Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

RALF BORNDÖRFER MARIKA NEUMANN MARC E. PFETSCH

Optimal Fares for Public Transport

Supported by the DFG Research Center MATHEON “Mathematics for key technologies” in Berlin

ZIB-Report 05-35 (August 2005)

Optimal Fares for Public Transport*

Ralf Borndörfer   Marika Neumann   Marc E. Pfetsch

August 1, 2005

Abstract

The *fare planning problem* for public transport is to design a system of fares that maximize the revenue. We introduce a nonlinear optimization model to approach this problem. It is based on a discrete choice logit model that expresses demand as a function of the fares. We illustrate our approach by computing and comparing two different fare systems for the intercity network of the Netherlands.

1 The Fare Planning Problem

The influence of fares on passenger behavior and revenues is traditionally studied from a macroscopic point of view. Classical topics are the analytic study of equilibria [7], price elasticities [3], and the prediction of passenger behavior [1]. The only approaches to fare optimization on a more detailed level that we are aware of are the work of Hamacher and Schöbel [5] on the optimal design of fare zones and the work of Kocur and Hendrikson [6] and De Borger et al. [4] who introduced a model for maximizing the revenue and the social welfare, respectively. In contrast to these approaches, our model for fare optimization takes different origins and destinations of travel into account, i.e., we consider a "network effect". Our aim in this article is to show that such a model can be a versatile tool for optimizing fare systems. While the general model has been introduced in [2], we focus here on the comparison of two examples.

Consider a traffic network with nodes (stations) V, *origin-destination pairs* (OD-pairs) $D \subseteq V \times V$, and a finite set $\mathcal{C}$ of *travel choices*; for examples see Section 2.1. Let $\boldsymbol{x} \in \mathbb{R}^n_+$ be a vector of fare variables $x_1, \ldots, x_n$, which we call *fares* in the following. Fares can be restricted to a polyhedron $P \subseteq \mathbb{R}^n_+$. Further, let $p^i_{st} : \mathbb{R}^n \to \mathbb{R}_+$, $\boldsymbol{x} \mapsto p^i_{st}(\boldsymbol{x})$ be the *price* for traveling from s to t and travel choice $i \in \mathcal{C}$. Similarly, let $d^i_{st}(\boldsymbol{x})$ determine the *demand* of

*Supported by the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin.
Address of the authors: Konrad-Zuse-Zentrum für Informationstechnik Berlin, Takustr. 7, 14195 Berlin, Germany; Email: `{borndoerfer, marika.neumann, pfetsch}@zib.de`

passengers for this combination. In our examples, demand functions and price functions are differentiable and $P = \mathbb{R}^n_+$. Given fares $\boldsymbol{x}$, the *revenue* $r(\boldsymbol{x})$ is:

$$r(\boldsymbol{x}) := \sum_{i \in \mathcal{C}} \sum_{(s,t) \in D} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) \,.$$

Our general model for the fare planning problem reads:

$$\text{(FPP)} \qquad \begin{array}{rl} \max & r(\boldsymbol{x}) \\ s.t. & \boldsymbol{x} \in P \,. \end{array}$$

(FPP) is a nonlinear program that may be quite hard to solve in general.

2 Discrete Choice Demand Functions

We use a discrete choice logit model to obtain realistic demand functions d^i_{st}. Our exposition assumes that the reader is familiar with such a construction. We refer to Ben-Akiva and Lerman [1] for a thorough exposition.

The model is as follows. A passenger traveling from s to t performs a random number of trips $X_{st} \in \mathbb{Z}_+$ during a time horizon T, i.e., X_{st} is a discrete random variable. We assume that $X_{st} \leq N$ and that the same travel alternative is chosen for all trips, i.e., passengers do not mix alternatives. For these trips, a passenger chooses among a finite set A of *alternatives* for the travel mode, e.g., single ticket, monthly ticket, bike, car travel, etc.

Associated with each alternative $a \in A$ and OD-pair $(s,t) \in D$ is a random *utility* variable U^a_{st} which may depend on the passenger. Each utility is the sum of an observable part, the *deterministic utility* V^a_{st}, and a random *disturbance term* ν^a_{st}. We consider the utility $U^a_{st}(\boldsymbol{x},k) = V^a_{st}(\boldsymbol{x},k) + \nu^a_{st}$, which depends on the fare system $\boldsymbol{x}$ and the number of trips k. Assuming that each passenger chooses the alternative with the highest utility, the probability of choosing alternative $a \in A$ (for given $\boldsymbol{x}$ and k) is

$$P^a_{st}(\boldsymbol{x},k) := \mathbb{P}\big[V^a_{st}(\boldsymbol{x},k) + \nu^a_{st} = \max_{b \in A}(V^b_{st}(\boldsymbol{x},k) + \nu^b_{st})\big].$$

In a *logit model*, the ν^a_{st} are Gumbel distributed and the probability for choosing alternative a for $(s,t) \in D$ can explicitly computed by the formula (see [1]):

$$P^a_{st}(\boldsymbol{x},k) = \frac{e^{\mu V^a_{st}(\boldsymbol{x},k)}}{\sum\limits_{b \in A} e^{\mu V^b_{st}(\boldsymbol{x},k)}} \,.$$

Here $\mu > 0$ is a scaling parameter for the disturbance terms ν^a_{st}.

We derive demand functions for (FPP) from this discrete choice model by defining the travel choices as $\mathcal{C} = A \times \{1, \ldots, N\}$ and setting

$$d^{a,k}_{st}(\boldsymbol{x}) \;=\; d_{st} \cdot P^a_{st}(\boldsymbol{x},k) \cdot \mathbb{P}[X_{st} = k], \tag{1}$$

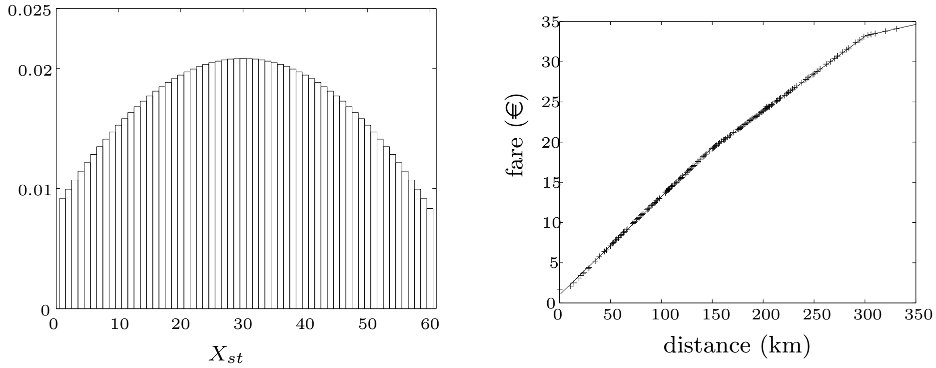

Figure 1: *Left:* Probabilities for the discrete random variables X_{st} representing the number of trips. *Right:* Samples for the fare system of NS Reizigers and corresponding piecewise linear approximation (with three pieces).

where d_{st} is the number of passengers that travel from s to t. The *expected* revenue (over the probability spaces for X_{st} and ν_{st}^a) can then be written as:

$$r(\boldsymbol{x}) = \sum_{a \in A'} \sum_{k=1}^{N} \sum_{(s,t) \in D} p_{st}^{a,k}(\boldsymbol{x}) \cdot d_{st}^{a,k}(\boldsymbol{x}),$$

where A' is the set of public transport alternatives and $p_{st}^{a,k}(\boldsymbol{x})$ is the price function. Note that $r(\boldsymbol{x})$ is differentiable if V_{st}^a and $p_{st}^{a,k}$ have this property; compare the examples in the next section.

2.1 Two Examples

We will demonstrate our approach by two examples. In the first example we work with alternatives "standard ticket" (S), "reduced ticket" (R), and "car" (C), i.e., $A = \{S, R, C\}$. In the second example we work with alternatives "monthly ticket" (M), "single ticket" (S), and "car" (C), i.e., $A = \{M, S, C\}$.

Both examples use a time horizon T of one month. We set the scaling parameter to $\mu = 0.01$. The (discrete) probabilities for the number of trips X_{st} are defined using the function $1 - \frac{1}{1500} \cdot (k - 30)^2$ and normalizing. The resulting probabilities are independent of the OD-pair $(s,t) \in D$ and are centered around 30 in an interval from 1 to $N := 60$, see Figure 1.

Our data for the intercity network of the Netherlands is taken from a publicly available GAMS model by Bussieck (`www.gams.com/modlib/libhtml/lop.htm`). It consists of a network containing 23 nodes (stations) and a corresponding upper-triangular origin-destination matrix (d_{st}^0) with 210 nonzero entries that account for a symmetric bidirectional traffic. We added to this data the currently valid fares, distances, and travel times taken from the internet site of the railway company NS Reizigers (`www.ns.nl`). It turns out that these fares are determined by a piecewise linear function with three

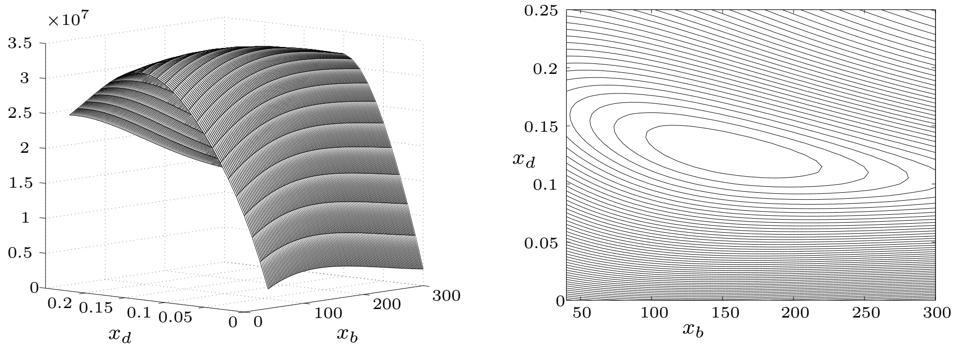

Figure 2: Example 1. *Left:* Total revenue. *Right:* A contour plot of the total revenue. The optimum is at $x_b \approx 153.31$ and $x_d \approx 0.13$.

pieces depending on distance, see Figure 1. With these data, the total demand is 91,791 and the current total revenue is 860,991 € per day.

Distances and travel times for alternative "car" were obtained from the routing planer Map24 (`www.map24.com`); we used the quickest route between the corresponding train stations. The price for alternative "car" is the sum of a fixed cost Q and distance dependent operating costs q, i.e., $p_{st}^{C,k}(\boldsymbol{x}) = Q + q \cdot \ell_{st}^{c} \cdot k$, which is constant; here, ℓ_{st}^{c} denotes the distance between s and t in kilometers for a car. We set $Q = 100$ € and $q = 0.1$ €.

We extrapolated the OD-matrix (d_{st}^0) in order to also include car traffic as follows. Using alternatives "car" as above and alternative "NS Reizigers" (with the current fares and travel times), we estimated for each OD-pair the percentage q_{st} of passengers using public transport applying (1) with $k = 30$. The total number of travelers between s and t is then $d_{st} = 100 \cdot d_{st}^0 / q_{st}$. The total number of passengers in (d_{st}) is 184,016.

2.1.1 Example 1: Standard Ticket, Reduced Ticket, and Car

We consider two fares x_d and x_b (hence $n = 2$). Namely, x_d is a distance fare per kilometer for standard tickets, and x_b is a basic fare that has to be paid once a month in order to buy reduced tickets that provide a 50% discount in comparison to standard tickets. We write $\boldsymbol{x} = (x_b, x_d)$ and set the prices for alternatives standard and reduced ticket to $p_{st}^{S,k}(\boldsymbol{x}) = x_d \cdot \ell_{st} \cdot k$ and $p_{st}^{R,k}(\boldsymbol{x}) = x_b + \frac{1}{2}\, x_d \cdot \ell_{st} \cdot k$, respectively, where ℓ_{st} denotes the shortest distance in the public transport network between s and t in kilometers.

We assume that the utilities are affine functions of prices and travel times t_{st}^a between s to t with alternative a. The utilities depend on the

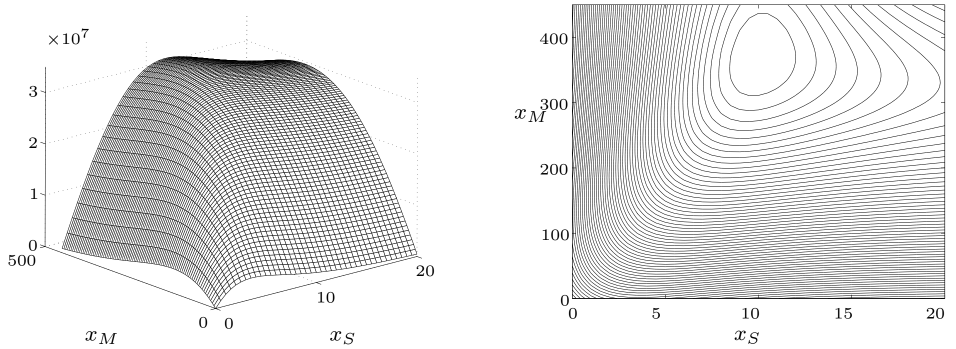

Figure 3: Example 2. *Left:* Total revenue. *Right:* A contour plot of the total revenue. The optimum is at $x_S \approx 10.54$ and $x_M \approx 368.85$.

number of trips k. More precisely, we set:

$$\begin{aligned}
U_{st}^S(x_b, x_d, k) &= -\delta_1 \cdot x_d \cdot \ell_{st} \cdot k - \delta_2 \cdot t_{st}^S \cdot k + \nu_{st}^S && \text{“standard ticket”}\\
U_{st}^R(x_b, x_d, k) &= -\delta_1 \left(x_b + \tfrac{1}{2}\, x_d \cdot \ell_{st} \cdot k\right) - \delta_2 \cdot t_{st}^R \cdot k + \nu_{st}^R && \text{“reduced ticket”}\\
U_{st}^C(x_b, x_d, k) &= -\delta_1 \left(Q + q \cdot \ell_{st}^c \cdot k\right) - \delta_2 \cdot t_{st}^C \cdot k + \nu_{st}^C && \text{“car”.}
\end{aligned}$$

Here, δ_1 and δ_2 are weight parameters; we use $\delta_1 = 1$ and $\delta_2 = 0.1$, i.e., 10 minutes of travel time are worth 1 €.

Altogether, the fare planning problem we want to consider has the form:

$$\begin{aligned}
\max \quad & \sum_{k=1}^{N} \sum_{(s,t)\in D} d_{st} \cdot \frac{\mathbb{P}[X_{st} = k]}{\displaystyle\sum_{b\in A} e^{\mu V_{st}^b(\boldsymbol{x},k)}} \cdot \Big[(x_d \cdot \ell_{st} \cdot k) \cdot e^{\mu V_{st}^S(\boldsymbol{x},k)} + \\
& \qquad\qquad\qquad\qquad (x_b + \tfrac{1}{2}\, x_d \cdot \ell_{st} \cdot k) \cdot e^{\mu V_{st}^R(\boldsymbol{x},k)} \Big] \\
& s.t. \quad \boldsymbol{x} \geq \mathbf{0}.
\end{aligned} \tag{2}$$

Note that the revenue function is differentiable.

2.1.2 Example 2: Single Ticket, Monthly Ticket, and Car

We consider the fares x_M (for the monthly ticket) and x_S (for the single ticket) and write $\boldsymbol{x} = (x_M, x_S)$. We set the cost for alternative “monthly ticket” and “single ticket” to $p_{st}^{M,k}(\boldsymbol{x}) = x_M$ and $p_{st}^{S,k}(\boldsymbol{x}) = x_S \cdot k$, respectively.

Analogously to the previous example we set the utility function as follows:

$$\begin{aligned}
U_{st}^M(x_M, x_S, k) &= -\delta_1 \cdot x_M - \delta_2 \cdot t_{st}^M \cdot k + \nu_{st}^M && \text{“monthly ticket”}\\
U_{st}^S(x_M, x_S, k) &= -\delta_1 \left(x_S \cdot k\right) - \delta_2 \cdot t_{st}^S \cdot k + \nu_{st}^S && \text{“single ticket”}\\
U_{st}^C(x_M, x_S, k) &= -\delta_1 \left(Q + q \cdot \ell_{st}^c \cdot k\right) - \delta_2 \cdot t_{st}^C \cdot k + \nu_{st}^C && \text{“car”.}
\end{aligned}$$

Here again, we use $\delta_1 = 1$ and $\delta_2 = 0.1$.

Table 1: Comparison of the results of Example 1 ("standard ticket, reduced ticket, and car") and Example 2 ("single ticket, monthly ticket, and car").

	revenue	demand	modal split				
Status quo	25,829,730.0	91,791	50.1%				
Example 1	34,201,767.8	126,786	68.9%	standard:	37.1%	reduced:	31.8%
Example 2	31,813,156.4	110,999	60.3%	single:	35.4%	monthly:	24.9%

Altogether the fare planning program for this example is

$$\begin{aligned} \max \quad & \sum_{k=1}^{N} \sum_{(s,t)\in D} d_{st} \cdot \frac{x_M \cdot e^{\mu V_{st}^M(\boldsymbol{x},k)} + x_S \cdot k \cdot e^{\mu V_{st}^S(\boldsymbol{x},k)}}{\sum\limits_{b\in A} e^{\mu V_{st}^b(\boldsymbol{x},k)}} \cdot \mathbb{P}[X_{st} = k] \\ \text{s.t.} \quad & \boldsymbol{x} \geq \mathbf{0}. \end{aligned} \tag{3}$$

2.1.3 Results

We solved models (2) and (3) using a Newton-type method in Matlab 7 and confirmed the results by the Nelder-Mead method. The optimal fares for Example 1 are $x_b \approx 153.31$ € and $x_d \approx 0.13$ €; see also Figure 2. The optimal fares for Example 2 are $x_S \approx 10.54$ € for the single ticket and $x_M \approx 368.85$ € for the monthly ticket; see also Figure 3. Table 1 compares revenue (per month), demand (per day), and modal split (percentage of passengers using public transport and the corresponding alternatives, respectively).

In Example 1, alternatives "standard" and "reduced ticket" attract more passengers for every OD-pair than the current fare system. The "reduced ticket", in particular, is used by passengers who often travel long distances. Similarly, in Example 2, passengers traveling long distances buy a "single ticket" if the number of trips is small and a "monthly ticket" if the number is high. In both examples, under the assumptions of this paper, optimized fares result in a higher revenue and larger demand than in the status quo, that is, we have managed to attract additional passengers to public transport and at the same time improved revenue.

References

[1] M. Ben-Akiva and S. R. Lerman, *Discrete Choice Analysis: Theory and Application to Travel Demand*, MIT-Press, Cambridge, 1985.

[2] R. Borndörfer, M. Neumann, and M. E. Pfetsch, *Fare planning for public transport*, Report 05-20, ZIB, 2005. `http://www.zib.de/Publications/abstracts/ZR-05-20`.

[3] J. F. Curtin, *Effect of fares on transit riding*, Highway Research Record 213, Washington D.C., 1968.

[4] B. De Borger, I. Mayeres, S. Proost, and S. Wouters, *Optimal pricing of urban passenger transport, A simulation exercise for Belgium*, J. Transport Economics and Policy, 30 (1996), pp. 31–54.

[5] H. W. Hamacher and A. Schöbel, *On fair zone designs in public transportation*, in Proc. of CASPT, vol. 430 of LNEMS, Springer-Verlag, 1995, pp. 8–22.

[6] G. Kocur and C. Hendrickson, *Design of local bus service with demand equilibrium*, Transportation Sci., 16 (1982), pp. 149–170.

[7] P. A. Pedersen, *On the optimal fare policies in urban transportation*, Transportation Res. Part B, 37 (2003), pp. 423–435.