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On weakly non-local, nilpotent, and super-recursion operators for $N = 1$ super-equations

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On weakly non-local, nilpotent, and super-recursion operators for $N = 1$ super-equations

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Abstract

We consider nonlinear, scaling-invariant $N = 1$ boson+fermion supersymmetric systems whose right-hand sides are homogeneous differential polynomials and satisfy some natural assumptions. We select the super-systems that admit infinitely many higher symmetries generated by recursion operators; we further restrict ourselves to the case when the dilaton dimensions of the bosonic and fermionic super-fields coincide and the weight of the time is half the weight of the spatial variable. We discover five systems that satisfy these assumptions; one system is transformed to the purely bosonic Burgers equation. We construct local, nilpotent, triangular, weakly non-local, and super-recursion operators for their symmetry algebras.
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Introduction. We consider the problem of a complete description of $N = 1$ nonlinear, scaling invariant evolutionary super-equations $\{f_t = \phi^f,\ b_t = \phi^b\}$ that admit infinitely many symmetries $\{f_s = F,\ b_s = B\}$ proliferated by recursion operators $\mathcal{R}$; here b is the bosonic super-field and f is the fermionic super-field. The axioms for selecting $N = 1$ nonlinear homogeneous polynomial evolutionary systems with higher symmetries were suggested [8] by V. V. Sokolov and A. S. Sorin; the axioms are discussed in [2].

By construction, the equations are scaling invariant: their right-hand sides are differential polynomials homogeneous w.r.t. a set of (half-)integer weights $[\theta] \equiv -\frac{1}{2}$, $[x] \equiv -1$, $[t] < 0$, $[f]$, $[b] > 0$; we also assume that the negative weight $[s]$ is (half-)integer. Here we denote by θ the super-variable and we put $\partial_\theta \equiv D_\theta + \theta\, D_x$ such that $\partial_\theta^2 = D_x$; here D_θ and D_x are the total derivatives w.r.t. θ and x, respectively. All notions and notation follow [4], see also [2] for details.

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In this paper, we investigate the properties of systems such that the weight of the time t is $[t] = -\frac{1}{2}$. We also assume $[f] = [b] = \frac{1}{2}$ (the weights may not be uniquely defined).

The first version of SsTools package [7] for REDUCE was used for finding the systems that satisfy the above axioms and possess higher symmetries under the bound $-5 \leqslant [s] \leqslant -\frac{1}{2}$. Five systems were thus discovered, see Table I below. Later, we used the second version of SsTools [3] for symmetry analysis of the super-systems in [8] and for constructing conservation laws and recursion operators for their symmetry algebras. The method of Cartan forms [4] for the recursion operators was applied. Within this approach, the recursions are regarded as symmetries of the linearized equations. Namely, we 'forget' the internal structure of the symmetry flows $f_s = F$, $b_s = B$ and operate with F and B as we do with the components of solutions of the linearized equations. The expressions $\mathcal{R} = R(F, B)$ are the recursion operators if each $\mathcal{R}$ satisfies the linearized equation again and if they are linear w.r.t. F, B, and their derivatives.

Let us introduce some notation. Assume $\mathcal{R}$ is a recursion for an equation and consider the symbol $^{\text{layers}}_{\text{ord}}\mathcal{R}^{\sharp}_{\text{weight}}$. The subscripts 'ord' and 'weight' denote the differential order and the weight of the recursion $\mathcal{R}$, respectively, and the superscript 'layers' (if non-empty) indicates the required number of layers of the nonlocal variables assigned to conservation laws. The symbol '$\sharp$' denotes the number of recursions for a given differential order, weight, and the nonlocalities. Further, we denote by L the local recursion operators, by N the nonlocal or weakly non-local [1, 6] recursions, the symbol Z denotes a nilpotent recursion whose powers equal zero except for a finite set, and Σ is a super-recursion that swaps the parities of the flows.

Now we list the five new super-equations and indicate their recursions. The weights of the recursion operators are calculated w.r.t. the standard values $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$.

(1)	$\begin{cases} f_t = \partial_\theta b, \\ b_t = b^2 + \partial_\theta f \end{cases}$	$^1_1 N^1_{-1}$		
(5)	$\begin{cases} f_t = \partial_\theta b + fb, \\ b_t = \partial_\theta f \end{cases}$	$^2_0 N^1_{-1\frac{1}{2}}$, $^2_{\frac{1}{2}} N^1_{-2}$, $^2_{\frac{1}{2}} N^1_{-2\frac{1}{2}}$, $^2_{\frac{1}{2}} N^1_{-3}$		
(6)	$\begin{cases} f_t = -\alpha\, fb, \\ b_t = b^2 + \partial_\theta f \end{cases}$	$\alpha = 2:$ $^1_{\frac{1}{2}} L^1_{-2}$, $^1_{\frac{1}{2}} L^1_{-2\frac{1}{2}}$, $^0_0 Z^1_{-2}$, $^0_0 \Sigma^1_{-2}$, $^0_0 \Sigma^1_{-2\frac{1}{2}}$, $^0_0 \Sigma^1_{-2\frac{1}{2}}$;		
		$\alpha = 1:$ $^0_0 Z^1_{-2\frac{1}{2}}$, $^1_{\frac{1}{2}} Z^1_{-3}$; $\alpha = 4:$ $^1_1 L^1_{-3\frac{1}{2}}$		

Table I.

It turns out that these equations exhibit practically the whole variety of properties that superPDE of mathematical physics possess. Let us discuss the properties of the equations present in Table I in more detail.

1. The Burgers equation.
First we construct an $N = 1$ supersymmetric representation of the Burgers equation and investigate its properties. We consider the system

$$f_t = \partial_\theta b, \qquad b_t = b^2 + \partial_\theta f. \qquad (1)$$

There is a unique set of weights $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$ in this case. Hence we conclude that the above system precedes the invariance w.r.t. the translation along x. Equation (1) admits the continuous sequence (3) of higher symmetries $f_s = \phi^f$, $b_s = \phi^b$ at all (half-)integer weights $[s] \leq -\frac{1}{2}$. Also, there is the continuous sequence (4) of supersymmetries for Eq. (1) at all (half-)integer weights $[\bar{s}] \leq -\frac{1}{2}$ of the fermionic 'time' $\bar{s}$.

System (1) is obviously reduced to the purely bosonic Burgers equation $b_x = b_{tt} - 2bb_t$. We emphasize that the role of the independent coordinates x and t is reversed w.r.t. the standard interpretation of t as the time and x as the spatial variable. The Cole–Hopf substitution $b = -u^{-1}u_t$ from the heat equation $u_x = u_{tt}$ is thus the solution for the bosonic component of (1).

Further, we introduce the bosonic nonlocality w of weight $[w] = 0$ by the rules $\partial_\theta w = -f$, $w_t = -b$. The variable w is a potential for both fields f and b. The nonlocality satisfies the potential Burgers equation $w_x = w_{tt} + w_t^2$ such that the formula $w = \ln u$ gives the solution; the relation $f = -\partial_\theta w$ determines the fermionic component in system (1).

Now we extend the set of dependent variables f, b, and w by the symmetry generators F, B, and W that satisfy the linearized relations upon the flows of the initial super-fields, respectively. In this setting, we obtain the recursion

$$\mathcal{R}_{[1]} = \begin{pmatrix} F_x - \partial_\theta f\, F + f_x\, W \\ B_x - \partial_\theta f\, B + b_x\, W \end{pmatrix} \iff R = \begin{pmatrix} D_x - \partial_\theta f + f_x\, \partial_\theta^{-1} & 0 \\ b_x\, \partial_\theta^{-1} & D_x - \partial_\theta f \end{pmatrix} \tag{2}$$

of weight $[s_R] = -1$. In agreement with [1], the above recursion is weakly non-local [6]. We recall that a recursion operator R is *weakly non-local* if each nonlocality ∂_θ^{-1} is preceded with a (shadow [4] of a nonlocal) symmetry φ_α and is followed by the gradient ψ_α of a conservation law: $R = \text{local part} + \sum_\alpha \varphi_\alpha \cdot \partial_\theta^{-1} \circ \psi_\alpha$. From [1] it follows that this property is automatically satisfied by all recursion operators which are constructed using one layer of the nonlocal variables assigned to conservation laws.

Recursion (2) generates two sequences of higher symmetries for system (1):

$$\begin{pmatrix} f_t \\ b_t \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta b_x - \partial_\theta f \partial_\theta b - f_x b \\ \partial_\theta f_x - (\partial_\theta f)^2 - b^2 \partial_\theta f + bb_x \end{pmatrix} \mapsto \cdots, \quad \begin{pmatrix} f_x \\ b_x \end{pmatrix} \mapsto \begin{pmatrix} f_{xx} - 2\partial_\theta f f_x \\ b_{xx} - 2\partial_\theta f b_x \end{pmatrix} \mapsto \cdots. \tag{3}$$

Also, recursion (2) produces two infinite sequences of supersymmetries for (1):

$$\begin{pmatrix} \partial_\theta f \\ \partial_\theta b \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta f_x - (\partial_\theta f)^2 - f_x f \\ \partial_\theta b_x - \partial_\theta f \partial_\theta b - b_x f \end{pmatrix} \mapsto \cdots, \quad \begin{pmatrix} f\partial_\theta b - b\, \partial_\theta f + b_x \\ b\partial_\theta b - f\, \partial_\theta f + f_x - fb^2 \end{pmatrix} \mapsto \cdots. \tag{4}$$

Remark 1. System (1) is not a supersymmetric extension of the Burgers equation; it is a supersymmetric representation of the Burgers equation. However, symmetries (3) and (4) are *not* reduced to the purely bosonic (x, t)-independent symmetries [5] of the Burgers equation (particularly, owing to the interchanged role of the variables x and t). We finally recall that the Burgers equation has infinitely many higher symmetries that depend explicitly on the base coordinates x, t but exceed the set of axioms [2] we use.

Two supersymmetric generalizations ($N = 0$ and $N = 2$) of the Burgers equation are constructed in [2]. The $N = 0$ extension relates it with integrable flows on associative algebras. The $N = 2$ Burgers equation contains a KdV-type component and admits an $N = 2$ modified KdV equation as a symmetry flow.

2. A system with nonlocal recursions.

The second system,

$$f_t = \partial_\theta b + fb, \qquad b_t = \partial_\theta f, \tag{5}$$

is also homogeneous w.r.t. a unique set of weights $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$. Similarly to the supersymmetric representation (1) for the Burgers equation, Eq. (5) admits symmetries (f_s, b_s) for all weights $[s] \leq -\frac{1}{2}$.

We conjecture that system (5) has only one conservation law that defines the fermionic variable w of weight 0 by $w_t = f$, $\partial_\theta w = b$. Then, many nonlocal conservation laws and hence many new variables appear. We use the fermionic variable v whose weight $[v] = \frac{3}{2}$ is minimal: we set $v_t = \partial_\theta b \cdot wfb + f_x wf$ and $\partial_\theta v = -\partial_\theta b \cdot fb + \partial_\theta f \cdot \partial_\theta b \cdot w + b_x wf$. Now, there are nontrivial solutions to the determining equations for recursion operators. First, we obtain the recursion of zero differential order with nonlocal coefficients:

$$R_{[-1\frac{1}{2}]} = \begin{pmatrix} -\partial_\theta b \cdot wfB + wvF + v \cdot B \\ \partial_\theta b \, wfF - vF + vw \cdot B \end{pmatrix}.$$

Also, we get a nonlocal operator with nonlocal coefficients,

$$R_{[-2]} = \begin{pmatrix} \partial_\theta b \, Vw - \partial_\theta f \, \partial_\theta B \, wf - \partial_\theta f \, \partial_\theta b \, Wf + \partial_\theta f \, \partial_\theta b \, Fw + \partial_\theta f \, V + Vwfb \\ \partial_\theta B \, \partial_\theta b \, wf + \partial_\theta b \, V - \partial_\theta b \, Fwfb + \partial_\theta f \, \partial_\theta b \, Vwf + \partial_\theta f \, Vw - Vfb \end{pmatrix}.$$

The coefficients of the recursions found for $[s_R] = -2\frac{1}{2}$ and $[s_R] = -3$ are also nonlocal.

3. A triplet of super-systems.

Finally, we consider the three systems

$$f_t = -\alpha fb, \qquad b_t = b^2 + \partial_\theta f \tag{6}$$

which differ by the values $\alpha = 1$, 2, and 4 of the parameter α and therefore exhibit rather different properties. The weights for the above equation are multiply defined, and we choose the tuple $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$ to be the primary 'reference system.'

Case $\alpha = 2$. First, we fix $\alpha = 2$ and consider Eq. (6): we get $f_t = -2fb$, $b_t = b^2 + \partial_\theta f$. The weights for symmetries are $[s] = -\frac{1}{2}$, $[s] = -1$, and then Eq. (6) admits a continuous chain of symmetry flows for all (half-)integer weights $[s] \leq -2\frac{1}{2}$. Surprisingly, no nonlocalities are needed to construct the recursion operators, although there are many conservation laws for this system. We obtain purely local recursion operators $\mathcal{R}$ that proliferate the symmetries: $\varphi = (F, B) \mapsto \varphi' = \mathcal{R}$ for any φ. The recursion

$$\mathcal{R}_{[-2]} = \begin{pmatrix} \frac{11}{2}\partial_\theta F \, \partial_\theta f \, f + 11\partial_\theta F \, fb^2 + \frac{3}{2}(\partial_\theta f)^2 F + 3\partial_\theta f \, Fb^2 + \frac{1}{2}f_x Ff \\[2ex] 11\partial_\theta B \, fb^2 + 8\partial_\theta b \, Fb^2 + 22\partial_\theta b \, fBb + 7(\partial_\theta f)^2 B + \\ 14\partial_\theta f \, Bb^2 + \frac{11}{2}\partial_\theta f \, \partial_\theta B \, f + \frac{5}{2}\partial_\theta f \, \partial_\theta b \, F + \frac{1}{2}b_x Ff + f_x Fb + 5f_x fB \end{pmatrix},$$

of weight $[s_R] = -2$ is triangular since R^f does not contain B. Also, we obtain the recursion of weight $2\frac{1}{2}$; its components are

$$\mathcal{R}^f_{[-2\frac{1}{2}]} = -2\partial_\theta b \, Ffb^2 - \partial_\theta F \, \partial_\theta f \, fb - \partial_\theta F \, fb^3 - \frac{1}{2}f_x Ffb - 2\partial_\theta f \, fBb^2,$$

$$\mathcal{R}^b_{[-2\frac{1}{2}]} = \partial_\theta B \, fb^3 + \partial_\theta b \, Fb^3 + \partial_\theta b \, fBb^2 + \frac{1}{8}\partial_\theta f_x Ff +$$
$$+ \frac{1}{2}\partial_\theta Fb^4 + \frac{1}{2}\partial_\theta F(\partial_\theta f)^2 + \partial_\theta F \, \partial_\theta f \, b^2 + \frac{1}{8}\partial_\theta F \, f_x f + (\partial_\theta f)^2 Bb +$$
$$+ \partial_\theta f \, Bb^3 + \partial_\theta f \, \partial_\theta B \, fb + \partial_\theta f \, \partial_\theta b \, Fb + \partial_\theta f \, \partial_\theta b \, fB + \frac{3}{8}\partial_\theta f \, F_x f +$$
$$+ \frac{1}{4}\partial_\theta f \, f_x F + \frac{1}{2}b_x Ffb + \frac{1}{2}F_x fb^2 + \frac{1}{4}f_x Fb^2 + \frac{1}{2}f_x fBb.$$

Further, we get a triangular nilpotent operator of weight -3 such that $\mathcal{R}^f_{[-3]} = 0$ and $\mathcal{R}^b_{[-3]} = (\partial_\theta f)^3 f F + 6(\partial_\theta f)^2 f b^2 F + 12\partial_\theta f\, f b^4 F + 8 f b^6 F$. The above recursion is a recurrence relation [2] which is well-defined for all symmetries of Eq. (6). Another local recursion for $[s] = -3$ is huge and therefore omitted.

For $\alpha = 2$, system (6) admits at least three super-recursions ${}^t(R^f, R^b)$ such that the parities of R^f and R^b are opposite to the odd parity for f (and hence for F) and to the even parity of b and B. This property is possible owing to the presence of the *odd* variable s_R. The triangular zero-order super-recursions are $\bar{\mathcal{R}}^f_{[-2]} = 4\partial_\theta f\, F f b + 8 F f b^3$, $\bar{\mathcal{R}}^b_{[-2]} = -4\partial_\theta b\, F f b + 2(\partial_\theta f)^2 F + 6\partial_\theta f\, F b^2 + 4\partial_\theta f\, f B b - f_x F f + 4 F b^4 + 8 f B b^3$ and

$$\bar{\mathcal{R}}_{[-2\frac{1}{2}]} = \begin{pmatrix} -\partial_\theta f\, f_x F - 2 f_x F b^2 \\ \partial_\theta b\, f_x F - \partial_\theta f\, b_x F + \partial_\theta f\, f_x B - 2 b_x F b^2 + 2 f_x B b^2 \end{pmatrix}$$

for weights $[s_R] = -2$ and $[s_R] = -2\frac{1}{2}$, respectively; the third super-recursion found for $[s_R] = -2\frac{1}{2}$ is very large. Quite naturally, system (6) has infinitely many supersymmetries if $\alpha = 2$.

Case $\alpha = 1$. For $\alpha = 1$ from (6) we obtain the system $f_t = -fb$, $b_t = b^2 + \partial_\theta f$. The default set of weights is the same as above: $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, and $[x] = -1$. The sequence of symmetries is not continuous and starts later than for the chain in the case $\alpha = 2$. We find out that there are symmetry flows if either $[s] = [t] = -\frac{1}{2}$ (the equation itself), $[s] = [x] = -1$ (the translation along x), or $[s] \leq -3\frac{1}{2}$ such that a continuous chain starts for all (half-)integer weights $[s]$.

Similarly to the previous case, no nonlocalities are needed to construct the recursions, which therefore are purely local. The recursion operator $\mathcal{R}^f_{[-2\frac{1}{2}]} = 0$, $\mathcal{R}^b_{[-2\frac{1}{2}]} = (\partial_\theta f)^2 F f + 3\partial_\theta f\, F f b^2 + \frac{9}{4} F f b^4$ of maximal weight $[s_R] = -2\frac{1}{2}$ is nilpotent: $\mathcal{R}^2 = 0$. For the succeeding weight $[s_R] = -3$, we obtain a nilpotent local recursion with components

$$\mathcal{R}^f_{[-3]} = \tfrac{5}{3}\partial_\theta F\,(\partial_\theta f)^2 f + \tfrac{5}{2}\partial_\theta F\,\partial_\theta f\, f b^2 - \tfrac{5}{3}(\partial_\theta f)^3 F - \tfrac{5}{2}(\partial_\theta f)^2 F b^2 +$$
$$+ 5\partial_\theta f\,\partial_\theta b\, F f b + \tfrac{20}{3}\partial_\theta f\, f_x F f + \tfrac{15}{2} f_x F f b^2,$$
$$\mathcal{R}^b_{[-3]} = \partial_\theta f_x\, F f b - \tfrac{105}{2}\partial_\theta F\,\partial_\theta b\, f b^2 - \tfrac{160}{3}\partial_\theta F\,\partial_\theta f\,\partial_\theta b\, f + 11\partial_\theta F\, f_x f b +$$
$$+ \tfrac{5}{3}(\partial_\theta f)^2\partial_\theta B f + \tfrac{5}{3}(\partial_\theta f)^2\partial_\theta b\, F + \tfrac{5}{2}\partial_\theta f\,\partial_\theta B\, f b^2 + \tfrac{5}{2}\partial_\theta f\,\partial_\theta b\, F b^2 -$$
$$- 55\partial_\theta f\,\partial_\theta b\, f B b + \tfrac{17}{3}\partial_\theta f\, b_x F f + \partial_\theta f\, f_x f B + \tfrac{23}{2} b_x F f b^2 + \tfrac{183}{2} f_x f B b^2.$$

It generates symmetries of system (6); the differential order of $\mathcal{R}_{[-3]}$ is positive.

Case $\alpha = 4$. Finally, let $\alpha = 4$; then system (6) acquires the form $f_t = -4fb$, $b_t = b^2 + \partial_\theta f$. Again, the basic set of weights is $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$, and system (6) admits the symmetries (f_s, b_s) such that their weights are $[s] = -\frac{1}{2}$, -1 or $[s] \leq -3\frac{1}{2}$ w.r.t. the basic set. This situation coincides with the case $\alpha = 1$. Again, no

nonlocalities are needed for constructing the recursion of minimal weight $[s_R] = -3\frac{1}{2}$:

$$\mathcal{R}^f_{[-3\frac{1}{2}]} = -12\partial_\theta b\, Ffb^4 - \partial_\theta F\,(\partial_\theta f)^2 fb - 4\partial_\theta F\,\partial_\theta f\, fb^3 - 3\partial_\theta F\, fb^5 -$$
$$- 4(\partial_\theta f)^2 fBb^2 - 4\partial_\theta f\,\partial_\theta b\, Ffb^2 - \tfrac{2}{3}\partial_\theta f\, f_x Ffb - 12\partial_\theta f\, fBb^4 - 2f_x Ffb^3,$$

$$\mathcal{R}^b_{[-3\frac{1}{2}]} = 3\partial_\theta B fb^5 + 3\partial_\theta b\, Fb^5 + 9\partial_\theta b\, fBb^4 + \tfrac{1}{9}\partial_\theta f_x\,\partial_\theta f\, Ff - \tfrac{1}{3}\partial_\theta f_x\, Ffb^2 + \tfrac{3}{4}\partial_\theta F\, b^6 +$$
$$+ \partial_\theta F\,\partial_\theta b\, fb^3 + \tfrac{1}{4}\partial_\theta F\,(\partial_\theta f)^3 + \tfrac{5}{4}\partial_\theta F\,(\partial_\theta f)^2 b^2 + \tfrac{7}{4}\partial_\theta F\,\partial_\theta f\, b^4 + \partial_\theta F\,\partial_\theta f\,\partial_\theta b\, fb +$$
$$+ \tfrac{5}{18}\partial_\theta F\,\partial_\theta f\, f_x f + \tfrac{1}{2}\partial_\theta F\, f_x fb^2 + (\partial_\theta f)^3 Bb + 4(\partial_\theta f)^2 Bb^3 + (\partial_\theta f)^2 \partial_\theta B\, fb +$$
$$+ (\partial_\theta f)^2 \partial_\theta b\, Fb + (\partial_\theta f)^2 \partial_\theta b\, fB + \tfrac{2}{9}(\partial_\theta f)^2 F_x f + \tfrac{1}{6}(\partial_\theta f)^2 f_x F + 3\partial_\theta f\, Bb^5 +$$
$$+ 4\partial_\theta f\,\partial_\theta B\, fb^3 + 4\partial_\theta f\,\partial_\theta b\, Fb^3 + 10\partial_\theta f\,\partial_\theta b\, fBb^2 + \tfrac{2}{3}\partial_\theta f\, b_x Ffb + \partial_\theta f\, F_x fb^2 +$$
$$+ \tfrac{2}{3}\partial_\theta f\, f_x Fb^2 + \tfrac{5}{3}\partial_\theta f\, f_x fBb + 2b_x Ffb^3 + F_x fb^4 + \tfrac{1}{2}f_x Fb^4 + f_x fBb^3.$$

No nilpotent recursion operators were found for system (6) if $\alpha = 4$.

Remark 2. We discovered that an essential part of recursion operators for supersymmetric PDE are nilpotent. At present, it is not clear how the nilpotent recursion operators contribute to the integrability of supersymmetric systems and what invariants they describe or symptomize. Further, we emphasize that this property does not always originate from the rule '$f \cdot f = 0$', but this is an immanent feature of the symmetry algebras. More generally, the nilpotent recursions are quite natural in the bosonic sector, too. We have

Example (I. S. Krasil'shchik, private communication). Consider a system of linear ordinary differential equations $\dot{\boldsymbol{x}} = A(t)\,\boldsymbol{x}$. Then any nilpotent constant matrix R that commutes with the matrix A is a recursion.

It would be of interest to construct an equation $\mathcal{E}$ that admits nilpotent differential recursion operators $\{R_1, \ldots \mid R_i^{n_i} = 0\}$ which generate an infinite sequence of symmetries $\varphi,\, R_{i_1}(\varphi),\, R_{i_2} \circ R_{i_1}(\varphi),\, \ldots$ for $\mathcal{E}$. Here we assume that at least two operators (without loss of generality, R_1 and R_2) do not commute and hence the flows never become zero.

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