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Supersymmetric representations and integrable super-extensions of the Burgers and Boussinesq equations

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SUPERSYMMETRIC REPRESENTATIONS AND INTEGRABLE SUPER-EXTENSIONS OF THE BURGERS AND BOUSSINESQ EQUATIONS

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Abstract. New evolutionary supersymmetric systems whose right-hand sides are homogeneous differential polynomials and which possess infinitely many higher symmetries are constructed. Their intrinsic geometry (symmetries, conservation laws, recursion operators, Hamiltonian structures, and exact solutions) is analyzed by using algebraic methods.

A supersymmetric $N = 1$ representation of the Burgers equation is obtained. An $N = 2$ KdV-component system that reduces to the Burgers equation in the diagonal $N = 1$ case $\theta^1 = \theta^2$ is found; the $N = 2$ Burgers equation admits and $N = 2$ modified KdV symmetry. A one-parametric family of $N = 0$ super-systems that extend the Burgers equation is described; we relate the systems within this family with the Burgers equation on associative algebras.

A supersymmetric boson+fermion representation of the dispersionless Boussinesq equation is investigated. We solve this equation explicitly and construct its integrable deformation that generates two infinite sequences of the Hamiltonians. The Boussinesq equation with dispersion is embedded in a one-parametric family of two-component systems with dissipation. We finally construct a three-parametric supersymmetric system that incorporates the Boussinesq equation with dispersion and dissipation but never retracts to it for any values of the parameters.

Introduction. In this paper we reveal the supersymmetric nature of the purely bosonic Burgers and Boussinesq equations, which describe the dissipative nonlinear evolution of rarified gas and the propagation of waves in a weakly nonlinear and weakly dispersive liquid [1, 2], respectively. We investigate the relation between $N = 0$ supersymmetric evolutionary equations that involve anticommuting functions, manifestly $N = 1, 2$ supersymmetric systems that contain the anticommuting independent variables, and evolutionary systems such that the unknown functions take their arguments to associative algebras [15] and which therefore generalize the concept of supersymmetry. We apply algebraic methods [9] to the study of geometry of supersymmetric PDE, and we use the SsTools package [8] for the computer algebra system Reduce in practical computations.

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We first obtain an $N = 1$ supersymmetric representation of the Burgers equation; we show that the resulting super-system admits infinitely many symmetries and supersymmetries that do not reduce to the previously known [10] flows for the purely bosonic scalar equation. We embed the Burgers equation in a family of $N = 0$ supersymmetric systems, which are parameterized by the real coupling constant α; for all α we also reconstruct the potential super-Burgers equations. We discover that the two-component Burgers-type systems within this family reduce to the Burgers equation written w.r.t. the super-field $u = b + \vartheta f$, where the variable ϑ anticommutes with the fermionic field f but, generally, does not anticommute with itself, obeying the identity $\vartheta \cdot \vartheta = \alpha \in \mathbb{R}$. If the coupling is $\alpha = 0$, then ϑ is supersymmetric; otherwise we obtain the Burgers equation on an associative algebra [15]. We also describe a scalar $N = 2$ generalization of the Burgers equation. The $N = 1$ diagonal reduction $\theta^1 = \theta^2$ of this generalization is the bosonic Burgers equation itself. We demonstrate that one of the four components in the $N = 2$ supersymmetric equation is of KdV-type. We find a higher symmetry of the $N = 2$ Burgers equation, and the symmetry is an $N = 2$ modified KdV equation.

Further, we solve the hydrodynamic-type dispersionless Boussinesq equation using the hodograph transformation; the solution is expressed by the Airy function. We also construct an integrable deformation for this equation. The deformation generates two infinite sequences of the Hamiltonians for contact symmetries. Next, we represent the dispersionless Boussinesq equation as a two-component Hamiltonian super-system. We prove that the supersymmetric representation admits two infinite sequences of commuting Hamiltonian symmetries whose differential orders equal 2 and which are nevertheless proliferated by a nonlocal recursion operator of differential order 1. By using the supersymmetric representation we conclude that the classical dispersionless Boussinesq equation admits two infinite sequences of supersymmetries besides the well-known purely bosonic symmetries.

We extend a supersymmetric representation for the Boussinesq equation with dispersion to a one-parametric family of Boussinesq systems with dissipation, and by using this extension we finally construct a family of coupled boson+fermion evolutionary super-systems that contain the Boussinesq equation but do not retract to it at any values of the parameters.

We also describe three dilaton analogues of the superKdV equation [12] and construct nonlocal recursion operators for their symmetry algebras, see Example 1 in the Introduction and Example 2 on p. 6.

The motivating idea of this research is the problem of a complete description of $N = 1$ nonlinear scaling-invariant evolutionary super-equations $\{f_t = \phi^f,\ b_t = \phi^b\}$ that admit infinitely many local symmetries proliferated by recursion operators; here b is the bosonic super-field and f is the fermionic super-field. We denote by θ the super-variable and we put $\partial_\theta \equiv D_\theta + \theta\, D_x$ such that $\partial_\theta^2 = D_x$ and $[\partial_\theta, \partial_\theta] = 2D_x$; here D_θ and D_x are the total derivatives w.r.t. θ and x, respectively. The following axioms suggested by V. V. Sokolov and A. S. Sorin were postulated:

(1) each equation admits at least one higher symmetry $\binom{f_s}{b_s}$;
(2) all equations are translation invariant and do not depend on the time t explicitly;
(3) none of the evolution equations involves only one field and hence none of the r.h.s. vanishes;
(4) at least one of the right-hand sides in either the evolution equation or its symmetry is nonlinear;

(5) at least one equation in a system or at least one component of its symmetry contains a fermion field or the superderivative ∂_θ;
(6) the time t and the parameters s along the integral trajectories of the symmetry fields are even (that is, commuting) variables;
(7) the equations are scaling invariant: their right-hand sides are differential polynomials homogeneous w.r.t. a set of (half-)integer weights $[\theta] \equiv -\frac{1}{2}$, $[x] \equiv -1$, $[t] < 0$, $[f]$, $[b] > 0$; we also assume that the negative weight $[s]$ is half-integer.

Axiom 8 for $N \geq 2$ supersymmetric equations that satisfy the above axioms is further described on p. 12.

Remark 1. From axiom 5 it follows that the admissible systems are either $N = 0$ super-equations that contain fermionic super-fields, or they are $N \geq 1$ supersymmetric if the derivation ∂_θ is present explicitly. For instance, the superKdV equation (2) by Mathieu [12] w.r.t. a fermionic super-field is $N = 1$, while its component form is $N = 0$, see [5]. An $N = 0$ two-component supersymmetric extension of the Burgers equation that can not be represented as a scalar $N = 1$ equation is obtained in Sec. 2.2, see Eq. (14) on p. 9.

The first version of the Reduce package SsTools by T. Wolf and W. Neun for super-calculus allowed to do symmetry investigations of supersymmetric equations and was used for finding $N \leq 1$ scalar fermionic and bosonic super-equations and coupled fermion+fermion, fermion+boson, and boson+boson supersymmetric evolutionary systems that satisfy the above axioms; a number of $N = 2$ scalar evolution PDE were also found. The bounds $0 < [f]$, $[b] \leq 5$ and $0 > [t] > [s] \geq -5$ were used. The experimental database [19] contains 1830 equations (the duplication of PDE that appeared owing to possible non-uniqueness of the weights is now eliminated) and their 4153 symmetries (plus the translations along x and t, and plus the scalings whose number is in fact infinite).

The classical integrable supersymmetric evolutionary systems as well as their generalizations and reductions are present in the database.

Example 1 (Mathieu's superKdV [12]). Let $N = 1$ and let the weight of the bosonic super-field be $[b] = 1$; further, let $[t] = -3$. Now we scan the cell which is assigned to these weights and which is filled in by the runs of SsTools. We then get the potential superKdV equation by Mathieu[1]

$$b_t = b_{xxx} + 3\partial_\theta(b_x \partial_\theta b). \tag{1}$$

Indeed, put $f(x, t; \theta) = \partial_\theta b$; then f is the fermionic super-field of the weight $[f] = \frac{3}{2}$ satisfying the superKdV equation ([12], see also [5] and [9, Ch. 6])

$$f_t = f_{xxx} + 3(f\partial_\theta f)_x. \tag{2}$$

The nonlocal fermionic conservation laws for Eq. (2), which are discussed in [13], are local structures for potential equation (1). We conjecture that the Hamiltonian deformation [3] of (2) that provides the recurrence relations between its nonlocal fermionic Hamiltonians is induced by a local deformation of Eq. (1).

Further, let the weights $[f] = \frac{3}{2}$ and $[t] = -3$ be fixed. Then we obtain four evolutionary supersymmetric equations that admit higher symmetries $f_s = \phi$ under the

[1]Throughout this text, the operator ∂_θ acts on the succeeding super-field unless stated otherwise explicitly.

assumption $[s] \geq -5$. The superKdV equation, see (2), is the first in this list. We also get the equation

$$f_t = f_{xxx} + f_x \partial_\theta f. \tag{3a}$$

The recursion operator for Eq. (3a) is constructed in Example 2 on p. 6. Third, we obtain the two-parametric dispersionless analogue of Eq. (2):

$$f_t = \alpha\, f \partial_\theta f_x + \beta\, f_x \partial_\theta f, \qquad \alpha, \beta = \text{const}\,. \tag{3b}$$

A computation by Yu. Naumov (Ivanovo State Power University) using SsTools demonstrates that Eq. (3b) admits the weakly nonlocal recursion operator

$$R = \alpha f \partial_\theta f\, \partial_\theta + \alpha f f_x - \alpha f \partial_\theta f_x\, \partial_\theta^{-1} - \beta f_x \partial_\theta f\, \partial_\theta^{-1} + \beta f_x\, \partial_\theta^{-1} \circ (f \partial_\theta + \partial_\theta f) \tag{4}$$

and two infinite sequences of symmetries that start from the translations f_x and f_t.

The fourth equation for the set of weights $[f] = \frac{3}{2}$, $[t] = -3$ is

$$f_t = \partial_\theta(f_x f). \tag{3c}$$

It admits the recursion $R = f\partial_\theta - f_x \partial_\theta^{-1}$ and an infinite sequence of supersymmetries that starts from the weight $[\bar{s}] = -8\frac{1}{2}$.

Remark 2. The symbols of the evolutionary supersymmetric equations that admit infinitely many symmetries are not necessarily constant. For example, Eq. (3c) can not be transformed to an equation $g_t = g_{xxx} + \ldots$ by a substitution $f = f[g]$. The proof is by *reductio ad absurdum*.

In this paper, we investigate the geometric properties of the boson+fermion systems under the additional assumption $[f] = [b]$ (for the primary sets of weights if they are multiply defined). From axioms 3 and 4 on p. 2 it follows that the triangular systems are regarded as trivial and therefore their properties are not analyzed. We emphasize that, generally, the two super-fields f, b can not be united to the fermionic super-field $\phi = f + \vartheta b$ such that $[\phi] = [f] = [b] - \frac{1}{2}$ or to the bosonic super-field $\beta = b + \vartheta f$ such that $[\beta] = [b] = [f] - \frac{1}{2}$ and such that a scalar equation w.r.t. ϕ or β holds (there is no contradiction with the diagonality assumption because the weights may not be uniquely defined).

We do not expose now the complete list of supersymmetric boson+fermion systems that satisfy the axioms on p. 2 and such that the dilaton dimensions $[f] = [b]$ coincide. In fact, the symmetries for a major part of these equations are proliferated by the recurrence relations (see p. 7); other equations that admit true recursions seem less physically important than the three supersymmetric variants of the Burgers and the Boussinesq equations we analyze.

Yet it is worthy to note some remarkable features of the five systems such that the dilaton dimension $[t] = -\frac{1}{2}$ of the time t is half the dimension of the spatial variable x (that is, the equations precede the translation invariance). It turned out that these five equations exhibit practically the whole variety of properties that superPDE of mathematical physics possess. Let us briefly summarize these features.

Three of the five evolutionary systems are given through

$$f_t = -\alpha f b, \quad b_t = b^2 + \partial_\theta f, \qquad \alpha = 1, 2, 4. \tag{5}$$

The equations differ by the values $\alpha = 1, 2, 4$ of the coefficient and demonstrate different geometrical properties. The geometry of the $\alpha = 2$ system is quite extensive: the system admits a continuous sequence of symmetries for all half-integer weights $[s] \leq -\frac{1}{2}$, a sequence of supersymmetries such that the parities of the dependent variables are opposite to the parities of their flows, four local recursions (one is nilpotent), and three local super-recursions. The equation for $\alpha = 1$ admits fewer structures, and the case $\alpha = 4$ for Eq. (5) is rather poor.

Another equation

$$f_t = \partial_\theta b + fb, \qquad b_t = \partial_\theta f \tag{6}$$

admits local symmetries for all half-integer weights $[s] \leq -\frac{1}{2}$. Equation (6) requires introduction of two layers of nonlocalities assigned to (non)local conservation laws. Four nonlocal recursion operators with nonlocal coefficients are then constructed for Eq. (6). The properties of systems (5, 6) will be considered in details in the succeeding paper [7].

The fifth system we mention is a supersymmetric representation of the Burgers equation, see (9); this system is C-integrable by using the Cole–Hopf substitution. We investigate its properties in Sec. 2.

The paper is organized as follows. In Sec. 1 we discuss two schemes for generating infinite sequences of higher symmetries of the evolutionary super-systems which are contained in the database [19]. In Sec. 2 we investigate supersymmetric representations and parametric super-extensions of the Burgers equation. Then in Sec. 3 we study a Hamiltonian deformation and a supersymmetric representation of the dispersionless Boussinesq equation. We also construct a parametric family of super-systems that incorporate the Boussinesq equation with dispersion and dissipation.

1. Recursion operators and recurrence relations

In this section we describe two principally different mechanisms for proliferation of symmetries of a PDE.

1.1. **Differential recursion operators.** We consider the (nonlocal) differential recursion operators first. The standard approach [5, 9] to recursion operators is regarding them as symmetries of the linearized equations. The essence of the method is the following. The ‘phantom variables’ (the Cartan forms) that satisfy the linearized equation are assigned to all the dependent variables in an equation $\mathcal{E}$; one may think that the internal structure of the symmetries is discarded and the (nonlocal) phantom variables imitate the (resp., nonlocal components of) symmetries for $\mathcal{E}$. In what follows, the capital letters F, B, etc. denote the variables associated with the fields f, b, respectively. Then any image $\mathcal{R} = R(\varphi)$ of a linear operator R that maps symmetries $\varphi = \{f_s = F, b_s = B\}$ of $\mathcal{E}$ to symmetries again is linear w.r.t. the right-hand sides F, B. One easily checks that $\mathcal{R}$ is then the r.h.s. of a symmetry flow

$$\frac{d}{ds_R}\begin{pmatrix} F \\ B \end{pmatrix} = \mathcal{R}$$

for the linearized equation $\mathrm{Lin}(\mathcal{E})$. If the initial equation $\mathcal{E}$ is evolutionary, then the phantom variables satisfy the well-known relations

$$F_{[t,s]} \doteq 0, \qquad B_{[t,s]} \doteq 0$$

that hold by virtue ($\doteq$) of the equations $\mathcal{E}$ and $\mathrm{Lin}(\mathcal{E})$. The method is reproduced literally in presence of nonlocalities w whose flows W are described by the corresponding components of nonlocal symmetries $\hat{\varphi} = (f_s = F,\ b_s = B,\ w_s = W)$. The recursion operators are then defined by the triples $\hat{\mathcal{R}} = (F_{s_R}, B_{s_R}, W_{s_R})$ and generate sequences of nonlocal symmetries. See [5, 9] for many examples.

We finally recall that not each symmetry φ can be extended to a nonlocal flow $\hat{\varphi}$ if the set $\{w\}$ of nonlocalities is already defined and, analogously, not all the pairs

$\mathcal{R} = (F_{s_R}, B_{s_R})$ generate a true recursion $\hat{\mathcal{R}}$. The pairs $\mathcal{R}$ are therefore called *shadows* [9] of nonlocal recursion operators. The shadows are sufficient for standard purposes since they describe the operators that map the local components of the flows. Hence in what follows we always set $W_{s_R} = 0$ (that is, we do not find the flows W_{s_R} that commute with the evolution W_t determined by the original system $\mathcal{E}$ and differential substitutions for w). Also, we describe the Cartan forms $\mathcal{R}$ rather than the differential operators R, and we use the term 'recursions' instead of the rigorous 'shadows of the generating Cartan forms for nonlocal recursion operators.'

Remark 3. The recursion operators considered in this paper are weakly non-local, see [11]; the reason why is revealed in [6]. Hence one can readily prove the locality of symmetry sequences generated by these recursions by using a supersymmetric version of the results in [17]. In the sequel, we use another method for the proof of locality. Namely, we construct an integrable deformation, see Sec. 3.1.1, and obtain local Hamiltonian functionals, whence we deduce the locality of the corresponding symmetry flows.

We finally note that a similar method of 'phantom variables' is applied for finding Hamiltonian and symplectic structures for PDE, see [4]. The (nonlocal) Hamiltonian structures for supersymmetrizations (32a), (33) are not extensively studied in this paper. The SsTools package is applicable for this investigation *as is*, since the theory is now transformed to standard algorithms of symmetry analysis.

Example 2. Consider analogue (3a) of the superKdV equation (2). We introduce the bosonic nonlocality v of weight $[v] = 1$ such that $\partial_\theta v = f$ and the fermionic nonlocality w such that $[w] = \frac{7}{2}$ and $\partial_\theta w = (\partial_\theta f)^2$. In this setting, we obtain the recursion

$$\mathcal{R}_{[2]} = \bigl(\partial_\theta f \cdot F + 3F_{xx} + f_x \cdot V + \tfrac{1}{2}W\bigr).$$

The above solution generates the sequence $f_x \mapsto f_t \mapsto \ldots$ of symmetries for Eq. (3a). The sequence starts with the translation along x and next contains the equation itself.

We now consider Eq. (3b) and introduce two bosonic nonlocal variables v and w such that $\partial_\theta v = f$ and $\partial_\theta w = f\partial_\theta f$. We then obtain the nonlocal recursion

$$\mathcal{R} = \alpha \cdot (\partial_\theta f_x f V - f\partial_\theta f \partial_\theta F + f_x f F) + \beta \cdot (f\partial_\theta f V - f_x W)\,.$$

The corresponding operator R is present in (4) on p. 4.

Next, Eq. (3c) is obviously a continuity relation. Therefore, we let v be the bosonic variable such that $\partial_\theta v = f$; hence we obtain the recursion $F_{s_R} = f\,\partial_\theta F - f_x\,V$.

Remark 4. The systems that admit several scaling symmetries and hence are homogeneous w.r.t. different weights allow to apply the breadth search method for recursions, which is the following. Let a recursion of weight $[s_R]$ w.r.t. a particular set of weights for the super-fields f, b and the time t be known. Now, recalculate its weight $[s'_R]$ w.r.t. another set and then find all recursion operators of weight $[s'_R]$. The list of solutions will incorporate the known recursion and, possibly, other operators. Generally, their weights will be different from the weight of the original recursion w.r.t. the initial set. Hence we repeat the reasonings for each new operator and thus select the weights $[s_R]$ such that nontrivial recursions exist. This method is a serious instrument for the control of calculations and elimination of errors. We used it while testing the second version of the SsTools package [8].

The second version of SsTools allows to reduce the search for nonlocal recursion operators to solving large overdetermined systems of nonlinear algebraic equations for the undetermined coefficients which are present in the weight-homogeneous ansatz for F_{s_R}, B_{s_R}. The algebraic systems are then solved using the program CRACK by T. Wolf [18]. The nonlocal variables, which are assigned to conservation laws if $N \leq 1$, were also obtained by SSTOOLS straightforwardly using the weight homogeneity assumptions.

Remark 5. The weights of the nonlocal variables constructed by using conserved currents for PDE are defined by obvious rules. Clearly, if the weight for a bosonic nonlocality is zero, then further assumptions about the maximal power of this variable in any ansatz should be made. Within this research we observed that the weights of the new super-fields necessary for constructing the recursions are never negative.

1.2. **Recurrence relations.** Consider an evolution equation $\mathcal{E} = \{u_t = \phi\}$ and let there be a differential function $q[f, b]$ of weight zero w.r.t. an admissible set of weights for $\mathcal{E}$. Suppose further that the flow $u_{s^n} = q^n \cdot \phi$ is a symmetry of $\mathcal{E}$ for any $n \in \mathbb{N}$; in a typical situation all the flows $\varphi_n = u_{s^n}$ commute with each other. Then, instead of an infinite sequence φ_n we have just one right-hand side $u_s = Q(q) \cdot \phi$ of fixed differential order and weight $[\phi]$; here Q is an arbitrary analytic function. In this case, we say that the sequence of the Taylor monomials φ_n is generated by a *recurrence relation*.

We note that the multiplication by q can be a zero-order recursion operator $R = q$ for the whole symmetry algebra $\operatorname{sym}\mathcal{E}$, otherwise the recurrence relation $\varphi_{n+1} = q \cdot \varphi_n$ generates the symmetries of $\mathcal{E}$ for the fixed 'seed' flows φ_0. The systems that admit recurrence relations for their symmetries can possess differential recursion operators as well.

Example 3. Consider the family of supersymmetric systems

$$f_t = b\partial_\theta b + f\partial_\theta f, \qquad b_t = \alpha f\partial_\theta b, \tag{7}$$

here $\alpha \in \mathbb{R}$ is arbitrary. We see that Eq. (7) is homogeneous w.r.t. the weights $[f] = [b] = \frac{1}{2}$, $[t] = -1$. The multiplication of the r.h.s. in (7) by b defines a recurrence relation for infinitely many symmetries. Indeed, the flows

$$f_t = b\,Q(b) \cdot \partial_\theta b + f\,Q(b) \cdot \partial_\theta f, \qquad b_t = \alpha f\,Q(b) \cdot \partial_\theta b$$

commute for all Qs and any constant α. Nevertheless, the operator $\left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)$ is not a recursion for Eq. (7) because it does not map an *arbitrary* symmetry to a symmetry.

Further, let $\alpha = 1$. The system

$$f_t = b\partial_\theta b + f\partial_\theta f, \qquad b_t = f\partial_\theta b \tag{7a}$$

admits the local zero-order recursions

$$\mathcal{R}^1_{[1\frac{1}{2}]} = \begin{pmatrix} \partial_\theta b b B + \partial_\theta b f F - \partial_\theta f f B \\ \partial_\theta b f B \end{pmatrix}, \qquad \mathcal{R}^2_{[2]} = \begin{pmatrix} b_x b F + f_x b B \\ b_x b B \end{pmatrix},$$
$$\mathcal{R}^3_{[3]} = \begin{pmatrix} \partial_\theta b b_x b B + \partial_\theta b b_x f F - \partial_\theta b f_x f B - \partial_\theta f b_x f B \\ \partial_\theta b\, b_x f B \end{pmatrix}.$$

An infinite number of local recursion operators for Eq. (7a) is obtained by multiplication of $\mathcal{R}^1$ by b^n, $n \in \mathbb{N}$. The recursions $\mathcal{R}^1$ and $\mathcal{R}^3$ are nilpotent: $\left(\mathcal{R}^1_{[1\frac{1}{2}]}\right)^4 = 0 = \left(\mathcal{R}^3_{[3]}\right)^4$.

If $\alpha = -1$, then Eq. (7) also admits infinitely many symmetries that do not originate from any recurrence relation because their differential orders grow.

The recurrence relation $\varphi_{n+1}(\varphi_n, q, n)$ can depend explicitly on the subscript n; then the generators of commuting flows contain the free functional parameters $Q(q)$, $Q'(q)$, etc.

Example 4. The flows

$$\begin{pmatrix} f_s(Q) \\ b_s(Q) \end{pmatrix} = \begin{pmatrix} \alpha f_x Q(b) + \gamma b_x f Q'(b) + \delta f b^2 Q'(b) \\ \alpha b_x Q(b) + \beta f_x f Q'(b) \end{pmatrix} \tag{8}$$

commute for arbitrary functions $Q(b)$ and constants α, β, γ, $\delta \in \mathbb{R}$. Indeed, for any $Q(b)$ and $S(b)$ we have

$$\left\{ \begin{pmatrix} f_\tau(Q) \\ b_\tau(Q) \end{pmatrix}, \begin{pmatrix} f_\sigma(S) \\ b_\sigma(S) \end{pmatrix} \right\} = 0.$$

The flows defined in (8) are also translation and scaling invariant.

In the sequel, we investigate the systems that are located on the diagonal $[f] = [b]$ and admit infinite sequences of (commuting) symmetries generated by recursion operators; we also analyze generalizations of these equations and properties of the new systems. The supersymmetric equations whose symmetries are proliferated by the recurrence relations are not discussed in this paper.

2. The Burgers equation

In this section we find three supersymmetric systems related with the Burgers equation. We construct an $N = 1$ supersymmetric representation of the Burgers equation itself, an $N = 0$ fermionic extension of the Burgers equation, and an $N = 2$ scalar super-equation whose $N = 1$ diagonal reduction ($\theta^1 = \theta^2$) is the purely bosonic Burgers equation. The $N = 2$ Burger equation contains a KdV-type component and admits an $N = 2$ modified KdV symmetry.

2.1. **Supersymmetric representation for the Burgers equation.** We first consider the system

$$f_t = \partial_\theta b, \qquad b_t = b^2 + \partial_\theta f. \tag{9}$$

There is a unique set of weights $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$ in this case. Hence we conclude that the above system precedes the invariance w.r.t. the translation along x. Equation (9) admits the continuous sequence (12) of higher symmetries $f_s = \phi^f$, $b_s = \phi^b$ at all (half-)integer weights $[s] \leq -\frac{1}{2}$. Also, there is the continuous sequence (13) of supersymmetries for Eq. (9) at all (half-)integer weights $[\bar{s}] \leq -\frac{1}{2}$ of the fermionic 'time' $\bar{s}$.

System (9) is obviously reduced to the purely bosonic Burgers equation

$$b_x = b_{tt} - 2bb_t. \tag{10}$$

We emphasize that the role of the independent coordinates x and t is reverse w.r.t. the standard interpretation of t as the time and x as the spatial variable. The Cole–Hopf substitution $b = -u^{-1}u_t$ from the heat equation

$$u_x = u_{tt}$$

is thus the solution for the bosonic component of (9).

We further introduce the bosonic nonlocality w of weight $[w] = 0$ by specifying its derivatives,

$$\partial_\theta w = -f, \quad w_t = -b;$$

the variable w is a potential for both fields f and b. The nonlocality satisfies the potential Burgers equation $w_x = w_{tt} + w_t^2$ such that the formula $w = \ln u$ gives the solution; the relation $f = -\partial_\theta w$ determines the fermionic component in system (9).

Now we extend the set of dependent variables f, b, and w by the symmetry generators F, B, and W that satisfy the linearized relations upon the flows of the initial superfields, respectively. In this setting, we obtain the recursion

$$\mathcal{R}_{[1]} = \begin{pmatrix} F_x - \partial_\theta f\, F + f_x\, W \\ B_x - \partial_\theta f\, B + b_x\, W \end{pmatrix} \tag{11a}$$

of weight $[s_R] = -1$. The operator assigned to $\mathcal{R}$ is

$$R = \begin{pmatrix} D_x - \partial_\theta f + f_x\, \partial_\theta^{-1} & 0 \\ b_x\, \partial_\theta^{-1} & D_x - \partial_\theta f \end{pmatrix}. \tag{11b}$$

In agreement with [6], the above recursion is weakly non-local [11]. We recall that a recursion operator R is *weakly non-local* if each nonlocality ∂_θ^{-1} is preceded with a (shadow [9] of a nonlocal) symmetry φ_α and is followed by the gradient ψ_α of a conservation law: $R = \text{local part} + \sum_\alpha \varphi_\alpha \cdot \partial_\theta^{-1} \circ \psi_\alpha$. From [6] it follows that this property is automatically satisfied by all recursion operators which are constructed using one layer of the nonlocal variables assigned to conservation laws. The weak nonnocality of recursion operators is essentially used in the proof of locality of the symmetry hierarchies they generate, see [17] and Remark 3 on p. 6.

Recursion (11) generates two sequences of higher symmetries for system (9):

$$\begin{pmatrix} f_t \\ b_t \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta b_x - \partial_\theta f \partial_\theta b - f_x b \\ \partial_\theta f_x - (\partial_\theta f)^2 - b^2 \partial_\theta f + b b_x \end{pmatrix} \mapsto \cdots, \qquad \begin{pmatrix} f_x \\ b_x \end{pmatrix} \mapsto \begin{pmatrix} f_{xx} - 2\partial_\theta f\, f_x \\ b_{xx} - 2\partial_\theta f\, b_x \end{pmatrix} \mapsto \cdots. \tag{12}$$

The same recursion (11) produces two infinite sequences of supersymmetries for the Burgers equation:

$$\begin{pmatrix} \partial_\theta f \\ \partial_\theta b \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta f_x - (\partial_\theta f)^2 - f_x f \\ \partial_\theta b_x - \partial_\theta f\, \partial_\theta b - b_x f \end{pmatrix} \mapsto \cdots, \qquad \begin{pmatrix} f \partial_\theta b - b\, \partial_\theta f + b_x \\ b \partial_\theta b - f\, \partial_\theta f + f_x - f b^2 \end{pmatrix} \mapsto \cdots. \tag{13}$$

Remark 6. System (9) is not a supersymmetric extension of Eq. (10); it is a supersymmetric representation of the Burgers equation. However, symmetries (12) and (13) are *not* reduced to the purely bosonic (x, t)-independent symmetries [10, §8.2] of the Burgers equation (particularly, owing to the interchanged role of the variables x and t).

We finally recall that the Burgers equation (10) has infinitely many higher symmetries that depend explicitly on the base coordinates x, t but exceed the set of axioms on p. 2.

2.2. **Supersymmetric extension of the Burgers equation.** The fermionic $N = 0$ extension of the Burgers equation

$$f_t = f_{xx} + (bf)_x, \quad b_t = b_{xx} + b b_x + \alpha\, f_x f, \qquad \alpha \in \mathbb{R}, \tag{14}$$

is a unique supersymmetric extension of the Burgers equation found by using SsTools. System (14) is homogeneous w.r.t. the uniquely defined set of weights $[f] = [b] = 1$, $[t] = -2$. The one-parametric family (14) admits the symmetries $\binom{f_s}{b_s}$ at all negative integer weights $[s] \le -1$. We also note that system (14) is the Burgers equation itself if the fermionic field f is set to zero.

2.2.1. *Trivial coupling in* (14): $\alpha = 0$. Let us suppose first that $\alpha = 0$. System (14) is triangular if the coupling constant α equals zero. The bosonic component is then linearized by the Cole–Hopf substitution

$$b = \frac{2q_x}{q}, \tag{15}$$

where the function $q(x,t)$ is a solution of the heat equation $q_t = q_{xx}$.

Remark 7. In this paper we do not investigate the relation between the scaling invariance of superPDE and their factorizations w.r.t. the scaling symmetries, and we do not study the physical significance of these projectivizations and the new arising supersymmetric equations. The Burgers equation as the factor of the heat equation w.r.t. its scaling symmetry, see (15), is a well-known example of this scheme used for generating new systems.

The fermionic component in (14) is linear w.r.t. the field $f(x,t)$, and hence the superposition principle is valid for it. The quantities

$$\int_{-\infty}^{+\infty} f(x,t)\,\mathrm{d}x = \text{const}, \qquad \int_{-\infty}^{+\infty} b(x,t)\,\mathrm{d}x = \text{const}$$

are integrals of motion for (14).

Remark 8. Let us introduce the bosonic super-field $u = b + \vartheta f$, here ϑ is a new variable that anticommutes with f and its derivatives w.r.t. x and such that $\vartheta \cdot \vartheta = 0$. Then from (14) it follows that $u(x,t;\vartheta)$ satisfies the Burgers equation

$$u_t = u_{xx} + uu_x. \tag{16}$$

Equation (16) has the well-known recursion operator

$$R = D_x + \tfrac{1}{2}u + \tfrac{1}{2}u_x D_x^{-1} \quad\Longleftrightarrow\quad \mathcal{R} = U_x + \tfrac{1}{2}uU + \tfrac{1}{2}u_x\, D_x^{-1}(U). \tag{17}$$

Let us construct an analogue of recursion (17) for system (14) if the coupling is $\alpha = 0$. We introduce the bosonic nonlocality v of weight zero by using $v_x = b$. The fermionic variable w is further assigned to the conserved current $D_t(f) = D_x(f_x + bf)$. We set $w_x = f$ such that $w \cdot w = 0$ and $[w] = 0$. The potentials v and w satisfy the system

$$w_t = w_{xx} + v_x w_x, \qquad v_t = v_{xx} + \tfrac{1}{2}v_x^2. \tag{18}$$

We make a technical assumption that the bosonic variables v and V of zero weight appear in the recursion for (14) at most linearly.

We then find out that in this setting there are two recursion operators of weight 1. The first operator,

$$\mathcal{R}^1_{[1]} = \begin{pmatrix} -\tfrac{1}{2}wB_x + \tfrac{1}{2}b_xW - \tfrac{1}{4}b_xwV + F_x + \tfrac{1}{2}f_xV + \tfrac{1}{2}bF - \tfrac{1}{4}wbB - \tfrac{1}{4}fbV \\ 2B_x + bB + b_xV \end{pmatrix},$$

is the direct supersymmetrization of twice the recursion (17) for Eq. (16). Simultaneously, we obtain the shadow recursion with nonlocal coefficients,

$$\mathcal{R}^2_{[1]} = \begin{pmatrix} wB_x + b_xW + \tfrac{1}{2}b_xwV + 2F_x + f_xV + bF + \tfrac{1}{2}wbB + \tfrac{1}{2}fbV + 2fB \\ 0 \end{pmatrix}.$$

Its differential order is 1; this recursion is not nilpotent.

Remark 9. System (14) models an unusual physical phenomenon. Recall that the Burgers equation describes the dissipative nonlinear one-dimensional dynamics of the density of rarified matter (e.g., the interstellar gas). Assume now that at any point $x \in \mathbb{R}$ there are two types of mass, the bosonic mass with density $b(x,t)$ and the invisible fermionic mass with density $w(x,t)$. Suppose that the initial numeric values $w(x,0)$ and $b(x,0)$ of the fermionic and bosonic densities, respectively, coincide at $t = 0$. Then the observable bosonic and the invisible fermionic densities will coincide for all $t > 0$ and the corresponding masses will be conserved w.r.t. the time.

If $\alpha \neq 0$, then the feed-back is switched on in (9). The ripple in the fermionic mass space is the cause for the bosonic mass to be changed. Indeed, the bosonic mass is no longer conserved unless $w(x,t) = \text{const}$ or, generally, unless the condition $\int_{-\infty}^{+\infty} w_{xx}(x,t) w_x(x,t)\,\mathrm{d}x = \text{const}$ holds for all $t > 0$. We see that the reaction of the fermionic component on the bosonic field depends on the incline w_x and curvature w_{xx} but not on the density w.

2.2.2. *Arbitrary coupling in* (14): $\alpha \in \mathbb{R}$. Now we reveal the geometric and algebraic nature of the phenomenon which is discussed in Remark 9 and that occurs if $\alpha \neq 0$.

Remark 10. In view of Remark 8, we introduce the variable ϑ of weight zero that anticommutes with the fermionic field f and its derivatives and such that $\vartheta \cdot \vartheta = \alpha$. As above, we set $u = b + \vartheta f$. We emphasize that ϑ is not an ordinary complex number whose square could easily be zero, negative, or positive (and ϑ would therefore be zero, imaginary, or negative, respectively). This is owing to the super-behaviour of ϑ. Indeed, for an observer located in the bosonic world ($f \equiv 0$) the variable ϑ does not exist.

Again, the super-field $u(x,t;\vartheta)$ satisfies the Burgers equation

$$u_t = u_{xx} + u u_x, \tag{16$'$}$$

but now Eq. (16′) is an equation on the associative algebra generated by u and its derivatives. This is because

$$u u_x = b b_x + \vartheta^2 f_x f + \vartheta \cdot (b_x f + b f_x) \neq b b_x - \vartheta^2 f_x f + \vartheta \cdot (b_x f + b f_x) = u_x u,$$

unless $\vartheta \cdot \vartheta = 0$. The geometry of equations on associative algebras has been studied recently, see [15], by using standard notions and computational algorithms.

Obviously, the fermionic nonlocality w such that $w_x = f$ is indifferent w.r.t. the value of the coupling constant α. We find out that system (14) always admits the conservation law that potentiates the bosonic variable b. We thus set

$$\tilde{v}_x = b + \tfrac{1}{2} f w \alpha, \qquad \tilde{v}_t = b_x + \tfrac{1}{2} b^2 + \tfrac{1}{2} f_x w \alpha + \tfrac{1}{2} f b w \alpha.$$

The weight of the nonlocality $\tilde{v}$ is zero. Surprisingly, the potential variable $\tilde{v}$ satisfies the same potential Burgers equation as the nonlocality v, see Eq. (18),

$$w_t = w_{xx} + \tilde{v}_x w_x, \qquad \tilde{v}_t = \tilde{v}_{xx} + \tfrac{1}{2} \tilde{v}_x^2. \tag{18$'$}$$

We conclude that Eq. (18′) potentiates the $N = 0$ super-system (14) for all $\alpha = \vartheta \cdot \vartheta$, although the algebraic nature of the Burgers equation (16) w.r.t. the super-field $u = b + \vartheta f$ is radically different from Eq. (16′) that describes a flow on an associative algebra.

In the nonlocal setting $\{f, w\} + \{b, \tilde{v}\}$ there are two supersymmetrizations of recursion (17). The first recursion of weight 1 for Eq. (9) is

$$\mathcal{R}^1_{[1]} = \begin{pmatrix} b_x W + 2F_x + f_x \tilde{V} + \frac{1}{2} f_x w W \alpha + bF + fB \\ \underline{2B_x + bB + b_x \tilde{V}} + \frac{1}{2} b_x w W \alpha + f_x W \alpha - f F \alpha \end{pmatrix},$$

here we underline the component that corresponds to (17). The second supersymmetric extension of weight 1 is

$$\begin{aligned} \mathcal{R}^{2,f}_{[1]} = {} & -wB_x - \tfrac{1}{2} b_x w \tilde{V} + \tfrac{1}{2} f_x w W \alpha - \tfrac{1}{4} w f b W \alpha - \tfrac{1}{2} w f F \alpha - \tfrac{1}{2} w b B \\ & - \tfrac{1}{2} f b \tilde{V} - fB, \\ \mathcal{R}^{2,b}_{[1]} = {} & \underline{2B_x + bB + b_x \tilde{V}} - w F_x \alpha + f_x W \alpha + \tfrac{1}{2} f_x w \tilde{V} + \tfrac{1}{2} f b W \alpha - \tfrac{1}{2} w b F \alpha \\ & - \tfrac{1}{2} w f b \tilde{V} \alpha - \tfrac{3}{2} w f B \alpha \end{aligned}$$

Remark 11. In the previous reasonings we treated system (14) as an $N = 0$ super-equation that involves the fermionic field but does not contain the super-derivative ∂_θ. Now we enlarge the (x, t, f, b) jet space with the anticommuting independent variable θ and the derivatives of f and b w.r.t. θ; we have ${\partial_\theta}^2 = D_x$. Physically speaking, we permit the consideration of conservation laws at half-integer weights for (14) and (18′). Indeed, there are many nonlocal fermionic conservation laws for Eq. (14); for example, we obtain the 'square roots' of the variables $\tilde{v}$ and w. We conjecture that there are infinitely many fermionic $N = 1$ conservation laws for Eq. (9). Also, there are many recursions that involve the nonlocalities assigned to fermionic conservation laws and which are nilpotent if $\alpha = 0$. This situation is analogous to the scheme that generates fermionic nonlocal Hamiltonians for the superKdV equation (2), see [13].

2.3. $N = 2$ **supersymmetric Burgers equation.** In the database [19] there is a scalar, third order $N = 2$ supersymmetrization of the Burgers equation; it is

$$b_t = \partial_{\theta^1}\partial_{\theta^2} b_x + b b_x, \qquad \partial_{\theta^i} = D_{\theta^i} + \theta^i\, D_x, \quad i = 1, 2. \tag{19}$$

We now admit that axiom 8 was used when constructing the evolutionary super-systems:

(8) each of the super-derivatives $\partial_{\theta^i} = D_{\theta^i} + \theta^i D_x$, $i = 1, \ldots, N$, occurs at least once in the r.h.s. of the evolutionary system if $N \geq 2$.

Equation (19) is reduced to the second order Burgers equation $b_t = b_{xx} + b b_x$ on the super-diagonal $\theta^1 = \theta^2$. Let us expand the super-field $b(x, t; \theta^1, \theta^2)$ in θ^1 and θ^2:

$$b = \beta(x, t) + \theta^1 \xi(x, t) + \theta^2 \eta(x, t) + \theta^1\theta^2 \gamma(x, t).$$

Hence from Eq. (19) we obtain the system for the components of b:

$$\begin{aligned} \beta_t &= -\gamma_x + \beta\beta_x, \qquad \xi_t = \eta_{xx} + (\beta\xi)_x, \qquad && \eta_t = -\xi_{xx} + (\beta\eta)_x, \\ & && \gamma_t = \beta_{xxx} + (\beta\gamma)_x - (\xi\eta)_x. \end{aligned} \tag{20}$$

We see that the fourth equation in (20) describing the evolution of γ is of KdV-type.

The $N = 2$ Burgers equation (19) is translation and scaling invariant w.r.t. a unique set of weights $[t] = -2$, $[b] = 1$. Equation (19) has the higher symmetry of weight $[s] = -3$,

$$b_s = \bigl(\partial_{\theta^1} b\, \partial_{\theta^2} b\bigr)_x + 2\bigl(b \partial_{\theta^1}\partial_{\theta^2} b\bigr)_x - \tfrac{4}{3} b_{xxx} + b^2 b_x. \tag{21}$$

The flow in (21) is an $N = 2$ modified KdV equation. A recursion operator for Eq. (19) is not found yet. Indeed, the nonlocalities for $N = 2$ systems do not originate from the conservation laws. The $N = 2$ extension of the Cole–Hopf transformation is also unknown. In the meantime we note that the database [19] contains the $N = 2$ extension

$$b_t = \alpha \cdot \partial_{\theta^1}\partial_{\theta^2} b_x + b_{xx}, \qquad \alpha \in \mathbb{R}, \tag{22}$$

of the heat equation $v_t = v_{xx}$; here we set $[b] = 1$ by definition, and we have $[t] = -2$. This extension has the symmetry of weight $[s] = -3$,

$$b_s = \bigl(\partial_{\theta^2} b \partial_{\theta^1} b\bigr)_x \cdot (1 + \alpha^2) - \tfrac{1}{2}\bigl(\partial_{\theta^1}\partial_{\theta^2} b\bigr)^2 \alpha + b_x \partial_{\theta^1}\partial_{\theta^2} b + \partial_{\theta^1}\partial_{\theta^2} b_{xx} + \tfrac{1}{2} b_x^2 \alpha.$$

The symmetry is nonlinear in b by axiom 4, and on the super-diagonal $\theta^1 = \theta^2$ it transforms to the $N = 0$ potential KdV equation

$$b_s = b_{xxx} + b_x^2, \tag{23}$$

which does not depend on α. We also note that any solution and any derivative of a solution for Eq. (22) is its symmetry since the equation at hand is linear.

3. The Boussinesq equation

In this section we discover a supersymmetric representation of the dispersionless Boussinesq equation; the supersymmetric system at hand admits two infinite commuting sequences of symmetries of constant differential order 2 which are generated by a weakly nonlocal recursion operator of differential order 1. We also embed the dispersionful Boussinesq equation in a one-parametric family of two-component super-systems, and we then find a supersymmetric analogue of the Boussinesq equation that depends on three parameters and contains the Boussinesq equation as a subsystem but does not retract to it for any values of the parameters.

3.1. **The dispersionless Boussinesq equation.** Now, we consider the two-component system

$$f_t = b\,\partial_\theta b, \qquad b_t = \partial_\theta f_x. \tag{24}$$

System (24) is a supersymmetric representation of the dispersionless Boussinesq equation

$$b_{tt} = \tfrac{1}{2}(b^2)_{xx}, \tag{25}$$

here b is the bosonic super-field. In what follows, we study the dispersionless Boussinesq equation (25) in the hydrodynamic representation. We solve it and construct a Hamiltonian deformation for this equation. Next, we transmit the properties of Hamiltonian symmetries for Eq. (25) onto supersymmetric representation (24).

3.1.1. *Two-component hydrodynamic-type dispersionless Boussinesq equation.* The bosonic two-component form of (25) is

$$b_t = c_x, \qquad c_t = bb_x. \tag{26}$$

Here we obviously have $c(x,t) = \partial_\theta f(x,t;\theta)$, that is, $f(x,t;\theta) = \phi(x,t) + \theta c(x,t)$. System (26) is an equation of hydrodynamic type.

Remark 12. The number of independent variables in (26) coincides with the number of unknown functions and equals two. Therefore, the system at hand is linearized by using the hodograph transformation $b(x,t)$, $c(x,t) \mapsto x(b,c)$, $t(b,c)$, see [16].

Indeed, we obtain the linear autonomous system

$$x_c = t_b, \qquad t_c = \frac{x_b}{b}, \tag{26a}$$

here c is the new time and b is the new spatial variable. The solution of hydrodynamic type system (26a) is found as follows. Cross-differentiating (26a), we single out the equation for $t = u(b,c)$:

$$u_{bb} = bu_{cc}. \tag{26b}$$

Denote by $\hat{u}(b,\xi)$ the Fourier transform of $u(b,c)$:

$$\hat{u}(b,\xi) = \int_{-\infty}^{+\infty} \exp(i\xi c) u(b,c)\,\mathrm{d}c, \qquad c \in \mathbb{R}.$$

Then the function $\hat{u}$ satisfies the equation $\hat{u}_{bb} + \xi^2 b \cdot \hat{u} = 0$. Recall that the Airy function

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(\lambda z + \frac{\lambda^3}{3}\right) \mathrm{d}\lambda$$

solves the Airy equation $y''(z) - zy(z) = 0$. Denote by $\mathrm{Bi}(z)$ a solution of the Airy equation which is linear independent of $\mathrm{Ai}(z)$. Therefore, we get

$$\hat{u}(b,\xi) = C_1(\xi) \cdot \mathrm{Ai}\left(-\xi^{2/3} b\right) + C_2(\xi) \cdot \mathrm{Bi}\left(-\xi^{2/3} b\right),$$

where C_1 and C_2 are arbitrary functions of ξ. Hence we obtain the general solution of Eq. (26b),

$$t = u(b,c) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}(b,\xi)\,\exp(-i\xi c)\,\mathrm{d}\xi.$$

Finally, the function $x(b,c)$ is found from (26a) in quadratures.

Now we construct a Hamiltonian deformation [3] for system (26). We obviously have

$$\begin{pmatrix} b \\ c \end{pmatrix}_t = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{bmatrix} \delta/\delta b \\ \delta/\delta c \end{bmatrix}_{(x,t)} \mathcal{H}(b,c), \tag{26}$$

where the density of the Hamiltonian $\mathcal{H}$ is $H = \frac{1}{6}b^3 + \frac{1}{2}c^2$. We proceed with a deformation

$$\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}_t = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{bmatrix} \delta/\delta w^1 \\ \delta/\delta w^2 \end{bmatrix}_{(x,t)} \bar{\mathcal{H}}(w^1,w^2)$$

of Eq. (26). We assume that the deformations of the dependent variables are

$$b = w^1 + \varepsilon W_1(\boldsymbol{w}) + \varepsilon^2 W_2(\boldsymbol{w}) + \cdots, \qquad c = w^2 + \varepsilon\Omega_1(\boldsymbol{w}) + \varepsilon^2\Omega_2(\boldsymbol{w}) + \cdots, \tag{27}$$

and we deform the Hamiltonian functional such that its density is

$$\bar{H} = H(\boldsymbol{w}) + \varepsilon H_1(\boldsymbol{w}) + \varepsilon^2 H_2(\boldsymbol{w}) + \cdots. \tag{28}$$

We also expand the fields $w^1(x,t;\varepsilon)$ and $w^2(x,t;\varepsilon)$ in ε:

$$w^i(x,t;\varepsilon) = \sum_{k=0}^{+\infty} w^i_k\,\varepsilon^k, \qquad i = 1,2.$$

Recall that equation (26) is in divergent form. Therefore, the Taylor coefficients w^i_k are termwise conserved, and from (27) we get the initial conditions $w^1_0 = b$, $w^2_0 = c$ and the recurrence relations for the conserved densities.

Now we truncate expansions (27), (28) to polynomials of sufficiently large degrees. By using the homogeneity reasonings, in view of $[\varepsilon] = -3$, we then reduce the deformation problem to an algebraic system for the undetermined coefficients in these expansions. We solve the algebraic system by using the program Crack [18] and finally obtain the deformation

$$\begin{aligned} b &= w^1 + \varepsilon w^1 w^2, \\ c &= w^2 + \tfrac{1}{3}\varepsilon\big(w^1\big)^3 + \varepsilon\big(w^2\big)^2 + \tfrac{1}{3}\varepsilon^2\big(w^2\big)^3, \\ \bar{H} &= \tfrac{1}{6}\big(w^1\big)^3 + \tfrac{1}{2}\big(w^2\big)^2 + \tfrac{1}{6}\varepsilon\big(w^2\big)^3. \end{aligned}$$

The initial terms in the two sequences of the conserved densities are

$$\begin{aligned} &w^1_0 = b, \qquad && w^1_1 = -bc, \qquad && w^1_2 = 2bc^2 + \tfrac{1}{3}b^4, \qquad && \cdots, \\ &w^2_0 = c, && w^2_1 = -c^2 - \tfrac{1}{3}b^3, && w^2_2 = \tfrac{5}{3}c^3 + \tfrac{5}{3}b^3 c, && \cdots. \end{aligned} \tag{29}$$

Taking into account that the Hamiltonian operator $\left(\begin{smallmatrix} 0 & D_x \\ D_x & 0 \end{smallmatrix}\right)$ maps gradients of conservation laws for Eq. (26) to its symmetries, see [9], we thus conclude that (29) describes two infinite sequences of Hamiltonians for (26). They determine infinitely many contact symmetry flows, which are local w.r.t. b and c. Moreover, each flow lies in the image of D_x by construction, and hence they induce local symmetry transformations of the super-variable f such that $c = \partial_\theta f$. We emphasize that we did not even need a recursion operator for (26) to obtain the contact symmetry flows and prove their locality.

Remark 13. The problem of constructing integrable deformations for homogeneous (super-)PDE is another practical application of the CRACK solver [18] for large overdetermined systems of nonlinear algebraic equation. This application has not been previously considered within the framework of [8, 18, 19].

3.1.2. *Supersymmetric representation of equation* (25). Now we discuss the Hamiltonian and supersymmetric properties of system (24). It is homogeneous w.r.t. multiply-defined weights; we let the primary set be $[f]=[b]=1$, $[t]=-1\frac{1}{2}$, $[x]=-1$. With respect to this set, system (24) admits two infinite sequences of Hamiltonian symmetry flows of unbounded weights -1, $-1\frac{1}{2}$, $-2\frac{1}{2}$, -3, ..., and constant differential order 2. These two sequences start with the flows

$$\begin{pmatrix} f_x \\ b_x \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta b\, b^2 + \partial_\theta f\, f_x \\ \partial_\theta f_x\, b + \partial_\theta f\, b_x \end{pmatrix} \mapsto \cdots, \qquad \begin{pmatrix} f_t \\ b_t \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta f\, \partial_\theta b\, b + \frac{1}{2} f_x b^2 \\ \partial_\theta f_x\, \partial_\theta f\, + \frac{1}{2} b_x b^2 \end{pmatrix} \mapsto \cdots. \tag{30}$$

The skew-adjoint operator $A = \left(\begin{smallmatrix} 0 & \partial_\theta \\ -\partial_\theta & 0 \end{smallmatrix}\right)$ is a Hamiltonian structure for the symmetries in (30), and, by Sec. 3.1.1, all the flows possess the Hamiltonian functionals. There are two Casimirs $H_0^{(1)} = b$, $H_0^{(2)} = \partial_\theta f$ for Eq. (24). From (29) we obtain the higher Hamiltonians with the densities

$$\begin{aligned} H_1^{(1)} &= b\partial_\theta f, & H_2^{(1)} &= \tfrac{1}{12} b^4 + \tfrac{1}{2} b(\partial_\theta f)^2, & &\cdots, \\ H_1^{(2)} &= \tfrac{1}{2}(\partial_\theta f)^2 + \tfrac{1}{6} b^3, & H_2^{(2)} &= \tfrac{1}{6}(\partial_\theta f)^3 + \tfrac{1}{6} b^3 \partial_\theta f, & &\cdots. \end{aligned} \tag{29$'$}$$

Let us construct the nonlocal recursion that maps the symmetries in (30). To this end, we introduce the nonlocality w such that $w_t = b^2/2$ and $\partial_\theta w = f$. We further let the variable v be such that $v_t = \partial_\theta f$, $v_x = b$. Then we obtain the nonlocal recursion

$$\mathcal{R}_{[1\frac{1}{2}]} = \begin{pmatrix} \partial_\theta b\, b\, V + \frac{1}{2} b^2\, \partial_\theta V + \frac{3}{4} \partial_\theta f\, F + \frac{3}{4} f_x\, W \\ \partial_\theta f_x\, V + \frac{1}{2} b\, \underline{\partial_\theta F} + \frac{3}{4} \partial_\theta f\, B + \frac{3}{4} b_x\, W \end{pmatrix} \tag{31}$$

of differential order 1. Indeed, the order of the flow $b_{s_{i+1}}$ equals the order of f_{s_i} plus 1 owing to the presence of the underlined differential operator $\partial_\theta = D_\theta + \theta D_x$ in $\mathcal{R}^b$.

Proposition. *Recursion* (31) *proliferates the symmetries of dispersionless Boussinesq equation* (24) *of constant differential order* 2. *The initial terms of the symmetry sequences are given in* (30), *and their Hamiltonians are* (29′).

The assertion follows from the results of Sec. 3.1.1.

We note that Hamiltonian symmetries (30) are an example of infinitely many flows that are not obtained by a recurrence multiplication scheme, see p. 7 for definition, although their differential order is constant. Indeed, they are contact in the coordinates b, c and are of second order w.r.t. f and b.

3.2. **The Boussinesq equation with dispersion and dissipation.** A supersymmetric representation of the dispersionless Boussinesq equation is embedded in the one-parametric family of supersymmetric systems

$$f_t = b\partial_\theta b + \partial_\theta b_{xx} - \alpha f_{xx}, \qquad b_t = \partial_\theta f_x + \alpha b_{xx}, \qquad \alpha \in \mathbb{R}. \tag{32a}$$

By definition, put $c = \partial_\theta f$. Then from (32a) we get the $N=0$ bosonic system

$$c_t = bb_x + b_{xxx} - \alpha c_{xx}, \qquad b_t = c_x + \alpha b_{xx}, \qquad \alpha \in \mathbb{R}. \tag{32b}$$

The above systems are homogeneous w.r.t. the weights $[b]=2$, $[f]=2\frac{1}{2}$, $[c]=3$, and $[t]=-2$. We see that Eq. (32a) has no nontrivial bosonic limit. Indeed, one can not set $f \equiv 0$ such that the fermionic equation remains consistent unless $b = \text{const}$.

If $\alpha = 0$, then (32) is the Boussinesq equation with dispersion. The terms involving α describe the dissipation. There is a well-known recursion operator of weight $[s_R] = -3$ for the Boussinesq equation without dissipation [6, 14]. The recursion for the fermionic component in (32a) is then, roughly speaking, the component corresponding to c in the recursion for (32b) conjugated by ∂_θ.

If α is non-zero, then system (32b) is not reduced to a scalar fourth order equation. In this case, the system is translation invariant and for all $\alpha \in \mathbb{R}$ it admits symmetries of weights $[s] = -4$ and $[s] = -5$; these flows are rather large and therefore omitted.

3.3. **The multi-parametric Boussinesq-type equation.** We finally construct the Boussinesq-type system using supersymmetric representation (32a) for the Boussinesq equation. From the database [19] we obtain

$$\begin{aligned} f_t &= \alpha\beta f b - \alpha\gamma b\partial_\theta b - \gamma^2\partial_\theta b_{xx} - \beta\gamma f_{xx}, \\ b_t &= \alpha\beta b^2 + \beta^2\partial_\theta f_x + \beta\gamma b_{xx}, \end{aligned} \tag{33}$$

here α, β, $\gamma \in \mathbb{R}$. This system is a dilaton analogue of the Boussinesq equation $b_{tt} = b_{xxxx} + (bb_x)_{xx}$ but does not retract to it for any value of the parameters. Similarly to Eq. (32a), system (33) does not have a nontrivial bosonic limit at $f \equiv 0$.

The translation invariance of Eq. (33) is obvious; we recall that $[x] = -1$ and $[t] = -2$. Also, Eq. (33) admits the higher symmetry

$$\begin{aligned} f_s = &-\partial_\theta bb_{xx}\gamma^3 + \partial_\theta b_x b_x\gamma^3 + \partial_\theta b_x\partial_\theta f\beta\gamma^2 - \partial_\theta f_x\partial_\theta b\beta\gamma^2 - \partial_\theta f_x f\beta^2\gamma + \partial_\theta f f_x\beta^2\gamma \\ &- b_{xx}f\beta\gamma^2 + b_x f_x\beta\gamma^2, \\ b_s = &-\partial_\theta b f_x\beta^2\gamma + \partial_\theta b_x f\beta^2\gamma + \partial_\theta b_x\partial_\theta b\beta\gamma^2 + f_x f\beta^3 \end{aligned}$$

such that $[s] = -4$, and there is the supersymmetry

$$\begin{aligned} f_{\bar s} &= \partial_\theta b f_x\beta\gamma^2 - \partial_\theta b_x f\beta\gamma^2 - \partial_\theta b_x\partial_\theta b\gamma^3 - b_x^2\gamma^3 - (\partial_\theta f)^2\beta^2\gamma - f_x f\beta^2\gamma - 2\partial_\theta f b_x\beta\gamma^2, \\ b_{\bar s} &= \partial_\theta bb_x\beta\gamma^2 + \partial_\theta f\partial_\theta b\beta^2\gamma + \partial_\theta f f\beta^2\gamma + b_x f\beta^2\gamma, \end{aligned}$$

here the weight of the supersymmetric parameter $\bar s$ is $[\bar s] = -3\frac{1}{2}$. No more symmetries exist for Eq. (33) if $[s] \geq -7$ and if the parity of s is arbitrary. Quite strangely, there are no conservation laws for Eq. (33) within the weights not greater than 11; hence no nonlocalities were constructed and no recursion is currently known for system (33).

Conclusion. In this paper, we obtained supersymmetric representations for purely bosonic equations of mathematical physics: system (9) is a representation for the Burgers equation (10), system (24) represents the dispersionless Boussinesq equation (25), and system (32a) corresponds to the Boussinesq equation with dispersion if $\alpha = 0$. These equations were solved by using various geometric techniques, and recursion operators for the symmetry algebras of their supersymmetric representations were obtained.

We believe that the supersymmetric extensions (9) and (33) are physically relevant. Indeed, the original evolutionary equations are obtained by using approximations related with the physics of the process, and the equations at hand may prove useful in description of super-fluid phenomena.

We note that these boson+fermion representations are remarkable by themselves. Indeed, the roles of the independent variables x and t are swapped in system (9): x is the time and t is the spatial coordinate in Eq. (10). The dispersionless Boussinesq system (24) admits two recursive sequences (30) of Hamiltonian symmetries whose differential orders are always 2, while the differential order of the nonlocal recursion (31) is strictly positive. The Boussinesq-type system (33) is multi-parametric and contains the Boussinesq equation with dispersion as a component but can not be reduced to it at any values of the parameters. We finally note that all the Boussinesq-type systems (24), (32a), (33), as well as representation (9) for the Burgers equation, do not have bosonic limits at $f \equiv 0$. We emphasize that supersymmetrizations (32a) and (33) are not the previously known $N \geq 1$ Boussinesq equations (see Refs. 11 in [13] for the $N = 2$ case), which are constructed by using $N \geq 1$ super-conformal extensions of the W_3-algebra. Moreover, we get apparently $N = 0$ Burgers and KdV equations, see (19) and (23), as $N = 1$ diagonal reductions of true $N = 2$ flows; hence the 'bosonic' equations are in fact already $N = 1$. This argument hints why the theory of $N = 1$ extensions for classical integrable systems is often less challenging than $N = 2$.

From the example of Burgers-type system (14) it follows that the 'direct $N \geq 1$ supersymmetrization' [12] based on replacing the derivatives with super-derivatives in a PDE is not a unique way to obtain supersymmetric extensions. We now see that the supersymmetric analogues can be $N = 0$ as well, containing the fermionic fields but no super-derivatives. We obtain three possible cases of the supersymmetric generalizations.

(1) We construct a new super-field that combines the old fields and new anticommuting variables, and the new field satisfies a manifestly supersymmetric $N \geq 1$ equation, see superKdV (2), for example.
(2) Analogously, we introduce a super-field, and the initial system results to an $N = 0$ equation. The geometric structures of the super-system are then inherited from the new equation by routine expansions in the super-variables, see (14) and (16).
(3) The new field involves a generalization of super-variables that anticommutes with the fermionic fields but not with itself, and the transformed system is an equation on an associative algebra [15], see (14) and Eq. (16′) on p. 11.

We conclude that the link between supersymmetric systems and equations on associative algebras produces a nontrivial generalization of supersymmetry and, *vice versa*, meaningful examples of integrable flows on the algebras may originate from known supersymmetric models.

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