Kalai [76] (for $d \geq 4$) and by Pfeifle and Ziegler [113] (for $d = 3$).

Kühnel and Lassmann [90] enumerated all combinatorial 3-manifolds with $n \leq 15$ vertices that have a vertex-transitive cyclic group action as well as all 3-manifolds with a vertex-transitive dihedral action for $n \leq 19$. (In their list appear two non-orientable manifolds IV_{17} and IV_{19} . These are homeomorphic to $\mathbb{R}\mathbf{P}^2 \times S^1$, as we were able to recognize with bistellar flips.) Enumeration results of all vertex-transitive triangulations of 3-manifolds with up to 15 vertices are listed in [80] and with 16 and 17 vertices in [102].

5 4-Manifolds

The unique 9-vertex triangulation $\mathbb{C}\mathbf{P}_9^2$ of Kühnel [87] of the complex projective plane $\mathbb{C}\mathbf{P}^2$ certainly is the most prominent combinatorial 4-manifold. By the Brehm-Kühnel bound (Theorem 2(a), [38]), it has the minimal number of vertices that a combinatorial 4-manifold, different from S^4 , can have.

Kühnel's triangulation $M^4 = M_3^4(11)$ ([83], [92]) of the product $S^3 \times S^1$ is vertex-minimal with 11 vertices. (The combinatorial manifolds $M_3^4(n)$ of Kühnel and Lassmann [92] are triangulations of $S^3 \times S^1$ for all $n \geq 11$.)

For the twisted sphere product $S^3 \times S^1$ it is conjectured (Conjecture 6) that at least 12 vertices are needed for a triangulation, while 11 vertices is the current best lower bound according to Corollary 4. A vertex-transitive 12-vertex triangulation ${}^4 12_1^{54}$ of $S^3 \times S^1$ with f -vector $(12, \underline{66}, 144, 150, 60)$ is described in [80]. We applied the bistellar flip program BISTELLAR [96] to the product triangulation of $S^3 \times S^1$ from Proposition 7:

Proposition 38 *There is a 12-vertex triangulation $(S^3 \times S^1)_{12}$ of $S^3 \times S^1$ with f -vector $(12, 60, 120, 120, 48)$.*

The facets of $(S^3 \times S^1)_{12}$ are

12678	12679	1268a	1269a	1278c	1279a	127ac	128ac
1348b	1348c	134bc	1358b	1358c	135bc	148bc	1578a
1578b	157ac	157bc	158ac	1678a	1679a	178bc	23467
2346b	23479	2349b	2367a	2369a	2369b	2379a	24679
2469b	2678a	278ac	34679	3469c	346bc	3489b	3489c
358bc	3679a	369bc	389bc	469bc	489bc	578ac	578bc,

with vertices 1–9, a , b , and c , respectively.

Conjecture 39 *The f -vector $(12, 60, 120, 120, 48)$ is component-wise minimal for combinatorial triangulations of $S^3 \times S^1$.*

In Table 9 we list the 4-dimensional manifolds for which we know triangulations with $n \leq 16$ vertices.

Kühnel's bound of Theorem 15 states that $\binom{n-4}{3} \geq 10(\chi(M) - 2)$ for every combinatorial 4-manifold M with n vertices (with equality if and only if M is 3-neighborly).

As a consequence, PL 4-manifolds M of Euler characteristic $\chi(M) = 4$, which include the manifolds $S^2 \times S^2$, $\mathbb{C}\mathbf{P}^2 \# \mathbb{C}\mathbf{P}^2$, and $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$ with homology $H_* = (\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z})$, need at least 10 vertices for a triangulation.

Table 9: Combinatorial 4-manifolds with $n \leq 16$ vertices and smallest known transitive triangulations with n_{vt} vertices (minimal if underlined).

Manifold	Homology	n	n_{vt}	Reference
S^4	$(\mathbb{Z}, 0, 0, 0, \mathbb{Z})$	<u>6</u>	<u>6</u>	
$\mathbb{C}\mathbf{P}^2$	$(\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z})$	<u>9</u>	<u>9</u>	[87], [88]
$S^3 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z})$	<u>11</u>	<u>11</u>	[83]
$S^2 \times S^2$	$(\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z})$	<u>11</u>	<u>12</u>	[122], [123]
$S^3 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}_2, 0)$	12	<u>12</u>	[80]
$\mathbb{C}\mathbf{P}^2 \# \mathbb{C}\mathbf{P}^2$	$(\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z})$	12	?	
$\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2 = S^2 \times S^2$	$(\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z})$	12	?	[101]
$(S^2 \times S^2) \# (S^2 \times S^2)$	$(\mathbb{Z}, 0, \mathbb{Z}^4, 0, \mathbb{Z})$	<u>12</u>	<u>12</u>	[80]
$\mathbb{C}\mathbf{P}^2 \# (\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2)$ $= \mathbb{C}\mathbf{P}^2 \# (S^2 \times S^2)$	$(\mathbb{Z}, 0, \mathbb{Z}^3, 0, \mathbb{Z})$	13	?	
$\mathbb{C}\mathbf{P}^2 \# (S^2 \times S^2) \#^2$	$(\mathbb{Z}, 0, \mathbb{Z}^5, 0, \mathbb{Z})$	13	?	
$(S^3 \times S^1) \# \mathbb{C}\mathbf{P}^2$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$	14	?	
$(S^3 \times S^1) \# \mathbb{C}\mathbf{P}^2$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2, 0)$	14	?	
$(\mathbb{C}\mathbf{P}^2 \# \mathbb{C}\mathbf{P}^2) \#^2$	$(\mathbb{Z}, 0, \mathbb{Z}^4, 0, \mathbb{Z})$	14	?	
$(S^2 \times S^2) \#^3$	$(\mathbb{Z}, 0, \mathbb{Z}^6, 0, \mathbb{Z})$	14	?	
$(S^3 \times S^1) \# (S^3 \times S^1)$	$(\mathbb{Z}, \mathbb{Z}^2, 0, \mathbb{Z}^2, \mathbb{Z})$	15	?	
$(S^3 \times S^1) \# (S^3 \times S^1)$	$(\mathbb{Z}, \mathbb{Z}^2, 0, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$	15	?	
$(S^3 \times S^1) \# (\mathbb{C}\mathbf{P}^2) \#^5$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^5, \mathbb{Z}_2, 0)$	15	15	[80]
$(S^3 \times S^1) \# (S^2 \times S^2)$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}, \mathbb{Z})$	16	?	
$(S^3 \times S^1) \# (S^2 \times S^2)$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}_2, 0)$	16	?	
$\mathbb{R}\mathbf{P}^4$	$(\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0)$	<u>16</u>	?	[101]
K3 surface	$(\mathbb{Z}, 0, \mathbb{Z}^{22}, 0, \mathbb{Z})$	<u>16</u>	<u>16</u>	[48]

Since $\binom{10-4}{3} = 10(4-2)$ for $n = 10$ and $\chi(M) = 4$, every 10-vertex triangulation of such a manifold would necessarily be 3-neighborly according to Theorem 15. However, Kühnel and Lassmann [88] proved that the boundary of the 5-simplex and \mathbb{CP}_9^2 are the only 3-neighborly combinatorial 4-manifolds with $n \leq 13$ vertices. Thus, at least 11 vertices are needed for a triangulation of a PL 4-manifold of Euler characteristic $\chi(M) = 4$.

Theorem 40 *The product $S^2 \times S^2$ can be triangulated vertex-minimally with $f = (11, \underline{55}, 150, 170, 68)$.*

Proof. We applied bistellar flips to the product triangulation of $S^2 \times S^2$ on $4 \cdot 4 = 16$ vertices and obtained an 11-vertex triangulation $(S^2 \times S^2)_{11}$ of $S^2 \times S^2$ with f -vector $(11, \underline{55}, 150, 170, 68)$ and facets:

12346	12347	12369	12379	12458	12459	12468	12479
12568	12569	13467	13567	13569	1357a	1359b	135ab
1379a	139ab	1458a	1459b	145ab	1467b	1468a	146ab
1479b	15678	1578a	1678b	168ab	178ab	179ab	23468
23478	2357a	2357b	235ab	2368a	2369a	2378b	2379a
238ab	24589	24789	2568b	2569a	256ab	25789	2578b
2579a	268ab	3467b	3468a	3469a	3469b	3478b	3489a
3489b	3567b	3569b	389ab	4569a	4569b	456ab	4589a
4789b	5678b	5789a	789ab.				

This triangulation is vertex-minimal by Theorem 15 and by the result of Kühnel and Lassmann [88] that there is no 3-neighborly triangulation of $S^2 \times S^2$ with 10 vertices. \square

Conjecture 41 *The f -vector $(11, \underline{55}, 150, 170, 68)$ is component-wise minimal for combinatorial triangulations of $S^2 \times S^2$.*

Vertex-transitive triangulations of $S^2 \times S^2$ with 12 vertices were first found by Sparla [122], [123] and Lassmann and Sparla [94]. Altogether, there are three such triangulations; see [80]. We applied bistellar flips to these as well and obtained further minimal triangulations of $S^2 \times S^2$ with 11 vertices, which are combinatorially distinct from the above example. All the examples that we found with 11 vertices are not symmetric.

Combinatorial triangulations of $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# -\mathbb{CP}^2$ with $9 + 9 - (4 + 1) = 13$ vertices can easily be obtained from Kühnel's 9-vertex triangulation \mathbb{CP}_9^2 of \mathbb{CP}^2 by taking two (disjoint) copies of \mathbb{CP}_9^2 , removing a 4-simplex (with $4+1$ vertices) each, and then gluing both parts together. Two combinatorially distinct triangulations of $\mathbb{CP}^2 \# -\mathbb{CP}^2$ with only 12 vertices occur as vertex-links of two simply connected 5-dimensional combinatorial pseudomanifolds with homology $H_* = (\mathbb{Z}, 0, 0, \mathbb{Z}^{13}, 0, \mathbb{Z})$; see [101]. With bistellar flips we obtained triangulations of $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# -\mathbb{CP}^2$ with smaller f -vector. The resulting lists of facets can be found online at [97].

Proposition 42 *The 4-manifolds $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# -\mathbb{CP}^2$ can be triangulated with $f = (12, 57, 148, 165, 66)$.*

Table 10: Combinatorial 5-manifolds with $n < 16$ vertices and smallest known transitive triangulations with n_{vt} vertices (minimal if underlined).

Manifold	Homology	n	n_{vt}	Reference
S^5	$(\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z})$	<u>7</u>	<u>7</u>	
$S^3 \times S^2$	$(\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z})$	<u>12</u>	<u>14</u>	[80], [99]
$SU(3)/SO(3)$	$(\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$	13	<u>13</u>	[80]
$S^4 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, 0, 0, \mathbb{Z}_2, 0)$	<u>13</u>	<u>13</u>	[83]
$S^4 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, 0, 0, \mathbb{Z}, \mathbb{Z})$	14	<u>14</u>	[92]

Conjecture 43 *The f -vector $(12, 57, 148, 165, 66)$ is component-wise minimal for combinatorial triangulations of $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ and $\mathbb{C}\mathbb{P}^2 \# -\mathbb{C}\mathbb{P}^2$.*

We formed further connected sums of the 4-manifolds $\mathbb{C}\mathbb{P}^2$, $S^3 \times S^1$, $S^3 \times S^1$, and $S^2 \times S^2$, applied bistellar flips to these, and obtained small triangulations as listed in Table 9.

By enumeration, we also found two vertex-transitive triangulations ${}^4 12_1^2$ and ${}^4 12_2^2$ of $(S^2 \times S^2) \# (S^2 \times S^2)$ with 12 vertices; see [80]. These triangulations are vertex-minimal according to Theorem 15. Moreover, a *tight* (in the sense of [86]) vertex-transitive 15-vertex triangulation ${}^4 15_1^4$ of $(S^3 \times S^1) \# (\mathbb{C}\mathbb{P}^2) \#^5$ was obtained; see [80]. We believe that this triangulation is vertex-minimal (cf. Conjecture 19).

A tight, vertex-minimal, vertex-transitive triangulation of the K3 surface was found by Casella and Kühnel [48]. For a survey on the known examples of tight triangulations see [93].

6 5-Manifolds

Small triangulations of higher-dimensional manifolds are still rare. Apart from S^d , triangulated as the boundary of the $(d + 1)$ -simplex, the Kühnel series [83], which contributes a vertex-minimal, vertex-transitive triangulation of the (twisted) S^{d-1} -bundle over S^1 in every dimension d , and the more general series $M_k^d(n)$ of Kühnel and Lassmann [92] for $1 \leq k \leq d - 1$ and $n \geq 2^{d-k}(k + 3) - 1$, we know of additional small triangulations only in dimensions $d \leq 8$. The 5-dimensional examples are listed in Table 10.

By Theorem 5 we have minimality for $S^4 \times S^1$ with 13 vertices. For the sphere product $S^4 \times S^1$ it is conjectured (cf. Conjecture 6) that 14 vertices is best possible.

Vertex-transitive triangulations of the simply connected sphere product $S^3 \times S^2$ with 14 vertices and of the simply connected homogeneous 5-manifold $SU(3)/SO(3)$ with 13 vertices were obtained by enumeration; see [80]. The unique vertex-transitive triangulation ${}^5 13_2^3$ of $SU(3)/SO(3)$ with 13 vertices

is 3-neighborly and tight (cf. [93]). Its vertex-minimality is conjectured in [80].

There are at least four combinatorially distinct vertex-transitive triangulations of $S^3 \times S^2$ with 14 vertices. By running the program BISTELLAR on the example ${}^5 14_9^3$ of [80] we obtained a 12-vertex triangulation $(S^3 \times S^2)_{12}^a$ of $S^3 \times S^2$ with f -vector $(12, \underline{66}, \underline{220}, 390, 336, 112)$ and facets:

12346a	12346b	123478	12347b	12348a	12357b	12357c	12359b
12359c	1236ab	12378c	1238ac	1239ab	1239ac	124678	12467b
124689	12469a	12489a	1257bc	1259bc	12678c	1267bc	12689c
1269ab	1269bc	1289ac	134678	13467b	13468a	13579b	13579c
13678c	1367bc	1368ac	136abc	1379ab	1379ac	137abc	145689
14568a	14569a	14589c	1458ac	1459ac	1489ac	15689b	1568ab
1569ab	1579ab	1579ac	157abc	1589bc	158abc	1689bc	168abc
23456a	23456c	23458a	23458b	2345bc	2346bc	23478b	235678
23567c	2356a0	23578b	2359bc	23678c	2368ac	236abc	239abc
24567a	24567c	24578a	24578b	2457bc	246789	24679a	2467bc
24789a	25678a	26789a	2689ac	269abc	345679	34567c	345689
34568a	34579c	34589b	3459bc	346789	3467bc	34789b	3479bc
356789	35789b	379abc	45679a	4578ab	4579ac	457abc	4589bc
458abc	4789ab	479abc	489abc	56789a	5689ab	5789ab	689abc.

Theorem 44 *The 12-vertex triangulation $(S^3 \times S^2)_{12}^a$ of $S^3 \times S^2$ has the minimal number of vertices that a combinatorial 5-manifold, different from S^5 , can have by the Brehm-Kühnel bound of Theorem 2. In particular, the Brehm-Kühnel lower bound is sharp in dimension 5.*

In addition to the example $(S^3 \times S^2)_{12}^a$, we found a second triangulation $(S^3 \times S^2)_{12}^b$ of $S^3 \times S^2$ with the same f -vector $(12, \underline{66}, \underline{220}, 390, 336, 112)$ by applying bistellar flips to the product triangulation of $S^3 \times S^2$ with 20 vertices. The two triangulations are combinatorially distinct: $(S^3 \times S^2)_{12}^a$ has Altshuler-Steinberg determinant 4471184572226676864, whereas the second example $(S^3 \times S^2)_{12}^b$ has determinant 4508595451809050112. For a list of facets of the second example see [93]. Both triangulations have no non-trivial symmetries: the Altshuler-Steinberg determinants of their 12 vertex links are pairwise distinct, respectively.

Three further minimal triangulations of $S^3 \times S^2$ with 12 vertices were found by starting with the vertex-transitive 14-vertex triangulations ${}^5 14_{13}^3$, ${}^5 14_{14}^3$, and ${}^5 14_{15}^3$ from [80].

With a new and much faster implementation by Nikolaus Witte of the bistellar flip program (accessible via the TOPAZ module of the polymake system [63]) another twenty examples were obtained by starting from the product triangulation of $S^3 \times S^2$. In fact, we started twenty-six times and each time achieved a minimal triangulation, but six of these examples appeared twice (up to relabeling the vertices).

Proposition 45 *There are at least 25 combinatorially distinct minimal triangulations of $S^3 \times S^2$ with 12 vertices.*

The 25 examples have different Altshuler-Steinberg determinants, their lists of facets can be found online at [97].

7 6-Manifolds

There are at least nine combinatorially different vertex-transitive 15-vertex triangulations of $S^3 \times S^3$; see [80]. Via bistellar flips (by starting with the triangulation ${}^6 15_2^7$ of $S^3 \times S^3$ from [80]) we obtained a 13-vertex triangulation $(S^3 \times S^3)_{13}^a$ of $S^3 \times S^3$ with f -vector (13, 78, 286, 715, 1014, 728, 208) and facets:

123456c	123456d	12345ab	12345ac	12345bd	12346cd	123479a	123479d
12347ad	12348ab	12348ad	12348bd	12349ac	12349cd	12356bc	12356bd
1235abc	1236bcd	12379ad	12389ab	12389ad	12389bd	1239abc	1239bcd
1245678	124567b	1245689	124569a	12456ac	12456bd	1245789	124579a
12457ab	124678b	124689b	12469ac	12469bd	12469cd	124789d	12478ab
12478ad	12489bd	125678c	12567bc	125689a	12568ac	125789a	12578ac
1257abc	12678ab	12678ac	1267abc	12689ab	1269abc	1269bcd	12789ad
1345679	134567d	134569a	13456ac	134579a	13457ad	1345abd	134679c
13467cd	13469ac	13479cd	1348abd	1356789	135678c	13567bc	13567bd
135689a	13568ac	135789c	13579ad	13579bc	13579bd	13589ad	13589bc
13589bd	1358abc	1358abd	136789c	1367bcd	13689ac	1379bcd	1389abc
1456789	14567bd	1457abd	146789c	14678bd	14678cd	14689bc	1468bcd
1469bcd	14789cd	1478abd	1489bcd	15789ad	15789cd	1578acd	1579bcd
157abcd	1589bcd	158abcd	1678abd	1678acd	167abcd	1689abc	168abcd
23456cd	23458bc	23458bd	23458cd	2345abc	23478ab	23478ad	23478bc
23478cd	23479ac	23479cd	2347abc	235678b	235678c	23567bc	235689b
235689d	23568cd	23569bd	23578bc	23589bd	23678ab	23678ad	23678cd
23679ab	23679ad	23679bd	2367bcd	23689ab	23689ad	2379abc	2379bcd
245678b	245689b	24569ac	24569bd	24569cd	245789c	24578bc	24579ac
2457abc	24589bd	24589cd	24789cd	25689ad	2568acd	2569acd	25789ad
25789cd	2578acd	2579acd	2678acd	2679abd	267abcd	269abcd	279abcd
345679a	34567ad	3456acd	3458abc	3458abd	3458acd	346789b	346789c
34678ab	34678ad	34678cd	34679ab	34689ab	34689ac	3468acd	34789bc
3479abc	3489abc	356789b	35679ad	35679bd	35689ad	3568acd	35789bc
456789b	45679ab	4567abd	4569abd	4569acd	45789bc	4579abc	4589bcd
458abcd	459abcd	4678abd	4689abc	468abcd	469abcd	5679abd	579abcd.

Theorem 46 *The 13-vertex triangulation $(S^3 \times S^3)_{13}^a$ of $S^3 \times S^3$ has the minimal number of vertices that a combinatorial 6-manifold, different from S^6 , can have by the Brehm-Kühnel bound of Theorem 2. In particular, the Brehm-Kühnel lower bound is sharp in dimension 6.*

Another 13-vertex triangulation $(S^3 \times S^3)_{13}^b$ of $S^3 \times S^3$ was obtained by starting the bistellar flip program on the vertex-transitive triangulation ${}^6 15_1^7$

Table 13: Combinatorial 6-manifolds with $n \leq 16$ vertices and smallest known transitive triangulations with n_{vt} vertices (minimal if underlined).

Manifold	n	n_{vt}	Reference
S^6	<u>8</u>	<u>8</u>	
$S^3 \times S^3$	<u>13</u>	15	[80]
$S^5 \times S^1$	<u>15</u>	<u>15</u>	[83]

of [80]. Both 13-vertex triangulations of $S^3 \times S^3$ are 4-neighborly and thus are tight, since equality holds in Proposition 4.6 of [86]; see also [93]. The facets of $(S^3 \times S^3)_{13}^b$ can be found in [93]. The two examples $(S^3 \times S^3)_{13}^a$ and $(S^3 \times S^3)_{13}^b$ have the same Altshuler-Steinberg determinant 745714154823444619853824. However, the Altshuler-Steinberg determinants of their vertex links are pairwise distinct. It follows that the two examples are asymmetric.

Two further tight 13-vertex triangulations of $S^3 \times S^3$ were obtained with bistellar flips by starting from the centrally-symmetric 16-vertex triangulations $\overset{6}{\times} 16_1^{cy}$ and $\overset{6}{\times} 16_1^{di}$ of [99]. The resulting minimal triangulations have the same Altshuler-Steinberg determinant as the two examples before, but again, the Altshuler-Steinberg determinants of their vertex links differ.

Proposition 47 *There are at least 4 combinatorially distinct minimal triangulations of $S^3 \times S^3$ with 13 vertices.*

The short list of 6-manifolds for which we know small triangulations is given in Table 13.

8 7-Manifolds

Vertex-transitive triangulations of 7-manifolds with up to 15 vertices are enumerated in [80] (with the exception of possible examples corresponding to the transitive actions of the groups \mathbb{Z}_{14} , D_7 , and \mathbb{Z}_{15} on 14 and 15 vertices). All resulting examples are triangulations of S^7 . (It is open whether there are vertex-transitive triangulations of 7-manifolds, different from S^7 , with a cyclic automorphism group on 15 vertices.)

Triangulations of centrally symmetric 7-manifolds with a vertex-transitive cyclic group action on 18 vertices are enumerated in [99]: There is one vertex-transitive centrally symmetric 18-vertex triangulation with cyclic symmetry of $S^4 \times S^3$ and one of $S^5 \times S^2$ each. These are the smallest triangulations for these manifolds that have been achieved so far; see Table 14.

Table 14: Combinatorial 7-manifolds with $n \leq 20$ vertices and smallest known vertex-transitive triangulations with n_{vt} vertices (minimal if underlined).

Manifold	n	n_{vt}	Reference
S^7	<u>9</u>	<u>9</u>	
$S^6 \times S^1$	<u>17</u>	<u>17</u>	[83]
$S^6 \times S^1$	18	18	[92]
$S^5 \times S^2$	18	18	[99]
$S^4 \times S^3$	18	18	[99]

Table 15: Combinatorial 8-manifolds with $n \leq 20$ vertices and smallest known vertex-transitive triangulations with n_{vt} vertices (minimal if underlined).

Manifold	n	n_{vt}	Reference
S^8	<u>10</u>	<u>10</u>	
$\sim \mathbb{H}\mathbf{P}^2$	<u>15</u>	<u>15</u>	[39]
$S^7 \times S^1$	<u>19</u>	<u>19</u>	[83]
$S^5 \times S^3$	20	20	[99]
$S^4 \times S^4$	20	20	[99]

9 8-Manifolds

According to the Brehm-Kühnel bound of Theorem 2(a), a combinatorial 8-manifold, different from S^8 , has at least 15 vertices. It is a ‘manifold like the quaternionic projective plane’ if it has 15 vertices. Such an example M_{15}^8 with a vertex-transitive A_5 -action and two further non-transitive examples, which are PL homeomorphic to the transitive one, were found by Brehm and Kühnel [39]. We denote their 8-manifold by $\sim \mathbb{H}\mathbf{P}^2$. With bistellar flips we found three further triangulations of $\sim \mathbb{H}\mathbf{P}^2$; see [97] for their lists of facets.

Proposition 48 *There are at least 6 combinatorially distinct vertex-minimal triangulations of the Brehm and Kühnel manifold $\sim \mathbb{H}\mathbf{P}^2$ with 15 vertices.*

Centrally symmetric 20-vertex triangulations of $S^4 \times S^4$ and of $S^5 \times S^3$ (one each) with dihedral symmetry are given in [99]. These are the smallest known triangulations for these two manifolds; see Table 14.

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